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On Another Class of Strongly Perfect Graphs

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Abstract: For a commutative ring R with unity, the associate ring graph, denoted by $AG(R)$, is a simple graph with vertices as nonzero elements of R and two distinct vertices are adjacent if they are associates. The graph $AG(R)$ contains components equal in number to the number of distinct orbits, except for the orbit of an element 0. Moreover, each component is a complete graph. An important finding is that this is a class of strongly perfect graphs. In this article we describe the structure of the associate ring graph of the ring of integers modulo n, denoted by $AG(\mathbb{Z}_n)$. We carried out computer experiments and provide a program for the same. We further characterize cases in which $AG(\mathbb{Z}_n)$, its complement $\overline{AG(\mathbb{Z}_n)}$, and their line graphs are planar, ring graphs, and outerplanar. We also discuss the properties of the associate ring graph of a commutative ring R with unity.

Keywords: associate ring graph; ring graph; line graph; planar; outerplanar

MSC: 05C10; 05C25; 05C85

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1. Introduction

In the past few decades, the study of graphs associated with rings has become a fascinating topic in algebraic graph theory. Several mathematicians have investigated the relationship between the ring-theoretical properties and the graph-theoretical properties of different graphs associated with a ring. Let R be a commutative ring with unity. Research on this topic started with an article by Beck [1] in 1988. He was the first to associate a graph, called the zero-divisor graph, denoted by $\Gamma(R)$, to a ring R. He was mainly concerned with the coloring of zero-divisor graphs. Since then, there has been a vast amount literature on this type of graph (see, [2–10]). There are also other graphs, namely, unit graphs [11], comaximal graphs [12], and total graphs [13] associated with rings. For more graphs associated with rings, see [14–17]. We refer to [18,19] for graph theory and [20] for ring theory for the definitions and notations not explicitly mentioned in this article.

In 2010, M. James Subhakar [21] introduced another graph to a ring and called it an associate ring graph. It is defined as a simple graph, denoted by $AG(R)$, with its vertices as nonzero elements of R and two distinct vertices are adjacent if and only if they are associates. The two elements $a, b \in R$ are associates in R if $b = a \cdot u$, for some unit $u \in R$. In this case, $a = bu^{-1}$, where u^{-1} denotes the inverse of u. The orbit of a, denoted by $Or(a)$, is the collection of associates of a. The binary relation $\sim$ on R, defined by $a \sim b$ if and only if a and b are associates, is an equivalence relation with the orbits of elements being the equivalence classes. Thus, $AG(R)$ is a disconnected graph with the number of components equal to the number of distinct orbits except for $Or(0) = \{0\}$. Moreover, each component is a complete graph and thus provides a class of strongly perfect graphs.

The complement of a graph G, denoted by $\overline{G}$, is a graph on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. A graph is planar if it can be embedded in the plane. A finite graph is planar if it does not contain a subgraph that is a subdivision of the complete bipartite graph $K_{3,3}$ or the complete graph K_5. A planar graph is an outerplanar if a plane embedding is such that all vertices lie on the boundary of the outer face. A vertex of a graph is a cut vertex if removal of that vertex increases the number of components in G. A maximal connected subgraph is called a block if it does not contain any cut vertex.

Given a graph H, we call a path P an H-path if it is a non-trivial path and meets the graph H exactly at its end vertices. A graph G is a ring graph if every block of G that is not a bridge or a vertex can be constructed via a cycle by successively adding H-paths of lengths of at least two that meet the graph H already constructed in the two adjacent vertices. Ring graphs were first introduced by Gitler et al. [22]. In [23], (Theorem 2.13, Proposition 2.17), the authors proved that every ring graph is a planar graph and that every outerplanar graph is a ring graph.

The line graph of a graph G, denoted by $L(G)$, is the graph in which the vertex set is in one-to-one correspondence with the edge set of G, such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. In 1963, Sadlacek characterized the planarity of line graphs as follows:

Theorem 1 ([24])**.** *The line graph of a graph G is planar if and only if G is planar, $\Delta(G) \leq 4$, and every vertex of degree 4 in G is a cut vertex, where $\Delta(G)$ denotes the maximum degree of a graph G.*

A weakly perfect graph is a graph for which $\omega(G) = \chi(G)$, where $\omega(G)$ and $\chi(G)$ is the clique number and the chromatic number of a graph G, respectively. A graph is perfect if for every induced subgraph H, $\omega(H) = \chi(H)$. Every perfect graph is a weakly perfect graph. A graph is strongly perfect if every induced subgraph has an independent set meeting all its maximal cliques. Every strongly perfect graph is perfect, but the converse is not necessarily true.

Theorem 2 ([25])**.** *Every P_4-free graph (i.e., a graph not containing the path graph P_4 as a vertex-induced subgraph) is a strongly perfect graph.*

For a positive integer n, Euler's totient function, denoted by $\phi(n)$, counts the positive integers up to a given integer n that are relatively prime to n.

This article is organized as follows:

In Section 2, we discuss the motivation behind studying associate ring graphs. Section 3 concerns the structure of an associate ring graph of a ring of integers modulo n. Section 4 provides an algorithm to create the same. In Sections 5–8, we characterize when $AG(\mathbb{Z}_n)$ and its complement, along with their line graphs, are planar, ring graphs, and outerplanar. Section 9 contains some results regarding the associate ring graph of a commutative ring with unity. Finally, Section 10 contains a Python 3.8 program to construct the graph $AG(\mathbb{Z}_n)$.

2. Motivation

A maximal complete subgraph of a graph is called a clique. In social network analysis (SNA), a clique may be treated as a group of people in which everybody knows each other. Thus, intuitively, a clique can be considered a cohesive group of tightly connected people (who are not tightly connected to people outside the group). The notion of a cohesive group is often used to explain and develop sociological theories analogous to clusters in graph-based data mining.

Since the associate ring graph is a union of complete graphs and cliques play a significant role in analyzing social networks in the present scenario, we were motivated to study associate ring graphs.

3. Structure of $AG(\mathbb{Z}_n)$

For a ring $\mathbb{Z}_n$, the associate ring graph denoted by $AG(\mathbb{Z}_n)$ is a simple graph with vertex set $\{1, 2, \ldots, n-1\}$ and two distinct vertices are adjacent if they are associates in $\mathbb{Z}_n$. We begin this section with two examples as a motivation to define the structure of $AG(\mathbb{Z}_n)$ for a general value of n.

Example 1. *Consider a ring $R = \mathbb{Z}_{20}$. The graph $AG(\mathbb{Z}_{20})$ is shown in Figure 1.*

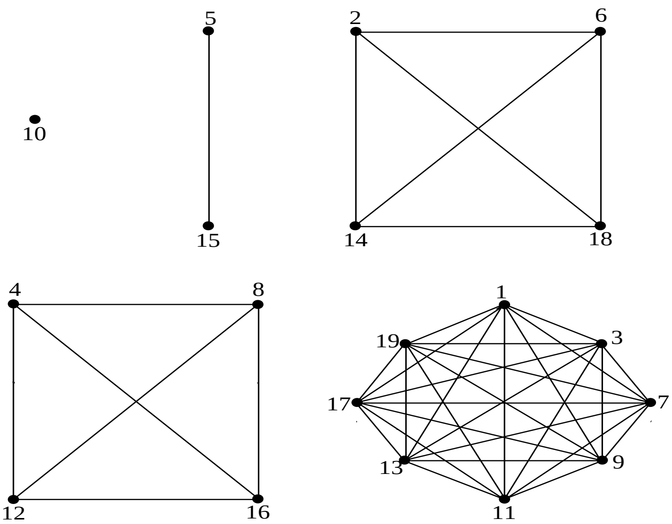


Figure 1. The associate ring graph $AG(\mathbb{Z}_{20})$.

In this case, the set of units $U(\mathbb{Z}_{20}) = \{1, 3, 7, 9, 11, 13, 17, 19\}$. The equivalence classes of $\mathbb{Z}_{20}$ formed using the equivalence relation defined by associates are as follows.

$$\begin{aligned} Or(1) &= \{1, 3, 7, 9, 11, 13, 17, 19\} \\ Or(2) &= \{2, 6, 14, 18\} \\ Or(4) &= \{4, 8, 12, 16\} \\ Or(5) &= \{5, 15\} \\ Or(10) &= \{10\} \end{aligned}$$

Clearly, $AG(\mathbb{Z}_{20}) = K_1 \cup K_2 \cup 2K_4 \cup K_8$.

Example 2. *Consider a ring $R = \mathbb{Z}_{15}$. The graph $AG(\mathbb{Z}_{15})$ is shown in Figure 2.*

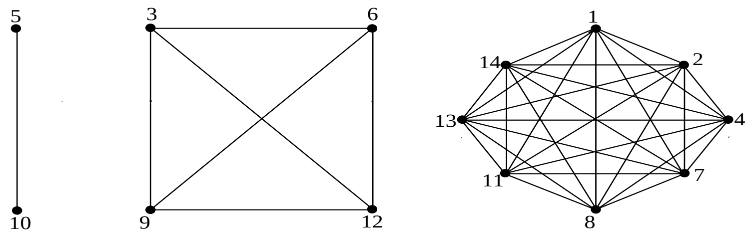


Figure 2. The associate ring graph $AG(\mathbb{Z}_{15})$.

In this case, the set of units $U(\mathbb{Z}_{15}) = \{1, 2, 4, 7, 8, 11, 13, 14\}$. The equivalence classes of $\mathbb{Z}_{15}$, formed using the equivalence relation defined by associates, are as follows.

$$Or(1) = \{1, 2, 4, 7, 8, 11, 13, 14\}$$
$$Or(3) = \{3, 6, 9, 12\}$$
$$Or(5) = \{5, 10\}$$

Clearly, $AG(\mathbb{Z}_{15}) = K_2 \cup K_4 \cup K_8$.

We now state a theorem for the structure of $AG(\mathbb{Z}_n)$ for $n > 1$.

Theorem 3. *Let $d_1, d_2, \ldots, d_k$ all be positive divisors of n in increasing order such that $d_1 > 1$. The structure of an associate ring graph is given by*

$$AG(\mathbb{Z}_n) = K_{\phi(d_1)} \cup K_{\phi(d_2)} \cup \cdots \cup K_{\phi(d_k)},$$

where ϕ denotes Euler's totient function.

Proof. Since the elements of an orbit share an order, for any $a \in \mathbb{Z}_n$, we have

$$Or(a) = \{b \in \mathbb{Z}_n : O(b) = O(a)\},$$

where $O(x)$ is the order of an element x in $\mathbb{Z}_n$. This implies that $|Or(a)|$ is equal to the number of elements of order $O(a)$. In turn, $|Or(a)| = \phi(O(a))$ and since it is true for every $a \in \mathbb{Z}_n$, for each positive divisor d of n, there exists an orbit with $\phi(d)$ elements. Furthermore, since the vertex set of an associate ring graph consists of nonzero elements, we exclude 1 from the list of positive divisors of n to obtain the result. □

Corollary 1. *The number of components in $AG(\mathbb{Z}_n)$ is equal to $\tau(n) - 1$.*

Proof. Since the number of positive divisors of a natural number n is given by $\tau(n)$ and by Theorem 3, against every positive divisor of n except 1, there exists a component (complete graph) in the associate ring graph $AG(\mathbb{Z}_n)$, the number of components in $AG(\mathbb{Z}_n)$= $\tau(n) - 1$. □

Corollary 2. *For an even value of n, the graph $AG(\mathbb{Z}_n)$ contains precisely one component of odd order. For odd values of n, all the components of $AG(\mathbb{Z}_n)$ are of even order.*

Proof. The proof is evident from Theorem 3. □

Corollary 3. *Let p, q and r denote distinct prime numbers. We have*

(i) $AG(\mathbb{Z}_p) = K_{\phi(p)} = K_{p-1}$.

(ii) $AG(\mathbb{Z}_{2p}) = K_{\phi(2)} \cup K_{\phi(p)} \cup K_{\phi(2p)} = K_1 \cup K_{p-1} \cup K_{p-1}$, for $p > 2$.

(iii) $AG(\mathbb{Z}_{p^2}) = K_{\phi(p)} \cup K_{\phi(p^2)} = K_{p-1} \cup K_{p(p-1)}$.

(iv) $AG(\mathbb{Z}_{p^3}) = K_{\phi(p)} \cup K_{\phi(p^2)} \cup K_{\phi(p^3)} = K_{p-1} \cup K_{p(p-1)} \cup K_{p^2(p-1)}$.

(v) $AG(\mathbb{Z}_{pq}) = K_{\phi(p)} \cup K_{\phi(q)} \cup K_{\phi(pq)} = K_{p-1} \cup K_{q-1} \cup K_{(p-1)(q-1)}$.

(vi) $AG(\mathbb{Z}_{pqr}) = K_{\phi(p)} \cup K_{\phi(q)} \cup K_{\phi(r)} \cup K_{\phi(pq)} \cup K_{\phi(qr)} \cup K_{\phi(pr)} \cup K_{\phi(pqr)}$
$= K_{p-1} \cup K_{q-1} \cup K_{r-1} \cup K_{(p-1)(q-1)} \cup K_{(q-1)(r-1)} \cup K_{(p-1)(r-1)}$
$\cup K_{(p-1)(q-1)(r-1)}$.

4. Algorithm to Create $AG(\mathbb{Z}_n)$

In this section, we present an algorithm to construct an associate ring graph of the ring of integers modulo n. The algorithm takes the value of n as an input and gives the graph $AG(\mathbb{Z}_n)$ as an output. The Python program for the same is given in Section 10.

Step 1. Input the value of n.
Step 2. Calculate all the positive divisors of n except 1.
Step 3. Corresponding to each divisor d of n except 1 (found in the previous step), we produce a set consisting of positive integers i less than n such that $\frac{n}{\gcd(n,i)} = d$.
Step 4. Draw the complete graphs corresponding to each set found in the previous step with the vertices as the set elements.
Step 5. The union of the graphs obtained in the last step is the graph $AG(\mathbb{Z}_n)$, which is exactly the same graph derived in Theorem 3.

Theorem 4. *The time complexity of the above algorithm is $O(n^2)$.*

Proof. In Step 2, for calculating the positive divisors of n, the time complexity is $O(n)$. In Step 3, calculating the greatest common divisor, the time complexity is $O(\log n)$. In this step, we also calculate the order of each element of the ring $\mathbb{Z}_n$. Thus, the time complexity of this step is $O(n \log n)$. In Step 4, for drawing subgraphs of $AG(\mathbb{Z}_n)$, the time complexity is $O(n^2)$.

Hence, overall, the complexity of computation involved in the above algorithm is $O(n^2)$, where n is the order of $\mathbb{Z}_n$. □

5. $AG(\mathbb{Z}_n)$ as a Planar Graph, a Ring Graph, and an Outerplanar Graph

In this section, we discuss the planarity of the graph $AG(\mathbb{Z}_n)$ for $n > 1$. We further characterize when they are ring graphs and outerplanar graphs. In this regard, we have the following results.

Lemma 1. *If $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}$, where p_1, p_2, p_3 are distinct prime numbers in increasing order and k_1, k_2, k_3 are positive integers, then $n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) \geq 8$.*

Proof. We have

$$\begin{aligned} n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) &= p_1^{k_1} p_2^{k_2} p_3^{k_3}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) \\ &= p_1^{k_1-1} p_2^{k_2-1} p_3^{k_3-1}(p_1 - 1)(p_2 - 1)(p_3 - 1). \end{aligned} \quad (1)$$

We divide the rest of the proof into the following three cases based on the value of n:

Case 1. If $n = 2 \times 3 \times 5$. Equation (1) yields the value 8 with $p_1 = 2, p_2 = 3, p_3 = 5$ and $k_1 = k_2 = k_3 = 1$.

Case 2. If $n = 2^{k_1} 3^{k_2} 5^{k_3}$ with at least one of the k_1, k_2 and k_3 is greater than 1. In this case, Equation (1) yields $2^{k_1-1} \times 3^{k_2-1} \times 5^{k_3-1} \times 2 \times 4$, which is greater than the value obtained in the previous case.

Case 3. If $n = q_1^{k_1} q_2^{k_2} q_3^{k_3}$, where q_1, q_2 and q_3 are distinct prime numbers in increasing order such that $q_1 \geq 2, q_2 \geq 3, q_3 > 5$ and k_1, k_2, k_3 are positive integers. In this case, Equation (1) yields $q_1^{k_1-1} q_2^{k_2-1} q_3^{k_3-1}(q_1 - 1)(q_2 - 1)(q_3 - 1)$. Since $q_1 \geq 2, q_2 \geq 3, q_3 > 5$, we have $q_1 - 1 \geq 1, q_1^{k_1-1} \geq 2^{k_1-1}, q_2 - 1 \geq 2, q_2^{k_2-1} \geq 3^{k_2-1}, q_3 - 1 > 4, q_3^{k_3-1} \geq 5^{k_3-1}$. Hence, the value obtained in this case is again greater than that obtained in the first case. □

Lemma 2. *If $n = p_1^{k_1} p_2^{k_2}$, where $p_1,\ p_2$ are distinct prime numbers in increasing order and $k_1,\ k_2$ are positive integers such that at least one of them is greater than 2, then $n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \geq 8$.*

Proof. We have

$$\begin{aligned} n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) &= p_1^{k_1} p_2^{k_2}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \\ &= p_1^{k_1-1} p_2^{k_2-1}(p_1 - 1)(p_2 - 1). \end{aligned} \quad (2)$$

We divide the proof into the following two cases:

Case 1. If $k_1 \geq 3$ and $k_2 < 3$, we further divide the proof into cases based on the value of n:

Case 1a. If $n = 2^3 \times 3$, Equation (2) yields the value 8 with $p_1 = 2, p_2 = 3, k_1 = 3$ and $k_2 = 1$.

Case 1b. If $n = 2^3 \times 3^2$, Equation (2) yields the value 24.

Case 1c. If $n = 2^{k_1} \times 3$, where $k_1 \geq 4$, Equation (2) yields $2^{k_1-1} \times 2$, which is greater than the value obtained in Case 1a.

Case 1d. If $n = 2^{k_1} \times 3^2$, where $k_1 \geq 4$, Equation (2) yields $2^{k_1-1} \times 3 \times 2$, which is again greater than the value obtained in Case 1a.

Case 1e. If $n = q_1^{k_1} q_2^{k_2}$, where q_1 and q_2 are distinct prime numbers in increasing order such that $q_1 \geq 2$ and $q_2 > 3$, Equation (2) yields $q_1^{k_1-1} q_2^{k_2-1}(q_1 - 1)(q_2 - 1)$. Since $q_1 \geq 2$ and $q_2 > 3$, we have $q_1 - 1 \geq 1, q_1^{k_1-1} \geq 2^{k_1-1}, q_2 - 1 > 2$, and $q_2^{k_2-1} \geq 3^{k_2-1}$. We have

$$
\begin{aligned}
n\left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right) &> 2^{k_1-1} \times 3^{k_2-1} \times 2 \\
&> 2^2 \times 3^0 \times 2 = 8 \qquad [\text{As } k_1 \geq 3 \text{ and } k_2 \in \{1, 2\}]
\end{aligned}
$$

Hence, the value obtained in this case is again greater than 8.

Case 2. If k_1 is any positive integer and $k_2 \geq 3$, we further divide the proof into cases based on the value of n:

Case 2a. If $n = 2 \times 3^3$, Equation (2) yields the value 18 with $p_1 = 2, p_2 = 3, k_1 = 1$ and $k_2 = 3$.

Case 2b. If $n = 2^{k_1} \times 3^{k_2}$, where $k_1 \geq 2$ and $k_2 \geq 3$, Equation (2) yields $2^{k_1-1} \times 3^{k_2-1} \times 2$, which is greater than the value obtained in Case 2a.

Case 2c. If $n = q_1^{k_1} q_2^{k_2}$, where q_1 and q_2 are distinct prime numbers in increasing order such that $q_1 \geq 2$ and $q_2 > 3$, Equation (2) yields $q_1^{k_1-1} q_2^{k_2-1}(q_1 - 1)(q_2 - 1)$. Since $q_1 \geq 2$ and $q_2 > 3$, we have $q_1 - 1 \geq 1, q_1^{k_1-1} \geq 2^{k_1-1}, q_2 - 1 > 2$, and $q_2^{k_2-1} > 3^{k_2-1}$. Hence, the value obtained in this case is again greater than that obtained in Case 2a.

By combining both the cases, the result follows. □

Lemma 3. *If $n = p_1 p_2$, where p_1 and p_2 are distinct prime numbers in increasing order such that $p_1 > 2$, then $(p_1 - 1)(p_2 - 1) \geq 8$.*

Proof. Since p_1 is a prime number greater than 2, we divide the proof into the following two cases based on the value of n:

Case 1. If $n = 3 \times 5$, then $(p_1 - 1)(p_2 - 1) = 8$ with $p_1 = 3$ and $p_2 = 5$.

Case 2. If $n = q_1 q_2$, where q_1 and q_2 are distinct prime numbers in increasing order such that $q_1 \geq 3$ and $q_2 > 5$. Since $q_1 \geq 3$ and $q_2 > 5$, we have $q_1 - 1 \geq 2$ and $q_2 - 1 > 4$. In turn, $(q_1 - 1)(q_2 - 1) > 8$. □

Lemma 4. *If $n = p^k$, where p is any prime number and k is a positive integer such that $k > 3$, then $p^{k-1}(p - 1) \geq 8$.*

Proof. We divide the proof into the following three cases based on the value of n:

Case 1. If $n = 2^4$, then $p^{k-1}(p - 1) = 8$ with $p = 2$ and $k = 4$.

Case 2. If $n = 2^k$, where $k \geq 5$, then $p^{k-1}(p - 1) = 2^{k-1} \geq 2^4 = 16$.

Case 3. If $n = q^k$, where q is a prime number greater than 2 and k is a positive integer such that $k \geq 4$. Since $q > 2$ and $k \geq 4$, we have $q - 1 > 1$ and $q^{k-1} > 2^{k-1} > 2^3 = 8$. In turn, $q^{k-1}(q - 1) > 8$. □

Theorem 5. *The graph $AG(\mathbb{Z}_n)$ is planar if and only if $n \in \{2, 3, 4, 5, 6, 8, 10, 12\}$.*

Proof. Suppose $AG(\mathbb{Z}_n)$ is planar and $n = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1, \ldots, p_r$ are distinct prime numbers in increasing order and $k_1, \ldots, k_r$ are positive integers.

We claim that $r \leq 2$. Suppose $r > 2$. In this case, the number $p_1^{k_1} p_2^{k_2} p_3^{k_3}$ is a positive divisor of n. According to Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains a complete graph of order $\phi(p_1^{k_1} p_2^{k_2} p_3^{k_3}) = p_1^{k_1-1} p_2^{k_2-1} p_3^{k_3-1}(p_1-1)(p_2-1)(p_3-1)$. According to Lemma 1, this value is greater than or equal to 8. Hence, this contradicts our assumption that $AG(\mathbb{Z}_n)$ is planar as it contains a complete graph K_5 as a subgraph. Therefore, $r \leq 2$. That is, n cannot have more than two prime factors. We divide our result into the following two cases:

Case 1. If $n = p_1^{k_1} p_2^{k_2}$, we now claim that $k_1, k_2 \leq 2$. On the contrary, if any of the k_1 or k_2 is greater than 2, then by Lemma 2, $n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \geq 8$. Again, we obtain a contradiction. Therefore, $k_1, k_2 \leq 2$. We further claim that $p_1 = 2$. On the contrary, suppose $p_1 > 2$. Since $p_1 p_2$ is a positive divisor of n, according to Theorem 3, $AG(\mathbb{Z}_n)$ contains a complete graph of order $\phi(p_1 p_2) = (p_1 - 1)(p_2 - 1)$. Based on Lemma 3, this value is greater than or equal to 8. Again, this contradicts our assumption. Therefore, $p_1 = 2$. We further divide the result into the following four cases where p denotes a prime number greater than 2.

Case 1a. If $n = 2p$, then $AG(\mathbb{Z}_n)$ is $K_1 \cup K_{p-1} \cup K_{p-1}$. If p is greater than 5, then it contradicts our assumption that $AG(\mathbb{Z}_n)$ is planar. Therefore, p must be equal to 3 or 5.

Case 1b. If $n = 2p^2$, then $AG(\mathbb{Z}_n)$ is $K_1 \cup K_{p-1} \cup K_{p(p-1)} \cup K_{p-1} \cup K_{p(p-1)}$. If p is greater than 2, then it contradicts our assumption that $AG(\mathbb{Z}_n)$ is planar.

Case 1c. If $n = 4p$, then $AG(\mathbb{Z}_n)$ is $K_1 \cup K_2 \cup K_{p-1} \cup K_{p-1} \cup K_{2(p-1)}$. If p is greater than 3, then it again contradicts our assumption that $AG(\mathbb{Z}_n)$ is planar. Therefore, p must be equal to 3.

Case 1d. If $n = 4p^2$, then $AG(\mathbb{Z}_n)$ is $K_1 \cup K_2 \cup K_{p-1} \cup K_{p(p-1)} \cup K_{p-1} \cup K_{2(p-1)} \cup K_{p(p-1)} \cup K_{2p(p-1)}$. If p is greater than 2, then it contradicts our assumption that $AG(\mathbb{Z}_n)$ is planar.

Case 2. If $n = p^k$, we now claim that $k \leq 3$. On the contrary, if $k > 3$, then p^4 is a positive divisor of n. That is, $AG(\mathbb{Z}_n)$ contains a complete graph of order $\phi(p^4) = p^3(p-1)$. Based on Lemma 4, this value is greater than or equal to 8. Again, this contradicts our assumption that $AG(\mathbb{Z}_n)$ is planar. Hence, $k \leq 3$. We further divide the result into the following three cases:

Case 2a. If $n = p$. In this case, $AG(\mathbb{Z}_n)$ is K_{p-1}. If p is greater than 5, then it contradicts that $AG(\mathbb{Z}_n)$ is planar. Therefore, p must be equal to 2 or 3 or 5.

Case 2b. If $n = p^2$. In this case, $AG(\mathbb{Z}_n)$ is $K_{p-1} \cup K_{p(p-1)}$. If p is greater than 2, then it contradicts that $AG(\mathbb{Z}_n)$ is planar. Therefore, p must be equal to 2.

Case 2c. If $n = p^3$. In this case, $AG(\mathbb{Z}_n)$ is $K_{p-1} \cup K_{p(p-1)} \cup K_{p^2(p-1)}$. Again, if p is greater than 2, then it contradicts our assumption. Therefore, p must be equal to 2.

Combining both the cases, $n \in \{2, 3, 4, 5, 6, 8, 10, 12\}$. On the other hand, if n belongs to this set, then $AG(\mathbb{Z}_n)$ is a union of complete graphs of which the orders are less than or equal to 4. Thus, the graph $AG(\mathbb{Z}_n)$ is planar. □

Theorem 6. *The graph $AG(\mathbb{Z}_n)$ is a ring graph if and only if $n \in \{2, 3, 4, 6\}$.*

Proof. Since every ring graph is planar, we only need to check for $n \in \{2, 3, 4, 5, 6, 8, 10, 12\}$. Consider the following cases:

Case 1. If $n = 2$. The graph $AG(\mathbb{Z}_2) = K_1$.
Case 2. If $n = 3$. The graph $AG(\mathbb{Z}_3) = K_2$.
Case 3. If $n = 4$. The graph $AG(\mathbb{Z}_4) = K_1 \cup K_2$.
Case 4. If $n = 5$. The graph $AG(\mathbb{Z}_5) = K_4$.
Case 5. If $n = 6$. The graph $AG(\mathbb{Z}_6) = K_1 \cup K_2 \cup K_2$.
Case 6. If $n = 8$. The graph $AG(\mathbb{Z}_8) = K_1 \cup K_2 \cup K_4$.
Case 7. If $n = 10$. The graph $AG(\mathbb{Z}_{10}) = K_1 \cup K_4 \cup K_4$.
Case 8. If $n = 12$. The graph $AG(\mathbb{Z}_{12}) = K_1 \cup K_2 \cup K_2 \cup K_2 \cup K_4$.
It is evident that $AG(\mathbb{Z}_n)$ is a ring graph if and only if $n \in \{2, 3, 4, 6\}$. □

Theorem 7. *The graph $AG(\mathbb{Z}_n)$ is outerplanar if and only if $n \in \{2,3,4,6\}$.*

Proof. Since every outerplanar graph is a ring graph, we only need to check for $n \in \{2,3,4,6\}$. It can be easily seen that for these values of n, all the vertices lie in the unbounded face. Therefore, $AG(\mathbb{Z}_n)$ is outerplanar if and only if $n \in \{2,3,4,6\}$. □

6. $L(AG(\mathbb{Z}_n))$ as a Planar Graph, a Ring Graph, and an Outerplanar Graph

We now discuss the line graph of $AG(\mathbb{Z}_n)$ for $n > 1$.

Theorem 8. *The line graph $L(AG(\mathbb{Z}_n))$ is planar if and only if $n \in \{2,3,4,5,6,8,10,12\}$.*

Proof. Since the planarity of a graph is the necessary condition for its line graph to be planar, we only need to check for $n \in \{2,3,4,5,6,8,10,12\}$. Consider the following cases:

Case 1. If $n = 2$, the graph $AG(\mathbb{Z}_2) = K_1$. In turn, $L(AG(\mathbb{Z}_2))$ is a null graph with no vertex.

Case 2. If $n = 3$, the graph $AG(\mathbb{Z}_3) = K_2$. The graph $L(AG(\mathbb{Z}_3)) = K_1$.

Case 3. If $n = 4$, the graph $AG(\mathbb{Z}_4) = K_1 \cup K_2$. The graph $L(AG(\mathbb{Z}_4))$ is a trivial graph.

Case 4. If $n = 5$, the graph $AG(\mathbb{Z}_5) = K_4$. In turn, $L(AG(\mathbb{Z}_5))$ is a Johnson graph $J(4,2)$ given in Figure 3.

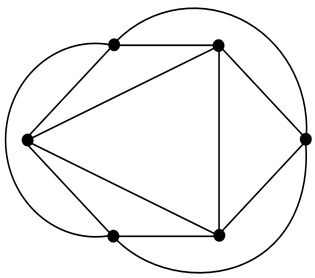


Figure 3. The graph $J(4,2)$.

Case 5. If $n = 6$, the graph $AG(\mathbb{Z}_6) = K_1 \cup K_2 \cup K_2$. In turn, $L(AG(\mathbb{Z}_6))$ is a null graph with two vertices.

Case 6. If $n = 8$, the graph $AG(\mathbb{Z}_8) = K_1 \cup K_2 \cup K_4$. In turn, $L(AG(\mathbb{Z}_8)) = K_1 \cup J(4,2)$.

Case 7. If $n = 10$, the graph $AG(\mathbb{Z}_{10}) = K_1 \cup K_4 \cup K_4$. In this case, $L(AG(\mathbb{Z}_{10})) = J(4,2) \cup J(4,2)$.

Case 8. If $n = 12$, the graph $AG(\mathbb{Z}_{12}) = K_1 \cup K_2 \cup K_2 \cup K_2 \cup K_4$. In this case, $L(AG(\mathbb{Z}_{12})) = K_1 \cup K_1 \cup K_1 \cup J(4,2)$.

It is evident that $AG(\mathbb{Z}_n)$ is planar if and only if $n \in \{2,3,4,5,6,8,10,12\}$. □

Theorem 9. *The line graph $L(AG(\mathbb{Z}_n))$ is a ring graph if and only if $n \in \{2,3,4,6\}$.*

Proof. Since every ring graph is planar, we only need to check for $n \in \{2,3,4,5,6,8,10,12\}$. Consider the following cases:

Case 1. If $n = 2$, the graph $L(AG(\mathbb{Z}_2))$ is a null graph with no vertex.

Case 2. If $n = 3$, the graph $L(AG(\mathbb{Z}_3)) = K_1$.

Case 3. If $n = 4$, the graph $L(AG(\mathbb{Z}_4))$ is a trivial graph.

Case 4. If $n = 5$, the graph $L(AG(\mathbb{Z}_5))$ is a Johnson graph $J(4,2)$ given in Figure 3.

Case 5. If $n = 6$, the graph $L(AG(\mathbb{Z}_6))$ is a null graph with two vertices.

Case 6. If $n = 8$, the graph $L(AG(\mathbb{Z}_8)) = K_1 \cup J(4,2)$.

Case 7. If $n = 10$, the graph $L(AG(\mathbb{Z}_{10})) = J(4,2) \cup J(4,2)$.

Case 8. If $n = 12$, the graph $L(AG(\mathbb{Z}_{12})) = K_1 \cup K_1 \cup K_1 \cup J(4,2)$.

It is evident that $L(AG(\mathbb{Z}_n))$ is a ring graph if and only if $n \in \{2,3,4,6\}$. □

Theorem 10. *The graph $L(AG(\mathbb{Z}_n))$ is outerplanar if and only if $n \in \{2,3,4,6\}$.*

Proof. Since every outerplanar graph is a ring graph, we only need to check for $n \in \{2,3,4,6\}$. It can be easily seen that for these values of n, all the vertices of $L(AG(\mathbb{Z}_n))$ lie on the unbounded face. Thus, $L(AG(\mathbb{Z}_n))$ is outerplanar if and only if $n \in \{2,3,4,6\}$. □

7. $\overline{AG(\mathbb{Z}_n)}$ as a Planar Graph, a Ring Graph, and an Outerplanar Graph

In this section, we discuss the planarity of the graphs $\overline{AG(\mathbb{Z}_n)}$ for $n > 1$. We further characterize when they are ring graphs and outerplanar graphs. In this regard, we have the following results.

Lemma 5. *If n contains more than two distinct prime factors, then* $AG(\mathbb{Z}_n)$ *contains* $K_1 \cup K_2 \cup K_4$ *as a subgraph.*

Proof. Suppose $n = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1, \ldots, p_r$ are distinct prime numbers in increasing order, $k_1, \ldots, k_r$ are positive integers and $r \geq 3$. We divide the proof into the following two cases based on the values of p_1, p_2, and p_3:

Case 1. If $p_1 = 2, p_2 = 3$, and $p_3 = 5$, in this case, the numbers $2, 3$, and 5 are positive divisors of n. According to Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_4$ as a subgraph.

Case 2. If $p_1 \geq 2, p_2 \geq 3$, and $p_3 > 5$, again, the numbers p_1, p_2, and p_3 are positive divisors of n. Based on Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_{p_1-1} \cup K_{p_2-1} \cup K_{p_3-1}$ as a subgraph. Since $p_1 \geq 2, p_2 \geq 3$, and $p_3 > 5$, we have $p_1 - 1 \geq 1, p_2 - 1 \geq 2$ and $p_3 - 1 > 4$. Hence, the graph $K_1 \cup K_2 \cup K_4$ is present as a subgraph in $AG(\mathbb{Z}_n)$. □

Remark 1. *The graph* $\overline{K_1 \cup K_2 \cup K_4}$ *is not planar (see Figure 4).*

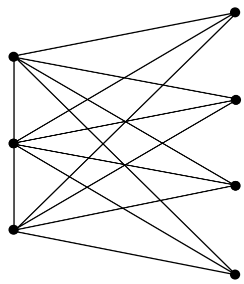


Figure 4. The graph $\overline{K_1 \cup K_2 \cup K_4}$.

Lemma 6. *If* $n = p_1^{k_1} p_2^{k_2}$, *where* p_1, p_2 *are distinct prime numbers in increasing order and* k_1, k_2 *are positive integers such that at least one of them is greater than 1, then* $AG(\mathbb{Z}_n)$ *contains* $K_1 \cup K_2 \cup K_2 \cup K_2$ *as a subgraph.*

Proof. We divide the proof into the following two cases:

Case 1. If $k_1 \geq 2$ and $k_2 = 1$, we further divide the proof into cases based on the value of n:

Case 1a. If $n = 2^2 \times 3$, the numbers $2, 4, 3$ and 6 are positive divisors of n. Based on Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_2 \cup K_2$ as a subgraph.

Case 1b. If $n = 2^{k_1} \times 3$, where $k_1 \geq 3$, again, the numbers $2, 4, 3$, and 6 are positive divisors of n. Based on Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_2 \cup K_2$ as a subgraph.

Case 1c. If $n = q_1^{k_1} q_2$, where q_1 and q_2 are distinct prime numbers in increasing order such that $q_1 \geq 2$ and $q_2 > 3$, again, the numbers q_1, q_1^2, q_2 and $q_1 q_2$ are positive divisors of n. According to Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_{q_1-1} \cup K_{q_1(q_1-1)} \cup K_{q_2-1} \cup K_{(q_1-1)(q_2-1)}$ as a subgraph. Since $q_1 \geq 2$ and $q_2 > 3$, we have $q_1 - 1 \geq 1, q_1(q_1-1) \geq 2, q_2 - 1 > 2$ and $(q_1-1)(q_2-1) > 2$. Hence, the graph $K_1 \cup K_2 \cup K_2 \cup K_2$ is present as a subgraph in $AG(\mathbb{Z}_n)$.

Case 2. If k_1 is any positive integer and $k_2 \geq 2$. We further divide the proof into cases based on the value of n:

Case 2a. If $n = 2 \times 3^2$, the numbers $2, 3, 9$, and 6 are positive divisors of n. Based on Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_6 \cup K_2$ as a subgraph. Hence, the graph $K_1 \cup K_2 \cup K_2 \cup K_2$ is present as a subgraph in $AG(\mathbb{Z}_n)$.

Case 2b. If $n = 2^2 \times 3^2$, again, the numbers $2, 3, 9$ and 6 are positive divisors of n. Based on Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_2 \cup K_2$ as a subgraph.

Case 2c. If $n = 2^{k_1} \times 3^{k_2}$, where $k_1 \geq 1$ and $k_2 \geq 3$, again, the numbers $2, 3, 9$ and 6 are positive divisors of n. Based on Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_2 \cup K_2$ as a subgraph.

Case 2d. If $n = q_1^{k_1} q_2^{k_2}$, where q_1 and q_2 are distinct prime numbers in increasing order such that $q_1 \geq 2$ and $q_2 > 3$, again, the numbers q_1, q_2, q_2^2 and $q_1 q_2$ are positive divisors of n. Based on Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_{q_1-1} \cup K_{q_2-1} \cup K_{q_2(q_2-1)} \cup K_{(q_1-1)(q_2-1)}$ as a subgraph. Since $q_1 \geq 2$ and $q_2 > 3$, we have $q_1 - 1 \geq 1, q_2 - 1 > 2, q_2(q_2 - 1) > 6$ and $(q_1 - 1)(q_2 - 1) > 2$. Hence, the graph $K_1 \cup K_2 \cup K_2 \cup K_2$ is present as a subgraph in $AG(\mathbb{Z}_n)$.

Hence, in all the cases, $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_2 \cup K_2$ as a subgraph. □

Remark 2. *The graph $\overline{K_1 \cup K_2 \cup K_2 \cup K_2}$ is not planar, being homeomorphic to K_5 (see Figure 5).*

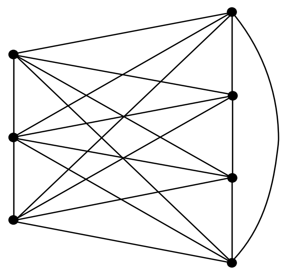


Figure 5. The graph $\overline{K_1 \cup K_2 \cup K_2 \cup K_2}$.

Lemma 7. *If $n = p_1 p_2$, where p_1 and p_2 are distinct prime numbers in increasing order such that $p_1 > 2$, then $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_4$ as a subgraph.*

Proof. Since p_1 is a prime number greater than 2, we divide the proof into the following two cases based on the value of n:

Case 1. If $n = 3 \times 5$, $AG(\mathbb{Z}_n) = K_2 \cup K_4 \cup K_8$. Hence, the graph $K_1 \cup K_2 \cup K_4$ is present as a subgraph in $AG(\mathbb{Z}_n)$.

Case 2. If $n = q_1 q_2$, where q_1 and q_2 are distinct prime numbers in increasing order such that $q_1 \geq 3$ and $q_2 > 5$, then $AG(\mathbb{Z}_n) = K_{q_1-1} \cup K_{q_2-1} \cup K_{(q_1-1)(q_2-1)}$. Since $q_1 \geq 3$ and $q_2 > 5$, we have $q_1 - 1 \geq 2, q_2 - 1 > 4$ and $(q_1 - 1)(q_2 - 1) > 8$. Again, the graph $K_1 \cup K_2 \cup K_4$ is present as a subgraph in $AG(\mathbb{Z}_n)$. □

Lemma 8. *If $n = p^k$, where p is any prime number and k is a positive integer such that $k > 2$, then $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_4$ as a subgraph.*

Proof. We divide the proof into the following three cases based on the value of n:

Case 1. If $n = 2^3$, then $AG(\mathbb{Z}_n) = K_1 \cup K_2 \cup K_4$.

Case 2. If $n = 2^k$, where $k \geq 4$, then the numbers $2, 4$, and 8 are positive divisors of n. According to Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_4$ as a subgraph.

Case 3. If $n = q^k$, where q is a prime number greater than 2 and k is a positive integer such that $k \geq 3$, then the numbers q, q^2 and q^3 are positive divisors of n. Again, based on Theorem 3, the graph $AG(\mathbb{Z}_n)$ contains $K_{q-1} \cup K_{q(q-1)} \cup K_{q^2(q-1)}$ as a subgraph. Since $q > 2$, we have $q - 1 > 1, q(q - 1) > 2$ and $q^2(q - 1) > 4$. Hence, the graph $K_1 \cup K_2 \cup K_4$ is present as a subgraph in $AG(\mathbb{Z}_n)$. □

Theorem 11. *The complement of $AG(\mathbb{Z}_n)$ is planar if and only if either n is a prime number or $n \in \{4, 6, 9\}$.*

Proof. Suppose $\overline{AG(\mathbb{Z}_n)}$ is planar and $n = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1, \ldots, p_r$ are distinct prime numbers in increasing order and $k_1, \ldots, k_r$ are positive integers.

We claim that $r \leq 2$. Suppose $r > 2$. Based on Lemma 5, the graph $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_4$ as a subgraph. In turn, $\overline{AG(\mathbb{Z}_n)}$ contains $\overline{K_1 \cup K_2 \cup K_4}$ as a subgraph. Consequently, $\overline{AG(\mathbb{Z}_n)}$ contains $K_{3,3}$ as a subgraph, which contradicts our assumption that $\overline{AG(\mathbb{Z}_n)}$ is planar. Therefore, $r \leq 2$. That is, n cannot have more than two prime factors. We divide our result into the following two cases:

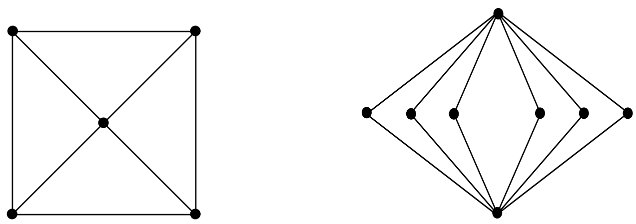
Case 1. If $n = p_1^{k_1} p_2^{k_2}$, we now claim that $k_1 = k_2 = 1$. On the contrary, if any of the k_1 or k_2 is greater than 1, then according to Lemma 6, $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_2 \cup K_2$ as a subgraph. In turn, $\overline{AG(\mathbb{Z}_n)}$ contains $\overline{K_1 \cup K_2 \cup K_2 \cup K_2}$ as a subgraph. Consequently, $\overline{AG(\mathbb{Z}_n)}$ contains $K_{3,3}$ as a subgraph, which contradicts our assumption that $\overline{AG(\mathbb{Z}_n)}$ is planar. Therefore, $k_1 = k_2 = 1$. That is, n is equal to the product of two primes p_1 and p_2. We further claim that $p_1 = 2$. On the contrary, suppose $p_1 > 2$. According to Lemma 7, $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_4$ as a subgraph. In turn, this contradicts the assumption. Therefore, $p_1 = 2$ and hence $n = 2p$, where p is a prime number greater than 2. In turn, $AG(\mathbb{Z}_n) = K_1 \cup K_{p-1} \cup K_{p-1}$. Now, if p is greater than 3, then $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_4$ as a subgraph. Consequently, this contradicts the assumption that $\overline{AG(\mathbb{Z}_n)}$ is planar. Thus, p must be equal to 3.

Case 2. If $n = p^k$. We now claim that $k \leq 2$. On the contrary, if $k > 2$, then according to Lemma 8, $AG(\mathbb{Z}_n)$ contains $K_1 \cup K_2 \cup K_4$ as a subgraph. In turn, this contradicts our assumption. Hence, $k \leq 2$. We further divide the result into the following two cases:

Case 2a. If $n = p$, then $AG(\mathbb{Z}_n)$ is K_{p-1}. In turn, $\overline{AG(\mathbb{Z}_n)}$ is a totally disconnected graph.

Case 2b. If $n = p^2$, then $AG(\mathbb{Z}_n)$ is $K_{p-1} \cup K_{p(p-1)}$. In turn, $\overline{AG(\mathbb{Z}_n)} = \overline{K_{p-1} \cup K_{p(p-1)}}$. If p is greater than 3, then this contradicts our assumption that $\overline{AG(\mathbb{Z}_n)}$ is planar. Therefore, p must be equal to 2 or 3.

Combining both cases, we find that either n is a prime number or $n \in \{4, 6, 9\}$. On the other hand, if n is a prime number, then $\overline{AG(\mathbb{Z}_n)}$ is a totally disconnected graph and hence it is planar. If $n = 4$, then $\overline{AG(\mathbb{Z}_n)}$ is a path graph on three vertices. The graphs $\overline{AG(\mathbb{Z}_6)}$ and $\overline{AG(\mathbb{Z}_9)}$ are given in Figure 6. It can be easily seen that the graphs are planar. □



(a) $\overline{AG(\mathbb{Z}_6)}$ **(b)** $\overline{AG(\mathbb{Z}_9)}$

Figure 6. The complement of the associate ring graphs.

Theorem 12. *The graph $\overline{AG(\mathbb{Z}_n)}$ is a ring graph if and only if either n is a prime number or $n = 4$.*

Proof. Since every ring graph is planar, we only need to check either for a prime number or $n \in \{4, 6, 9\}$. We have the following cases:

Case 1. If $n = p$, where p is a prime number, then the graph $\overline{AG(\mathbb{Z}_n)}$ is totally disconnected.

Case 2. If $n = 4$, the graph $\overline{AG(\mathbb{Z}_4)}$ is a path graph on three vertices.

Case 3. If $n = 6$, the graph $\overline{AG(\mathbb{Z}_6)}$ is given in Figure 6a.

Case 4. If $n = 9$, the graph $\overline{AG(\mathbb{Z}_9)}$ is given in Figure 6b.

It is evident that $\overline{AG(\mathbb{Z}_n)}$ is a ring graph if and only if either n is a prime number or $n = 4$. □

Theorem 13. *The graph $\overline{AG(\mathbb{Z}_n)}$ is outerplanar if and only if n is either a prime number or $n = 4$.*

Proof. Since every outerplanar graph is a ring graph, we check either for a prime number or $n = 4$. It can be easily seen that for these values of n, all the vertices of $\overline{AG(\mathbb{Z}_n)}$ lie on the unbounded face. Thus, $\overline{AG(\mathbb{Z}_n)}$ is outerplanar if and only if n is a prime number or $n = 4$. □

8. $L(\overline{AG(\mathbb{Z}_n)})$ as a Planar Graph, a Ring Graph, and an Outerplanar Graph

We now discuss the line graph of $\overline{AG(\mathbb{Z}_n)}$ for $n > 1$.

Theorem 14. *The line graph $L(\overline{AG(\mathbb{Z}_n)})$ is planar if and only if n is either a prime number or $n = 4$.*

Proof. Since the planarity of a graph is the necessary condition for its line graph to be planar, we only need to check for a prime number or $n \in \{4, 6, 9\}$. We have the following cases:

Case 1. If $n = p$, where p is a prime number, then the graph $\overline{AG(\mathbb{Z}_n)}$ is totally disconnected. In turn, $L(\overline{AG(\mathbb{Z}_2)})$ is a null graph with no vertex.

Case 2. If $n = 4$, the graph $\overline{AG(\mathbb{Z}_4)}$ is a path graph on three vertices. The graph $L(\overline{AG(\mathbb{Z}_3)}) = K_2$.

Case 3. If $n = 6$, the graph $\overline{AG(\mathbb{Z}_6)}$ is given in Figure 6a. The vertex with a degree equal to 4 is not a cut vertex. Therefore, based on Theorem 1, $L(\overline{AG(\mathbb{Z}_6)})$ is not planar.

Case 4. If $n = 9$, the graph $\overline{AG(\mathbb{Z}_9)}$ is given in Figure 6b. Again, the vertex with a degree equal to 4 is not a cut vertex. Therefore, according to Theorem 1, $L(\overline{AG(\mathbb{Z}_9)})$ is not planar.

Hence, $L(\overline{AG(\mathbb{Z}_n)})$ is planar if and only if either n is a prime number or $n = 4$. □

Theorem 15. *The line graph $L(\overline{AG(\mathbb{Z}_n)})$ is a ring graph if and only if n is either a prime number or $n = 4$.*

Proof. Since every ring graph is planar, we only need to check for a prime number or $n = 4$. Since, in both cases, $L(\overline{AG(\mathbb{Z}_n)})$ is acyclic, the result is trivially true. □

Theorem 16. *The graph $L(\overline{AG(\mathbb{Z}_n)})$ is outerplanar if and only if either n is a prime number or $n = 4$.*

Proof. Since every outerplanar graph is a ring graph, we check for a prime number or $n = 4$. It can be easily seen that for these values of n, all the vertices of $L(\overline{AG(\mathbb{Z}_n)})$ lie on the unbounded face. Thus, $L(\overline{AG(\mathbb{Z}_n)})$ is outerplanar if and only if either n is a prime number or $n = 4$. □

9. Properties of the Associate Ring Graph $AG(R)$

This section discusses some properties of an associate ring graph of a commutative ring R with unity.

Theorem 17. *Let R be a ring. The diameter of an associate ring graph is given by*

$$diam(AG(R)) = \begin{cases} 1, & \text{if } R \text{ is a field}, \\ \infty, & \text{otherwise}. \end{cases}$$

Proof. If R is a field, then $AG(R)$ is a complete graph. Therefore, the diameter of $AG(R)$ is 1. On the other hand, if R is not a field, then $AG(R)$ has at least two components, so the diameter of $AG(R)$ is infinite. □

Corollary 4. *The diameter of $AG(\mathbb{Z}_n)$ is 1 if and only if n is a prime number.*

Lemma 9. *If $n = p_1^{k_1} p_2^{k_2}$, where p_1, p_2 are distinct prime numbers in increasing order and k_1, k_2 are positive integers such that at least one of them is greater than 1, then $n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \geq 4$.*

Proof. The proof is exactly on the same lines as that of Lemma 2. □

Lemma 10. *If $n = p^k$, where p is any prime number and k is a positive integer such that $k > 2$, then $p^{k-1}(p-1) \geq 4$.*

Proof. The proof is exactly on the same lines as that of Lemma 4. □

Theorem 18. *The girth of an associate ring graph $AG(\mathbb{Z}_n)$ is given by*

$$g(AG(\mathbb{Z}_n)) = \begin{cases} \infty, & \text{if } n = 2, 3, 4, \text{ and } 6: \\ 3, & \text{otherwise.} \end{cases}$$

Proof. According to Theorem 3, for each positive divisor d of n except 1, there exists a complete graph on $\phi(d)$ vertices as a component in $AG(\mathbb{Z}_n)$. Clearly, if this value is greater than 2, then $g(AG(\mathbb{Z}_n)) = 3$. So we only need to check for which values of n Euler's totient function applied to the divisors of n gives a value less than or equal to 2. In this regards, let $n = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1, \ldots, p_r$ are distinct prime numbers in increasing order and $k_1, \ldots, k_r$ are positive integers.

We claim that $r \leq 2$. Suppose $r > 2$. In this case, the number $p_1^{k_1} p_2^{k_2} p_3^{k_3}$ is a positive divisor of n. Based on Lemma 1, $\phi(p_1^{k_1} p_2^{k_2} p_3^{k_3})$ is greater than or equal to 8. In this case, $AG(\mathbb{Z}_n)$ contains a complete graph of order 4 as a subgraph. That is, $g(AG(\mathbb{Z}_n)) = 3$. Therefore, $r \leq 2$. That is, n cannot have more than two prime factors. We divide our result into the following two cases:

Case 1. If $n = p_1^{k_1} p_2^{k_2}$, we now claim that $k_1 = k_2 = 1$. On the contrary, if any of the k_1 or k_2 is greater than 1, then based on Lemma 9, $\phi(n) \geq 4$. Again, $g(AG(\mathbb{Z}_n)) = 3$. Therefore, $k_1, k_2 \leq 1$. We further claim that $p_1 = 2$. On the contrary, suppose $p_1 > 2$. Since $p_1 p_2$ is a positive divisor of n, according to Theorem 3, $AG(\mathbb{Z}_n)$ contains a complete graph of order $\phi(p_1 p_2) = (p_1 - 1)(p_2 - 1)$. Based on Lemma 3, this value is greater than or equal to 8. Therefore, $p_1 = 2$. Now, if p_2 is greater than 3, then $AG(\mathbb{Z}_n)$ contains a complete graph of order 4 as a subgraph. Therefore, p_2 must be equal to 3.

Case 2. If $n = p^k$, we now claim that $k \leq 2$. On the contrary, if $k > 2$, then p^3 is a positive divisor of n. That is, $AG(\mathbb{Z}_n)$ contains a complete graph of order $\phi(p^3) = p^2(p-1)$. According to Lemma 10, this value is greater than or equal to 4. Hence, $k \leq 2$. We further divide the result into the following three cases:

Case 2a. If $n = p$, then $AG(\mathbb{Z}_n)$ is K_{p-1}. If p is greater than 3, then $AG(\mathbb{Z}_n)$ contains a complete graph of order 4 as a subgraph. Therefore, p must be equal to 2 or 3.

Case 2b. If $n = p^2$, then $AG(\mathbb{Z}_n)$ is $K_{p-1} \cup K_{p(p-1)}$. If p is greater than 2, then $AG(\mathbb{Z}_n)$ contains a complete graph of order 4 as a subgraph. Therefore, p must be equal to 2.

Combining both the cases, $g(AG(\mathbb{Z}_n)) = \infty$ if and only if $n \in \{2, 3, 4, 6\}$. On the other hand, if n belongs to this set, then $AG(\mathbb{Z}_n)$ is a union of complete graphs the orders of which are less than or equal to 2. Thus, $g(AG(\mathbb{Z}_n)) = \infty$. □

Theorem 19. *Let R be a ring. The chromatic number of $AG(R)$ is equal to the clique number of $AG(R)$.*

Proof. Since every component of the associate ring graph $AG(R)$ of a ring R is complete, the chromatic number of $AG(R)$ is equal to the clique number of $AG(R)$. □

Theorem 20. *For an associate ring graph $AG(\mathbb{Z}_n)$,*

$$\chi(AG(\mathbb{Z}_n)) = \omega(AG(\mathbb{Z}_n)) = \phi(n),$$

where $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of a graph G, respectively.

Proof. Based on Theorem 3, for each positive divisor d of n except 1, there exists a complete graph on $\phi(d)$ vertices as a component in $AG(\mathbb{Z}_n)$. Among all the divisors, n corresponds to the largest value. Therefore, the chromatic number and the clique number of a graph $AG(\mathbb{Z}_n)$ are equal to $\phi(n)$. □

Corollary 5. *For a prime p, $\omega(AG(\mathbb{Z}_p))$ is $p-1$.*

Theorem 21. *The associate ring graph $AG(\mathbb{Z}_n)$ is a strongly perfect graph.*

Proof. Based on Theorem 3, every induced subgraph of associate ring graph $AG(\mathbb{Z}_n)$ is a P_4-free graph and hence according to Theorem 2, the associate ring graph $AG(\mathbb{Z}_n)$ is a strongly perfect graph. □

Theorem 22. *If R is an integral domain with $|U(R)| = r$, then $AG(R)$ is $(r-1)$-regular graph, where $U(R)$ denotes the number of units in R and r is a positive integer.*

Proof. Let R be an integral domain with $|U(R)| = r$. Since R contains no nonzero zero-divisor, for every $a \in R$, $|Or(a)| = |U(R)|$. Therefore, each component is a complete graph on r vertices. Hence, $AG(R)$ is $(r-1)$-regular. □

Theorem 23. *Let $d_1, d_2, \ldots, and d_k$ all be positive divisors of n in increasing order such that $d_1 > 1$. Then the number of edges in $AG(\mathbb{Z}_n)$ is equal to*

$$\frac{1}{2}(n-1)(n-2) - \sum_{1 \le i < j \le k} \phi(d_i)\phi(d_j),$$

where ϕ denotes Euler's totient function.

Proof. Let v be a vertex of $AG(\mathbb{Z}_n)$ and let $d(v)$ denote the degree of v. We first evaluate $\sum\limits_{v \in V(AG(\mathbb{Z}_n))} d(v)$. Using Theorem 3, we have

$$\begin{aligned}
\sum_{v \in V(AG(\mathbb{Z}_n))} d(v) &= \sum_{v \in V(K_{\phi(d_1)})} d(v) + \sum_{v \in V(K_{\phi(d_2)})} d(v) + \ldots + \sum_{v \in V(K_{\phi(d_k)})} d(v) \\
&= \phi(d_1)(\phi(d_1) - 1) + \phi(d_2)(\phi(d_2) - 1) + \ldots + \phi(d_k)(\phi(d_k) - 1)
\end{aligned}$$

$$= (\phi(d_1))^2 + (\phi(d_2))^2 + \ldots (\phi(d_k))^2 - (\phi(d_1)\phi(d_2) + \phi(d_1)\phi(d_3) + \ldots$$

$$+ \phi(d_1)\phi(d_k) + \phi(d_2)\phi(d_3) + \ldots + \phi(d_2)\phi(d_k) + \ldots$$
$$+ \phi(d_{k-1})\phi(d_k)) - (\phi(d_1) + \phi(d_2) + \ldots + \phi(d_k))$$

Since, $\sum_{d|n} \phi(d) = n$, we have

$$= (n-1)^2 - 2\sum_{1\leq i<j\leq k} \phi(d_i)\phi(d_j) - (n-1)$$
$$= (n-1)(n-2) - 2\sum_{1\leq i<j\leq k} \phi(d_i)\phi(d_j)$$

Now, since each edge contributes two to the degrees, the number of edges is equal to $\frac{1}{2}\sum_{v\in V(AG(\mathbb{Z}_n))} d(v)$, and we have

$$= \frac{1}{2}(n-1)(n-2) - \sum_{1\leq i<j\leq k} \phi(d_i)\phi(d_j).$$

□

A bijective mapping of a finite set S into itself is called a permutation. The set of all permutations on a set with n elements satisfies the axioms of a group and this group is called a permutation group of degree n. An automorphism of a graph G is an isomorphism of G onto itself. Thus, every automorphism of G is a permutation of the vertex set which preserves the relation "is a neighbor of". Obviously, the bijection takes a vertex to a vertex of the same degree.

It is also clear that the composition of two automorphisms is an automorphism; so the automorphisms of G form a permutation group on the vertex set of G. We call this the group of automorphisms of G and write $Aut(G)$.

Lemma 11 ([19]). *If G is a connected graph, then*

$$Aut(nG) = S_n[Aut(G)].$$

Lemma 12 ([19]). *If G_1 and G_2 are disjoint, connected, nonisomorphic graphs, then*

$$Aut(G_1 \cup G_2) = Aut(G_1) + Aut(G_2).$$

The following theorem determines the structure of the automorphism group of $AG(\mathbb{Z}_n)$.

Theorem 24. *If $n \geq 2$ is a positive integer, then the automorphism group of $AG(\mathbb{Z}_n)$ is isomorphic to $Aut(AG(\mathbb{Z}_n)) \cong S_{n_1}[S_{|G_1|}] + S_{n_2}[S_{|G_2|}] + \ldots + S_{n_r}[S_{|G_r|}]$.*

Proof. According to Theorem 3, graph $AG(\mathbb{Z}_n)$ can be written as $AG(\mathbb{Z}_n) = n_1G_1 \cup n_2G_2 \cup \ldots \cup n_rG_r$, where all G_is are complete subgraphs of $AG(\mathbb{Z}_n)$ and n_i is the number of components of $AG(\mathbb{Z}_n)$ isomorphic to G_i. Applying the above lemmas, we have the result

$$Aut(AG(\mathbb{Z}_n)) \cong S_{n_1}[Aut(G_1)] + S_{n_2}[Aut(G_2)] + \ldots + S_{n_r}[Aut(G_r)].$$

Since we know that $Aut(K_n) \cong S_n$. Then, we have

$$Aut(AG(\mathbb{Z}_n)) \cong S_{n_1}[S_{|G_1|}] + S_{n_2}[S_{|G_2|}] + \ldots + S_{n_r}[S_{|G_r|}].$$

□

For an associate ring graph $AG(\mathbb{Z}_n)$, by relabeling the vertices as $v_i = i-1$, the adjacency matrix associated with $AG(\mathbb{Z}_n)$ is an $n \times n$ matrix $AG(\mathbb{Z}_n) = [aij]$ of which the rows (and columns) correspond to integers $0, 1, \ldots, n-1$ and is such that $a_{ii} = 0$ and for $i \neq j$, $a_{ij} = 1$ if i and j are associates; otherwise they are 0. The spectrum of an adjacency matrix is a list of its eigenvalues along with their multiplicities. The spectrum of a graph G, denoted by $Spec(G)$, is a spectrum of its adjacency matrix.

Theorem 25. *Let $d_1, d_2, \ldots, d_k$ be all positive divisors of n in increasing order such that $d_1 > 1$. For an associate ring graph $AG(\mathbb{Z}_n)$,*

$$Spec(AG(\mathbb{Z}_n)) = \begin{pmatrix} \phi(d_1) - 1 & \phi(d_2) - 1 & \cdots & \phi(d_k) - 1 & -1 \\ 1 & 1 & \cdots & 1 & n - \tau(n) \end{pmatrix},$$

where τ computes the number of positive divisors of n.

Proof. Based on Theorem 3 and the fact that the spectrum of the union of disjoint graphs is the union of the spectrum of individual graphs, the result holds. □

10. Python 3.8 Program for Finding the Structure of $AG(\mathbb{Z}_n)$

In this section, we present a Python 3.8 program to construct the graph $AG(\mathbb{Z}_n)$. In the following, we present are commands from the Python workspace. We used the libraries networkx, matplotlib, and numpy.

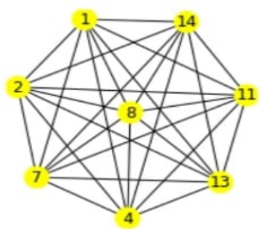
```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt

# Define a function 'gcd' that returns the greatest common
   divisor of numbers 'a' and 'b'.
def gcd(a,b):
    if a==0:
        return b
    return gcd(b%a,a)

# Define a function 'divisors' that returns positive divisors of
   a number 'n' except 1.
def divisors(n):
    d = []
    for i in range(2,n+1):
        if n%i==0:
            d.append(i)
    return d

# Define a function 'prime_divisors' that returns prime divisors
   of a number 'n'.
def prime_divisors(n):
    l = []
    t = n
    for i in range(2,t+1):
        if n%i==0:
            l.append(i)
        while n%i==0:
            n = n//i
    return l

n = int(input("The value of n is ")) # Input the value of 'n'.
print("The positive divisors of",n,"except 1 are",end=" ")
d = divisors(n)
print(d,end=".")
```

```
print()

# Calculate Euler's totient function of every element of 'd'.
eulerTotient = []
for j in range(len(d)):
    l = prime_divisors(d[j])
    phiJ = d[j]
    for i in l:
        phiJ = phiJ*(i-1)
    for i in l:
        phiJ = phiJ//i
    eulerTotient.append(phiJ)

components = len(d)
print("The total number of components in the associate ring graph
     of a ring of integers modulo "+str(n)+" is "+str(components)+
    ".")
print()

print("The",components,"components of the associate ring graph of
     a ring of integers modulo",n,"are of the following orders,
    respectively"+".")
for i in range(len(eulerTotient)):
    temp = (d[i],eulerTotient[i])
    print('\u03C6(',end="")
    print(d[i],end="")
    print(") = ",end=" ")
    print(eulerTotient[i],end="")
    if i!=components-1:
        print(",",end="")
    else:
        print(".",end="")
    print()

print()

# Calculate the orders of the elements of a ring of integers
   modulo 'n' and put it into a dictionary 'D'.
D = {}
for i in range(1,n):
    order = n//gcd(i,n)
    if order in D.keys():
        D[order].append(i)
    else:
        D.setdefault(order,[i])

# Plot the components of the associate ring graph of a ring of
   integers modulo 'n'.
for order,vertices in D.items():
    print("A component of order",len(vertices),"with vertex set "
    ,end="")
    print("Or("+str(vertices[0])+") = "+"{",end="")
    length = len(vertices)
    i = 0
```

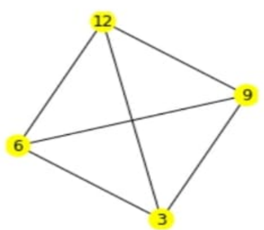
```
    for vertex in vertices:
        print(vertex,end="")
        if i!=length-1:
            print("",end=",")
        i+=1
    print('}',end=" ")
    print("is given below.")
    g = nx.complete_graph(vertices)
    color_map = ['yellow']*len(vertices)
    plt.figure(3,figsize=(2.5,2.5))
    nx.draw(g,node_color=color_map,with_labels=1)
    plt.show()
print("The union of the above graphs is the associate ring graph
    of a ring of integers modulo "+str(n)+".")
```

An output is given when above Python program is run for $n = 15$.

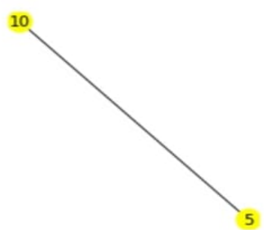
```
The value of n is 15
The positive divisors of 15 except 1 are [3, 5, 15].
The total number of components in the associate ring graph of a
ring of integers modulo 15 is 3.

The 3 components of the associate ring graph of a ring of intege
rs modulo 15 are of the following orders, respectively.
φ(3) =  2,
φ(5) =  4,
φ(15) =  8.

A component of order 8 with vertex set Or(1) = {1,2,4,7,8,11,13,
14} is given below.
```




```
A component of order 4 with vertex set Or(3) = {3,6,9,12} is giv
en below.
```




```
A component of order 2 with vertex set Or(5) = {5,10} is given b
elow.
```




```
The union of the above graphs is the associate ring graph of a r
ing of integers modulo 15.
```

11. Conclusions

In this article, we have studied the associate ring graph of the ring of integers modulo n. In general, $AG(\mathbb{Z}_n)$ is a disconnected graph and each component of it is complete. This property of the associate ring graph interconnects it with social network analysis. In the following scenario, the clique problem emerges. Consider a social network in which the edges indicate mutual acquaintances and the vertices represent persons. Then a clique is a collection of people who all know each other, and algorithms for discovering cliques can be used to find these groupings of common friends. The clique problem has numerous applications in bioinformatics and computational chemistry, in addition to social networks.

Moreover, this definition provides another class of strongly perfect graphs. We have also characterized when $AG(\mathbb{Z}_n)$, its complement $\overline{AG(\mathbb{Z}_n)}$, and its line graphs are planar, ring, and outerplanar graphs. We have also provided an algorithm and a Python 3.8 code for the complete structure of $AG(\mathbb{Z}_n)$. Some other properties of associate ring graphs can also be investigated and the definition can be extended to other ring structures as well.

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