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A New Parameter-Uniform Discretization of Semilinear Singularly Perturbed Problems

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Abstract: In this paper, we present a numerical approach to solving singularly perturbed semilinear convection-diffusion problems. The nonlinear part of the problem is linearized via the quasilinearization technique. We then design and implement a fitted operator finite difference method to solve the sequence of linear singularly perturbed problems that emerges from the quasilinearization process. We carry out a rigorous analysis to attest to the convergence of the proposed procedure and notice that the method is first-order uniformly convergent. Some numerical evaluations are implemented on model examples to confirm the proposed theoretical results and to show the efficiency of the method.

Keywords: singularly perturbed problems; semilinear differential equation; quasilinearization; boundary layer; fitted operator finite difference method; uniform convergence

MSC: 65L10; 65L11; 65L12

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1. Introduction

Differential problems in which a small parameter, often referred to as a perturbation parameter, multiply the highest derivatives are called singularly perturbed differential problems. These problems arise in different fields of study such as fluid dynamics, magnetohydrodynamics, aerodynamics, oceanography, quantum mechanics, plasma dynamics, chemical reactions, and liquid crystal modeling [1–3]. As an example, the heat and mass transport phenomena [4] are described by singularly perturbed differential equations in which the diffusion coefficient is regarded as a perturbation parameter.

Classical numerical methods have often failed to solve singularly perturbed problems. This is because one or more boundaries or interior layers may arise as the perturbation parameter approaches zero, thereby give undesirable results. To overcome this problem, constructed numerical methods such as finite difference methods, in the form of fitted mesh and fitted operator finite difference methods, finite element methods, and spline methods are adopted. These methods are used on layer-adapted meshes such as the Shishkin mesh, which is easy to construct, the Bakhvalov mesh, which gives superior accuracy, the Bakhvalov–Shishkin mesh, and the Vulanovic–Shishkin mesh (see [5–10]). In this paper, we consider the singularly perturbed semilinear convection-diffusion problems

$$\varepsilon y''(x) + a(x)y'(x) + f(x, y(x)) = 0, \quad x \in \Omega := (0, 1), \tag{1}$$

subject to the following boundary conditions:

$$y(0) = A, \qquad y(1) = B, \tag{2}$$

where ε is the perturbation parameter such that $0 < \varepsilon << 1$, A and B are given constants, and the functions $a(x)$ and $f(x, y(x))$ are sufficiently smooth in the intervals Ω and $C^2\ (\Omega \times R)$, respectively, satisfying

$$a(x) \geq \alpha > 0, \qquad \forall\ x \in \Omega, \tag{3}$$

where α is a positive constant.

We obtain the reduced problem of Equation (1) by setting ε to zero, given as

$$a(x)y'(x) + f(x, y(x)) = 0. \tag{4}$$

Under these conditions, Equations (1) and (2) and the reduced problem in Equation (4) have a unique solution. This unique solution to Equations (1) and (2) exhibits a boundary layer at the origin of the interval Ω as the perturbation parameter ε approaches zero (i.e., at $x = 0$) (see [11–14]).

Not much work has been conducted on convection-diffusion semilinear singularly perturbed problems related to Equation (1). Cimen and Amiraliyev [15] constructed an exponential finite difference scheme, which flourishes by the method of integral identities to solve a singularly perturbed semilinear delay differential equation. They obtained a first-order uniform convergence in the discrete maximum norm. Niijima and Stynes [16,17] separately solved a singularly perturbed boundary value problem of the form in Equation (1). They adopted the use of finite difference schemes, and each obtained an almost first-order uniform accuracy in a discrete L_1 norm. Cakir and Arslan [18] used a fitted mesh finite difference scheme constructed on a Shishkin mesh to solve a singularly perturbed semilinear problem with integral boundary conditions. Their proposed scheme was found to be first-order uniformly convergent in the discrete maximum norm. Cakir and Amiraliyev [19] constructed a uniform finite difference scheme on a Shishkin-type mesh to solve a singularly perturbed semilinear convection-diffusion three-point boundary value problem. This method was shown to be first-order uniform convergent in the discrete maximum norm. Igor and Pack [20] solved a singularly perturbed semilinear convection-diffusion problem with discontinuous data using a difference scheme on local Green's functions. The authors achieved a first-order uniform convergent scheme on arbitrary meshes.

Linß [21] constructed a fitted mesh finite difference scheme on a Shishkin mesh to solve singularly perturbed convection-diffusion with a boundary layer of attractive turning points. He achieved an almost first-order convergence. Shishkin and Shishkina [22] examined a Dirichlet problem on a vertical strip for a singularly perturbed semilinear convection-diffusion problem. The authors used an iterative monotone difference scheme to solve the problem and obtained a first-order uniform convergent results. They then improved the order of convergence to second-order uniform (and improved the accuracy) using a Richardson scheme. Linß and Vulanović [23] solved a semilinear convection-diffusion problem with attractive boundary turning points by constructing a fitted mesh finite difference method on a Shishkin mesh type. This method was established to be of first-order uniform convergence.

More recently, further works were completed on semilinear singularly perturbed scalar boundary value problems [24], scalar parabolic problems [25], or on systems of such problems [26–29]. Again, the methods adopted in these works are essentially the fitted finite difference methods based on Shishkin meshes.

Based on the literature, we observed that authors have mostly exploited the fitted mesh finite difference schemes as well as some other methods to solve singularly perturbed semilinear convection-diffusion problems. However, none of them, to the best of our knowledge, have proposed a numerical method in the framework of nonstandard finite difference (NSFD) methods to solve such problems.

In this paper, we propose an NSFD scheme to solve singularly perturbed semilinear convection-diffusion problems. This scheme falls under the category of fitted operator finite difference methods (FOFDMs), as they are known in previously published works such as [30–32]. We transform the semilinear problem into a sequence of linear equations via the quasilinearization technique. We then construct a fitted operator finite difference scheme on the transformed problem. We show that the proposed method is ε-uniform convergent to the first order. Unlike its fitted mesh counterpart, not only do the fitted operator finite difference methods provide a simpler platform for analysis owing to their use of a uniform mesh, but their error bounds are not adversely affected by a logarithmic factor, as pointed out in [33].

The rest of this paper is structured as follows. In Section 2, we transform the semilinear problem into a system of linear singularly perturbed problems by the quasilinearization technique. In Section 3, we analyze some properties of the system of linear problems. In Section 4, we construct a fitted operator finite difference scheme, which is analyzed in Section 5. In Section 6, we present numerical examples to demonstrate the ε-uniformity of the proposed method. Finally, we present the conclusion in Section 7.

2. Quasilinearization

We use the quasilinearization technique to transform the semilinear singularly perturbed convection-diffusion problem into a sequence of linear equations. We choose a reliable initial approximation for the function $y(x)$ in $f(x, y(x))$, and by a Taylor series, we expand $f(x, y(x))$ around the chosen initial approximation and obtain

$$f(x, y^{(k+1)}(x)) = f(x, y^{(k)}(x)) + (y^{(k+1)}(x) - y^{(k)}(x))\left(\frac{\partial f^{(k)}}{\partial y}\right)_{(x, y^{(0)})} + \dots \tag{5}$$

By putting Equation (5) into Equations (1) and (2), we have

$$\varepsilon y''^{(k+1)}(x) + a(x)y'^{(k+1)}(x) + \left(\frac{\partial f^{(k)}}{\partial y}\right)y^{(k+1)} = -f(x, y^{(k)}(x)) + \left(\frac{\partial f^{(k)}}{\partial y}\right)y^{(k)}(x), \tag{6}$$

$$y^{(k+1)}(0) = A, \qquad y^{(k+1)}(1) = B, \tag{7}$$

for the iteration index $k = 0, 1, 2 \dots$.

Notice that Equation (6) is linear in $y^{(k+1)}$. Therefore, we solve the sequence of linear Equations (6) and (7) in place of the semilinear problem in Equations (1) and (2) by the fitted operator finite difference method that will be introduced in Section 4.

For the solution of the semilinear boundary value problem, we require that

$$\max_{k \to \infty} y^{(k)}(x) = y^{*}(x), \tag{8}$$

where $y^{*}(x)$ is the solution of the semilinear problem. Numerically, we require that

$$|y^{(k+1)}(x) - y^{(k)}(x)| < \lambda, \tag{9}$$

where λ is a small tolerance chosen by us. Then, $y^{(k+1)}$ is the approximate solution of the semilinear problem.

3. Some Properties of the Linear Problem

We rewrite Equations (6) and (7) as

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = F(x), \tag{10}$$

where

$$b(x) = \frac{\partial f^{(k)}}{\partial y}, \qquad F(x) = -f(x, y^{(k)}(x)) + \frac{\partial f^{(k)}}{\partial y}y^{(k)}(x),$$

and $y^{(k+1)}(x) = u(x)$ such that

$$u(0) = A, \qquad u(1) = B. \tag{11}$$

We present some important properties for the solution of Equations (10) and (11) which will be useful in the subsequent section for the analysis of relevant numerical solutions. Without a loss of generality, we assume that f is a decreasing function of y:

Lemma 1. *Continuous minimum principle: Assume that $\nu(x)$ is a sufficiently smooth function which satisfies $\nu(0) \geq 0$ and $\nu(1) \geq 0$. Then, $\mathcal{L}\nu(x) \leq 0,\ \forall x \in \Omega$ implies that $\nu(x) \geq 0, \forall x \in \bar{\Omega}$.*

Proof. Let ν be a value such that $\nu(x^*) = \min_{x\in\Omega} \nu(x)$, and assume that $\nu(x^*) < 0$. Clearly, $x^* \notin \{0,1\}$, and therefore $\nu'(x^*) = 0$ and $\nu''(x^*) \geq 0$. Moreover, there is

$$\mathcal{L}\nu(x^*) = \varepsilon\nu''(x^*) + a(x^*)\nu'(x^*) + b(x^*)\nu(x^*) \geq 0,$$

which is a contradiction. It follows that $\nu(x^*) \geq 0$, and thus $\nu(x) \geq 0,\ \forall\ \ x \in \Omega$. □

Lemma 2. *Uniform stability estimate: Let u(x) be the solution of Equations (10) and (11). Then, we have*

$$||u(x)|| \leq \alpha^{-1}||F|| + \max(|A|,|B|), \quad \forall\ \ x \in \Omega. \tag{12}$$

Proof. We construct two barrier functions $\Psi^{\pm}$ defined by

$$\Psi^{\pm}(x) = \alpha^{-1}||F|| + \max(|A|,|B|) \pm u(x).$$

Then, it can be said that

$$\begin{aligned}
\Psi^{\pm}(0) &= \alpha^{-1}||F|| + \max(|A|,|B|) \pm u(0) \\
&= \alpha^{-1}||F|| + \max(|A|,|B|) \pm A \\
&\geq 0; \\
\Psi^{\pm}(1) &= \alpha^{-1}||F||| + \max(|A|,|B|) \pm u(1) \\
&= \alpha^{-1}||F|| + \max(|A|,|B|) \pm B \\
&\geq 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{L}\Psi^{\pm}(x) &= \varepsilon(\Psi^{\pm}(x))'' + a(x)(\Psi^{\pm}(x))' + b(x)\Psi^{\pm}(x) \\
&= b(x)[\alpha^{-1}||F|| + \max(|A|,|B|) \pm \mathcal{L}u(x)] \\
&= b(x)[\alpha^{-1}||F|| + \max(|A|,|B|) \pm f(x)] \\
&\geq b(x)[\alpha^{-1}||F|| + \max(|A|,|B|)] \\
&\geq 0, \quad \text{since} \quad ||F|| \geq F(x).
\end{aligned}$$

Through Lemma 1, we obtain $\Psi^{\pm}(x) \geq 0,\ \ \forall\ x \in \Omega$. □

Lemma 3. *By letting u(x) be the solution of Equations (10) and (11) and $a(x), b(x)$, and $F(x)$ be smooth functions, then*

$$|v^{(i)}(x)| \leq C(1 + \varepsilon^{-i}e^{-\alpha x/\varepsilon}), \qquad i = 1,\ldots 4,\ \ x \in (0,1), \tag{13}$$

where α and C are positive constants independent of ε.

The proof can be seen in [7].

4. Construction of the FOFDM

In this section, we design a fitted operator finite different scheme base on the Mickens rules [34,35]. We denote the approximations of u_j at the grid point x_j by the unknown U_j. We partitioned the domain $\Omega := [0,1]$ into N subintervals of a length h such that

$$x_o = 0, \quad x_j = x_o + jh, \quad j = 1(1)N-1, \quad h_j = x_j - x_{j-1}, \quad x_N = 1.$$

We denote the set of these mesh points by Ω^N. We then discretized Equations (10) and (11) as

$$\varepsilon\delta^2 U_j + a_j D^+ U_j + b_j U_j = F_j, \qquad j = 1, \ldots N-1, \tag{14}$$

with the boundary condition

$$U_0 = A, \qquad U_N = B, \tag{15}$$

where

$$D^+ U_j = \frac{U_{j+1} - U_j}{h}, \quad \delta^2 U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{\phi_j^2},$$

and ϕ_j^2 is the denominator function given by

$$\phi_j^2 = \frac{\varepsilon h}{a_j}\left(\exp\left(\frac{a_j h}{\varepsilon}\right) - 1\right). \tag{16}$$

The difference equations consist of $N-1$ equations for $N+1$ unknowns $U_0, U_1, \ldots, U_N$, where U_0 and U_N are given boundary conditions. We write Equations (14) and (15) in matrix form as

$$AU = G$$

where $U = [U_1, U_2, \ldots, U_{N-1}]^T$ and A is a tridiagonal matrix whose entries are of the form

$$\begin{aligned} A_{ij} &= r_j^-, \ i = j+1, \ j = 1, \ldots, N-2, &(17)\\ A_{ij} &= r_j^c, \ i = j, \ j = 1, \ldots, N-1, &(18)\\ A_{ij} &= r_j^+, \ i = j-1, \ j = 2, \ldots, N-1, &(19) \end{aligned}$$

where

$$r_j^- = \frac{\varepsilon}{\phi_j^2}, \qquad r_j^c = -\frac{2\varepsilon}{\phi_j^2} - \frac{a_j}{h} + b_j, \qquad r_j^+ = \frac{\varepsilon}{\phi_j^2} + \frac{a_j}{h},$$

and G is obtained as

$$\begin{aligned} G_1 &= F_1 - (r_1^-)U_0, &(20)\\ G_j &= F_j, \qquad j = 2, \ldots, N-2, &(21)\\ G_{N-1} &= F_{N-1} - (r_{N-1}^+)U_N. &(22) \end{aligned}$$

Thus, the unknown $U_1, U_2, \ldots U_{N-1}$ is solved. The following lemma are relevant in the convergence analysis of this method:

Lemma 4. *Discrete minimum principle: Let η_j be a discrete function defined on Ω^N and satisfying $\eta_o \geq 0$, $\eta_N \geq 0$. Then, $\mathcal{L}^N \eta_j \leq 0$, $\forall\, 1 \leq j \leq N-1$ implies $\eta_j \geq 0$, $\forall\, 0 \leq j \leq N$.*

Proof. Let k be a value such that $\eta_k = \min_{x \in \Omega^N} \eta_j$, and assume $\eta_k < 0$. Clearly, $k \neq 0$, $k \neq N$, $\eta_{k+1} - \eta_k \geq 0$, and $\eta_k - \eta_{k-1} \leq 0$. It follows that

$$\begin{aligned} \mathcal{L}^N \eta_j &= \frac{\varepsilon}{\phi_k^2}(\eta_{k+1} - 2\eta_k + \eta_{k-1}) + \frac{a_k}{h}(\eta_{k+1} - \eta_k) + b_k \eta_k &(23)\\ &= \frac{\varepsilon}{\phi_k^2}[(\eta_{k+1} - \eta_k) - (\eta_k - \eta_{k-1})] + \frac{a_k}{h}(\eta_{k+1} - \eta_k) + b_k \eta_k \geq 0 &(24) \end{aligned}$$

Thus, $\mathcal{L}^N \eta_j \leq 0$, $1 \leq k \leq N-1$, which is a contradiction. Hence, $\eta_j \geq 0$, $\forall\, x \in \Omega$. □

Lemma 5. *Uniform stability estimate: If μ_i is in any mesh function such that $\mu_o = \mu_N = 0$, then*

$$|\mu_i| \leq \frac{1}{\alpha} \max_{1 \leq j \leq N-1} |\mathcal{L}^N \mu_j|, \qquad 0 \leq i \leq N. \tag{25}$$

Proof. Put $Z_i = \frac{1}{\alpha} \max_{1 \leq j \leq N-1} |\mathcal{L}^N \mu_i|$ for $1 \leq i \leq N-1$. Introduce two mesh functions $\psi^{\pm}$ defined by

$$\psi_i^{\pm} = Zx_i \pm \mu_i$$

Clearly, $\psi_o^{\pm} = 0,\ \psi_N^{\pm} = 0$ and $\forall\ 1 \leq i \leq N-1$:

$$\mathcal{L}^N \psi_i^{\pm} = Za_i + \mathcal{L}^N \mu_i \leq 0.$$

Since $a_i > \alpha$, Lemma 4 implies that $\psi_i^{\pm} \geq 0, \quad \forall \quad 0 \leq i \leq N$, and this completes the proof. □

5. Convergence Analysis

In this section, we analyze the convergence property of the proposed method described in the previous section. The truncation error at the grid point x_i is

$$\begin{aligned} \mathcal{L}^h(u-U)_i &= (\mathcal{L} - \mathcal{L}^h)u_i && (26) \\ &= \varepsilon u_i'' + a_i u_i' - \frac{\varepsilon}{\phi_i^2}(u_{i+1} - 2u_i + u_{i-1}) - \frac{a_i}{h}(u_{i+1} - u_i). && (27) \end{aligned}$$

By taking the Taylor series expansion of u_{i+1} and u_{i-1} and the truncated Taylor series expansion of $\phi_i^{-2} = \dfrac{1}{h^2} + \dfrac{a_i}{2\varepsilon h} - \dfrac{a_i^2}{12\varepsilon^2}$, we obtain

$$\begin{aligned} \mathcal{L}^h(u-U)_i &= \varepsilon u_i'' - \left(h^2 u_i'' + \frac{h^4}{12} u_i^{iv}(\xi_i)\right) \times \left(\frac{\varepsilon}{h^2} + \frac{a_i}{2h} - \frac{a_i^2}{12\varepsilon}\right) - a_i\left(\frac{h}{2} u_i''\right) \\ &= \varepsilon u_i'' - \left(h^2 u_i'' + \frac{h^4}{12} u_i^{iv}(\xi_i)\right) \times \left(\frac{\varepsilon}{h^2} + \frac{1}{2}\left(\frac{a_{i+1} - a_i}{h}\right) - \frac{a_i^2}{12\varepsilon}\right) - a_i\left(\frac{h}{2} u_i''\right) \\ &= -\frac{a_i h u''}{2} + \left(\frac{\varepsilon u^{iv}}{12}(\xi_i) - \frac{a_i^2 u''}{12\varepsilon} + \frac{a_{i(x)} u''}{2}\right) h^2 + \left(\frac{a_{i(x)} u^{iv}}{24}(\xi_i) - \frac{a_i^2 u^{iv}}{144\varepsilon}(\xi_i)\right) h^4, \end{aligned}$$

where $\xi_i \in (x_{i-1}, x_{i+1})$. By applying the boundary of the solution and its derivative (see Lemma 3) along with Lemma 5.2 in [36], we obtain

$$\mathcal{L}^h(u-U)_i \leq -\frac{a_i h}{2} + \left(\frac{\varepsilon}{12} - \frac{a_i^2}{12\varepsilon} + \frac{a_{i(x)}}{2}\right) h^2 + \left(\frac{a_{i(x)}}{24} - \frac{a_i^2}{144\varepsilon}\right) h^4.$$

From the relation $h > h^2 > h^4$, we have

$$|\mathcal{L}^h(u-U)_i| \leq Ch.$$

By applying the uniform stability estimate (Lemma 5), we obtain

$$\max_{0 \leq j \leq N} |(u-U)_i| \leq Ch. \tag{28}$$

Theorem 1. *Let $u(x)$ be the solution of Equations (10) and (11) and $U(x)$ be the numerical approximation of Equations (14) and (15). If $a(x), b(x)$ and $F(x)$ are sufficiently smooth functions, then the truncation error is given by*

$$\max_{0 \leq j \leq N} |(u-U)_j| \leq Ch, \tag{29}$$

where C is a constant independent of ε and h. This establishes that the numerical method developed is first-order uniformly convergent.

6. Numerical Results

In this section, we consider four test examples of singularly perturbed semilinear convection-diffusion problems to confirm our theoretical findings and to illustrate the performance of the proposed method in practice. We compute the maximum error and the rate of convergence and display the results in tables for different values of N and ε. Because the exact solution of Example 3 does not behave well for ε values close to 1, we chose $N = 2^i$ and $\varepsilon = 10^{-j}$ for $i \geq 4$ and $j \geq 2$. In the case where the exact solution is known, the point-wise maximum error is given by

$$E_{N,\varepsilon} = \max_{0\leq j\leq N} |u_j - U_j|, \tag{30}$$

where U is the approximate solution and u is the exact solution. In cases where the exact solution is unknown, we compute the maximum point-wise error using the double mesh principle [37]:

$$E_{N,\varepsilon} = \max_{0\leq j\leq N} |U_j^{\varepsilon,N} - U_{2j}^{\varepsilon,2N}|, \tag{31}$$

where U_N and U_{2N} are the numerical solutions computed on the meshes Ω^N and Ω^{2N}, respectively.

The rates of convergence are computed using the formula

$$R_{N,\varepsilon} = \log_2(E_{N_k}/E_{2N_k}), \tag{32}$$

In the iteration process, the initial guess is $U_N^{(0)} = (A, 0, 0, \ldots, 0, B)$, and the stopping criterion is

$$\max_k |U^{(k+1)} - U^{(k)}| \leq 10^{-10}, \qquad k = 1, 2, \ldots \tag{33}$$

Example 1. *Consider the following singularly perturbed semilinear problem [38]:*

$$\varepsilon u'' + 2u' + \exp(u) = 0, \qquad x \in [0,1],$$

$$u(0) = 0, \qquad u(1) = -\frac{\ln 2}{\exp(2/\varepsilon)}.$$

The exact solution is

$$u(x) = \ln\left(\frac{2}{1+x}\right) - \exp\left(\frac{-2x}{\varepsilon}\right)\ln 2.$$

In this case, the exact value is known, the maximum error, and the rate of convergence are obtained with the formula described in Equations (30) and (32).

The quasilinearization process equations are

$$\varepsilon u''^{(k+1)}(x) + 2u'^{(k+1)}(x) + \exp(u^{(k)}(x))u^{(k+1)}(x) = \exp(u^{(k)}(x))(u^{(k)}(x) - 1), \tag{34}$$

$$u^{(k)}(0) = 0, \qquad u^{(k)}(1) = 0.$$

Example 2. *Consider the following singularly perturbed semilinear problem [39]:*

$$\varepsilon u'' + u' + u^2 = 0, \qquad x \in [0,1],$$

$$u(0) = 0, \quad u(1) = 1/2.$$

In this case, the exact value is unknown, and mthe aximum error and rate of convergence are obtained with the formula describe in Equations (31) and (32). The quasilinearization process equations are

$$\varepsilon u''^{(k+1)}(x) + u'^{(k+1)}(x) + 2u^{(k)}(x)u^{(k+1)}(x) = (u^{(k)}(x))^2, \tag{35}$$

$$u^{(k)}(0) = 0, \qquad u^{(k)}(1) = 1/2.$$

Example 3. *Consider the following singularly perturbed semilinear problem [40]:*

$$\varepsilon u'' + (2x+1)u' + u^2 = 0, \qquad x \in [0,1],$$

$$u(0) = 1, \quad u(1) = 1.$$

The exact value is unknown, and the maximum error and rate of convergence are obtained with the formula described in Equations (31) and (32). The quasilinearization process equations are

$$\varepsilon u''^{(k+1)}(x) + (2x+1)u'^{(k+1)}(x) + 2u^{(k)}(x)u^{(k+1)}(x) = (u^{(k)}(x))^2, \tag{36}$$

$$u^{(k)}(0) = 1, \qquad u^{(k)}(1) = 1.$$

Example 4. *Consider the following singularly perturbed semilinear problem [41]:*

$$\varepsilon u'' + u' = u\exp(u), \qquad x \in [0,1],$$

$$u(0) = 1, \qquad u(1) = 0.$$

The exact value is unknown, and the maximum error and rate of convergence are also obtained with the formula described in Equations (31) and (32). The quasilinear process equations are

$$\varepsilon u''^{(k+1)}(x) + u'^{(k+1)}(x) + \exp(u^{(k)}(x))(u^k(x)+1)(x)u^{(k+1)}(x) = (u^{(k)}(x))\exp(u^{(k)}(x)), \tag{37}$$

$$u^{(k)}(0) = 1, \qquad u^{(k)}(1) = 0.$$

In each of the four examples, the solution has a boundary layer at the left side of the interval Ω. Tables 1–4 present the point-wise maximum error E_N and the rate of convergence R_N for different values of ε and N. The results shown in the tables reveal that the proposed method is of first-order uniform convergence, as projected by the theoretical analysis. Figure 1 provides the plots of the exact and numerical solutions of Example 1 for $\varepsilon = 10^{-2}$ and $N = 512$, intuitively showing that this numerical solution is a "good" approximation of the exact solution. In Figure 2, for the fixed number of subintervals $N = 256$, we plot the numerical solution for Example 1 for different values of ε, showing that the impact of ε on the numerical solution disappears as ε approaches 0, thus confirming the ε-uniform aspect of the proposed method. These conclusions were arrived at when observing the tabulated results. When ε is small, the nodal maximum errors and the rate of convergence remain unaffected by the change in value of this parameter.

Table 1. Results for Example 1: maximum errors and convergence rates for FOFDM.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-2}	1.60×10^{-2}	9.36×10^{-3}	5.37×10^{-3}	3.82×10^{-3}	3.37×10^{-3}	3.25×10^{-3}	3.22×10^{-3}
	0.77	0.80	0.49	0.18	0.05	0.01	
10^{-3}	1.60×10^{-2}	9.33×10^{-3}	5.03×10^{-3}	2.61×10^{-3}	1.33×10^{-3}	6.95×10^{-4}	4.45×10^{-4}
	0.78	0.89	0.95	0.97	0.94	0.64	
10^{-4}	1.60×10^{-2}	9.33×10^{-3}	5.03×10^{-3}	2.61×10^{-3}	1.33×10^{-3}	6.71×10^{-4}	3.37×10^{-4}
	0.78	0.89	0.95	0.97	0.99	0.99	
10^{-4}	1.60×10^{-2}	9.33×10^{-3}	5.03×10^{-3}	2.61×10^{-3}	1.33×10^{-3}	6.71×10^{-4}	3.37×10^{-4}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-20}	1.60×10^{-2}	9.33×10^{-3}	5.03×10^{-3}	2.61×10^{-3}	1.33×10^{-3}	6.71×10^{-4}	3.37×10^{-4}
	0.78	0.89	0.95	0.97	0.99	0.99	
E_N	1.60×10^{-2}	9.33×10^{-3}	5.03×10^{-3}	2.61×10^{-3}	1.33×10^{-3}	6.71×10^{-4}	3.37×10^{-4}
R_N	0.78	0.89	0.95	0.97	0.99	0.99	

Table 2. Results for Example 2: maximum errors and convergence rate for FOFDM.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-2}	4.03×10^{-2}	1.85×10^{-2}	5.58×10^{-3}	1.10×10^{-3}	1.71×10^{-4}	3.52×10^{-5}	7.68×10^{-6}
	1.13	1.72	2.35	2.68	2.28	2.20	
10^{-3}	4.01×10^{-2}	2.03×10^{-2}	1.02×10^{-2}	5.12×10^{-3}	2.48×10^{-3}	8.71×10^{-4}	1.93×10^{-4}
	0.98	0.99	0.99	1.05	1.51	2.18	
10^{-4}	4.01×10^{-2}	2.03×10^{-2}	1.02×10^{-2}	5.10×10^{-3}	2.55×10^{-3}	1.28×10^{-3}	6.39×10^{-4}
	0.98	0.99	1.00	1.00	1.00	1.00	
10^{-5}	4.01×10^{-2}	2.03×10^{-2}	1.02×10^{-2}	5.10×10^{-3}	2.55×10^{-3}	1.28×10^{-3}	6.38×10^{-4}
	0.98	0.99	1.00	1.00	1.00	1.00	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-20}	4.01×10^{-2}	2.03×10^{-2}	1.02×10^{-2}	5.10×10^{-3}	2.55×10^{-3}	1.28×10^{-3}	6.38×10^{-4}
	0.98	0.99	1.00	1.00	1.00	1.00	
E_N	4.01×10^{-2}	2.03×10^{-2}	1.02×10^{-2}	5.10×10^{-3}	2.55×10^{-3}	1.28×10^{-3}	6.38×10^{-4}
R_N	0.98	0.99	1.00	1.00	1.00	1.00	

Table 3. Results for Example 3: maximum errors and convergence rate for FOFDM.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-2}	2.39×10^{-2}	1.16×10^{-2}	6.45×10^{-2}	1.24×10^{-1}	1.14×10^{-1}	7.52×10^{-2}	4.29×10^{-2}
	1.04	−2.47	−0.95	0.13	0.60	0.81	
10^{-3}	2.38×10^{-2}	1.11×10^{-2}	5.38×10^{-3}	2.65×10^{-3}	1.33×10^{-3}	3.29×10^{-2}	1.09×10^{-1}
	1.10	1.05	1.02	0.99	−4.62	−1.72	
10^{-4}	2.38×10^{-2}	1.11×10^{-2}	5.38×10^{-3}	2.65×10^{-3}	1.31×10^{-3}	6.54×10^{-4}	3.26×10^{-4}
	1.10	1.05	1.02	1.01	1.01	1.00	
10^{-5}	2.38×10^{-2}	1.11×10^{-2}	5.38×10^{-3}	2.65×10^{-3}	1.31×10^{-3}	6.54×10^{-4}	3.26×10^{-4}
	1.10	1.05	1.02	1.01	1.01	1.00	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-20}	2.38×10^{-2}	1.11×10^{-2}	5.38×10^{-3}	2.65×10^{-3}	1.31×10^{-3}	6.54×10^{-4}	3.26×10^{-4}
	1.10	1.05	1.02	1.01	1.01	1.00	
E_N	2.38×10^{-2}	1.11×10^{-2}	5.38×10^{-3}	2.65×10^{-3}	1.31×10^{-3}	6.54×10^{-4}	3.26×10^{-4}
R_N	1.10	1.05	1.02	1.01	1.01	1.00	

Table 4. Results for Example 4: maximum errors and convergence rates for FOFDM.

ε	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{-2}	2.36×10^{-1}	7.94×10^{-2}	2.04×10^{-2}	3.67×10^{-3}	9.17×10^{-4}	3.31×10^{-4}	1.43×10^{-4}
	1.51	1.96	2.47	2.00	1.47	1.21	
10^{-3}	2.43×10^{-1}	9.97×10^{-2}	4.58×10^{-2}	2.17×10^{-2}	9.48×10^{-3}	3.05×10^{-3}	6.41×10^{-4}
	1.28	1.12	1.08	1.20	1.63	2.25	
10^{-4}	2.43×10^{-1}	9.97×10^{-2}	4.58×10^{-2}	2.20×10^{-2}	1.08×10^{-2}	5.36×10^{-3}	2.65×10^{-3}
	1.28	1.12	1.06	1.03	1.01	1.01	
10^{-5}	2.43×10^{-1}	9.97×10^{-2}	4.58×10^{-2}	2.20×10^{-2}	1.08×10^{-2}	5.36×10^{-3}	2.67×10^{-3}
	1.28	1.12	1.06	1.03	1.01	1.01	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	2.43×10^{-1}	9.97×10^{-2}	4.58×10^{-2}	2.20×10^{-2}	1.08×10^{-2}	5.36×10^{-3}	2.67×10^{-3}
	1.28	1.12	1.06	1.03	1.01	1.01	
E_N	2.43×10^{-1}	9.97×10^{-2}	4.58×10^{-2}	2.20×10^{-2}	1.08×10^{-2}	5.36×10^{-3}	2.67×10^{-3}
R_N	1.28	1.12	1.06	1.03	1.01	1.01	

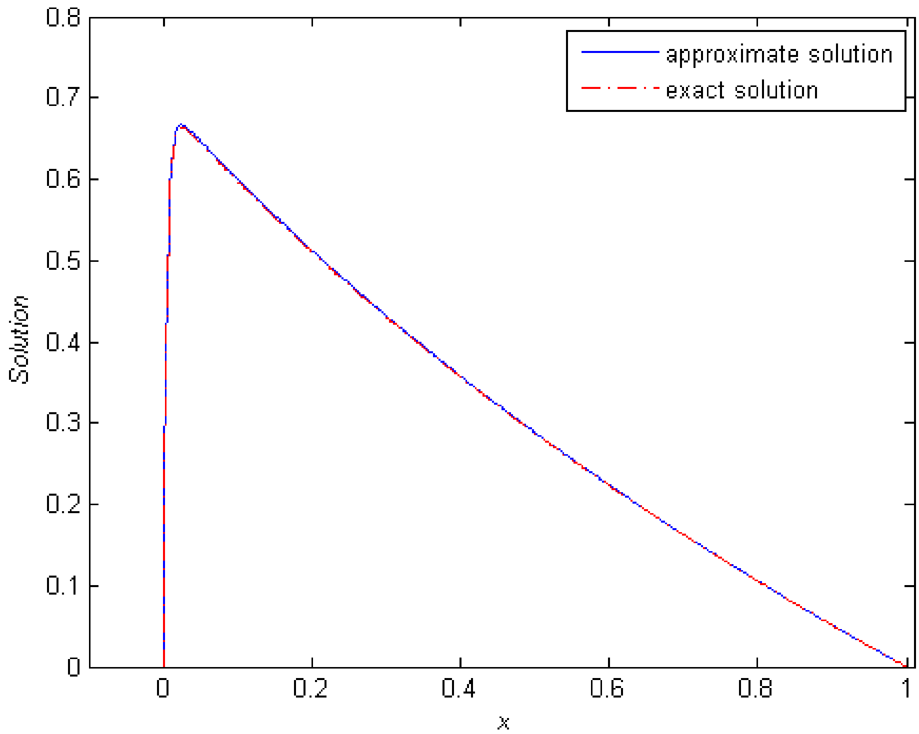
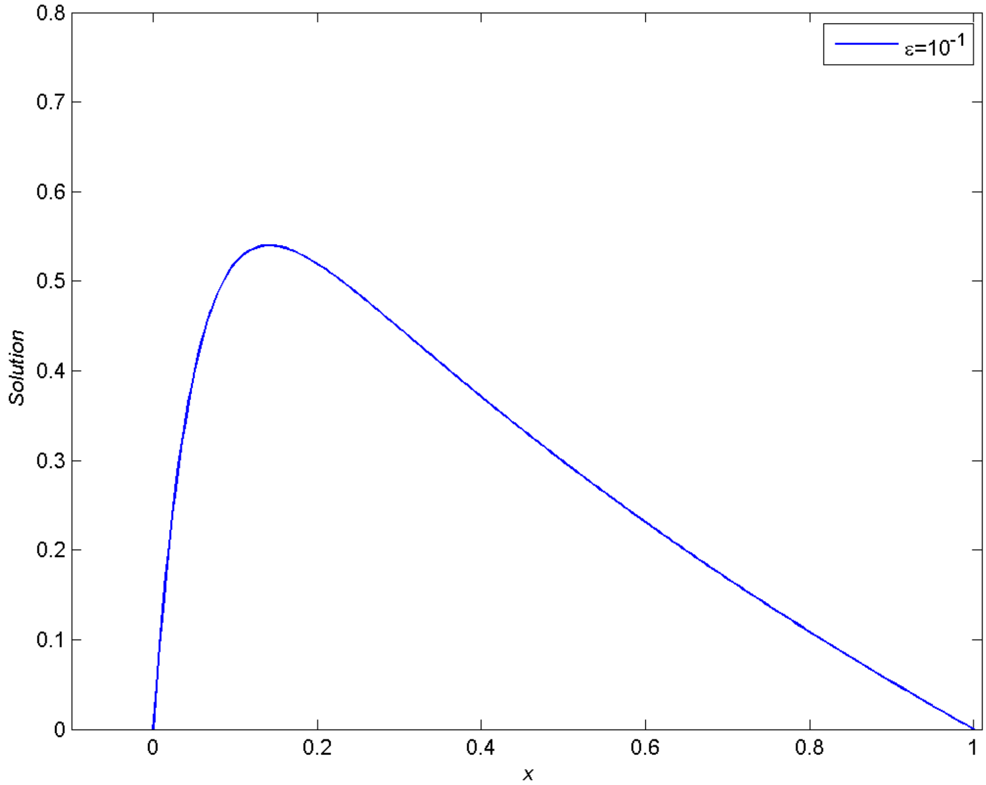
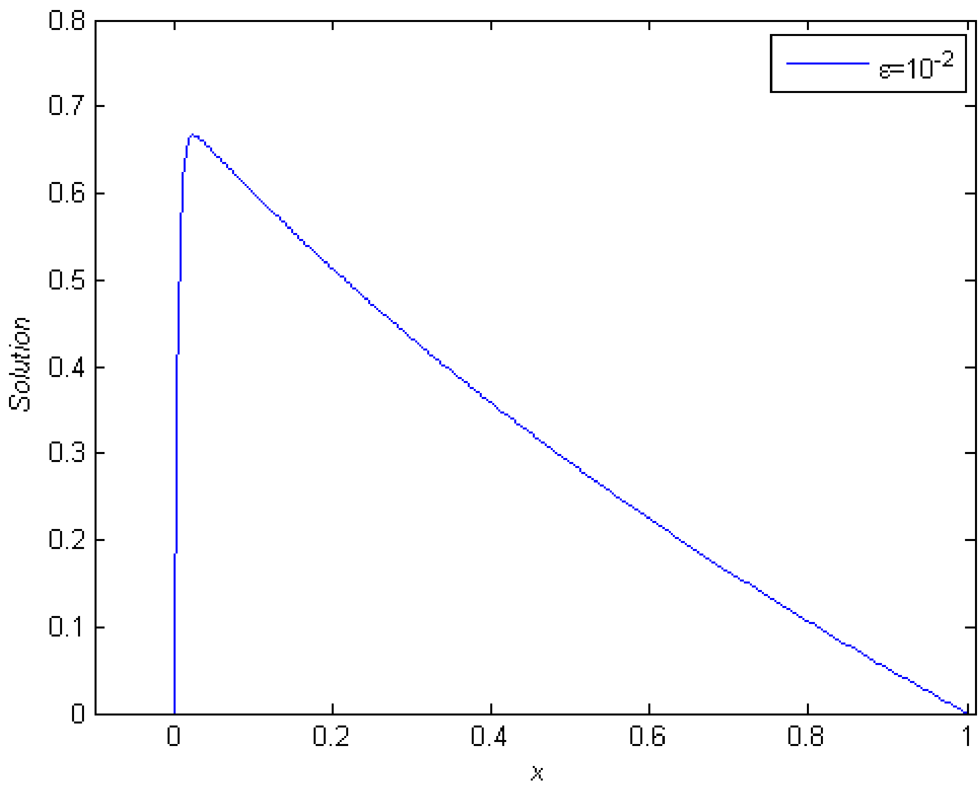


Figure 1. Comparison of the exact and approximate solutions of Example 1 for $\varepsilon = 10^{-2}$ and $N = 512$.

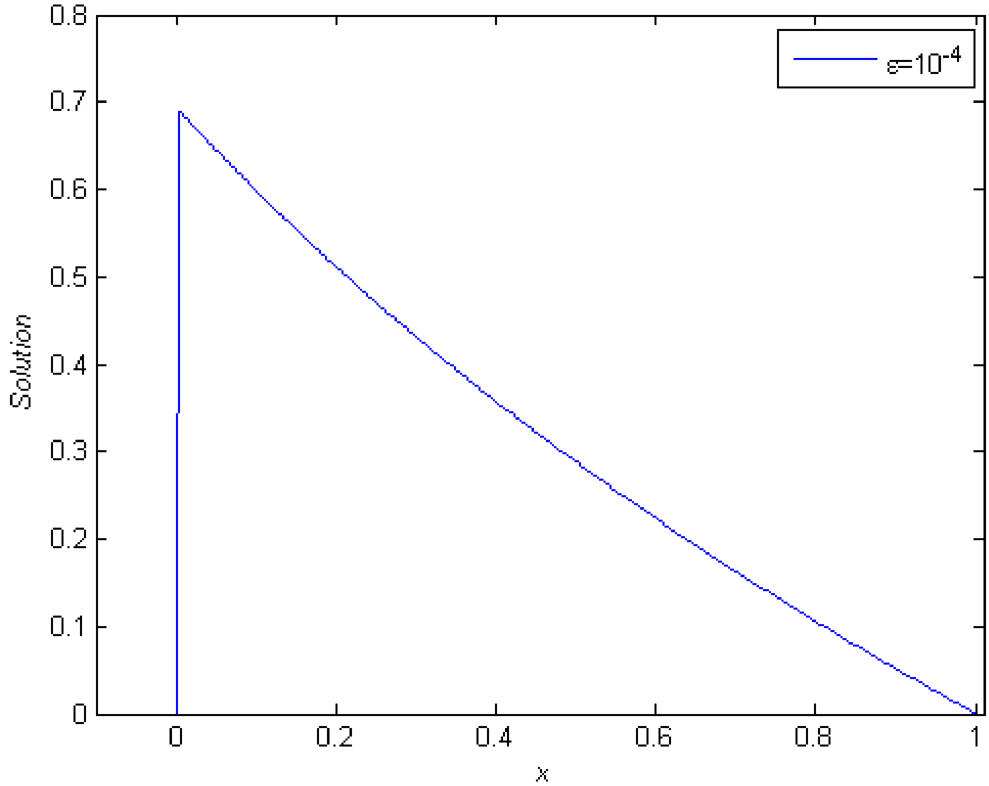


(**a**)

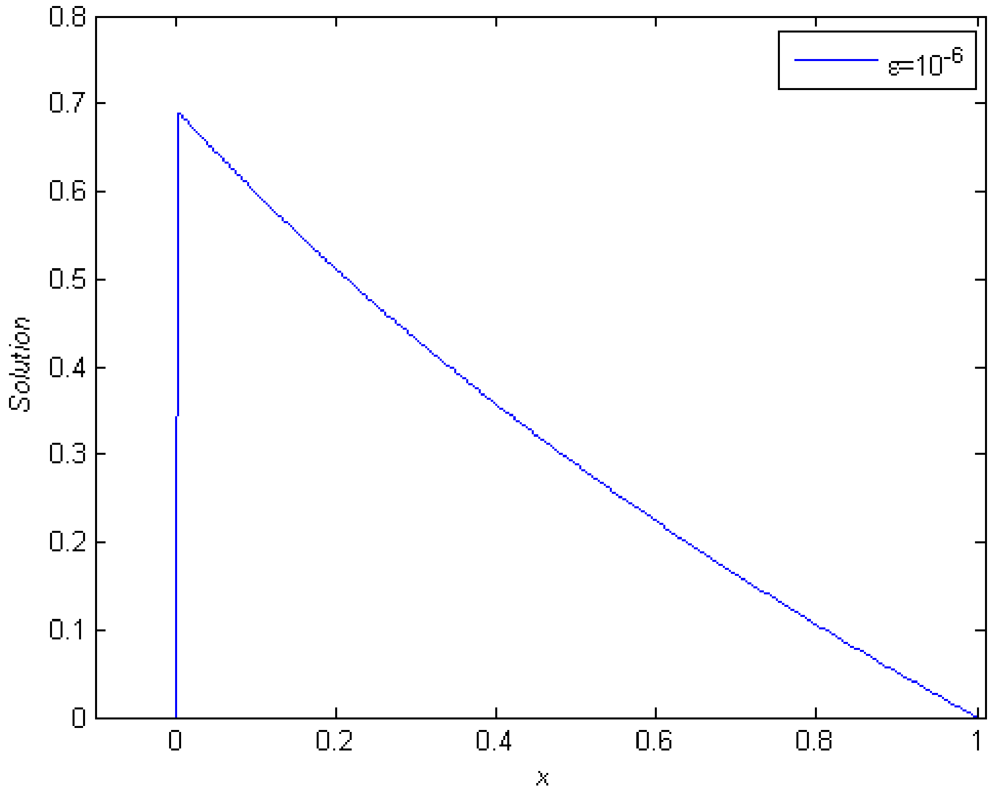


(**b**)

Figure 2. *Cont*.



(**c**)



(**d**)

Figure 2. Plots of the approximate solution of Example 1 for (**a**) $\varepsilon = 10^{-1}$, (**b**) $\varepsilon = 10^{-2}$, (**c**) $\varepsilon = 10^{-4}$, and (**d**) $\varepsilon = 10^{-6}$ with $N = 256$.

7. Conclusions

We designed a fitted operator finite difference method for the numerical solution of a singularly perturbed semilinear convection-diffusion problem. The semilinear problem is transformed into a system of linear problems via the quasilinearization technique. The analysis and the numerical illustration conform to an agreement of a first-order ε-uniform convergence rate independent of the perturbation parameter. We present the maximum error and rate of convergence for different values of ε and N in tables. This work presents an alternative to numerical approaches for this class of problems. This is, to the best of our knowledge, the first time that singularly perturbed semilinear convection-diffusion two-point boundary value problems are solved using fitted operator finite difference methods.

The few previous works on singularly perturbed semilinear problems considered majorly fitted mesh methods based on Shishkin meshes. Although uniformly convergent, these methods suffer the drawback of presenting error bounds that depend on a logarithmic factor. This factor contributes to lowering both the accuracy and the rate of convergence. A further advantage of fitted operator finite difference methods is the simplicity of their analysis due to their use of uniform meshes.

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