mathematics

MDPI

Article

On the Boundary Value Problem of Nonlinear Fractional Integro-Differential Equations

Chenkuan Li [1,*], Reza Saadati [2,*], Rekha Srivastava [3] and Joshua Beaudin [1]

[1] Department of Mathematics and Computer Science, Brandon University, Brandon, MB R7A 6A9, Canada; beaudijd31@brandonu.ca
[2] School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 13114-16846, Iran
[3] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; rekhas@math.uvic.ca
[*] Correspondence: lic@brandonu.ca (C.L.); rsaadati@eml.cc (R.S.)

Abstract: Using Banach's contractive principle and the Laray–Schauder fixed point theorem, we study the uniqueness and existence of solutions to a nonlinear two-term fractional integro-differential equation with the boundary condition based on Babenko's approach and the Mittag–Leffler function. The current work also corrects major errors in the published paper dealing with a one-term differential equation. Furthermore, we provide examples to illustrate the application of our main theorems.

Keywords: Liouville–Caputo integro-differential equation; Laray–Schauder fixed point theorem; Banach's contractive principle; Mittag–Leffler function; Babenko's approach

MSC: 34B15; 34A12; 26A33

Citation: Li, C.; Saadati, R.; Srivastava, R.; Beaudin, J. On the Boundary Value Problem of Nonlinear Fractional Integro-Differential Equations. *Mathematics* **2022**, *10*, 1971. https://doi.org/10.3390/math10121971

Academic Editor: Christopher Goodrich

Received: 1 May 2022
Accepted: 5 June 2022
Published: 8 June 2022

1. Introduction

Let $\ell \in N = \{1, 2, 3, \cdots\}$. We consider the uniqueness and existence for the following nonlinear integro-differential Equation (NI-D equation) with the boundary condition:

$$\begin{cases} {}_{LC}D_p^{\kappa}\Psi(\omega) + \mu\, I_p^{\zeta}\Psi(\omega) = \Theta(\omega, \Psi(\omega)), \quad \ell - 1 < \kappa \le \ell,\ \zeta \ge 0,\ \ \omega \in [p, q], \\ \Psi(p) = \Psi'(p) = \cdots = \Psi^{(\ell-2)}(p) = 0,\ \ \psi^{(\ell-1)}(q) = 0, \end{cases} \tag{1}$$

where Ψ is an unknown function, μ is a constant, $0 \le p < q < +\infty$, ${}_{LC}D_p^{\kappa}$ is the Liouville–Caputo fractional derivative of order κ, and I_p^{ζ} is the Riemann–Liouville fractional integral of order ζ. In particular, for $\ell = 1$, Equation (1) was found to be

$$\begin{cases} {}_{LC}D_p^{\kappa}\Psi(\omega) + \mu\, I_p^{\zeta}\Psi(\omega) = \Theta(\omega, \Psi(\omega)), \quad 0 < \kappa \le 1,\ \zeta \ge 0,\ \ \omega \in [p, q], \\ \Psi(q) = 0. \end{cases}$$

Fractional differential and integral equations provide powerful tools in describing and modeling many phenomenons in various fields of science and engineering, such as control theory, porous media, memory and electromagnetics [1–4]. There has been a great deal of research published on the existence and uniqueness of fractional differential and integral equations involving Riemann–Liouville or Liouville–Caputo operators with initial conditions or boundary value problems [5–13].

In 2022, Rezapour et al. [14] investigated the existence of solutions for a category of the multi-point boundary value problem involving a p-Laplacian differential operator with the generalized fractional derivatives depending on another function. The authors in [15] considered the existence, uniqueness and stability of a positive solution in relation to a fractional version of a variable order thermostat model equipped with nonlocal boundary values in the Caputo sense using Guo–Krasnoselskii's fixed point theorem on cones.

In 2021, Turab et al. [16] looked into the existence of solutions for a class of nonlinear boundary value problems on a hexasilinane graph with applications in chemical formulas. The authors in [17] dealt with the existence and Ulam–Hyers stability (UHs) of Caputo-type fuzzy fractional differential equations (FFDEs) with time-delays by applying Schauder's fixed point theorem and a hypothetical condition. In 2017, Sun et al. [18] studied the existence and uniqueness for the following system of FDEs with a boundary value based on Banach's contractive principle (BCP) and the Laray–Schauder fixed point theorem (L-SFTP):

$$\begin{cases} {}_{LC}D_p^{\kappa}\Psi(\omega) = \Theta(\omega, \Psi(\omega)), \quad \ell - 1 < \kappa \leq \ell, \quad \omega \in [p, q], \\ \Psi^{(\rho)}(p) = \omega_\rho, \ \ \rho = 0, 1, 2, \cdots, \ell - 2; \quad \Psi^{(\ell-1)}(q) = \omega_q, \end{cases}$$

where $\omega_0, \omega_1, \cdots, \omega_{\ell-2}, \omega_q$ are real constants and $\Theta : [p, q] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Their work relies on Lemma 2.4 in the paper, which states the following:

Lemma 1. *Let $\ell - 1 < \kappa < \ell$, $\ell = [\kappa] + 1$, where $[\kappa]$ denotes the integer part of κ and let $\Phi : [p, q] \to \mathbb{R}$ be continuous. A function $\Psi(\omega)$ is a solution to the FDE with the boundary value*

$$\begin{cases} {}_{LC}D_p^{\kappa}\Psi(\omega) = \Phi(\omega), \ \ \omega \in [p, q], \\ \Psi^{(\rho)}(p) = \omega_\rho, \ \ \rho = 0, 1, 2, \cdots, \ell - 2; \quad \Psi^{(\ell-1)}(q) = \omega_q, \end{cases} \tag{2}$$

if and only if

$$\begin{aligned} \Psi(\omega) \ &= \ \frac{1}{\Gamma(\kappa)} \int_p^\omega (\omega - \varsigma)^{\kappa-1} \Phi(\varsigma) d\varsigma + \left[\frac{\omega_q}{(\ell-1)!} + \frac{\Phi(p)(q-p)^{\kappa-\ell+1}}{(\ell-2)!\Gamma(\kappa-\ell+2)} \right] (\omega - p)^{\ell-1} \\ &\quad - \frac{(\omega-p)^{\ell-1}}{(\ell-1)!\Gamma(\kappa-\ell+1)} \int_p^q (q-\varsigma)^{\kappa-\ell} \Phi(\varsigma) d\varsigma + \sum_{\rho=0}^{\ell-2} \frac{\omega_\rho}{\rho!} (\omega - p)^\rho. \end{aligned}$$

However, the authors consider this lemma is to be incorrect, and the term

$$\frac{\Phi(p)(q-p)^{\kappa-\ell+1}}{(\ell-2)!\Gamma(\kappa-\ell+2)} (\omega - p)^{\ell-1}$$

should not appear in the lemma. Indeed,

$$\Psi(\omega) = \mu_0 + \mu_1(\omega - p) + \cdots + \mu_{\ell-1}(\omega - p)^{\ell-1} + \frac{1}{\Gamma(\kappa)} \int_p^\omega (\omega - \varsigma)^{\kappa-1} \Phi(\varsigma) d\varsigma,$$

plus the boundary condition

$$\Psi^{(\rho)}(p) = \omega_\rho, \ \ \rho = 0, 1, 2, \cdots, \ell - 2; \quad \Psi^{(\ell-1)}(q) = \omega_q,$$

only implies that

$$\begin{aligned} \mu_\rho &= \frac{\omega_\rho}{\rho!}, \ \ \rho = 0, 1, 2, \cdots, \ell - 2, \\ \mu_{\ell-1} &= \frac{\omega_q}{(\ell-1)!} - \frac{1}{(\ell-1)!\Gamma(\kappa-\ell+1)} \int_p^q (q-\varsigma)^{\kappa-\ell} \Phi(\varsigma) d\varsigma. \end{aligned}$$

To move forward, we begin by introducing several differential and integral operators, a Banach space $C^\ell[p, q]$, the Mittag–Leffler function (the M-L function) as well as Babenko's approach (BA) in Section 2. Then, we present sufficient conditions for the existence and uniqueness of the solutions using the BCP and the L-SFPT, with illustrative examples to show the applications of the main theorems in Section 3. Finally, we summarize the entire paper in Section 4.

2. Preliminaries

We define the Banach space $C^\ell[p,q]$ for $\ell \in N$ as

$$C^\ell[p,q] = \left\{\Psi(\omega) : [p,q] \to \mathbb{R} \text{ such that } \Psi^{(\ell)}(\omega) \text{ is continuous on } [p,q]\right\}$$

with the norm

$$\|\Psi\| = \max_{\omega\in[p,q]} |\Psi(\omega)| < +\infty.$$

Clearly, $C^\ell[p,q] \subset C[p,q]$.

The Riemann–Liouville fractional integral I_p^κ of order $\kappa \in \mathbb{R}^+$ is defined for function $\Psi(\omega)$ as (see [1,2])

$$(I_p^\kappa\Psi)(\omega) = \frac{1}{\Gamma(\kappa)} \int_p^\omega (\omega - v)^{\kappa-1}\Psi(v)dv,$$

if the integral exists. In particular,

$$(I_p^0\Psi)(\omega) = \Psi(\omega),$$

from [19]. In fact,

$$(I_p^0\Psi)(\omega) = \delta(\omega) * \Psi(\omega) = \Psi(\omega),$$

where $\delta(\omega)$ is the Dirac delta function, which is an identity in terms of convolution.

Let $\ell - 1 < \kappa \leq \ell$. The Liouville–Caputo derivative of fractional order $\kappa \in \mathbb{R}^+$ of function $\Psi(\omega)$ is defined as

$$({}_{LC}D_p^\kappa\Psi)(\omega) = I_p^{\ell-\kappa}\frac{d^\ell}{d\omega^\ell}\Psi(\omega) = \frac{1}{\Gamma(\ell-\kappa)} \int_p^\omega (\omega - v)^{\ell-\kappa-1}\Psi^{(\ell)}(v)dv,$$

if the integral exists.

The two-parameter Mittag–Leffler function [3] is defined by

$$E_{\kappa,\zeta}(g) = \sum_{\rho=0}^{\infty} \frac{g^\rho}{\Gamma(\kappa\rho+\zeta)}, \ g \in C, \ \kappa,\zeta > 0.$$

BA [20] is a useful instrument in solving differential and integral equations with initial conditions by treating bounded integral operators as normal variables. The method itself is close to the Laplace transform while dealing with differential equations with constant coefficients, but it can be applied to differential and integral equations with variable coefficients [21,22]. Evidently, it is always necessary to prove the convergence of solution series, otherwise the solution is not well-defined. To demonstrate this technique in detail, we present the following example to solve Abel's integral equation, as well as Lemma 2, which will play an important role in the subsequent section to define the nonlinear mappings.

Consider Abel's integral equation for $\alpha > 0$ and a constant c

$$\Psi(\omega) - \frac{c}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1}\Psi(\tau)d\tau = \Phi(\omega),$$

where Φ is a continuous function. Clearly,

$$\Psi(\omega) - \frac{c}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1}\Psi(\tau)d\tau = (1 - cI_0^\alpha)\Psi(\omega) = \Phi(\omega).$$

Treating the factor $(1 - cI_0^\alpha)$ as a normal variable, we come to

$$\begin{aligned}
\Psi(\omega) &= (1 - cI_0^\alpha)^{-1}\Phi(\omega) = \sum_{k=0}^{\infty} c^k I_0^{\alpha k}\Phi(\omega) = \Phi(\omega) + \sum_{k=1}^{\infty} c^k I_0^{\alpha k}\Phi(\omega) \\
&= \Phi(\omega) + \sum_{k=0}^{\infty} c^{k+1} I_0^{\alpha k+\alpha}\Phi(\omega) \\
&= \Phi(\omega) + c\sum_{k=0}^{\infty} \frac{c^k}{\Gamma(\alpha k+\alpha)} \int_0^\omega (\omega-\tau)^{\alpha k+\alpha-1}\Phi(\tau)d\tau \\
&= \Phi(\omega) + c\int_0^\omega (\omega-\tau)^{\alpha-1} \sum_{k=0}^{\infty} \frac{c^k}{\Gamma(\alpha k+\alpha)}(\omega-\tau)^{\alpha k}\Phi(\tau)d\tau \\
&= \Phi(\omega) + c\int_0^\omega (\omega-\tau)^{\alpha-1} E_{\alpha,\alpha}(c(\omega-\tau)^\alpha)\Phi(\tau)d\tau,
\end{aligned}$$

which is well-defined.

The following lemma is another application of BA.

Lemma 2. *Let $\Phi : [p,q] \to \mathbb{R}$ be continuous. A function $\Psi(\omega)$ is a solution of the FDE with the boundary value*

$$\begin{cases} {}_{LC}D_p^\kappa\Psi(\omega) + \mu I_p^\zeta\Psi(\omega) = \Phi(\omega), \ \ \omega \in [p,q], \\ \Psi^{(\rho)}(p) = 0, \ \ \rho = 0,1,2,\cdots,\ell-2; \quad \Psi^{(\ell-1)}(q) = 0, \end{cases}$$

if and only if $\Psi(\omega)$ is a solution of the fractional integral equation

$$\begin{aligned}
&\Psi(\omega) = \sum_{\rho=0}^{\infty}(-1)^\rho\mu^\rho I_p^{(\kappa+\zeta)\rho+\kappa}\Phi(\omega) \\
&+\frac{\mu}{(\ell-1)!\,\Gamma(\kappa-\ell+1+\zeta)}\int_p^q (q-v)^{\kappa-\ell+\zeta}\Psi(v)dv\cdot\sum_{\rho=0}^{\infty}(-1)^\rho\mu^\rho I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1} \\
&-\frac{1}{(\ell-1)!\,\Gamma(\kappa-\ell+1)}\int_p^q (q-v)^{\kappa-\ell}\Phi(v)dv\cdot\sum_{\rho=0}^{\infty}(-1)^\rho\mu^\rho I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1}, \qquad (3)
\end{aligned}$$

where $\ell-1<\kappa<\ell$, $\ell=[\kappa]+1$.

Proof. Applying the operator I_p^κ to both sides of the equation

$${}_{LC}D_p^\kappa\Psi(\omega) + \mu I_p^\zeta\Psi(\omega) = \Phi(\omega)$$

and using the condition $\Psi^{(\rho)}(p) = 0$, for $\rho = 0,1,2,\cdots,\ell-2$, we find

$$\Psi(\omega) + \mu I_p^{\kappa+\zeta}\Psi(\omega) = \mu_{\ell-1}(\omega-p)^{\ell-1} + I_p^\kappa\Phi(\omega).$$

Dedifferentiating the above equation $\ell-1$ times and setting $\omega = q$, we derive that

$$\begin{aligned}
&\Psi^{(\ell-1)}(q) + \frac{\mu}{\Gamma(\kappa-\ell+1+\zeta)}\int_p^q (q-v)^{\kappa+\zeta-\ell}\Psi(v)dv \\
&= \mu_{\ell-1}(\ell-1)! + \frac{1}{\Gamma(\kappa-\ell+1)}\int_p^q (q-v)^{\kappa-\ell}\Phi(v)dv.
\end{aligned}$$

Therefore,

$$\mu_{\ell-1} = \frac{\mu}{(\ell-1)!\Gamma(\kappa-\ell+1+\zeta)}\int_p^q (q-v)^{\kappa+\zeta-\ell}\Psi(v)dv - \frac{1}{(\ell-1)!\Gamma(\kappa-\ell+1)}\int_p^q (q-v)^{\kappa-\ell}\Phi(v)dv,$$

by noting that $\Psi^{(\ell-1)}(q) = 0$. In summary, we have

$$\begin{aligned} &\left(1+\mu I_p^{\kappa+\zeta}\right)\Psi(\omega) = \\ &\frac{\mu\,(\omega-p)^{\ell-1}}{(\ell-1)!\Gamma(\kappa-\ell+1+\zeta)}\int_p^q (q-v)^{\kappa+\zeta-\ell}\Psi(v)dv \\ &- \frac{(\omega-p)^{\ell-1}}{(\ell-1)!\Gamma(\kappa-\ell+1)}\int_p^q (q-v)^{\kappa-\ell}\Phi(v)dv \\ &+ I_p^{\kappa}\Phi(\omega). \end{aligned}$$

Using BA (treating the factor $\left(1+\mu I_p^{\kappa+\zeta}\right)$ as a variable), we come to

$$\begin{aligned} \Psi(\omega) &= \sum_{\rho=0}^{\infty}(-1)^\rho \mu^\rho I_p^{(\kappa+\zeta)\rho+\kappa}\Phi(\omega) \\ &+ \frac{\mu}{(\ell-1)!\,\Gamma(\kappa-\ell+1+\zeta)}\int_p^q (q-v)^{\kappa-\ell+\zeta}\Psi(v)dv\cdot\sum_{\rho=0}^{\infty}(-1)^\rho\mu^\rho I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1} \\ &- \frac{1}{(\ell-1)!\,\Gamma(\kappa-\ell+1)}\int_p^q (q-v)^{\kappa-\ell}\Phi(v)dv\cdot\sum_{\rho=0}^{\infty}(-1)^\rho\mu^\rho I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1}, \end{aligned}$$

utilizing

$$\left(1+\mu I_p^{\kappa+\zeta}\right)^{-1} = \sum_{\rho=0}^{\infty}(-1)^\rho\mu^\rho I_p^{(\kappa+\zeta)\rho}.$$

We note that all the above steps are reversible since BA is.

It remains to be shown that all series on the right-hand side of Equation (3) are convergent in terms of the norm in $C[p,q]$. Clearly,

$$\begin{aligned} &\left\|\sum_{\rho=0}^{\infty}(-1)^\rho\mu^\rho I_p^{(\kappa+\zeta)\rho+\kappa}\Phi(\omega)\right\| \leq \sum_{\rho=0}^{\infty}|\mu|^\rho\frac{\|\Phi\|}{\Gamma((\kappa+\zeta)\rho+\kappa)}\int_p^\omega(\omega-v)^{(\kappa+\zeta)\rho+\kappa-1}dv \\ &\leq \|\Phi\|(q-p)^\kappa\sum_{\rho=0}^{\infty}|\mu|^\rho\frac{(q-p)^{(\kappa+\zeta)\rho}}{\Gamma((\kappa+\zeta)\rho+\kappa+1)} \\ &= \|\Phi\|(q-p)^\kappa\, E_{\kappa+\zeta,\,\kappa+1}(|\mu|(q-p)^{\kappa+\zeta}) < +\infty, \end{aligned}$$

by the fact that $\Phi \in C[p,q]$. Similarly, we have

$$\begin{aligned} &\left\|\sum_{\rho=0}^{\infty}(-1)^\rho\mu^\rho I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1}\right\| \leq (q-p)^{\ell-1}\sum_{\rho=0}^{\infty}\frac{\left(|\mu|(q-p)^{\kappa+\zeta}\right)^\rho}{\Gamma((\kappa+\zeta)\rho+1)} \\ &= (q-p)^{\ell-1}E_{\kappa+\zeta,1}\left(|\mu|(q-p)^{\kappa+\zeta}\right) < +\infty. \end{aligned}$$

This completes the proof of Lemma 2. □

Remark 1. *(i) In particular, for $\mu = 0$, the FDE with boundary value*

$$\begin{cases} {}_{LC}D_p^{\kappa}\Psi(\omega) = \Phi(\omega), \ \ \omega \in [p,q], \\ \Psi^{(\rho)}(p) = 0, \ \ \rho = 0,1,2,\cdots,\ell-2; \quad \Psi^{(\ell-1)}(q) = 0, \end{cases}$$

has the solution

$$\Psi(\omega) = I_p^{\kappa}\Phi(\omega) - \frac{(\omega-p)^{\ell-1}}{(\ell-1)!\,\Gamma(\kappa-\ell+1)}\int_p^q (q-v)^{\kappa-\ell}\Phi(v)dv.$$

(ii) Clearly, the FDE with boundary value

$$\begin{cases} {}_{LC}D_p^{\kappa}\Psi(\omega) + \mu I_p^{\zeta}\Psi(\omega) = \Phi(\omega), \ \ \omega \in [p,q], \\ \Psi^{(\rho)}(p) = \omega_{\rho}, \ \ \rho = 0,1,2,\cdots,\ell-2; \quad \Psi^{(\ell-1)}(q) = \omega_q, \end{cases}$$

can be solved along the same lines. This is clearly a generalization of Equation (2).
(iii) Lemma 2 still holds for $\kappa = \ell$ using the same computation.

The following theorems will be used in Section 3 to study the existence and uniqueness.

Theorem 1. (Banach's contractive principle). *If $T : X \to X$ is a contraction mapping on a complete metric space (X,d), then there is exactly one solution of $T(x) = x$ for $x \in X$.*

Theorem 2. (Leray–Schauder's alternative). *Consider the continuous and compact function T of a Banach space S into itself. The boundedness of*

$$\{x \in S : x = \lambda T x \text{ for some } 0 \le \lambda \le 1\}$$

implies that T has a fixed point.

3. Existence and Uniqueness of Solutions

Theorem 3. *Assume that $\Theta : [p,q] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the Lipschitz condition for a constant $\mathcal{Q} \ge 0$:*

$$|\Theta(\omega,\varkappa_1) - \Theta(\omega,\varkappa_2)| \le \mathcal{Q}|\varkappa_1 - \varkappa_2|,$$

where $\varkappa_1, \varkappa_2 \in \mathbb{R}$. Furthermore, we suppose

$$\sigma = \mathcal{Q}(q-p)^{\kappa}\, E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) + \left[\frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)} + \frac{\mathcal{Q}(q-p)^{\kappa}}{(\ell-1)!\,\Gamma(\kappa-\ell+2)}\right] E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) < 1.$$

Then, the FD system (1) has a unique solution in the space $C^{\ell-1}[p,q]$.

Proof. From Lemma 2, we define a nonlinear mapping Y over the space $C^{\ell-1}[p,q]$ as

$$\begin{aligned}(\mathrm{Y}\Psi)(\omega) &= \sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{k} I_p^{(\kappa+\zeta)\rho+\kappa}\Theta(\omega,\Psi(\omega)) \\ &+ \frac{\mu}{(\ell-1)!\,\Gamma(\kappa-\ell+1+\zeta)}\int_p^q (q-v)^{\kappa-\ell+\zeta}\Psi(v)dv \cdot \sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho} I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1} \\ &- \frac{1}{(n\ell-1)!\,\Gamma(\kappa-\ell+1)}\int_p^q (q-v)^{\kappa-\ell}\Theta(v,\Psi(v))dv \cdot \sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho} I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1}.\end{aligned}$$

Clearly, $\frac{d^{\ell-1}}{d\omega^{\ell-1}}(\Upsilon\Psi)(\omega)$ is continuous on $[p,q]$ since $\ell-1<\kappa$ and Θ is continuous. Moreover,

$$\begin{aligned}\|\Upsilon\Psi\| \quad \leq \quad & \|\Theta(\omega,\Psi(\omega))\|(q-p)^{\kappa}\, E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) \\ & +\frac{|\mu|\,\|\Psi\|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) \\ & +\frac{\|\Theta(\omega,\Psi(\omega))\|\,(q-p)^{\kappa}}{(\ell-1)!\,\Gamma(\kappa-\ell+2)}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big),\end{aligned}$$

from the proof of Lemma 2. Clearly,

$$\begin{aligned}|\Theta(\omega,\Psi(\omega))| \quad = \quad & |\Theta(\omega,\Psi(\omega))-\Theta(\omega,0)+\Theta(\omega,0)| \leq |\Theta(\omega,\Psi(\omega))-\Theta(\omega,0)|+|\Theta(\omega,0)| \\ & \leq \mathcal{Q}|\Psi(\omega)-0|+|\Theta(\omega,0)|,\end{aligned}$$

which infers that

$$\|\Theta(\omega,\Psi(\omega))\| \leq \mathcal{Q}\|\Psi\| + \max_{\omega\in[p,q]}|\Theta(\omega,0)| < +\infty.$$

Thus Υ is a mapping from $C^{\ell-1}[p,q]$ to itself. To prove that Υ is contractive, we notice that, for $\Psi,\Omega\in C^{\ell-1}[p,q]$

$$\begin{aligned}&|(\Upsilon\Psi)(\omega)-(\Upsilon\Omega)(\omega)| \leq \left|\sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho}I_p^{(\kappa+\zeta)\rho+\kappa}[\Theta(\omega,\Psi(\omega))-\Theta(\omega,\Omega(\omega))]\right| \\ &+\left|\frac{\mu}{(\ell-1)!\,\Gamma(\kappa-\ell+1+\zeta)}\int_p^q(q-v)^{\kappa-\ell+\zeta}[\Psi(v)-\Omega(v)]dv\right. \\ &\left.\cdot\sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho}I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1}\right| \\ &+\left|\frac{1}{(\ell-1)!\,\Gamma(\kappa-\ell+1)}\int_p^q(q-v)^{\kappa-\ell}[\Theta(v,\Psi(v))-\Theta(v,\Omega(v))]dv\right. \\ &\left.\cdot\sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho}I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1}\right| \triangleq I_1+I_2+I_3.\end{aligned}$$

Clearly, for I_1, we derive that

$$\begin{aligned}|I_1| \quad \leq \quad & \mathcal{Q}\|\Psi-\Omega\|\sum_{\rho=0}^{\infty}|\mu|^{\rho}\frac{1}{\Gamma((\kappa+\zeta)\rho+\kappa)}\int_p^{\omega}(\omega-v)^{(\kappa+\zeta)\rho+\kappa-1}dv \\ & \leq \mathcal{Q}(q-p)^{\kappa}\|\Psi-\Omega\|\sum_{\rho=0}^{\infty}|\mu|^{\rho}\frac{(q-p)^{(\kappa+\zeta)\rho}}{\Gamma((\kappa+\zeta)\rho+\kappa+1)} \\ & = \mathcal{Q}(q-p)^{\kappa}\, E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)\|\Psi-\Omega\|.\end{aligned}$$

Regarding I_2,

$$|I_2| \leq \frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}\,E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)\|\Psi-\Omega\|,$$

by using

$$\left\|\sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho}I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1}\right\| \leq (q-p)^{\ell-1}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)$$

from Lemma 2. Finally,

$$|I_3| \leq \frac{\mathcal{Q}(q-p)^{\kappa}}{(\ell-1)!\,\Gamma(\kappa-\ell+2)}\, E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)\,\|\Psi-\Omega\|.$$

It follows from the above that

$$\begin{aligned}
&\|\Upsilon\Psi-\Upsilon\Omega\| \leq \mathcal{Q}(q-p)^{\kappa}\, E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)\,\|\Psi-\Omega\| \\
&+\left[\frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}+\frac{\mathcal{Q}(q-p)^{\kappa}}{(\ell-1)!\,\Gamma(\kappa-\ell+2)}\right] E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) \\
&\cdot\|\Psi-\Omega\| = \sigma\|\Psi-\Omega\|.
\end{aligned}$$

Since $\sigma < 1$, Υ is contractive. By BCP, the FD system (1) has a unique solution in the space $C^{\ell-1}[p,q]$. This completes the proof of Theorem 3. □

Example 1. *The following FDE with boundary value*

$$\begin{cases} {}_C D_{0.6}^{4.6}\Psi(\omega) + \frac{1}{20.59}\, I_{0.6}^{1.1}\Psi(\omega) = 1.3\sin\Psi(\omega) + e^{\omega} + \omega^2, & \omega \in [0.6, 2.3], \\ \Psi(0.6) = \Psi'(0.6) = \Psi''(0.6) = \Psi'''(0.6) = 0, & \Psi^{(4)}(2.3) = 0, \end{cases}$$

has a unique solution in the space $C^4[0.6, 2.3]$.

Proof. Clearly, the function

$$\Theta(\omega, \varkappa) = 1.3\sin\varkappa + e^{\omega} + \omega^2$$

satisfies

$$|\Theta(\omega,\varkappa_1) - \Theta(\omega,\varkappa_2)| \leq 1.3|\varkappa_1 - \varkappa_2|.$$

Hence, we compute the σ value in Theorem 3 as

$$\sigma \;=\; 1.3 * 1.7^{4.6}\sum_{\rho=0}^{\infty}\frac{\left(\frac{1.7^{5.7}}{20.59}\right)^{\rho}}{\Gamma(5.7\rho+5.6)} + \left[\frac{\frac{1.7^{5.7}}{20.59}}{4!\,\Gamma(2.7)} + \frac{1.3 * 1.7^{4.6}}{4!\,\Gamma(1.6)}\right]\sum_{\rho=0}^{\infty}\frac{\left(\frac{1.7^{5.7}}{20.59}\right)^{\rho}}{\Gamma(5.7\rho+1)}.$$

Using online calculators from the site https://www.wolframalpha.com/ (accessed on 25 March 2022), we obtain

$$\begin{aligned}
&1.7^{4.6}\sum_{\rho=0}^{\infty}\frac{\left(\frac{1.7^{5.7}}{20.59}\right)^{\rho}}{\Gamma(5.7\rho+5.6)} \approx 0.186558, \\
&\frac{\frac{1.7^{5.7}}{20.59}}{4!\,\Gamma(2.7)} \approx 0.0269681, \quad \frac{1.7^{4.6}}{4!\,\Gamma(1.6)} \approx 0.535491, \text{ and} \\
&\sum_{\rho=0}^{\infty}\frac{\left(\frac{1.7^{5.7}}{20.59}\right)^{\rho}}{\Gamma(5.7\rho+1)} \approx 1.00242.
\end{aligned}$$

Therefore $\sigma \leq 0.9673916 < 1$. By Theorem 3, the FD system has a unique solution in $C^4[0.6, 2.3]$. This completes the proof of Example 1. □

We are now ready to present the following theorem regarding existence of solutions to the FDE (1).

Theorem 4. *Assume* $\Theta : [p,q] \times \mathbb{R} \to \mathbb{R}$ *is a continuous and bounded function and*

$$\frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}\, E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) < 1.$$

Then, the FDE (1) with a boundary value has at least one solution in the space $C^{\ell-1}[p,q]$.

Proof. Again, we consider the nonlinear mapping Υ from $C^{\ell-1}[p,q]$ to itself by

$$\begin{aligned}
(\Upsilon\Psi)(\omega) &= \sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho} I_p^{(\kappa+\zeta)\rho+\kappa}\Theta(\omega,\Psi(\omega)) \\
&+\frac{\mu}{(\ell-1)!\,\Gamma(\kappa-\ell+1+\zeta)}\int_p^q (q-v)^{\kappa-\ell+\zeta}\Psi(v)dv\cdot\sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho}I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1} \\
&-\frac{1}{(\ell-1)!\,\Gamma(\kappa-\ell+1)}\int_p^q (q-v)^{\kappa-\ell}\Theta(v,\Psi(v))dv\cdot\sum_{\rho=0}^{\infty}(-1)^{\rho}\mu^{\rho}I_p^{(\kappa+\zeta)\rho}(\omega-p)^{\ell-1}.
\end{aligned}$$

We first claim (i) that Υ is continuous. Indeed, we find from the proof of Theorem 3

$$\begin{aligned}
|(\Upsilon\Psi)(\omega)-(\Upsilon\Omega)(\omega)| &\le (q-p)^{\kappa}\,E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) \\
&\quad\cdot\sup_{\omega\in[p,q]}|\Theta(\omega,\Psi(\omega))-\Theta(\omega,\Omega(\omega))| \\
&+\frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}\,E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)\|\Psi-\Omega\| \\
&+\frac{(q-p)^{\kappa}}{(\ell-1)!\,\Gamma(\kappa-\ell+2)}\,E_{\kappa+\zeta,1}\Big(|c|(b-a)^{\alpha+\beta}\Big) \\
&\quad\cdot\sup_{\omega\in[p,q]}|\Theta(\omega,\Psi(\omega))-\Theta(\omega,\Omega(\omega))|,
\end{aligned}$$

and we deduce that Υ is continuous since Θ is continuous.

(ii) Υ maps bounded sets to bounded sets in $C^{\ell-1}[p,q]$.

Since Θ is bounded, there exists a constant $\mathcal{Z} > 0$ such that

$$|\Theta(\omega,\Psi(\omega))| \le \mathcal{Z}$$

for all $\omega \in [p,q]$ and $\Psi \in C^{\ell-1}[p,q]$. Let $\mathcal{H}$ be a bounded set in $C^{\ell-1}[p,q]$. Then, there exists $\mathcal{Z}_1 > 0$ such that

$$\|u\| \le \mathcal{Z}_1,$$

for all $u \in \mathcal{H}$. Clearly, from the above,

$$\begin{aligned}
\|\Upsilon\psi\| &\le \mathcal{Z}(q-p)^{\kappa}\,E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) \\
&+\frac{|\mu|\,\mathcal{Z}_1(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) \\
&+\frac{\mathcal{Z}\,(q-p)^{\kappa}}{(\ell-1)!\,\Gamma(\kappa-\ell+2)}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big),
\end{aligned}$$

which is uniformly bounded.

(iii) Υ is completely continuous from $C[p,q]$ to itself.

By the Arzela–Ascoli theorem, it remains to be shown that Υ is equicontinuous over the bounded set $\mathcal{H} \subset C^{\ell-1}[p,q]$. For $v_1, v_2 \in [p,q]$ and $\Psi \in \mathcal{H}$, suppose that $v_1 < v_2$. Then, we have

$$
\begin{aligned}
&|(\Upsilon\Psi)(v_2)-(\Upsilon\Psi)(v_1)| \leq \sum_{\rho=0}^{\infty}|\mu|^{\rho}\frac{1}{\Gamma((\kappa+\zeta)\rho+\kappa)} \\
&\cdot\left|\int_p^{v_2}(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv-\int_p^{v_1}(v_1-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv\right| \\
&+\frac{|\mu|\,\mathcal{Z}_1}{(\ell-1)!\,\Gamma(\kappa-\ell+1+\zeta)}\int_p^q(q-v)^{\kappa-\ell+\zeta}dv\sum_{\rho=0}^{\infty}|\mu|^{\rho}\frac{1}{\Gamma((\kappa+\zeta)\rho)} \\
&\cdot\left|\int_p^{v_2}(v_2-v)^{(\kappa+\zeta)\rho-1}(v-p)^{\ell-1}dv-\int_p^{v_1}(v_1-v)^{(\kappa+\zeta)\rho-1}(v-p)^{\ell-1}dv\right| \\
&+\frac{\mathcal{Z}}{(\ell-1)!\,\Gamma(\kappa-\ell+1)}\int_p^q(q-v)^{\kappa-\ell}dv\sum_{\rho=0}^{\infty}|\mu|^{\rho}\frac{1}{\Gamma((\kappa+\zeta)\rho)} \\
&\cdot\left|\int_p^{v_2}(v_2-v)^{(\kappa+\zeta)\rho-1}(v-p)^{\ell-1}dv-\int_p^{v_1}(v_1-v)^{(\kappa+\kappa)\rho-1}(v-p)^{\ell-1}dv\right| \\
&\triangleq I_{11}+I_{22}+I_{33}.
\end{aligned}
$$

Let us estimate I_{11} first. Clearly,

$$
\begin{aligned}
&\int_p^{v_2}(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv \\
&=\int_p^{v_1}(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv+\int_{v_1}^{v_2}(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_p^{v_2}(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv-\int_p^{v_1}(v_1-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv \\
&=\int_p^{v_1}[(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}-(v_1-v)^{(\alpha+\beta)k+\alpha-1}]\Theta(v,\Psi(v))dv \\
&+\int_{v_1}^{v_2}(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv.
\end{aligned}
$$

It follows that, for all $\rho\geq 0$,

$$
\begin{aligned}
&\left|\int_p^{v_1}[(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}-(v_1-v)^{(\kappa+\zeta)\rho+\kappa-1}]\Theta(v,\Psi(v))dv\right| \\
&\leq\mathcal{Z}\left[\frac{(v_2-p)^{(\kappa+\zeta)\rho+\kappa}}{(\kappa+\zeta)\rho+\kappa}-\frac{(v_1-p)^{(\kappa+\zeta)\rho+\kappa}}{(\kappa+\zeta)\rho+\kappa}\right].
\end{aligned}
$$

By the mean value theorem,

$$
\frac{(v_2-p)^{(\kappa+\zeta)\rho+\kappa}-(v_1-p)^{(\kappa+\zeta)\rho+\kappa}}{(v_2-p)-(v_1-p)}=[(\kappa+\zeta)\rho+\kappa]\tau^{(\kappa+\zeta)\rho+\kappa-1},
$$

where $\tau\in(v_1-p,v_2-p)$. This derives that

$$
\left|\int_p^{v_1}[(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}-(v_1-v)^{(\alpha+\zeta)\rho+\kappa-1}]\Theta(v,\Psi(v))dv\right|\leq\mathcal{Z}(v_2-v_1)(q-p)^{(\kappa+\zeta)\rho+\kappa-1}.
$$

In addition,

$$
\begin{aligned}
&\left|\int_{v_1}^{v_2}(v_2-v)^{(\kappa+\zeta)\rho+\kappa-1}\Theta(v,\Psi(v))dv\right|\leq\mathcal{Z}\frac{(v_2-v_1)^{(\kappa+\zeta)\rho+\kappa}}{(\kappa+\zeta)\rho+\kappa} \\
&\leq\mathcal{Z}(v_2-v_1)^{\kappa}\frac{(q-p)^{(\kappa+\zeta)\rho}}{(\kappa+\zeta)\rho+\kappa}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|I_{11}| &= \sum_{\rho=0}^{\infty} |\mu|^{\rho} \frac{1}{\Gamma((\kappa+\zeta)\rho+\kappa)} \\
&\cdot \left| \int_p^{v_2} (v_2 - v)^{(\kappa+\zeta)\rho+\kappa-1} \Theta(v, \Psi(v)) dv - \int_p^{v_1} (v_1 - v)^{(\kappa+\zeta)\rho+\kappa-1} \Theta(v, \Psi(v)) dv \right| \\
&\leq \mathcal{Z}(v_2 - v_1)(q-p)^{\kappa-1} E_{\kappa+\zeta,\kappa}\left(|\mu|(q-p)^{\kappa+\zeta}\right) \\
&\quad + \mathcal{Z}(v_2 - v_1)^{\kappa} E_{\kappa+\zeta,\kappa+1}\left(|\mu|(q-p)^{\kappa+\zeta}\right).
\end{aligned}
$$

Let us consider the term I_{22}. Evidently,

$$
\frac{|\mu|\, \mathcal{Z}_1}{(\ell-1)!\,\Gamma(\kappa-\ell+1+\zeta)} \int_p^q (q-v)^{\kappa-\ell+\zeta} dv = \frac{|\mu|\, \mathcal{Z}_1 (q-p)^{\kappa-\ell+1+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)},
$$

and

$$
\begin{aligned}
&\sum_{\rho=0}^{\infty} |\mu|^{\rho} \frac{1}{\Gamma((\kappa+\zeta)\rho)} \\
&\cdot \left| \int_p^{v_2} (v_2 - v)^{(\kappa+\zeta)\rho-1} (v-p)^{\ell-1} dv - \int_p^{v_1} (v_1 - v)^{(\kappa+\zeta)\rho-1} (v-p)^{\ell-1} dv \right| \\
&= (v_2 - p)^{\ell-1} - (v_1 - p)^{\ell-1} + \sum_{\rho=1}^{\infty} |\mu|^{\rho} \frac{1}{\Gamma((\kappa+\zeta)\rho)} \\
&\cdot \left| \int_p^{v_2} (v_2 - v)^{(\kappa+\zeta)\rho-1} (v-p)^{\ell-1} dv - \int_p^{v_1} (v_1 - v)^{(\kappa+\zeta)\rho-1} (v-p)^{\ell-1} dv \right| \\
&= (v_2 - p)^{\ell-1} - (v_1 - p)^{\ell-1} + |\mu| \sum_{\rho=0}^{\infty} |\mu|^{\rho} \frac{1}{\Gamma((\kappa+\zeta)\rho+\kappa+\zeta)} \\
&\cdot \left| \int_p^{v_2} (v_2 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1} (v-p)^{\ell-1} dv - \int_p^{v_1} (v_1 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1} (v-p)^{\ell-1} dv \right|.
\end{aligned}
$$

Like the term I_{11}, we have

$$
\begin{aligned}
&\int_p^{v_2} (v_2 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1} (v-p)^{\ell-1} dv - \int_p^{v_1} (v_1 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1} (v-p)^{\ell-1} dv \\
&= \int_p^{v_1} [(v_2 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1} - (v_1 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1}] (v-p)^{\ell-1} dv \\
&+ \int_{v_1}^{v_2} (v_2 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1} (v-p)^{\ell-1} dv,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| \int_p^{v_1} [(v_2 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1} - (v_1 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1}] (v-p)^{\ell-1} dv \right| \\
&\leq (q-p)^{\ell-1} (v_2 - v_1)(q-p)^{(\kappa+\zeta)\rho+\kappa+\zeta-1},
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\left| \int_{v_1}^{v_2} (v_2 - v)^{(\kappa+\zeta)\rho+\kappa+\zeta-1} (v-p)^{\ell-1} dv \right| \leq (q-p)^{\ell-1} \frac{(v_2 - v_1)^{(\kappa+\zeta)\rho+\kappa+\zeta}}{(\kappa+\zeta)\rho+\kappa+\zeta} \\
&\leq (q-p)^{\ell-1} (v_2 - v_1)^{\kappa+\zeta} \frac{(q-p)^{(\kappa+\zeta)\rho}}{(\kappa+\zeta)\rho+\kappa+\zeta}.
\end{aligned}
$$

In summary,

$$|I_{22}| \leq \frac{|\mu|\, \mathcal{Z}_1 (q-p)^{\kappa-\ell+1+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}$$
$$\cdot\Big[(v_2-p)^{\ell-1}-(v_1-p)^{\ell-1}+(q-p)^{\ell-2+\kappa+\zeta}|\mu|(v_2-v_1)E_{\kappa+\zeta,\,\kappa+\zeta}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)$$
$$+(q-p)^{\ell-1}|\mu|(v_2-v_1)^{\kappa+\zeta}E_{\kappa+\zeta,\,\kappa+\zeta+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)\Big].$$

I_{33} follows similarly. Therefore, Υ is completely continuous.

(iv) Finally, we prove that the set

$$\mathcal{Y} = \{\Psi \in C^{\ell-1}[p,q] : \Psi = \beta\Upsilon\Psi \text{ for some } 0 < \beta < 1\}$$

is bounded. For any $\Psi \in \mathcal{Y}$, $\Psi = \beta\Upsilon\Psi$. This infers that

$$\|\Psi\| \leq \|\Upsilon\Psi\| \leq \mathcal{Z}(q-p)^{\kappa}\, E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)$$
$$+\frac{|\mu|\,\|\Psi\|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)$$
$$+\frac{\mathcal{Z}\,(q-p)^{\kappa}}{(\ell-1)!\,\Gamma(\kappa-\ell+2)}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big).$$

Let

$$\sigma_1 = 1-\frac{|\mu|\,(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big) > 0,$$

which claims that

$$\|\Psi\| \leq \frac{1}{\sigma_1}\mathcal{Z}(q-p)^{\kappa}\, E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)$$
$$+\frac{\mathcal{Z}\,(q-p)^{\kappa}}{\sigma_1\,(\ell-1)!\,\Gamma(\kappa-\ell+2)}E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big).$$

Hence, $\mathcal{Y}$ is bounded. By L-SFPT, the FDE (1) with boundary value has at least one solution in the space $C^{\ell-1}[p,q] \subset C[p,q]$. This completes the proof of Theorem 4. □

Example 2. *The following FDE with boundary value*

$$\begin{cases} {}_C D_1^{3.1}\Psi(\omega) - \; I_1^{1.2}\Psi(\omega) = 3\cos(\omega\Psi^2(\omega)) + \dfrac{1}{1+\omega^2}, & \omega \in [1,2], \\ \Psi(1) = \Psi'(1) = \Psi''(1) = 0, \;\; \Psi^{(3)}(2) = 0, \end{cases}$$

has at least one solution in the space $C^3[1,2]$.

Proof. Clearly,

$$\Theta(\omega,\varkappa) = 3\cos(\omega\varkappa^2) + \frac{1}{1+\omega^2},$$

is a bounded function over $[1,2]\times\mathbb{R}$. By Theorem 4, we need to evaluate the value of

$$\frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}\, E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)$$
$$= \frac{1}{3!\Gamma(2.3)}\sum_{\rho=0}^{\infty}\frac{1}{\Gamma(4.3\rho+1)} \approx 0.146604 < 1,$$

by the online calculator. Therefore, by Theorem 4, the fractional value problem has a solution in the space $C^3[1,2]$. □

Remark 2. *Clearly, from*

$$\sigma = \mathcal{Q}(q-p)^{\kappa}\, E_{\kappa+\zeta,\,\kappa+1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)$$
$$+\left[\frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}+\frac{\mathcal{Q}(q-p)^{\kappa}}{(\ell-1)!\,\Gamma(\kappa-\ell+2)}\right]E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)<1,$$

in Theorem 3, we imply that

$$\frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}\,E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)<1,$$

in Theorem 4. However, a Lipschitz function over $[p,q]\times\mathbb{R}$ may not be a bounded function. Conversely, a bounded function over $[p,q]\times\mathbb{R}$ may not be a Lipschitz function. Furthermore, it seems difficult to study the uniqueness of Example 2, since we cannot claim that

$$\Theta(\omega,\varkappa)=3\cos(\omega\varkappa^2)+\frac{1}{1+\omega^2},$$

is a Lipschitz function due to the factor $\varkappa^2$.

As a final example, in the following, we discuss both the existence and uniqueness of a solution simultaneously.

Example 3. *The following FDE with boundary value*

$$\begin{cases} {}_C D^{4.6}_{0.6}\Psi(\omega)+\frac{1}{20.59}\,I^{1.1}_{0.6}\Psi(\omega)=0.5\sin(\omega\Psi(\omega))+\cos(\omega^2), \quad \omega\in[0.6,2.3],\\ \Psi(0.6)=\Psi'(0.6)=\Psi''(0.6)=\Psi'''(0.6)=0, \quad \Psi^{(4)}(2.3)=0, \end{cases}$$

has a unique solution in the space $C^4[0.6,2.3]$.

Proof. Clearly, the function $\Theta(\omega,\varkappa)=0.5\sin(\omega\varkappa)+\cos(\omega^2)$ is bounded and satisfies the condition

$$|\Theta(\omega,\varkappa_1)-\Theta(\omega,\varkappa_2)|\le 1.3|\varkappa_1-\varkappa_2|.$$

It follows from Example 1 that $\sigma<1$. Then,

$$\frac{|\mu|(q-p)^{\kappa+\zeta}}{(\ell-1)!\,\Gamma(\kappa-\ell+2+\zeta)}\,E_{\kappa+\zeta,1}\Big(|\mu|(q-p)^{\kappa+\zeta}\Big)<1,$$

in Theorem 4 by Remark 2 . Hence, the system has a solution in the space $C^4[0.6,2.3]$ by Theorem 4. In addition, the solution is unique by Theorem 3. □

4. Conclusions

We studied the uniqueness and existence of solutions to the nonlinear two-term fractional integro-differential equation with a boundary condition by using Babenko's approach, the Mittag–Leffler function, Banach's contractive principle and the Laray–Schauder fixed point theorem. The current work also indicated key errors in the paper (Applied Mathematics, 2017, 8, 312–323) in handling a one-term differential equation. Furthermore, we provided three examples to demonstrate the application of our main theorems using the online Mittag–Leffler calculator. Clearly, it would be interesting and challenging to study the same system with a variable coefficient $\mu(x)$.

Author Contributions: C.L., methodology, writing—original draft preparation, supervision and project administration. R.S. (Reza Saadati),writing—original draft preparation and project administration. R.S. (Rekha Srivastava), writing–original draft preparation and project administration.

J.B., methodology and writing—original draft preparation. All authors have read and agreed to the published version of the manuscript.

Funding: Chenkuan Li is supported by the Natural Sciences and Engineering Research Council of Canada (Grant No. 2019-03907).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
2. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: Philadelphia, PA, USA, 1993.
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
4. Oldham, K.B.; Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA, 1974.
5. Li, C.; Srivastava, H. Uniqueness of solutions of the generalized Abel integral equations in Banach spaces. *Fractal Fract.* **2021**, *5*, 105. [CrossRef]
6. Guo, Z.; Liu, M.; Wang, D. Solutions of nonlinear fractional integro-differential equations with boundary conditions. *Bull. TICMI* **2012**, *16*, 58–65.
7. Burton, T.A.; Furumochi, T. Krasnoselskii's fixed point theorem and stability. *Nonlinear Anal. Theory Methods Appl.* **2002**, *49*, 445–454. [CrossRef]
8. Yu, C.; Gao, G. Existence of fractional differential equations. *J. Math. Anal. Appl.* **2005**, *310*, 26–29. [CrossRef]
9. Guezane-Lakoud, A.; Ramdane, S. Existence of solutions for a system of mixed fractional differential equations. *J. Taibah Univ. Sci.* **2018**, *12*, 421–426. [CrossRef]
10. Long, H.V.; Dong, N.P. An extension of Krasnoselskii's fixed point theorem and its application to nonlocal problems for implicit fractional differential systems with uncertainty. *J. Fixed Point Theory Appl.* **2018**, *20*, 37. [CrossRef]
11. Sudsutad, W.; Tariboon, J. Existence results of fractional integro-differential equations with m-point multi-term fractional order integral boundary conditions. *Bound. Value Probl.* **2012**, *2012*, 94. [CrossRef]
12. Nabil, T. Krasnoselskii N-Tupled Fixed Point Theorem with Applications to Fractional Nonlinear Dynamical System. *Adv. Math. Phys.* **2019**, *2019*, 6763842. [CrossRef]
13. Burton, T.A.; Zhang, B.O. Fixed points and fractional differential equations: Examples. *Fixed Point Theory* **2013**, *14*, 313–326.
14. Rezapour, S.; Abbas, M.I.; Etemad, A.; Dien, N.M. On a multi-point p-Laplacian fractional differential equation with generalized fractional derivatives. *Math. Methods Appl. Sci.* **2022**. [CrossRef]
15. Rezapour, S.; Souid, M.S.; Bouazza, Z.; Hussain, A.; Etemad, S. On the fractional variable order thermostat model: Existence theory on cones via piece-wise constant functions. *J. Funct. Spaces* **2022**, *2022*, 8053620. [CrossRef]
16. Turab, A.; Mitrovic, Z.D.; Savic, A. Existence of solutions for a class of nonlinear boundary value problems on the hexasilinane graph. *Adv. Differ. Equ.* **2021**, *2021*, 494. [CrossRef]
17. Wang, X.; Luo, D.; Zhu, Q. Ulam-Hyers stability of caputo type fuzzy fractional differential equations with time-delays. *Chaos Solitons Fractals* **2022**, *156*, 111822. [CrossRef]
18. Sun, Y.; Zeng, Z.; Song, J. Existence and uniqueness for the boundary value problems of nonlinear fractional differential equations. *Appl. Math.* **2017**, *8*, 312–323. [CrossRef]
19. Li, C. Several results of fractional derivatives in $\mathcal{D}'(R_+)$. *Fract. Calc. Appl. Anal.* **2015**, *18*, 192–207. [CrossRef]
20. Babenko, Y.I. *Heat and Mass Transfer*; Khimiya: Leningrad, Russia, 1986. (In Russian)
21. Li, C.; Beaudin, J. On the nonlinear integro-differential equations. *Fractal Fract.* **2021**, *5*, 82. [CrossRef]
22. Li, C.; Plowman, H. Solutions of the generalized Abel's integral equations of the second kind with variable coefficients. *Axioms* **2019**, *8*, 137. [CrossRef]