Article

Classical and Bayesian Inference of the Inverse Nakagami Distribution Based on Progressive Type-II Censored Samples

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Abstract: This paper explores statistical inferences when the lifetime of product follows the inverse Nakagami distribution using progressive Type-II censored data. Likelihood-based and maximum product of spacing (MPS)-based methods are considered for estimating the parameters of the model. In addition, approximate confidence intervals are constructed via the asymptotic theory using both likelihood and product spacing functions. Based on traditional likelihood and the product of spacing functions, Bayesian estimates are also considered under a squared error loss function using non-informative priors, and Gibbs sampling based on the MCMC algorithm is proposed to approximate the Bayes estimates, where the highest posterior density credible intervals of the parameters are obtained. Numerical studies are presented to compare the proposed estimators using Monte Carlo simulations. To demonstrate the proposed methodology in a real-life scenario, a well-known data set on agricultural machine elevators with high defect rates is also analyzed for illustration.

Keywords: inverse Nakagami model; progressively censored data; maximum likelihood estimation; maximum product of spacing approach; Bayesian inference

MSC: 62F10; 62F15

1. Introduction

Due to non availability of information/data, experiments are often conducted with censoring schemes where the testing is ceased before the failure of all units. Generally speaking, censoring means that there is only a portion of exact failure times of observed units are known under studies. Two common censoring schemes are Type-I censoring and Type-II censoring. In Type-I censoring, the test is terminated at a predetermined time, whereas in Type-II censoring experiment stops at a predetermined number of failures. However, such conventional censoring schemes sometimes may take a longer time due to the product characteristic of being highly reliable with a long lifetime.

Therefore, in order to find a more efficient way to collect failure information in lifetime studies, many other censoring schemes have been proposed, and progressive Type-II censoring is the most popular in practice. In this scheme, the detailed scenarios can be described as follows. Suppose that \( n \) independent identical components are placed in a life test, and only \( m(\leq n) \) failures are observed with a prefixed censoring scheme \( R = (r_1, r_2, \ldots, r_m) \). When the first failure time, say \( x_{1:mn} \), has occurred, \( r_1 \) survival components are randomly removed from the test.

When the second failure time, say \( x_{2:mn} \), has occurred, \( r_2 \) survival components are randomly removed from the test. Following a similar procedure until the \( m \)th failure, say \( x_{m:mn} \), has occurred, the remaining \( r_m \) survival components are removed from the test, and the experiment stops. Therefore, \( x_{1:mn} \leq x_{2:mn} \leq \cdots \leq x_{m:mn} \) are called progressive Type-II censored samples with the censored scheme \( R = (r_1, r_2, \ldots, r_m) \). It is noted that
\[ n = m + \sum_{i=1}^{m} r_i \] and that many censoring schemes are its special cases, including Type-II censoring and the complete sample.

The conventional Type-I censoring, Type-II censoring and various progressive censoring scenarios have been extensively implemented and discussed in wide practical situations. See, for example, some recent works of Ducros and Pamphile [1], Zhao et al. [2], Jia [3], Kohansal and Shoeae [4], Mahto et al. [5], Luo et al. [6], Han and Kundu [7] and the references therein. For more details about conventional and progressive censoring schemes, one may refer to the monographs of Lawless [8], Balakrishnan and Aggarwala [9].

Let \( x_{i:m:n} \) be a progressive Type-II censored data with censoring scheme \((r_1, r_2, \ldots, r_m)\) from the population with cumulative distribution function (CDF) \( F(x) \) and probability density function (PDF) \( f(x) \), then the joint density function can be written as

\[
L(x_1, x_2, \ldots, x_m) = A \prod_{i=1}^{m} f(x_i) [1 - F(x_i)]^{r_i},
\]

with \( A = n(n-1-r_1) \cdots (n-r_1 - \cdots - r_{m-1} - m+1) \) being regular constants.

The Nakagami distribution is a well-known model in communications engineering, medical image processing, hydrologic engineering and seismological analysis (see, e.g., Wang et al. [10], Tsui et al. [11], Sarkar [12], Nakahara and Carcole [13]). As an extension and application of existing lifetime models, the corresponding inverse distributions have provided a potential comprehension of common models in data fitting (see, Basheer [14], Punzo [15], Ghitany et al. [16]). The inverse Nakagami distribution (IND) was introduced by Louzada [17] as an extension of the Nakagami distribution proposed by Nakagami [18]. A random variable \( X \) is said to have IND with the CDF

\[
F(x; \alpha, \theta) = \frac{1}{\Gamma(\alpha)} \Gamma\left(\alpha, \frac{\theta}{x^2}\right), \quad \alpha > 0.5, \; \theta > 0, \; x > 0,
\]

and the PDF as expressed as follows

\[
f(x; \alpha, \beta) = \frac{2}{\Gamma(\alpha)} \left(\frac{\alpha}{\beta}\right)^{\alpha} x^{-2\alpha-1} e^{-\frac{\alpha}{\beta} x}, \quad \alpha > 0.5, \; \beta > 0, \; x > 0,
\]

where \( \Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt \) is the complete gamma function.

In statistical inference, the maximum product of spacing estimation (MPS) technique was initially proposed by Cheng and Amin [19] and then further discussed by Ranneby [20] where the MPS method is treated as an alternative approach to traditional maximum likelihood estimation (MLE) for inferential studies. Recently, the MPS technique has attracted great attention in data analysis due to its potential advantage over the classical ML procedure. For instance, Singh et al. [21,22], Shao [23], El-Sherpieny et al. [24], Volovskiy and Kamps [25], Kawanishi [26] have discussed the MPS method under various data types when the lifetime distributions were the Weibull, power Lomax, generalized inverted exponential, generalized power Weibull, Lindley and generalized Rayleigh models, respectively. Anatolyev and Kosenok [27] studied the invariance property of the MPS and stated that the MPSEs have more efficient properties than the MLEs for skewed distributions or in small sample cases for heavy-tailed distributions.

Due to the IND’s importance and practicability, our aim in this paper is to develop an estimation procedure for IND parameters when progressive Type-II censored data is available. The MLEs and maximum product spacing estimates (MPSEs) as well as the approximate confidence intervals (ACIs) for both cases are obtained. In addition, Bayesian point and interval estimates are obtained based on likelihood functions from MLE and MPSE scenarios, respectively. As far as we know, no attempt has been made on estimation of IND parameters based on progressive Type-II censored data using the considered methods of estimation.
The rest of article is organized as follows. Classical likelihood based estimations, including MLE and ACI, are discussed in Section 2. Section 3 presented parameter estimation as well as ACI using the MPS technique. Bayesian inference is proposed in Section 4 based on likelihood and maximum product spacing functions, respectively, and associated Monte Carlo algorithms are proposed for complex posterior computation. Simulation studies and a real data example are presented in Section 5 for illustration. Our concluding remarks are presented in Section 6.

2. Classical Likelihood Estimation

In this section, the MLEs and ACIs of the parameters \( \alpha \) and \( \theta \) are obtained based on the classical likelihood function.

Suppose \( X_1 < X_2 < \cdots < X_m \) are progressive Type-II censored variables from IND with censoring scheme \( R = (r_1, r_2, \ldots, r_m) \), the likelihood function can be written from (1)–(3) as

\[
L(\alpha, \theta | x) \propto \Gamma(\alpha)^{-m} \theta^m e^{-\theta \sum_{i=1}^{m} x_i^{-2}} \prod_{i=1}^{m} x_i^{-2\alpha} \times \prod_{i=1}^{m} \left[ \frac{\gamma\left(\alpha, \theta x_i^{-2}\right)}{\Gamma(\alpha)} \right]^{r_i},
\]

where \( \gamma(s, x) = \int_0^x t^{s-1} e^{-t}dt \) is the lower incomplete gamma function. Ignoring the constant term, the log-likelihood function of \( \alpha \) and \( \theta \) is given by

\[
\ell(\alpha, \theta | x) = -m \ln \Gamma(\alpha) + m \alpha \ln \theta - \theta \sum_{i=1}^{m} x_i^{-2} - 2\alpha \sum_{i=1}^{m} \ln x_i + \sum_{i=1}^{m} r_i \left[ \ln \gamma\left(\alpha, \theta x_i^{-2}\right) - \ln \Gamma(\alpha) \right].
\]

2.1. Point Estimation Based on MLE Approach

Upon differentiating (5) with respect to \( \alpha \) and \( \theta \), the MLEs of \( \alpha \) and \( \theta \) can be obtained from the following equations

\[
\frac{\partial \ell(\alpha, \theta | x)}{\partial \alpha} = -m \eta(\alpha) + m \ln \theta - 2 \sum_{i=1}^{m} \ln x_i + \sum_{i=1}^{m} r_i \psi'_{\alpha} = 0,
\]

\[
\frac{\partial \ell(\alpha, \theta | x)}{\partial \theta} = m \frac{\alpha}{\theta} - \sum_{i=1}^{m} x_i^{-2} + \sum_{i=1}^{m} r_i \psi'_{\theta} = 0,
\]

where \( \eta(\alpha) = \frac{\partial \ln \Gamma(\alpha)}{\partial \alpha} \) is the digamma function, \( \psi(x_i) = \gamma\left(\alpha, \theta x_i^{-2}\right) \) and

\[
\psi'_{\alpha} = \frac{\partial \ln \psi(x_i)}{\partial \alpha} = \int_0^{\theta x_i^{-2}} \frac{\gamma(s, x) - 1}{\psi(x_i)} ds, \quad \psi'_{\theta} = \frac{\partial \ln \psi(x_i)}{\partial \theta} = \frac{\theta^{\alpha-1} e^{-\theta x_i^{-2}} x_i^{-2\alpha}}{\psi(x_i)}.
\]

It is noted that there are no closed forms for the MLEs \( \hat{\alpha} \) and \( \hat{\theta} \) of \( \alpha \) and \( \theta \); thus, numerical techniques, such as the Newton–Raphson method, could be implemented for computation.

2.2. ACIs Based on MLEs

The second derivatives of \( \ell(\alpha, \theta) \) can be obtained directly as follows

\[
\frac{\partial^2 \ell(\alpha, \theta | x)}{\partial \alpha^2} = -m \eta''(\alpha) + \sum_{i=1}^{m} r_i \psi''_{\alpha}, \quad \frac{\partial^2 \ell(\alpha, \theta | x)}{\partial \theta^2} = -m \frac{\alpha}{\theta^2} + \sum_{i=1}^{m} r_i \psi''_{\theta},
\]
where \( \eta'(\alpha) \) is the trigamma function, and

\[
\psi'' = \frac{\partial^2 \ln \psi(x_i)}{\partial \alpha^2} = \frac{\psi(x_i) \int_0^{\theta x_i^{-2}} \frac{\mu-1}{\sigma^2} \ln t \, dt - \left( \int_0^{\theta x_i^{-2}} \frac{\mu-1}{\sigma^2} \ln t \, dt \right)^2}{\psi^2(x_i)},
\]

\[
\psi'' = \frac{\partial^2 \ln \psi(x_i)}{\partial \theta^2} = \frac{x_i^{-2a} e^{-\theta x_i^{-2}} \theta^{a-1} \left[ (a - 1) \theta^{-1} - x_i^{-2} \right] \psi(x_i) - \left( \theta^a e^{-\theta x_i^{-2}} x_i^{-2} \right)^2}{\psi^2(x_i)},
\]

\[
\psi'' = \frac{\partial^2 \ln \psi(x_i)}{\partial \alpha \partial \theta} = \frac{x_i^{-2a} e^{-\theta x_i^{-2}} \theta^{a-1} \ln(\theta x_i^{-2}) \psi(x_i) - \theta^a e^{-\theta x_i^{-2}} x_i^{-2a} \int_0^{\theta x_i^{-2}} \frac{\mu-1}{\sigma^2} \ln t \, dt}{\psi^2(x_i)}.
\]

The approximate asymptotic variance–covariance matrix of the MLEs \( \hat{\alpha} \) and \( \hat{\theta} \) can be obtained through the inverse of the observed Fisher information matrix as

\[
I^{-1}_0(\hat{\alpha}, \hat{\theta}) = \left[ \begin{array}{cc}
-\frac{\partial^2 \ln L(\alpha, \theta)}{\partial \alpha^2} & -\frac{\partial^2 \ln L(\alpha, \theta)}{\partial \alpha \partial \theta} \\
-\frac{\partial^2 \ln L(\alpha, \theta)}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ln L(\alpha, \theta)}{\partial \theta^2}
\end{array} \right]^{-1}_{(\alpha, \theta)=(\hat{\alpha}, \hat{\theta})} = \left[ \begin{array}{cc}
\text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\theta}) \\
\text{cov}(\hat{\alpha}, \hat{\theta}) & \text{var}(\hat{\theta})
\end{array} \right].
\]

Under mild regularity conditions, the asymptotic distribution of the MLE is known that \((\hat{\alpha}, \hat{\theta}) \sim N((\alpha, \theta), I_0(\hat{\alpha}, \hat{\theta}))\). Thus, for arbitrary \( 0 < \tau < 1 \), the \((1 - \tau)\%\) ACIs of the parameters \( \alpha \) and \( \theta \) can be obtained as follows

\[
\hat{\alpha} \pm z_{\tau/2} \sqrt{\text{var}(\hat{\alpha})} \quad \text{and} \quad \hat{\theta} \pm z_{\tau/2} \sqrt{\text{var}(\hat{\theta})},
\]

where \( z_{\tau/2} \) is the upper \((\tau/2)\)th percentile point of the standard normal distribution.

3. Maximum Product of Spacing Estimation

In this section, MPSEs are constructed for the IND parameters \( \alpha \) and \( \theta \) under progressively censored data.

3.1. Point Estimation Based on MPS Approach

Following Ng et al. [28], the product spacings under the progressive Type-II censoring scheme can be constructed as

\[
S(\Theta) \propto \prod_{i=1}^m U_i [1 - F(x_i; \Theta)]^{r_i},
\]

with

\[
U_i = \begin{cases} 
F(x_i; \Theta), & \text{if } i = 1; \\
F(x_{i-1}; \Theta) - F(x_{i-1}; \Theta), & \text{if } i = 2, 3, \ldots, m; \\
1 - F(x_m; \Theta), & \text{if } i = m + 1.
\end{cases}
\]
where $F(x; \Theta)$ with $\Theta$ being the parameter vector of the lifetime distribution of the considered population. Therefore, for the progressively Type-II censored data from IND (2), the MPS function can be expressed as follows

$$S(\alpha, \theta) = \prod_{i=1}^{m+1} \left[ \frac{\Gamma(\alpha, \theta x_i^{-2}) - \Gamma(\alpha, \theta x_{i-1}^{-2})}{\Gamma(\alpha)} \right] \times \prod_{i=1}^{m} \left[ \frac{\gamma(\alpha, \theta x_i^{-2})}{\Gamma(\alpha)} \right]^{r_i} \tag{8}$$

and the associated logarithm of MPS function can be written as

$$\ln S(\alpha, \theta) = \sum_{i=1}^{m+1} \ln \left[ \frac{\Gamma(\alpha, \theta x_i^{-2}) - \Gamma(\alpha, \theta x_{i-1}^{-2})}{\Gamma(\alpha)} \right] - (m + 1) \ln \Gamma(\alpha)$$

$$+ \sum_{i=1}^{m} r_i \ln \gamma(\alpha, \theta x_i^{-2}) - \sum_{i=1}^{m} r_i \ln \Gamma(\alpha) \tag{9}$$

By taking derivatives of (9), the MPSEs of $\alpha$ and $\theta$ can be obtained from following non-linear equations, such as

$$\frac{\partial \ln S(\alpha, \theta)}{\partial \alpha} = \sum_{i=1}^{m+1} \frac{\zeta_a'(x_i) - \zeta_a'(x_{i-1})}{\zeta(x_i) - \zeta(x_{i-1})} + \sum_{i=1}^{m} r_i \psi_a' - (n + 1) \eta(\alpha),$$

$$\frac{\partial \ln S(\alpha, \theta)}{\partial \theta} = \sum_{i=1}^{m+1} \frac{\zeta_\theta'(x_i) - \zeta_\theta'(x_{i-1})}{\zeta(x_i) - \zeta(x_{i-1})} + \sum_{i=1}^{m} r_i \psi_\theta' \tag{10}$$

where $\zeta(x_i) = \Gamma(\alpha, \theta x_i^{-2})$ and

$$\zeta_a'(x_i) = \frac{\partial \zeta(x_i)}{\partial \alpha} = \int_{\theta x_i^{-2}}^\infty \frac{t^{\alpha - 1} \ln t}{c^\alpha} dt,$$

$$\zeta_\theta'(x_i) = \frac{\partial \zeta(x_i)}{\partial \theta} = -\theta^{\alpha - 1} x_i^{-2} e^{-\theta x_i^{-2}}.$$

Similarly, the MPSEs $\hat{\alpha}$ and $\hat{\theta}$ could be also obtained by using a numerical approach.

3.2. ACIs Based on MPSEs

In this subsection, the ACIs are constructed for the model parameters $\alpha$ and $\theta$ based on MPSEs.

The second derivatives of the log-MPS function (9) are presented as follows

$$\frac{\partial^2 \ln S(\alpha, \theta)}{\partial \alpha^2} = \sum_{i=1}^{m+1} \left[ \frac{\zeta_a''(x_i) - \zeta_a''(x_{i-1})}{\zeta'(x_i) - \zeta'(x_{i-1})} \right] \left[ \frac{\zeta(x_i) - \zeta(x_{i-1})}{\zeta(x_i) - \zeta(x_{i-1})} - \frac{\zeta_a'(x_i) - \zeta_a'(x_{i-1})}{\zeta(x_i) - \zeta(x_{i-1})} \right]^2$$

$$- (n + 1) \eta'(\alpha) + \sum_{i=1}^{m} r_i \psi_a'',$$

$$\frac{\partial^2 \ln S(\alpha, \theta)}{\partial \theta^2} = \sum_{i=1}^{m+1} \left[ \frac{\zeta_\theta''(x_i) - \zeta_\theta''(x_{i-1})}{\zeta'(x_i) - \zeta'(x_{i-1})} \right] \left( \frac{\zeta(x_i) - \zeta(x_{i-1})}{\zeta(x_i) - \zeta(x_{i-1})} - \frac{\zeta_\theta'(x_i) - \zeta_\theta'(x_{i-1})}{\zeta(x_i) - \zeta(x_{i-1})} \right)^2$$

$$+ \sum_{i=1}^{m} r_i \psi_\theta'',$$

and

$$\frac{\partial^2 \ln S(\alpha, \theta)}{\partial \alpha \partial \theta} = \sum_{i=1}^{m+1} \left\{ \frac{\zeta_\theta''(x_i) - \zeta_\theta''(x_{i-1})}{\zeta'(x_i) - \zeta'(x_{i-1})} \left[ \frac{\zeta(x_i) - \zeta(x_{i-1})}{\zeta(x_i) - \zeta(x_{i-1})} - \frac{\zeta_\theta'(x_i) - \zeta_\theta'(x_{i-1})}{\zeta(x_i) - \zeta(x_{i-1})} \right] \right\}$$

$$+ \sum_{i=1}^{m} r_i \psi_{a\theta}'',$$
where
\[ \zeta''_a(x_i) = \frac{\partial^2 \zeta_a'(x_i)}{\partial a^2} = \int_0^\infty t^{a-1} \frac{1}{e^t} dt, \]
\[ \zeta''_\theta(x_i) = \frac{\partial^2 \zeta_\theta'(x_i)}{\partial \theta^2} = -x_i^{-2}\theta \exp(-\theta x_i^2), \]
\[ \zeta''_{a\theta}(x_i) = \frac{\partial^2 \zeta_a'(x_i)}{\partial \theta \partial a} = -\theta^a - x_i^{-2}\theta \exp(-\theta x_i^2) \ln(\theta x_i^2). \]

Following similar line as conventional ACIs in Section 2.2, the asymptotic variance-covariance matrix of \( \hat{a} \) and \( \hat{\theta} \) under MPS scenario can be written as
\[ I_1(\hat{a}, \hat{\theta}) = \begin{bmatrix} -\frac{\partial^2 \ln S(a, \theta)}{\partial a^2} & -\frac{\partial^2 \ln S(a, \theta)}{\partial a \partial \theta} \\ -\frac{\partial^2 \ln S(a, \theta)}{\partial \theta \partial a} & -\frac{\partial^2 \ln S(a, \theta)}{\partial \theta^2} \end{bmatrix}^{-1} \begin{bmatrix} \text{var}(\hat{a}) & \text{cov}(\hat{a}, \hat{\theta}) \\ \text{cov}(\hat{a}, \hat{\theta}) & \text{var}(\hat{\theta}) \end{bmatrix}. \]

Therefore, based on the asymptotic properties of the MPSEs as in the case of the MLEs, it follows that \( (\hat{a}, \hat{\theta}) \sim N((a, \theta), I_0(\hat{a}, \hat{\theta})) \). Therefore, the \( (1 - \tau)\% \) ACIs of \( a \) and \( \theta \) are given, respectively, by
\[ \hat{a} \pm z_{\tau/2} \sqrt{\text{var}(\hat{a})} \quad \text{and} \quad \hat{\theta} \pm z_{\tau/2} \sqrt{\text{var}(\hat{\theta})}. \]

4. Bayesian Inference

Modern products may exhibit high reliable and long life-cycles, which sometimes yield limited availability of data in lifetime experiments where a limited sample size may heavily affect the accuracy of the inferential results. Therefore, the Bayesian inference is provided as an alternative to the traditional likelihood method due to its ability to integrate extra information into the inferential approach.

In this section, Bayesian inference is discussed for the unknown parameters based on the conventional likelihood function and MSP function, respectively. In this paper, an objective prior is considered for \( (a, \theta) \) as follows (see Louzada et al. [17])
\[ \pi(a, \theta) \propto \sqrt{\frac{\alpha \eta'(a)}{\theta}}. \]

4.1. Posterior Density Using the Likelihood Function

Based on the traditional likelihood function (5), the posterior density of \( a \) and \( \theta \) can be written as
\[ \Omega(a, \theta|x) \propto \pi(a, \theta)L(a, \theta) \]
\[ \propto \sqrt{\frac{\alpha \eta'(a)}{\theta}} \Gamma(a)^{-m - \beta m} e^{-\theta \sum_{i=1}^m x_i^2} \prod_{i=1}^m x_i^{-2\beta} \prod_{i=1}^m \left[ \frac{\gamma(\alpha, \theta, x_i^2)}{\Gamma(\alpha)} \right]^{-\gamma_a}. \]

For an arbitrary function \( \varphi(a, \theta) \) of parameters \( a \) and \( \theta \), the Bayes estimator \( \hat{\varphi}(a, \theta) \) under squared error loss function is given by
\[ \hat{\varphi}(a, \theta) = \frac{\int_0^\infty \int_0^\infty \varphi(a, \theta) \pi(a, \theta) L(a, \theta) d\alpha d\theta}{\int_0^\infty \int_0^\infty \pi(a, \theta) L(a, \theta) d\alpha d\theta}. \]

Clearly, the corresponding Bayes estimator of \( \varphi(a, \theta) \) is expressed as the ratio of two integrals, which does not have a specific form. To solve that problem, we utilize the MCMC method to obtain the Bayesian point estimates and the highest posterior density (HPD) credible intervals.
From the joint posterior density function (13), the conditional posterior distributions of $\alpha$ and $\theta$ can be expressed, respectively, as

$$
\Omega(\alpha|x, \theta) = \sqrt{\alpha \eta'(\alpha)} - 1 \Gamma(\alpha)^{-m} \prod_{i=1}^{m} x_i^{-2\alpha} \times \prod_{i=1}^{m} \left[ \frac{\gamma(\alpha, \theta x_i^{-2})}{\Gamma(\alpha)} \right]^\tau_i,
$$

and

$$
\Omega(\theta|x, \alpha) = \frac{1}{\theta} \theta^m e^{-\theta \sum_{i=1}^{m} x_i^{-2}} \times \prod_{i=1}^{m} \gamma(\alpha, \theta x_i^{-2})^\tau_i.
$$

Since the conditional posterior distribution of $\alpha$ and $\theta$ cannot be reduced analytically to some well-known distributions, therefore, in order to find the Bayes estimates, the Gibbs sampling based on MCMC algorithm is used to generate random samples from these posterior distributions. The following Algorithm 1 is provided below.

**Algorithm 1: Bayesian estimation via Gibbs sampling.**

**Step 1** Let $k = 1$ and start with initial guess point of $\theta$, say $\theta^0$.

**Step 2** Sample $\alpha^i$ from $\Omega(\alpha|x, \theta^{i-1})$ with a proposal distribution.

**Step 3** Like Step 2, sample $\theta^i$ from $\Omega(\theta|x, \alpha^i)$.

**Step 4** Let $k = k + 1$.

**Step 5** Repeat Step 2 to Step 4 $N$ times, obtain $N$ random samples of $\alpha$ and $\theta$ as $\alpha^1, \alpha^2, \ldots, \alpha^N$ and $\theta^1, \theta^2, \ldots, \theta^N$.

**Step 6** Let $\lambda^k = (\alpha^k, \theta^k)$, then the Bayes estimates of $\phi_{SEL}(\alpha, \theta)$ under squared error loss functions can be calculate as

$$
\hat{\phi}(\alpha, \theta) = \frac{1}{N - K} \sum_{i=K+1}^{K} \lambda^i,
$$

where $K$ is the burn-in times. Take $\lambda^k$, $k = K + 1, \ldots, N$ be arranged in ascending order as $\lambda^{(K+1)}, \lambda^{(K+2)}, \ldots, \lambda^{(N)}$. The corresponding $(1 - \tau)\%$ bayes HPD credible interval can be constructed as

$$
\left[ \lambda^{(i^*)}, \lambda^{(i^*+(N-K)(1-\tau)-1)} \right],
$$

with $k^*$ being derived by

$$
\lambda^{(i^*+(N-K)(1-\tau)-1)} - \lambda^{(i^*)} = \min_{i=K+1} \left[ \lambda^{(i+(N-K)(1-\tau)-1)} - \lambda^{(i)} \right].
$$

4.2. Posterior Density Using Maximum Product Spacing Function

Similarly, based on the MPS function (8), the MPS-based posterior density function of $\alpha$ and $\theta$ is given by

$$
\Psi(\alpha, \theta|x) = \frac{\sqrt{\alpha \eta'(\alpha)}}{\theta} \prod_{i=1}^{m+1} \left[ \frac{\Gamma(\alpha, \theta x_i^{-2}) - \Gamma(\alpha, \theta x_{i-1}^{-2})}{\Gamma(\alpha)} \right] \prod_{i=1}^{m} \left[ \frac{\gamma(\alpha, \theta x_i^{-2})}{\Gamma(\alpha)} \right]^\tau_i, \quad (15)
$$
and the MPS-based conditional posterior distributions of \( \alpha \) and \( \theta \), respectively, are

\[
\Psi(\alpha|x, \theta) = \sqrt{\alpha} \rho(\alpha) - 1 \prod_{i=1}^{m+1} \left[ \frac{\Gamma(a, \theta x_i^{-2}) - \Gamma(a, \theta x_i^{-2})}{\Gamma(a)} \right] \prod_{i=1}^{m} \left[ \frac{\gamma(a, \theta x_i^{-2})}{\Gamma(a)} \right]^{r_i},
\]

and

\[
\Psi(\theta|x, \alpha) = \frac{1}{\theta} \prod_{i=1}^{m+1} \left[ \frac{\Gamma(a, \theta x_i^{-2}) - \Gamma(a, \theta x_i^{-2})}{\Gamma(a)} \right] \prod_{i=1}^{m} \gamma(a, \theta x_i^{-2})^{r_i}.
\]

Further, following the similar line as Algorithm 1, the MPS-based Bayesian point and interval estimates could be obtained for arbitrary parameter \( \varphi(\alpha, \theta) \); the details are omitted here to save space.

5. Numerical Analysis

5.1. Simulation Studies

In this section, extensive simulation studies are conducted for investigating the performance of the proposed methods. For comparing the accuracy of different results, the performance of different point and interval estimates are evaluated based on the following criteria quantities:

(a) The mean square error (MSE) for point estimate \( \hat{\lambda} \) of \( \lambda = \alpha \) and \( \theta \), respectively, computed by \( \frac{1}{N} \sum (\hat{\lambda} - \lambda)^2 \).

(b) The average bias (AB) for point estimate \( \hat{\lambda} \) defined by \( \frac{1}{N} \sum |\hat{\lambda} - \lambda| \).

(c) The average width (AW) of 100(1 - \( \tau \))\% intervals of \( \lambda \).

Moreover, for different sample size \( n, m \), following types of censoring scheme (CSs) are considered in this part as

- **CS I**: \( r_1 = r_2 = \cdots = r_{m-1} = 0 \) and \( r_m = n - m \);
- **CS II**: \( r_1 = n - m \) and \( r_2 = \cdots = r_m = 0 \);
- **CS III**: \( \{ r_1 = \cdots = r_{n-2} = r_{n-1} = 1, r_{n-1} = \cdots = r_m = 0 \}, \) if \( n \leq 2m \);
  \( \{ r_1 = \cdots = r_{n-1} = 1, r_m = n - 2m + 1 \}, \) if \( n > 2m \).

Since the closed form of the inverse function of IND CDF does not exist, to generate the progressively Type-II censored data from IND, one could use the principle of progressive Type-II censoring scheme and the variable transformation, the manner proposed by Louzada et al. [17] and Balakrishnan and Kundu [29]. For different choices of sample sizes \( n, m \) and the values of the parameters \( \alpha, \theta \), the numerical analysis are conducted based on 1000 repeats.

In addition, the simulation work was conducted in R software where the maximum likelihood and maximum product of spacing estimates are obtained by using the global optimization search algorithm in R software, and interested readers can ask for the program on request. All the results of the criteria quantities are listed in Tables 1–4, where significance level \( \tau = 0.05 \) for all interval estimates. Moreover, for Bayesian estimation, the Gibbs sampling size \( N \) is taken to be 10,000 with a burn-in time of \( K = 5000 \). In addition, for showing the performance of different results in a visual representation, the associated numerical results are shown in Figures 1 and 2.

Based on the results listed in Tables 1 and 3, one can observe the following:

- As the effective sample size \( n \) or \( m \) or their combination increases, ABs and MSEs of the MLEs and MPSEs as well as Bayesian estimates become smaller, which indicates the consistency property of the proposed estimates.
- Under fixed schemes, MPSEs outperform the method of MLEs in terms of ABs and MSEs. A similar phenomenon also appears for the MPSE using the Bayesian method, and these are superior to traditional MLE under various CSs I, II and III, respectively.
The Bayesian estimates perform better as compared to the method of MLE in terms of the criteria quantities in general.

Table 1. The ABs and MSEs (within bracket) for the model parameters at \((\alpha, \theta) = (1.4, 1.0)\).

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>CS</th>
<th>MLE</th>
<th>MPSE</th>
<th>BLE</th>
<th>BPE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>(\theta)</td>
<td>(\alpha)</td>
<td>(\theta)</td>
<td>(\alpha)</td>
</tr>
<tr>
<td>30</td>
<td>15</td>
<td>I</td>
<td>0.7235</td>
<td>0.6936</td>
<td>0.5762</td>
<td>0.5811</td>
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<tr>
<td></td>
<td></td>
<td>II</td>
<td>0.7512</td>
<td>0.7016</td>
<td>0.6218</td>
<td>0.6023</td>
</tr>
<tr>
<td></td>
<td></td>
<td>III</td>
<td>0.7367</td>
<td>0.6738</td>
<td>0.6091</td>
<td>0.5975</td>
</tr>
</tbody>
</table>

Table 2. The AWs for the model parameters at \((\alpha, \theta) = (1.4, 1.0)\).

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>CS</th>
<th>ACI</th>
<th>HPD</th>
</tr>
</thead>
<tbody>
<tr>
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<td>MPSE</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>(\alpha)</td>
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<td>I</td>
<td>1.8914</td>
<td>1.5902</td>
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<td></td>
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<td>2.0215</td>
<td>1.5692</td>
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<td></td>
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<td>III</td>
<td>1.9272</td>
<td>1.5936</td>
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<tr>
<td>20</td>
<td>15</td>
<td>I</td>
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<td>1.3852</td>
</tr>
<tr>
<td></td>
<td></td>
<td>II</td>
<td>1.4676</td>
<td>1.3739</td>
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<tr>
<td></td>
<td></td>
<td>III</td>
<td>1.4984</td>
<td>1.4042</td>
</tr>
<tr>
<td>60</td>
<td>30</td>
<td>I</td>
<td>0.9793</td>
<td>0.9354</td>
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<td></td>
<td>II</td>
<td>1.1064</td>
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<tr>
<td></td>
<td></td>
<td>III</td>
<td>0.9892</td>
<td>0.9128</td>
</tr>
<tr>
<td>45</td>
<td>15</td>
<td>I</td>
<td>0.6851</td>
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<td>0.6948</td>
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</table>

We also observed from Tables 2 and 4 that

- The AWs of all likelihood and MPSE-based ACIs and the Bayesian HPD credible intervals decrease when the effective sample sizes increase.
- The MPSE-based ACIs perform better comparing with traditional likelihood-based ACIs; whereas similar superiority also appeared between Bayes HPD credible interval estimates based on likelihood and MPS functions in terms of AW.
- The AW of the intervals obtained from the Bayesian approach are generally shorter than those of the ACIs.
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Table 3. The ABs and MSEs (within bracket) for the model parameters at \((\alpha, \theta) = (2.1, 1.3)\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(m)</th>
<th>CS</th>
<th>(MLE)</th>
<th>(MPSE)</th>
<th>(BLE)</th>
<th>(BPE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>20</td>
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<td>0.3292</td>
<td>0.4183</td>
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</tr>
<tr>
<td></td>
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<tr>
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<td>0.2180</td>
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<tr>
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<td></td>
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<tr>
<td>45</td>
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<td>0.1438</td>
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<td>0.2405</td>
<td>0.2135</td>
<td>0.1896</td>
<td>0.1476</td>
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</table>

To sum up, from the listed simulation results, the efficiency of the MPS method is superior to traditional likelihood-based point and interval estimates under classical and Bayesian analysis, respectively. This further indicates that the MPSEs provide potential good alternatives to likelihood estimation under progressive censoring when the latent lifetime of items follows IND.

Table 4. The AWs for the model parameters at \((\alpha, \theta) = (2.1, 1.3)\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(m)</th>
<th>CS</th>
<th>(ACI)</th>
<th>(HPD)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>MLE</td>
<td>MPSE</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(\theta)</td>
<td>(\alpha)</td>
</tr>
<tr>
<td>30</td>
<td>15</td>
<td>I</td>
<td>1.3783</td>
<td>0.9325</td>
</tr>
<tr>
<td></td>
<td></td>
<td>II</td>
<td>1.1642</td>
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<tr>
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<tr>
<td>20</td>
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<td>0.9327</td>
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</tr>
<tr>
<td>60</td>
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<td>I</td>
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<td></td>
<td>III</td>
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<td></td>
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<td>0.3991</td>
</tr>
</tbody>
</table>
5.2. Real Life Example

To demonstrate the adaptability of the proposed methodologies to real phenomena, in this section, we re-analyze one real data set presented in Table 5 related to agricultural machine elevators showing high defect rates after a short repair time, which compromises the cost of production. For this data set, Louzada et al. [17] illustrated that compared with traditional Weibull, Gamma and log-normal distributions, the IND provides a best fit in terms of various goodness-of-fit criteria quantities with MLEs $\hat{\alpha} = 0.3230$ and $\hat{\theta} = 0.9466$ under complete data.
From the complete real data from Table 5, two sets of progressively Type-II censored data are provided in Table 6 with traditional Type-II scenarios. The associated point and interval estimates are listed in Table 7 with the significance level $\tau = 0.05$ for interval estimates. The bias between the point estimates and the MLEs of $\alpha$ and $\theta$ obtained under complete samples are also provided and are shown in square brackets in Table 7.

From the tabulated results in Table 7, it is observed the MLE and MPSE performed similar in terms of the bias, and the Bayes results appear similar under the data sets (i) and (ii). Further, if we consider the corresponding estimated CDFs, respectively, via all the estimated results, it is also noted that they match quite well under each data cases; see Figure 3 for illustration.

In addition, the complicated forms of the traditional and MPS likelihood equations make them difficult to establish the existence and uniqueness of the associated MLEs and MPSEs. However, with modern computing technology, two variables could be solved numerically. For illustration, the contour maps of $(\theta, \alpha)$ are presented in Figures 4 and 5 for the likelihood and MPS likelihood functions, respectively.

In addition, for the Bayesian MCMC approach, trace plots with their sample mean and 95% credible intervals and corresponding histograms of the associated chain values for $\alpha$ and $\theta$ are listed in Figures 6–9 based on the data sets (i) and (ii), respectively. The plots indicate good mixing performance.

### Table 5. Complete agricultural data of machine motors.

<table>
<thead>
<tr>
<th>1</th>
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<th>1</th>
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<td>2</td>
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<td>3</td>
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<td>3</td>
</tr>
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<td>8</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 6. Progressively Type-II censored samples from agricultural machine motor data.

| data (i): $(n, m) = (64, 60)$ and $R = (0, \ldots, 4)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 |
| 4 | 5 | 5 | 5 | 5 | 5 | 5 | 7 | 8 | 8 | 9 | 9 |
| 11 | 12 | 12 | 13 | 16 | 17 | 17 | 18 | 18 | 19 | 22 | 24 |
| data (ii): $(n, m) = (64, 44)$ and $R = (0, \ldots, 20)$ |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 4 |
| 4 | 5 | 5 | 5 | 5 | 5 | 5 | 7 | 8 | 8 | 9 | 9 |

### Table 7. Point and interval estimates of $\alpha$ and $\theta$ under real-world data.

<table>
<thead>
<tr>
<th>Point Estimates</th>
<th>Interval Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>MLE</td>
<td>MPSE</td>
</tr>
<tr>
<td>data (i)</td>
<td></td>
</tr>
<tr>
<td>0.3270 [0.0443]</td>
<td>0.9858 [0.2653]</td>
</tr>
<tr>
<td>0.2994 [0.0546]</td>
<td>0.9036 [0.2914]</td>
</tr>
<tr>
<td>data (ii)</td>
<td></td>
</tr>
<tr>
<td>0.2997 [0.0521]</td>
<td>0.9878 [0.2547]</td>
</tr>
<tr>
<td>0.2765 [0.0435]</td>
<td>0.8634 [0.2718]</td>
</tr>
<tr>
<td>0.2776 [0.0427]</td>
<td>0.8184 [0.2536]</td>
</tr>
</tbody>
</table>

Note: The ESEs of point estimates and interval lengths are provided in square brackets.
Figure 3. Plots of CDF of the IND using different estimates under data sets (i) and (ii).

Figure 4. Contour plots of $\theta$ and $\alpha$ using the likelihood function (left) and MPE function (right) based on data set (i).

Figure 5. Contour plots of $\theta$ and $\alpha$ using the likelihood function (left) and MPE function (right) based on data set (ii).
Figure 6. Trace plots (the posterior mean (dashed line) with 95% credible intervals (solid lines)) and corresponding histograms of BLEs of $\alpha$ and $\theta$ for data set I.

Figure 7. Cont.
Figure 7. Trace plots (the posterior mean (dashed line) with 95% credible intervals (solid lines)) and corresponding histograms of BPEs of $\alpha$ and $\theta$ for data set I.

Figure 8. Trace plots (the posterior mean (dashed line) with 95% credible intervals (solid lines)) and corresponding histograms of BLEs of $\alpha$ and $\theta$ for data set II.
Figure 9. Trace plots (the posterior mean (dashed line) with 95% credible intervals (solid lines)) and corresponding histograms of BPEs of $\alpha$ and $\theta$ for data set II.

6. Conclusions

In this paper, we discussed statistical inference of the inverse Nakagami distribution when the failure times are progressively censored. In addition, the traditional maximum likelihood estimation and MPS approaches were introduced for the IND parameters. Extensive simulations and real-world examples were performed to investigate the performance of different methods. The numerical results demonstrated that the MPS method outperformed the ML method in terms of the criteria quantities under both classical and Bayesian perspectives. We concluded that the MPSEs provide a reasonable alternative for estimating IND parameters compared with the usual likelihood approach. Due to the scenario of the MPS method, although the inferential work was discussed for IND progressively censored data, the work could be extended to other types of censoring schemes, such as Type-I censoring, Type-II censoring and progressive (hybrid) censoring as well as interval censoring, which will be explored in the future.

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References