Canonical Forms for Reachable Systems over Von Neumann Regular Rings

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Abstract: If \((A, B)\) is a reachable linear system over a commutative von Neumann regular ring \(R\), a finite collection of idempotent elements is defined, constituting a complete set of invariants for the feedback equivalence. This collection allows us to construct explicitly a canonical form. Relations are given among this set of idempotents and various other families of feedback invariants. For systems of fixed sizes, the set of feedback equivalent classes of reachable systems is put into 1-1 correspondence with an appropriate partition of \(\text{Spec}(R)\) into open and closed sets. Furthermore, it is proved that a commutative ring \(R\) is von Neumann regular if and only if every reachable system over \(R\) is a finite direct sum of Brunovsky systems, for a suitable decomposition of \(R\).

Keywords: systems over commutative rings; feedback equivalence; von neumann regular rings

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1. Introduction

Many systems in physics and engineering are described by a state vector \(x\) and an input or control vector \(u\), together with a linear evolution equation in time \(t\). Depending on whether \(t\) is considered continuous or discrete, this equation can be, respectively:

\[
x(t) = Ax(t) + Bu(t) \quad \text{or} \quad x(t+1) = Ax(t) + Bu(t),
\]

for matrices \(A, B\) of suitable sizes. In the classical Control Theory, \(A\) and \(B\) have coefficients in \(\mathbb{R}, \mathbb{C}\) or some other field \(\mathbb{F}\) (see [1,2]). More generally, scalars can be taken in a commutative ring \(R\), which is a useful tool for studying delay differential equations, families of systems depending smoothly on a parameter, digital systems or coding theory, among other applications (see [3–6]). As a concrete example, taking scalars in the polynomial ring \(\mathbb{R}[\delta]\) is a suitable model for studying differential equations when a delay \(\delta\) is introduced [5], which has applications in various areas of life sciences, for example, population dynamics, epidemiology, immunology, physiology, and neural networks. This leads to the following definition (see [1,3]).

Definition 1. Let \(R\) be a commutative ring with unit 1.

- An \(m\)-input, \(n\)-dimensional linear dynamical system over \(R\), or shortly a system of size \((n,m)\), is a pair of matrices \(\Sigma = (A, B)\), where \(A = (a_{ij}) \in R^{n \times n}\) and \(B = (b_{ij}) \in R^{n \times m}\).
- The system \(\Sigma\) is reachable or controllable if the columns of \([B|AB| \ldots |A^{n-1}B]\) span \(R^n\).
- Two systems \((A, B)\) and \((A', B')\) of size \((n,m)\) are feedback equivalent if for some (non necessarily unique) matrices \(P, Q, K\) of appropriate sizes, with \(P, Q\) invertible, one has \((A', B') = (P(A + BK)P^{-1}, PBQ)\). In this case, there exist matrices \(P' = P^{-1}\), \(Q' = Q^{-1}\) and \(K' = -Q^{-1}KP^{-1}\) such that \((A, B) = (P'(A' + B'K')P'^{-1}, P'B'Q')\). Furthermore, given two feedback equivalent systems \(\Sigma, \Sigma'\), one has that \(\Sigma\) is reachable if and only if \(\Sigma'\) is reachable.

If \(R\) is a field, it is known [1] that any reachable system is feedback equivalent to a Brunovsky canonical form, and the Kronecker indices are a complete set of invariants for the feedback equivalence. For an arbitrary commutative ring \(R\), one cannot expect...
every reachable system to have always a Brunovsky canonical form, in fact, this is possible only if \( R \) is a field (see [7], p. 87). For rings \( R \) or reachable systems \((A, B)\) with additional properties, several partial results have been given, see, e.g., [8–10]. In particular, von Neumann regular rings have attracted much attention recently, as a class of rings where things work almost as good as in the case of fields, see [11–15].

Von Neumann regular rings are especially important because of the following motivating situation. When dealing with parameter-depending systems, one has to work typically with systems with scalars in a ring of functions, see [4]. Suppose that the ring \( R \) of invariants ([11], Remark 2.2).

### 2. Preliminaries

Let \( \Sigma = (A, B) \) be a system of size \((n, m)\) over a commutative ring \( R \). For each \( i = 1, \ldots, n \), consider the \( n \times i m \) matrix \( C_i^R = [B|A] \cdots |A^{i-1}|B] \). The image of \( C_i^R \) is denoted by \( N_i^R \) and the quotient \( R^n / N_i^R = M_i^R \). Both families of \( R \)-modules \( \{N_i^R\} \) and \( \{M_i^R\} \) are invariant under feedback ([10], Lemma 2.1), but they do not form a complete set of invariants ([11], Remark 2.2).

We recall here some known facts that will be needed later:

For a general reading of linear systems over commutative rings, see [3]. To see the importance of von Neumann regular rings in systems theory, see [11,13,15] and the references therein.

In Section 4, we extend the results of ([11] Theorem 4.1) and ([15] Theorem 5), by constructing explicitly a canonical form for \( \Sigma \) in terms of the invariant factors \( \{d_{ij}\} \). Furthermore, we characterize von Neumann regular as those commutative rings for which every reachable system is a finite direct sum of Brunovsky systems, the precise formulation is given in Theorem 4. Moreover, for a given von Neumann regular ring \( R \) and for fixed sizes \( n, m \), we establish a 1-1 correspondence between feedback equivalence classes of systems of size \((n, m)\) and certain decompositions of \( \text{Spec}(R) \) into \( p(n, m) \) disjoint open-closed sets, where \( p(n, m) \) is the number of partitions of \( n \) into \( m \) nonnegative parts, see [17]. If \( R \) is Noetherian, then \#Spec \( (R) \) is finite, and the number of feedback classes is \( p(n, m) \#\text{Spec}(R) \), in accordance with ([13], Proposition 4.2). We prove that this result cannot be generalized to the non-Noetherian case.

The paper ends with some examples and conclusions.

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### 2. Preliminaries

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We recall here some known facts that will be needed later:
• (F1) Brunovsky canonical forms. If a system \((A, B)\) of size \((n, m)\) is in Brunovsky canonical form, we can associate to it a partition of the number \(n\) into \(m\) nonnegative parts, i.e., a set of Kronecker indices \(\kappa = (\kappa_1, \ldots, \kappa_m)\), with \(\kappa_1 \geq \cdots \geq \kappa_m \geq 0\) and \(\kappa_1 + \cdots + \kappa_m = n\). If \(s = \max\{i : \kappa_i > 0\}\), we recall the Brunovsky canonical form \((A, B)\) with boxes of sizes \(\kappa_1, \ldots, \kappa_s\):

\[
A = \text{diag}(A_1, \ldots, A_s), \quad B = \text{diag}(B_1, \ldots, B_s, 0],
\]

where \(A_i = [e^{(2)}, \ldots, e^{(s)}], B_i = e^{(1)}\), with \(e^{(j)}\) the \(j\)-th canonical basis vector in \(R^{s_i}\) for \(1 \leq i \leq s\). Note, the \(k\)-th canonical basis vector in \(R^{s_i}\) is the conjugate partition of \(\kappa_i\). It is known that for each \(i\), \(\xi_i\) is the number of indices \(\kappa_i\) of size \(\geq i\) (and, dually, \(\kappa_{ij}\) is the number of \(\kappa_{ij}\)’s which are \(\geq i\)), the sum \(\zeta_i = \xi_1 + \cdots + \xi_i\) is equal to \(\text{rank}(N_{ij}^2)\) (this makes sense because \(N_{ij}^2\) is free, by \([10]\), Proposition 2.2), and the difference \(\zeta_i = \xi_i - \xi_{i+1}\) is the number of \(\kappa_{ij}\)’s of size exactly \(i\). In fact, each of the families of indices \((\kappa_i), (\xi_i), (\zeta_i)\) or \((\xi_i)\) is sufficient to recover the remaining sets. We refer the reader to \([1,9]\) or \([11]\) for more details about these calculations. In order to indicate that certain indices are derived from a partition \(\kappa\), we will use the notation \((\xi_i^\kappa), (\zeta_i^\kappa), (\xi_i^\kappa)\).

• (F2) Idempotents. If \(e\) is an idempotent element of \(R\), the ideal \(eR\) has a ring structure with unit element \(e\). If \(a\) divides \(e\) and \(b\) divides \(e\), then \(ab\) divides \(e\). If \(e\) divides \(c\) then \(ec = c\) and \(e(1-c) = -c\). Two idempotents generating the same principal ideal must be equal. If \(e, f\) are orthogonal idempotents (\(ef = 0\)) then \(e + f\) is idempotent and generates the ideal \(eR + fR\).

• (F3) Von Neumann regular rings. A commutative ring \(R\) is called von Neumann regular (or absolutely flat, \([18]\), Ch. 2, Ex. 27) if for any \(a\) in \(R\) there exists \(x \in R\) such that \(a^2x = a\). Some characterizations of von Neumann regular rings are, among others (see \([19,20]\) for details): every prime ideal is maximal and \(R\) has no nonzero nilpotents (in particular, the nilradical and the Jacobson radical of \(R\) are zero), every element of \(R\) is the product of a unit by an idempotent, every ideal is radical, every localization at a prime ideal is a field, every finitely generated ideal is principal with an idempotent generator. The most important examples of von Neumann regular rings in systems theory are products of fields (e.g., \(\mathbb{Z}_n\), where \(n\) is a squarefree integer), and rings of continuous functions \(C(X, \mathbb{R})\), where \(X\) is a \(P\)-space \([16]\).

2.1. The Partition of the Spectrum

Let \(R\) be a von Neumann regular ring and consider a reachable system \(\Sigma = (A, B)\) over \(R\). Since every prime ideal is maximal, \(\text{Spec}(R)\) coincides with the set of maximal ideals, \(\text{Max}(R)\). For every maximal ideal \(m\) of \(R\), consider the system \(\Sigma(m)\) over the residue field \(R/m\) obtained via the natural map \(R \rightarrow R/m\). By \([1]\), \(\Sigma(m)\) is feedback equivalent to a Brunovsky canonical form given by some Kronecker indices \(\kappa\). We view \(\kappa\) as an element of the set \(P(n, m)\) of partitions of \(n\) with \(m\) parts \(\geq 0\), whose cardinality is \(p(n, m)\) \([17]\). The assignment \(m \mapsto \kappa\) gives a map

\[
\varphi_{\Sigma} : \text{Spec}(R) \rightarrow P(n, m).
\]

As \(\kappa\) varies in \(P(n, m)\), the sets \(\varphi_{\Sigma}^{-1}(\kappa)\) form a set partition of \(\text{Spec}(R)\) into \(p(n, m)\) parts (some of them may possibly be empty).

2.2. The Invariant Factors

Recall that von Neumann regular rings are elementary divisor rings \([21]\). For each \(i = 1, \ldots, n\), the matrix \(G^\kappa_i = [B|AB| \cdots |A^{i-1}B]\) is equivalent to a Smith normal form \(\text{diag}\{d_{ij}, d_{ij}, \ldots, d_{ij}\}\), with the divisibility conditions \(d_{ij} | d_{ij} | \cdots | d_{ij}\) (by convention \(d_{ij} = 0\) if \(j > \min\{m, n\}\)). For each \(i, j\), the element \(d_{ij}\) can be assumed to be idempotent, because every element is the product of a unit by an idempotent \([21]\). Furthermore, the \(j\)-th determinantal ideal of the matrix \(G^\kappa_i\) is generated by the product \(d_{ij} \cdots d_{ij}\), which is equal to \(d_{ij}\) (apply repeatedly (F2)). This way, we associate to each reachable system \(\Sigma\) a collection
of idempotents \( \{d_{ij}\} \), for \( i, j = 1, \ldots, n \), which we shall call the invariant factors. If \( \Sigma \) is a Brunovsky system, then all the ideals \( \mathcal{U}_i(G_\Sigma^\Sigma) \) are \((0)\) or \( R \) (see [10]), hence each \( d_{ij} \) is 0 or 1, accordingly.

The feedback equivalence class of a reachable system can be completely determined either by the previously defined map \( \varphi_\Sigma \) or by the invariant factors. The next result confirms this, by gathering some facts essentially contained in [10,11,15].

**Proposition 1.** Let \( R \) be a commutative von Neumann regular ring and let \( \Sigma = (A, B) \) and \( \Sigma' = (A', B') \) be two reachable systems over \( R \), with associated invariant factors \( \{d_{ij}\} \) and \( \{d'_{ij}\} \), respectively, defined as above. Then, the following statements are equivalent.

(i) \( \Sigma, \Sigma' \) are feedback equivalent over the ring \( R \).

(ii) \( \Sigma(m), \Sigma'(m) \) are feedback equivalent over the field \( R / m \), for every maximal ideal \( m \) of \( R \), or equivalently, the maps \( \varphi_\Sigma \) and \( \varphi_{\Sigma'} \) are equal.

(iii) \( \Sigma, \Sigma' \) have equal invariant factors, i.e., \( d_{ij} = d'_{ij} \) for all \( i, j = 1, \ldots, n \).

(iv) The \( R \)-modules \( N_\Sigma^\Sigma, N_{\Sigma'}^\Sigma \) are isomorphic for all \( i = 1, \ldots, n \).

**Proof.** The equivalence \( (i) \Leftrightarrow (ii) \) is proved in ([11], Theorem 4.1), and \( (i) \Leftrightarrow (iv) \) follows from ([10], Lemma 2.1) and ([15], Proposition 7).

\( (ii) \Rightarrow (iii) \): By ([11] Theorem 3.2), the ideals \( \mathcal{U}_i(G_\Sigma^\Sigma), \mathcal{U}_i(G_{\Sigma'}^\Sigma) \) have the same radical for all \( i, j \), but every ideal of \( R \) is radical [19], hence \( d_{ij}R = d'_{ij}R \) for all \( i, j \), and \( (iii) \) follows from (F2).

\( (iii) \Rightarrow (iv) \): Since the matrices \( G_\Sigma^\Sigma, G_{\Sigma'}^\Sigma \) have equal Smith normal forms, they are equivalent, and hence their image modules are isomorphic. \( \square \)

### 3. The Idempotent Decomposition

Throughout this section \( R \) will be a commutative von Neumann regular ring and \( \Sigma = (A, B) \) is a reachable system of size \((n, m)\) over \( R \). With the two previous sets of invariants, we are able to characterize exactly when \( \Sigma \) is feedback equivalent to a particular canonical form \( \kappa \): for all \( i \), the first \( r_i^\Sigma \) invariant factors are 1 and the remaining 0, that is to say:

\[
\Sigma \text{ is feedback equivalent over } R \text{ to the canonical form } \kappa \text{ if and only if } d_{ij} = 1 \text{ for all } j \leq r_i^\Sigma, \text{ otherwise } d_{ij} = 0. \tag{2}
\]

For each maximal ideal \( m \), the elements \( \{d_{ij} \mod m\} \) are the idempotent invariant factors of \( \Sigma(m) \) over the field \( R / m \), which has no nontrivial idempotents, therefore it is clear that:

\[
\Sigma(m) \text{ is feedback equivalent over } R / m \text{ to the canonical form } \kappa \text{ if and only if } d_{ij} \equiv 1 \mod m \text{ for all } j \leq r_i^\Sigma, \text{ otherwise } d_{ij} \equiv 0 \mod m. \tag{3}
\]

As before, let \( \Sigma = (A, B) \) be a reachable system over a von Neumann regular ring \( R \). In ([15], Theorem 5 and Corollary 6) a family \( \{e_i\} \) of orthogonal idempotents with sum 1 is obtained, such that for each \( i \), the system \( e_i \Sigma \) is feedback equivalent over the ring \( e_iR \) to a Brunovsky canonical form. However, the construction of these idempotents is the result of an inductive procedure, and no explicit formula is given in terms of the invariant factors \( \{d_{ij}\} \) or the spectrum partition. We shall fill this gap soon, defining the idempotents in a very natural way, and relating them to the other families of invariants.

Let \( e \in R \) be an idempotent. Since the ring \( eR \) is von Neumann regular, in the same manner as in (2) and (3), we have that:

\[
e \Sigma \text{ is feedback equivalent over } eR \text{ to the canonical form } \kappa \text{ if and only if } ed_{ij} = e \text{ for all } j \leq r_i^\Sigma, \text{ otherwise } ed_{ij} = 0. \tag{4}
\]
A natural way of obtaining (4) for some $i$ is by making $\epsilon$ multiple of $d_{i,r_i^\epsilon + 1}$ and orthogonal to $d_{i,r_i^\epsilon + 1}$. Consider the difference $d_{i,r_i^\epsilon} - d_{i,r_i^\epsilon + 1}$, which by (F2) can also be expressed as $d_{i,r_i^\epsilon}(1 - d_{i,r_i^\epsilon + 1})$: such an element is a multiple of $d_{i,r_i^\epsilon}$ (and hence a multiple of all $d_{ij}$ with $j \leq r_i^\epsilon$), and is orthogonal to $d_{i,r_i^\epsilon + 1}$ (and hence orthogonal to all $d_{ij}$ with $j > r_i^\epsilon$). Analogous conditions should hold for all $i$.

This motivates us to define for each partition $\kappa \in P(n,m)$ the following idempotent:

$$e_\kappa = \prod_{j=1}^n (d_{i,j^\kappa} - d_{i,j^\kappa + 1})$$  \hspace{1cm} (5)

By the above construction, each $e_\kappa$ satisfies (4), and the following theorem confirms that the collection $\{e_\kappa\}$ is a complete set of invariants for the feedback equivalence class of $\Sigma$.

**Theorem 2.** With the preceding notations, the following statements hold.

(i) For each $e_\kappa$ defined as in (5), the system $e_\kappa \Sigma$ is feedback equivalent (over the ring $e_\kappa R$) to the Brunovsky canonical form $\kappa$.

(ii) The elements $e_\kappa$ are pairwise orthogonal and their sum is 1, in particular $R \cong \prod e_\kappa R$.

(iii) Two reachable systems are equivalent if and only if their associated idempotents coincide.

**Proof.** (i) For each partition $\kappa$, by construction the element $e_\kappa$ satisfies (4), therefore $e_\kappa \Sigma$ is equivalent over $e_\kappa R$ to the canonical form $\kappa$.

(ii) If $\kappa, \kappa'$ are two different partitions, by (F1) they must differ in at least one of the cumulative sums $\{r_i\}$. Without loss of generality, we may assume that $r_i^\kappa < r_i^{\kappa'}$. Then, $e_\kappa$ has the factor $1 - d_{i,r_i^\kappa + 1}$ and $e_{\kappa'}$ is a multiple of $d_{i,r_i^\kappa}$, which is a multiple of $d_{i,r_i^\kappa + 1}$, since $r_i^\kappa + 1 \leq r_i^{\kappa'}$. Consequently, $e_\kappa e_{\kappa'} = 0$ because $d_{i,r_i^\kappa + 1}(1 - d_{i,r_i^\kappa + 1}) = 0$.

In order to prove that the elements $e_\kappa$ generate $R$, it suffices to check that no maximal ideal contains all of them. For any maximal ideal $m$, pick $\kappa = \psi_\Sigma(m)$. We claim that $e_\kappa \notin m$. Indeed, by putting $j = r_i^\kappa$ and $j = r_i^{\kappa'} + 1$ in (3) we obtain the following congruences mod $m$, for all $i$:

$$d_{i,r_i^\kappa} - d_{i,r_i^{\kappa'} + 1} \equiv 1 - 0 \equiv 1 \mod m,$$

and therefore by (5) $e_\kappa \equiv 1^n \equiv 1 \mod m$, so $e_\kappa \notin m$, as we claimed.

Finally, by (F2) we have that the element $\sum e_\kappa$ is idempotent and generates the ideal $\sum(e_\kappa R) = 1 \cdot R$, hence it follows that $\sum e_\kappa = 1$.

(iii) It remains to prove that two systems $\Sigma, \Sigma'$ are feedback equivalent if and only if the idempotents constructed from $\Sigma, \Sigma'$ coincide, i.e., $e_\kappa = e_\kappa'$ for all $\kappa$. The ‘only if’ part is a direct consequence of Proposition 1 and (5). Conversely, if $e_\kappa = e_\kappa'$, then $e_\kappa \Sigma, e_\kappa \Sigma'$ are feedback equivalent over the ring $e_\kappa R$ (both with Brunovsky canonical form $\kappa$), therefore reasoning with the direct product $R \cong \prod e_\kappa R$ yields the equivalence of $\Sigma, \Sigma'$ over $R$. $\square$

We are now ready to establish the promised relations among the three sets of invariants.

3.1. From Partition of Spectrum to Invariant Factors

By putting $\kappa = \psi_\Sigma(m)$ in (3), one obtains each $d_{ij}$ by solving a family of congruences:

$$d_{ij} \equiv 1 \mod m \text{ if } j \leq r_i^{\psi_\Sigma(m)}, \text{ otherwise}$$

$$d_{ij} \equiv 0 \mod m.$$  \hspace{1cm} (6)

We already know that a solution exists, now we are able to recover $d_{ij}$ for all $i, j = 1, \ldots, n$. In fact, the solution is unique, because for any two candidates $d_{ij}, d'_{ij}$, their difference must belong to the intersection of all maximal ideals, which by (F3) is zero.
3.2. From Invariant Factors to Partition of Spectrum

Knowing all the values of \(d_{ij}\), one obtains \(\varphi_\Sigma\) as follows:

For each \(m \in \text{Spec}(R)\) and \(i = 1, \ldots, n\), compute \(r_i = \max\{j, d_{ij} \equiv 1 \mod m\}\), then \(\varphi_\Sigma(m) = \kappa\), the unique partition satisfying \(r_i^\kappa = r_i\) for all \(i = 1, \ldots, n\). (7)

3.3. From Partition of Spectrum to Idempotent Decomposition

We know from the proof of Theorem 2 that for each maximal ideal \(m\), if we take \(\kappa = \varphi_\Sigma(m)\), then \(e_\kappa \equiv 1 \mod m\). However, if \(\kappa' \neq \kappa\), then from the equality \(e_\kappa e_{\kappa'} = 0\) we get \(e_{\kappa'} \in m\) for all \(\kappa' \neq \kappa\). Consequently, the obtention of each \(e_\kappa\) is performed as follows:

\[ e_\kappa \equiv 1 \mod m \text{ if } \varphi_\Sigma(m) = \kappa, \text{ otherwise } e_\kappa \equiv 0 \mod m. \] (8)

Again, as in (6) we know that there exists a unique solution.

3.4. From Idempotent Decomposition to Partition of Spectrum

Assuming the family \(\{e_\kappa\}\) is known, then \(\varphi_\Sigma(m) = \kappa\) if and only if \(e_\kappa \notin m\), or equivalently

\[ \varphi_\Sigma^{-1}(\kappa) = \{m, e_\kappa \notin m\}. \] (9)

3.5. From Invariant Factors to Idempotent Decomposition

This passage is immediate, by (5):

\[ e_\kappa = \prod_{i=1}^{n} \left( d_{ij}r_i^\kappa - d_{ij}r_i^{\kappa+1} \right) \] (10)

3.6. From Idempotent Decomposition to Invariant Factors

Because \(d_{ij} = d_{ij}(\sum e_\kappa) = \sum(e_\kappa d_{ij})\), by using (4) we immediately obtain

\[ d_{ij} = \sum_{\kappa: j \leq r_i^\kappa} e_\kappa. \] (11)

We close this section by exhibiting the connections among \(\{e_\kappa\}\) and other known families of invariants, which are also complete sets of invariants for the feedback equivalence.

3.7. Relation among Idempotent Decomposition and the R-Modules \(M_i^\Sigma\)

In ([8], Corollary 11), the R-modules \(M_i^\Sigma\) are used to construct a complete set of feedback invariants for special systems. We give here a relation between the sets \(\{M_i^\Sigma\}\) and \(\{e_\kappa\}\).

Assuming the isomorphism class of \(M_i^\Sigma\) is known for all \(i = 1, \ldots, n\), then for every maximal ideal \(m\) we obtain the dimensions of \(M_i^\Sigma(m)\) as \(R/m\)-vector spaces, and by ([11], Proposition 2.5) these numbers determine completely the Brunovsky canonical form of \(\Sigma(m)\). That is to say, we are able to obtain \(\varphi_\Sigma(m)\) for each \(m\), i.e., we reconstruct the partition of the spectrum. From this, we obtain our idempotent decomposition by following (8).

Conversely, each \(M_i^\Sigma\) has the following finite presentation: \(R^m \xrightarrow{G_i^\Sigma} R^n \rightarrow M_i^\Sigma \rightarrow 0\).

Since the matrix \(G_i^\Sigma\) has a Smith normal form \(\text{diag}(d_{1i}, \ldots, d_{ni})\), one has \(M_i^\Sigma \cong \bigoplus_{j=1}^{n} R/d_{ij}R\), where by (11) we know that \(d_{ij} = \sum_{\kappa: j \leq r_i^\kappa} e_\kappa\). Thus, we obtain the structure of the modules \(M_i^\Sigma\) in terms of the idempotents \(\{e_\kappa\}\).
3.8. Relation among Idempotent Decomposition and the R-Modules \( Z^\Sigma_i \)

Assume we know the structure of the R-modules \( Z^\Sigma_i \) (see the precise definition in [9]).

For each maximal ideal \( m \), the \( R/m \)-vector space \( Z^\Sigma_i(m) \) has dimension \( \xi_i = \xi_i - \xi_{i+1} \) (see [9], p. 1137), where \( (\xi_i) \) are the indices of \( \kappa \), the conjugate partition of the canonical form of \( \Sigma(m) \). In particular, by (F1) the Brunovsky canonical form \( \kappa \) can be recovered for each \( m \), i.e., we have determined \( \varphi_{\Sigma}(m) \) and therefore we can construct our idempotents as described in (8).

Conversely, for each \( e_\kappa \) we know that \( e_\kappa \Sigma \) is equivalent over \( e_\kappa R \) to a Brunovsky form \( \kappa \), and so \( Z^\Sigma_i \) must be a free \( e_\kappa R \)-module of rank \( \xi_i \), from which it follows that \( Z^\Sigma_i \cong \bigoplus_k (e_\kappa R)^{\xi_i} \).

4. Canonical Forms and Number of Feedback Classes

The classification of reachable systems over von Neumann regular rings is solved in ([15], Theorem 5 and Corollary 6), as the result of an inductive procedure.

On the other hand, if for a given system \( \Sigma \) we know a canonical form for \( \Sigma(m) \) over each residue field \( R/m \), by ([11], Theorem 4.1) we have enough information to recover the feedback equivalence class of \( \Sigma \) over \( R \). Therefore, it seems natural to try to construct a canonical form for \( \Sigma \) over \( R \) by lifting all the canonical forms modulo \( m \). This would lead us to solving a system of (possibly infinite) congruences. Our next result shows that both procedures in [11,15] “converge” to the same canonical form.

**Theorem 3.** For each partition \( \kappa \), denote by \( (A_\kappa, B_\kappa) \) the Brunovsky canonical form \( \kappa \) over the ring \( R \), so \( (e_\kappa A_\kappa, e_\kappa B_\kappa) \) is the corresponding canonical form over the ring \( e_\kappa R \). Given a system \( \Sigma = (A, B) \) and its associated family of idempotents \( (e_\kappa) \), one has that the matrix pair

\[
\begin{align*}
\big( \hat{A} = \sum_k e_\kappa A_\kappa, & \quad \hat{B} = \sum_k e_\kappa B_\kappa \big)
\end{align*}
\]

is a feedback canonical form for \( (A, B) \).

Moreover, if for each maximal ideal \( m \) we take \( \kappa = \varphi_{\Sigma}(m) \), it follows that

\[
\hat{A} \equiv A_\kappa \mod m, \quad \hat{B} \equiv B_\kappa \mod m.
\]

That is to say, the canonical form \( (\hat{A}, \hat{B}) \) constructed from the idempotent decomposition is exactly the same as the one we would have obtained by solving congruences.

**Proof.** The first part is clear, by applying Theorem 2 and the isomorphism \( R \cong \prod_k e_\kappa R \).

For the second statement, in view of (8), when reducing \( (\hat{A}, \hat{B}) modulo m \), only one \( e_\kappa \) “survives” (the one corresponding to \( \varphi_{\Sigma}(m) = \kappa \)), with \( e_\kappa \equiv 1 \mod m \), and for all \( \kappa' \neq \kappa \) we have \( e'_\kappa \equiv 0 \mod m \), hence (13) holds. \( \square \)

**Remark 1.** As a general strategy, if \#Spec(\( R \)) is finite and sufficiently small compared with the number of partitions \( p(n, m) \), the most efficient method for obtaining canonical forms appears to be the residual approach: work in each residue field \( R/m \), and lift the solutions to \( R \). In all other cases, we recommend to first compute the invariant factors \( d_{ij} \), then define each \( e_\kappa \) as in (10), and finally obtain \( (\hat{A}, \hat{B}) \) as in (12).

**Remark 2** (The cases \( m = 1 \) and \( n = 1 \)). For single-input systems \( (m = 1) \), there exists a unique canonical form \( (\hat{A}, \hat{B}) \) for all reachable systems over arbitrary commutative rings, see, e.g., ([13], Theorem 3.2). With our notations, this corresponds to the unique trivial partition \( \kappa = (n) \) of \( n \) with \( m = 1 \) nonnegative parts, and the associated idempotent is \( e_\kappa = 1 \). Similarly, for one-dimensional reachable systems \((n = 1)\) there is only one canonical form, as can be seen in the proof of ([15], Theorem 5).
Remark 3 (The case \( m = 2 \)). Let \((A, B)\) be a generic reachable system of size \((n, 2)\) over a regular ring \(R\). Note that the only possible partitions of \(n\) into \(m = 2\) parts are \((n - i) + i\), for \(i = 0, \ldots, \alpha = \left\lfloor \frac{n}{2} \right\rfloor\). With the notations of Theorem 2, let us call \(e_i\) the (possibly zero) idempotent associated with the partition \((n - i, i)\). If \(e_0 \neq 0\), the canonical form \((e_0 A, e_0 B)\) over the ring \(e_0 R\) is the classical canonical form for single-input systems, with the second column of \(e_0 B\) zero. For each \(i \geq 1\) with \(e_i \neq 0\), the canonical form \((e_i A, e_i B)\) over \(e_i R\) has Brunovsky boxes of sizes \(n - i\) and \(i\) (see (F1)), \(i.e., e_i B\) has unit elements \(e_i\) in positions \((1, 1)\) and \((n - i + 1, 2)\), and \(e_i A\) has units \(e_i\) all along the main subdiagonal, except in the rows \(1\) and \(n - i + 1\). In particular, for each \(j = 1, \ldots, n\), the sum of the \(j\)-th rows of \(e_i A\) and \(e_i B\) is \(e_i\). Then, \((A, B)\) is feedback equivalent to the superposition of the canonical forms corresponding to these \([n + 2 \choose 2]\) partitions, which gives:

\[
\begin{bmatrix}
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & 1 - e_n & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & 0 & 1 - e_1 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & e_{n} \\
\vdots & \vdots \\
0 & e_{1}
\end{bmatrix}
\]

after making some straightforward simplifications: \(e_i = \sum_{j=0}^{\alpha} e_j = 1\) and \(\sum_{i \neq j} e_i = 1 - e_j\). This completes the case \(m = 2\).

4.1. Characterization of Von Neumann Regular Rings

We know that many properties of linear systems that hold over von Neumann regular rings, actually characterize this class of rings among all commutative rings, see, \(e.g., \) Theorem 4.1 [11], Theorem 6 [12], Theorem 3.2 [13] and Theorem 4 [14]. We will prove that this is also the case with the idempotent decomposition obtained in Theorem 2.

**Theorem 4.** For a commutative ring \(R\) with \(1\), the following statements are equivalent:

(i) \(R\) is von Neumann regular.

(ii) For every reachable system \(\Sigma\), the ring \(R\) is isomorphic to a finite direct product \(\prod_{i=1}^{k} R_i\), such that for each \(i\) the system \(\Sigma_i\) (obtained from \(\Sigma\) via the natural map \(\pi_i : R \to R_i\)) is feedback equivalent over \(R_i\) to a Brunovsky canonical form.

**Proof.** (i) \(\Rightarrow\) (ii) is already proved in Theorem 2.

(ii) \(\Rightarrow\) (i). It is sufficient to prove that for any two elements \(a, b\) in \(R\), the ideal \(I = aR + bR\) is principal and generated by an idempotent element. Consider the reachable system \(\Sigma = (A, B)\) over \(R\) of sizes \((n = 2, m = 3)\) given by the matrices:

\[
A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \end{bmatrix}.
\]

By (ii), there exists an isomorphism \(f : R \to \prod_{i=1}^{k} R_i\). For each \(i = 1, \ldots, t\), if \(1_i\) is the unit of \(R_i\), then \(e_i = f^{-1}(0, \ldots, 0, 1_i, 0, \ldots, 0)\) is an idempotent of \(R\) and \(R_i\) can be identified with the principal ideal \(e_i R\) of \(R\), which is given a ring structure with unit element \(e_i\). In particular, the elements \(e_1, \ldots, e_t\) are orthogonal idempotents with sum 1.

Note that \(U_2(B) = 1\) and for each \(i = 1, \ldots, t\) one has \(U_2(e_i B) = e_i I\), which is equal to \(e_i R\) or \(0\), because the determinantal ideals are invariant under feedback, and in systems
equivalent to a Brunovsky canonical form (see (F1)) they are either zero or the whole ring (see [10]).

Now, we can partition \( \{1, \ldots, l\} \) into a disjoint union of two finite sets \( V, W \), with \( e_i I = e_i R \) for all \( i \in V \) and \( e_i I = (0) \) for all \( i \in W \). If we define \( e = \sum_{i \in V} e_i \), then by (F2) \( e \) is idempotent, and clearly \( e I = e R \). Similarly, \( 1 - e = \sum_{i \in W} e_i \) and \( (1 - e) I = (0) \). Finally, from \( I = e I + (1 - e) I \) we obtain \( I = e R \), proving that \( I \) is principal with an idempotent generator \( e \), as we wanted to prove. □

4.2. Number of Feedback Classes

Let \( R \) be a von Neumann regular ring \( R \) and let \( n, m \) be fixed integers. We denote by \( FE_R(n, m) \) the set of feedback equivalence classes of reachable systems of size \((n, m)\) over \( R \). If either \( n = 1 \) or \( m = 1 \), we know by Remark 2 that all reachable systems are feedback equivalent, so from now on we may assume \( n, m \geq 2 \). Let \( P(n, m) \) be the set of partitions of \( n \) with \( m \) parts \( \geq 0 \), whose cardinality is \( p(n, m) \). Furthermore, denote by \( P(n, m)^{\text{Spec}(R)} \) the set of all possible maps from \( \text{Spec}(R) \) to \( P(n, m) \). With these notations, we define:

\[
\psi : FE_R(n, m) \rightarrow P(n, m)^{\text{Spec}(R)}
\]

such that \( \psi(\Sigma) = \varphi_\Sigma \), with \( \varphi_\Sigma \) as in (1). Note that \( \psi \) is well defined and injective, because of the equivalence (i) \( \Leftrightarrow \) (ii) in Proposition 1. Furthermore, by Theorem 2 we can put \( FE_R(n, m) \) into 1-1 correspondence with the subsets of \( p(n, m) \) idempotents of \( R \) with the property of being pairwise orthogonal with sum 1.

On the other hand, in (9) we saw that for a given partition \( \kappa \), \( \varphi_\Sigma^{-1}(\kappa) \) is equal to \( \{m, e, \notin m\} \), i.e., a basic open set in the Zariski topology, this set is denoted by \( D(e) \) or \( X(e) \) (see [18]). Moreover, by ([18], Ch. 3, Ex. 11), the set \( D(e) \) is also closed; therefore, the set partition of \( \text{Spec}(R) \) cannot be arbitrary, it must be a disjoint union of sets which are closed and open. This raises the question of when \( \psi \) is bijective.

When \( R \) is Noetherian, an immediate adaptation of ([13], Proposition 4.2) shows that \( \psi \) is bijective, and the number of feedback classes of systems of size \((n, m)\) is equal to \( p(n, m)^{\text{Spec}(R)} \). As we will see, the above cited result of [13] is the best possible, in the following sense: if \( R \) is not Noetherian, we cannot expect \( \psi \) to be bijective, nor the number of feedback equivalence classes to be finite.

Theorem 5. With the preceding notations, the following statements are equivalent.

(i) \( \text{The map } \psi \text{ is bijective.} \)

(ii) \( \text{Spec}(R) \) is a discrete topological space, with the Zariski topology.

(iii) \( \text{Spec}(R) \) is finite.

(iv) \( \text{The cardinality of } FE_R(n, m) \text{ is finite.} \)

(v) \( \text{R is a Noetherian ring.} \)

Proof. (i) \( \Rightarrow \) (ii): If (i) is true, then for any maximal ideal \( m \) of \( R \) there exists a reachable system \( \Sigma \) for which \( \varphi_\Sigma^{-1}(\kappa) \) is the one-point set \( \{m\} \) of \( \text{Spec}(R) \). By (9), there exist an idempotent \( e = e_\kappa \) such that \( \varphi_\Sigma^{-1}(\kappa) = D(e_\kappa) \), i.e., the set \( \{m\} \) is open, and (ii) follows.

(ii) \( \Rightarrow \) (iii): Since \( \text{Spec}(R) \) is quasi-compact [18], Ch. 1, Ex. 17.

(iii) \( \Rightarrow \) (iv): Since \( P(n, m) \) is finite and \( \psi \) is injective, then assuming (iii) it is clear that \( \#FE_R(n, m) \leq p(n, m)^{\text{Spec}(R)} < \infty \).

(iv) \( \Rightarrow \) (v): In this case the number of possible idempotent decompositions is finite, hence in particular \( R \) can only have a finite number of idempotents. However, every element of \( R \) is the product of a unit by an idempotent, which means that every ideal of \( R \) is finitely generated by some idempotent elements, i.e., \( R \) is Noetherian.

(v) \( \Rightarrow \) (i): By ([13] Proposition 4.2), \( \#FE_R(n, m) = p(n, m)^{\text{Spec}(R)} \), which in turn is equal to the cardinality of \( P(n, m)^{\text{Spec}(R)} \), and (i) holds. □
Remark 6. For small values of $n, m$, sometimes we are able to describe completely all canonical forms, without determining the structure of $\text{Spec}(R)$.

For example, if $n = 3, m = 2$, by Remark 3 the canonical forms are

$$
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 - e_1 & 0
\end{pmatrix},
\quad
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & e_1
\end{pmatrix},
$$

exactly one for each idempotent $e_1$. Thus, the set of feedback equivalent classes of systems of size $(3, 2)$ is in 1-1 correspondence with the idempotent algebra of $R$, which is the whole ring $R$, if $R$ is boolean. An example of an infinite boolean ring is given, e.g., by any infinite direct product of copies of $\mathbb{Z}_2$, and there is one canonical form for each element of $R$.

The following example illustrates many of our previous results.

Example 1 (Determining explicitly the canonical form). Consider the system $\Sigma$ of size $(6, 4)$ over the ring $R = \mathbb{Z}/210\mathbb{Z}$, given by

$$
A = \begin{bmatrix}
18 & 164 & 148 & 32 & 148 & 131 \\
92 & 36 & 139 & 156 & 137 & 184 \\
202 & 107 & 107 & 50 & 17 & 59 \\
103 & 50 & 146 & 41 & 195 & 120 \\
100 & 37 & 15 & 171 & 30 & 119 \\
186 & 112 & 69 & 51 & 115 & 116
\end{bmatrix},
\quad
B = \begin{bmatrix}
41 & 26 & 101 & 9 \\
126 & 128 & 86 & 58 \\
86 & 12 & 174 & 140 \\
121 & 164 & 59 & 92 \\
82 & 86 & 8 & 206 \\
8 & 42 & 90 & 200
\end{bmatrix}.
$$

We begin with a straightforward calculation of the Smith normal forms of the matrices $G_\kappa^\Sigma = [B|AB]|\cdots|A^iB]$, for $i = 1, \ldots, n = 6$, which yields the following invariant factors:

<table>
<thead>
<tr>
<th>$G_i$</th>
<th>$d_{i1}$</th>
<th>$d_{i2}$</th>
<th>$d_{i3}$</th>
<th>$d_{i4}$</th>
<th>$d_{i5}$</th>
<th>$d_{i6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>1</td>
<td>1</td>
<td>36</td>
<td>36</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$G_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>106</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>$G_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>$G_4$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>106</td>
</tr>
<tr>
<td>$G_5$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_6$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

There are 9 partitions of the integer $n = 6$ with $m = 4$ nonnegative parts, concretely: \{6000, 5100, 4200, 3300, 4110, 3210, 3111, 2220, 2221\}. Note that by (5), the corresponding $e_\kappa$ will always be zero whenever $d_{i1} r_1 = d_{i1} r_1 + 1$. In the above table, this situations arises if in some row there are two consecutive invariant factors equal. For example, look at the equality $d_{i3} = d_{i4} = 36$. Any partition for which $r_1 = 3$ would force us to choose the zero term $36 - 36$. This occurs, e.g., for $\kappa = (2, 2, 2)$, with dual partition $\kappa^T = (3, 3)$ and $r_1 = 3, r_2 = \cdots = r_6 = 6$.

Now, we show in detail the calculation of some nonzero $e_\kappa$. If $\kappa = (5, 1, 0, 0)$, its dual is $\kappa^T = (2, 1, 1, 1)$ with cumulative sums $r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 5, r_5 = 6, r_6 = 6$. By using (5) and the values of $d_{ij}$ shown before, one has:

$$
e_\kappa = (d_{12} - d_{13})(d_{23} - d_{24})(d_{34} - d_{35})(d_{45} - d_{46})(d_{56} - d_{57})(d_{66} - d_{67}) =
$$

The table in Figure 1 shows the complete calculation of $e_\kappa$ for each partition $\kappa$. As a result, only three nonzero idempotents appear: 105, 70 and 36, associated with the Brunovsky canonical forms with partitions $(5, 1, 0, 0)$, $(3, 3, 0, 0)$ and $(2, 2, 1, 1)$, respectively, of $n = 6$. Thus, by Theorem 3 a
canonical form for \((A, B)\) over \(R\) is obtained by adding the above canonical forms over the rings \(105R, 70R\) and \(36R\):

\[
\begin{bmatrix}
105 & 0 & 0 & 0 \\
36 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + 
\begin{bmatrix}
70 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + 
\begin{bmatrix}
36 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
105 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + 
\begin{bmatrix}
70 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + 
\begin{bmatrix}
36 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

\[
\begin{array}{c|c|c}
\text{Diagram} & (x_i, (r_i)) & \text{Idempotent} \\
\hline
***** & x = 6000, x^T = 111111 & 123456 \\
& x = 5100, x^T = 211110 & 234566 \\
& x = 4200, x^T = 221100 & 245666 \\
& x = 4110, x^T = 311100 & 345666 \\
& x = 3300, x^T = 222000 & 246666 \\
& x = 3210, x^T = 321000 & 356666 \\
& x = 3111, x^T = 411000 & 456666 \\
& x = 2220, x^T = 330000 & 366666 \\
& x = 2211, x^T = 420000 & 466666
\end{array}
\]

Figure 1. Calculation of the \(e_x\) in terms of the \(d_{ij}\) in Example 1.

To conclude this example, note that \(R \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}\), the product of four finite fields. The residual canonical forms are the following:

<table>
<thead>
<tr>
<th>modulo</th>
<th>partition</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,1,0,0)</td>
<td>(3,3,0,0)</td>
<td>(2,2,1,1)</td>
<td>(2,2,1,1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The idempotent \(e\) associated with the partition \((2,2,1,1)\) is, by (8), the solution to the congruences

\[
\{e \equiv 1 \mod 5, \ e \equiv 1 \mod 7, \ e \equiv 0 \mod 2, \ e \equiv 0 \mod 3\},
\]

i.e., \(e \equiv 36 \mod 210\). Similarly, we recover 105 and 70 as the idempotents corresponding to the partitions \((5,1,0,0)\) and \((3,3,0,0)\), respectively.
5. Conclusions

Given a commutative von Neumann regular ring $R$ and a reachable system $\Sigma = (A, B)$ over $R$, we have been able to construct a new complete set of feedback invariants and an explicit canonical form, thus extending the results of [15]. Furthermore, we have clarified the relation between this new family of invariants and other families studied in [8,9,11,15]. Moreover, a characterization of von Neumann regular rings is obtained, in terms of systems over rings, as was done in [11–14]. In addition, the set of feedback equivalent classes of reachable systems of fixed sizes is put into 1-1 correspondence with an appropriate partition of $\text{Spec}(R)$ into open and closed sets. When $R$ is Noetherian, the number of feedback classes is finite, as proved in ([13], Proposition 4.2), and we have shown that an extension of this result to the non Noetherian case is not possible.

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