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Monotonicity Results for Nabla Riemann–Liouville Fractional Differences

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Abstract: Positivity analysis is used with some basic conditions to analyse monotonicity across all discrete fractional disciplines. This article addresses the monotonicity of the discrete nabla fractional differences of the Riemann–Liouville type by considering the positivity of $\left({}^{RL}_{b_0}\nabla^{\theta}g\right)(z)$ combined with a condition on $g(b_0+2)$, $g(b_0+3)$ and $g(b_0+4)$, successively. The article ends with a relationship between the discrete nabla fractional and integer differences of the Riemann–Liouville type, which serves to show the monotonicity of the discrete fractional difference $\left({}^{RL}_{b_0}\nabla^{\theta}g\right)(z)$.

Keywords: discrete fractional calculus; discrete nabla Riemann–Liouville fractional differences; monotonicity analysis

MSC: 26A48; 26A51; 33B10; 39A12; 39B62

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1. Introduction

In recent years, discrete fractional operators (sums and differences) have turned out to be modern tools in the modelling of many phenomena of mathematical analysis [1–3], electric circuits [4,5], medical sciences [6], material sciences and mechanics (see, for details, [7–9]). From among the several meanings and avenues of such studies, we choose to mention the discrete delta/nabla fractional operators of the Riemann–Liouville, Liouville–Caputo or other types (see, for details, [10,11]), from singular and non-singular kernel operators to the definitions based upon the time-scale theory. Some of these definitions are equivalent, even though they seem to be completely different, and they have been established by different authors (see, for example, [12–15]).

The positivity and monotonicity analyses have proven to be useful tools in discrete fractional calculus theory:

For the set $\{b_0, b_0+1, b_0+2, \dots\}$ denoted by $\mathbb{J}_{b_0}$ with $b_0 \in \mathbb{R}$, let g be defined on $\mathbb{J}_{b_0}$. Then, the function g will be monotonically increasing if $(\nabla g)(z)$ is positive; that is:

$$(\nabla g)(z) := g(z) - g(z-1) \geqq 0,$$

for each $z \in \mathbb{J}_{b_0+1}$.

In the context of discrete fractional calculus, the development of new positivity and monotonicity analyses is a source of interesting mathematical problems (see, for example, [16–21]). In recent years, several papers have been published devoted exclusively to the study of the problem of the monotonicity of discrete nabla/delta fractional operators with a certain kernel (and often under additional assumptions about the function). The interested reader may be referred, for example, to the developments reported in [22–28].

Our results in this paper concern the analysis of monotonicity for the discrete nabla fractional differences of Riemann–Liouville-type under the conditions that $\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \geqq 0$ and the ones coming from one of the following:

- $g(b_0+2) \geqq \frac{\theta}{\ell-1} g(b_0+1)$ for $\ell \in \mathbb{J}_3$ in Theorem 1.
- $g(b_0+3) \geqq \frac{\theta}{\ell-2} g(b_0+2) + \frac{\theta(\ell-\theta-2)}{(\ell-1)(\ell-2)} g(b_0+1)$ for $\ell \in \mathbb{J}_4$ in Theorem 2.
- $g(b_0+4) \geqq \frac{\theta}{\ell-2} g(b_0+3) + \frac{\theta(\ell-\theta-3)}{(\ell-2)(\ell-3)} g(b_0+2) + \frac{\theta(\ell-\theta-2)(\ell-\theta-3)}{(\ell-1)(\ell-2)(\ell-3)} g(b_0+1)$ for $\ell \in \mathbb{J}_5$ in Theorem 3.

Furthermore, we will show that the discrete nabla fractional Riemann–Liouville difference $\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \geqq 0$ for each $z \in \mathbb{J}_{b_0+1}$ by considering the relationship between the discrete nabla fractional and integer differences. It is worth mentioning that our results are motivated by the results in [29], wherein somewhat analogous results were investigated for the discrete delta Riemann–Liouville fractional differences.

The paper is divided into another three sections as follows. Section 2 considers preliminaries on discrete fractional operators of the Riemann–Liouville type and a main lemma, which we need in the next section. Our main results are presented in Section 3, which is separated into two subsections: In Section 3.1, we consider our three main theorems, which establish the monotonicity analysis of the discrete fractional operators. The relationship between the discrete nabla fractional and integer differences will be examined in Section 3.2, which will show how it will be used to establish the positivity of the discrete nabla fractional Riemann–Liouville operators. At the end of each theorem, a corollary is made. Section 4 provides a specific example, which confirms the applicability of our results. The article ends with concluding remarks and brief considerations of several discrete fractional modelling extensions, which can be applicable in the future to obtain monotonicity analysis for other types of discrete fractional operators in Section 5.

2. Preliminaries and a Lemma

Here, we provide some background material regarding the discrete nabla fractional operators toward the proof of our main achievements. As such, the main lemma is given.

Definition 1 (see [13,14,30]). *Let us denote the set $\{b_0, b_0+1, b_0+2, \dots\}$ by $\mathbb{J}_{b_0}$ and with the starting point $a \in \mathbb{R}$. Assume that g is defined on $\mathbb{J}_{b_0}$. Then, the ∇ Riemann–Liouville fractional sum of order θ (> 0) is expressed as follows:*

$$\left({}_{b_0}\nabla^{-\theta} g\right)(z) = \sum_{r=b_0+1}^{z} \frac{(z-r+1)^{[\theta-1]}}{\Gamma(\theta)} g(r) \quad \text{for } z \text{ in } \mathbb{J}_{b_0+1}, \tag{1}$$

where $z^{(\theta)}$ is defined by

$$z^{[\theta]} = \frac{\Gamma(z+\theta)}{\Gamma(z)} \quad \text{for } z \text{ and } \theta \text{ in } \mathbb{R}, \tag{2}$$

and it yields zero at a pole. It is also worth recalling that

$$\nabla z^{[\theta]} = \theta z^{[\theta-1]}. \tag{3}$$

Definition 2 (**see** [30])**.** *Let g be defined on $\mathbb{J}_{b_0}$. Then the ∇ Riemann–Liouville fractional difference of order θ ($\ell - 1 < \theta < \ell$) is defined by*

$$\begin{aligned}\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) &= \left(\nabla^{\ell}\, {}_{b_0}\nabla^{-(\ell-\theta)} g\right)(z) \\ &= \nabla^{\ell}\left[\sum_{r=b_0+1}^{z} \frac{(z-r+1)^{[\ell-\theta-1]}}{\Gamma(\ell-\theta)} g(r)\right] \quad \text{for } z \text{ in } \mathbb{J}_{b_0+1},\ \ell \text{ in } \mathbb{J}_1.\end{aligned}$$

Recently, Liu et al. [31] established an equivalent definition to Definition 2, as follows.

Definition 3 (**see** [31])**.** *Let $\ell - 1 < \theta < \ell$. Then the ∇ Riemann–Liouville fractional difference of order θ can be expressed as follows:*

$$\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) = \sum_{r=b_0+1}^{z} \frac{(z-r+1)^{[-\theta-1]}}{\Gamma(-\theta)} g(r) \quad \text{for } z \text{ in } \mathbb{J}_{b_0+\ell},\ \ell \text{ in } \mathbb{J}_1.$$

In order to begin our work later, we state and prove the following main lemma.

Lemma 1. *For g defined on $\mathbb{J}_{b_0}$, the ∇ Riemann–Liouville fractional difference of order θ ($1 < \theta < 2$) can be expressed as follows*:

$$\begin{aligned}\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) &= (\nabla g)(z) + \frac{(z-b_0)^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) \\ &\quad + \sum_{r=b_0+2}^{z-1} \frac{(z-r+1)^{[-\theta]}}{\Gamma(1-\theta)} (\nabla g)(r) \quad \text{for } z \text{ in } \mathbb{J}_{b_0+3}.\end{aligned} \tag{4}$$

In addition, it is essential to observe that

$$\frac{(z-r+1)^{[-\theta]}}{\Gamma(1-\theta)} < 0, \tag{5}$$

for each $r = b_0+2, b_0+3, \ldots, z-1$ and $z \in \mathbb{J}_{b_0+3}$.

Proof. According to Definition 3, we note for $1 < \theta < 2$ and $z \in \mathbb{J}_{b_0+2}$ that

$$\begin{aligned}\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) &= \sum_{r=b_0+1}^{z} \frac{(z-r+1)^{[-\theta-1]}}{\Gamma(-\theta)} g(r) \\ &\overset{\text{by (3)}}{=} \sum_{r=b_0+1}^{z} \frac{\nabla (z-r+1)^{[-\theta]}}{\Gamma(1-\theta)} g(r) \\ &= \sum_{r=b_0+1}^{z} \frac{(z-r+1)^{[-\theta]}}{\Gamma(1-\theta)} g(r) - \sum_{r=b_0+1}^{z-1} \frac{(z-r)^{[-\theta]}}{\Gamma(1-\theta)} g(r) \\ &= \frac{1}{\Gamma(1-\theta)}\left[(z-b_0)^{[-\theta]} g(b_0+1) + \sum_{r=b_0+2}^{z} (z-r+1)^{[-\theta]} g(r)\right.\end{aligned}$$

$$
\begin{aligned}
&\left. - \sum_{r=b_0+2}^{z} \left(z-r+1\right)^{[-\theta]} g(r-1) \right] \\
&= \left(\nabla g\right)(z) + \frac{\left(z-b_0\right)^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) + \sum_{r=b_0+2}^{z-1} \frac{\left(z-r+1\right)^{[-\theta]}}{\Gamma(1-\theta)} \left(\nabla g\right)(r),
\end{aligned}
$$

which completes the proof of (4). For $r = b_0+2, b_0+3, \ldots, z-1$ with $z \in \mathbb{J}_{b_0+3}$, we have

$$
\begin{aligned}
\frac{\left(z-r+1\right)^{[-\theta]}}{\Gamma(1-\theta)} &= \frac{\Gamma\left(z-r-\theta+1\right)}{\Gamma(1-\theta)\Gamma(z-r+1)} \\
&= \frac{(z-r-\theta)(z-r-\theta-1)\cdots(2-\theta)(1-\theta)}{(z-r)!},
\end{aligned}
$$

which is clearly positive for $\theta \in (1,2)$. Therefore, the second part of the lemma is proved. Hence, the proof of the lemma is complete. □

3. Main Results

This section is divided into two main subsections.

3.1. Monotonicity Results

This section is devoted to the study of the monotonicity analysis of the discrete fractional Riemann–Liouville differences.

Theorem 1. *Suppose that* $g : \mathbb{J}_{b_0} \longrightarrow \mathbb{R}$ *satisfies each of the following conditions*:

$$
\begin{aligned}
&\text{(i)} \quad \left({}_{b_0}^{RL}\nabla^{\theta} g\right)(z) \geqq 0 \quad \text{for each } z \in \mathbb{J}_{b_0+3}, \\
&\text{(ii)} \quad g(b_0+2) \geqq \frac{\theta}{\ell-1} g(b_0+1) \quad \text{for } \ell \in \mathbb{J}_3,
\end{aligned}
$$

for $\theta \in (1,2)$. *Then* $\left(\nabla g\right)(z) \geqq 0$ *for* $z \in \mathbb{J}_{b_0+3}$.

Proof. According to the assumption that $\left({}_{b_0}^{RL}\nabla^{\theta} g\right)(z) \geqq 0$ and the identity (4), one can see, for $z \in \mathbb{J}_{b_0+3}$, that

$$
\left(\nabla g\right)(z) \geqq -\frac{\left(z-b_0\right)^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) - \sum_{r=b_0+2}^{z-1} \frac{\left(z-r+1\right)^{[-\theta]}}{\Gamma(1-\theta)} \left(\nabla g\right)(r).
$$

If we set $z := b_0 + \ell$ for $\ell \in \mathbb{J}_3$, it follows that

$$
\left(\nabla g\right)(b_0+\ell) \geqq -\frac{\ell^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) - \sum_{r=b_0+2}^{b_0+\ell-1} \frac{\left(b_0+\ell-r+1\right)^{[-\theta]}}{\Gamma(1-\theta)} \left(\nabla g\right)(r). \tag{6}
$$

We proceed with the proof using the principle of mathematical induction on ℓ for the inequality (6). Indeed, for $\ell = 3$, we have

$$
\begin{aligned}
(\nabla g)(b_0+3) &\geqq -\frac{3^{[-\theta]}}{\Gamma(1-\theta)}g(b_0+1) - \frac{2^{[-\theta]}}{\Gamma(1-\theta)}(\nabla g)(b_0+2) \\
&= \frac{-1}{\Gamma(1-\theta)}\left[\frac{\Gamma(3-\theta)}{\Gamma(3)}g(b_0+1) + \frac{\Gamma(2-\theta)}{\Gamma(2)}(\nabla g)(b_0+2)\right] \\
&= \frac{-\Gamma(2-\theta)}{\Gamma(1-\theta)}\left[\frac{2-\theta}{2}g(b_0+1) + g(b_0+2) - g(b_0+1)\right] \\
&= \underbrace{(\theta-1)}_{>0}\underbrace{\left[\frac{-\theta}{2}g(b_0+1) + g(b_0+2)\right]}_{\geqq 0 \text{ per condition (ii)}} \geqq 0.
\end{aligned}
$$

Suppose that $(\nabla g)(b_0+\jmath) \geqq 0$ for $\jmath = 3,4,\ldots,\ell-1$ and $\ell \in \mathbb{J}_4$. Then, we shall show that $(\nabla g)(b_0+\ell) \geqq 0$. By making use of (6), we obtain

$$
\begin{aligned}
(\nabla g)(b_0+\ell) &\geqq -\frac{\ell^{[-\theta]}}{\Gamma(1-\theta)}g(b_0+1) - \sum_{r=b_0+2}^{b_0+\ell-1}\frac{(b_0+\ell-r+1)^{[-\theta]}}{\Gamma(1-\theta)}(\nabla g)(r) \\
&= -\frac{\ell^{[-\theta]}}{\Gamma(1-\theta)}g(b_0+1) - \frac{(\ell-1)^{[-\theta]}}{\Gamma(1-\theta)}(\nabla g)(b_0+2) \\
&\quad - \sum_{r=b_0+3}^{b_0+\ell-1}\underbrace{\frac{(b_0+\ell-r+1)^{[-\theta]}}{\Gamma(1-\theta)}}_{<0 \text{ per } (5)}\underbrace{(\nabla g)(r)}_{\geqq 0 \text{ per our claim}} \\
&\geqq -\frac{\ell^{[-\theta]}}{\Gamma(1-\theta)}g(b_0+1) - \frac{(\ell-1)^{[-\theta]}}{\Gamma(1-\theta)}(\nabla g)(b_0+2) \\
&= \frac{-1}{\Gamma(1-\theta)}\left[\frac{\Gamma(\ell-\theta)}{\Gamma(\ell)}g(b_0+1) + \frac{\Gamma(\ell-\theta-1)}{\Gamma(\ell-1)}(\nabla g)(b_0+2)\right] \\
&= \frac{-\Gamma(\ell-\theta-1)}{\Gamma(1-\theta)\Gamma(\ell-1)}\underbrace{\left[-\frac{\theta}{\ell-1}g(b_0+1) + g(b_0+2)\right]}_{\geqq 0 \text{ per condition (ii)}} \geqq 0,
\end{aligned}
$$

where we have used that

$$
\frac{-\Gamma(\ell-\theta-1)}{\Gamma(1-\theta)\Gamma(\ell-1)} = -\frac{(\ell-\theta-2)(\ell-\theta-3)\cdots(2-\theta)(1-\theta)}{(\ell-2)!} > 0, \tag{7}
$$

for $\theta \in (1,2)$ and $\ell \geqq 3$. Thus, the proof is complete. □

Corollary 1. *If the function $g : \mathbb{J}_{b_0} \longrightarrow \mathbb{R}$ satisfies*

(i) $\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \leqq 0$ for each $z \in \mathbb{J}_{b_0+3}$,

(ii) $g(b_0+2) \leqq \dfrac{\theta}{\ell-1}g(b_0+1)$ for $\ell \in \mathbb{J}_3$,

for $\theta \in (1,2)$, then, $(\nabla g)(z) \leqq 0$ for $z \in \mathbb{J}_{b_0+2}$.

Proof. Define $h := -g$. Thus, the proof follows immediately from Theorem 1 applying for the function g. □

Theorem 2. *Assume that* $g : \mathbb{J}_{b_0} \longrightarrow \mathbb{R}$ *satisfies each of the following conditions*:

$$\text{(i)} \quad \left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \geqq 0 \quad \text{for each } z \in \mathbb{J}_{b_0+3},$$

$$\text{(ii)} \quad g(b_0+3) \geqq \frac{\theta}{\ell-2} g(b_0+2) + \frac{\theta(\ell-\theta-2)}{(\ell-1)(\ell-2)} g(b_0+1) \quad \text{for } \ell \in \mathbb{J}_4,$$

for $\theta \in (1,2)$. *Then* $(\nabla g)(z) \geqq 0$ *for* $z \in \mathbb{J}_{b_0+4}$.

Proof. The proof will make use of the principle of mathematical induction on ℓ for the inequality (6). In fact, for $\ell = 4$, it follows that

$$\begin{aligned}
&(\nabla g)(b_0+4) \\
&\geqq -\frac{4^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) - \frac{3^{[-\theta]}}{\Gamma(1-\theta)} (\nabla g)(b_0+2) - \frac{2^{[-\theta]}}{\Gamma(1-\theta)} (\nabla g)(b_0+3) \\
&= \frac{-1}{\Gamma(1-\theta)} \left[\frac{\Gamma(4-\theta)}{\Gamma(4)} g(b_0+1) + \frac{\Gamma(3-\theta)}{\Gamma(3)} (\nabla g)(b_0+2) + \frac{\Gamma(2-\theta)}{\Gamma(2)} (\nabla g)(b_0+3) \right] \\
&= \frac{-\Gamma(2-\theta)}{\Gamma(1-\theta)} \left[\frac{(2-\theta)(3-\theta)}{6} g(b_0+1) + \frac{2-\theta}{2} \{ g(b_0+2) - g(b_0+1) \} \right. \\
&\left. + g(b_0+3) - g(b_0+2) \right] \\
&= \underbrace{(\theta-1)}_{>0} \underbrace{\left[-\frac{\theta(2-\theta)}{6} g(b_0+1) - \frac{\theta}{2} g(b_0+2) + g(b_0+3) \right]}_{\geqq 0 \text{ per condition (ii)}} \geqq 0.
\end{aligned}$$

If we let $(\nabla g)(b_0+\jmath) \geqq 0$ for $\jmath = 4, 5, \ldots, \ell-1$ and $\ell \in \mathbb{J}_5$, then, we try to show that $(\nabla g)(b_0+\ell) \geqq 0$, with the help of (6), we can deduce

$$\begin{aligned}
&(\nabla g)(b_0+\ell) \\
&\geqq -\frac{\ell^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) - \frac{(\ell-1)^{[-\theta]}}{\Gamma(1-\theta)} (\nabla g)(b_0+2) - \frac{(\ell-2)^{[-\theta]}}{\Gamma(1-\theta)} (\nabla g)(b_0+3) \\
&\underbrace{- \sum_{r=b_0+4}^{b_0+\ell-1} \frac{(b_0+\ell-r+1)^{[-\theta]}}{\Gamma(1-\theta)}}_{\geqq 0 \text{ per (5) and our claim}} \\
&\geqq -\frac{\ell^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) - \frac{(\ell-1)^{[-\theta]}}{\Gamma(1-\theta)} (\nabla g)(b_0+2) - \frac{(\ell-2)^{[-\theta]}}{\Gamma(1-\theta)} (\nabla g)(b_0+3) \\
&= \frac{-1}{\Gamma(1-\theta)} \left[\frac{\Gamma(\ell-\theta)}{\Gamma(\ell)} g(b_0+1) + \frac{\Gamma(\ell-\theta-1)}{\Gamma(\ell-1)} (\nabla g)(b_0+2) \right. \\
&\left. + \frac{\Gamma(\ell-\theta-2)}{\Gamma(\ell-2)} (\nabla g)(b_0+3) \right] \\
&= \underbrace{\frac{-\Gamma(\ell-\theta-2)}{\Gamma(1-\theta)\Gamma(\ell-2)}}_{>0 \text{ per (7)}} \underbrace{\left[-\frac{\theta(\ell-\theta-2)}{(\ell-1)(\ell-2)} g(b_0+1) - \frac{\theta}{\ell-2} g(b_0+2) + g(b_0+3) \right]}_{\geqq 0 \text{ per condition (ii)}} \geqq 0,
\end{aligned}$$

which completes the proof. □

Corollary 2. *If the function* $g : \mathbb{J}_{b_0} \longrightarrow \mathbb{R}$ *satisfies*

$$\text{(i)} \quad \left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \leqq 0 \quad \text{for each } z \in \mathbb{J}_{b_0+3},$$

$$\text{(ii)} \quad g(b_0+3) \leqq \frac{\theta}{\ell-2} g(b_0+2) + \frac{\theta(\ell-\theta-2)}{(\ell-1)(\ell-2)} g(b_0+1) \quad \text{for } \ell \in \mathbb{J}_4,$$

for $\theta \in (1,2)$*, then,* $\big(\nabla g\big)(z) \leqq 0$ *for* $z \in \mathbb{J}_{b_0+5}$.

Proof. The proof follows immediately from Theorem 2 applied to the function $h := -g$. □

Theorem 3. *Assume that* $g : \mathbb{J}_{b_0} \longrightarrow \mathbb{R}$ *satisfies each of the following conditions*:

$$\text{(i)} \quad \left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \geqq 0 \quad \text{for each } z \in \mathbb{J}_{b_0+3},$$

$$\text{(ii)} \quad g(b_0+4) \geqq \frac{\theta}{\ell-2} g(b_0+3) + \frac{\theta(\ell-\theta-3)}{(\ell-2)(\ell-3)} g(b_0+2) + \frac{\theta(\ell-\theta-2)(\ell-\theta-3)}{(\ell-1)(\ell-2)(\ell-3)} g(b_0+1) \quad \text{for } \ell \in \mathbb{J}_5,$$

for $\theta \in (1,2)$*. Then* $\big(\nabla g\big)(z) \geqq 0$ *for* $z \in \mathbb{J}_{b_0+5}$.

Proof. Again, we will prove this theorem using the principle of mathematical induction induction on ℓ. Thus, (6) at $\ell = 5$ leads to

$$\begin{aligned}
&\big(\nabla g\big)(b_0+5) \\
&\geqq -\frac{5^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) - \frac{4^{[-\theta]}}{\Gamma(1-\theta)} \big(\nabla g\big)(b_0+2) - \frac{3^{[-\theta]}}{\Gamma(1-\theta)} \big(\nabla g\big)(b_0+3) \\
&- \frac{2^{[-\theta]}}{\Gamma(1-\theta)} \big(\nabla g\big)(b_0+4) \\
&= \frac{-\Gamma(2-\theta)}{\Gamma(1-\theta)} \left[\frac{(2-\theta)(3-\theta)(4-\theta)}{24} g(b_0+1) + \frac{(2-\theta)(3-\theta)}{6} \big(\nabla g\big)(b_0+2) \right. \\
&\left. + \frac{2-\theta}{2} \big(\nabla g\big)(b_0+3) + \big(\nabla g\big)(b_0+4) \right] \\
&= \underbrace{(\theta-1)}_{>0} \underbrace{\left[-\frac{\theta(2-\theta)(3-\theta)}{24} g(b_0+1) - \frac{\theta(2-\theta)}{6} g(b_0+2) - \frac{\theta}{2} g(b_0+3) + g(b_0+4) \right]}_{\geqq 0 \text{ per condition (ii)}} \geqq 0.
\end{aligned}$$

Now, we assume that $\big(\nabla g\big)(b_0+\jmath) \geqq 0$ for $\jmath = 5, 6, \ldots, \ell-1$ and $\ell \in \mathbb{J}_6$. Then, we have to show that $\big(\nabla g\big)(b_0+\ell) \geqq 0$. In view of (6), we can deduce

$$\begin{aligned}
&\big(\nabla g\big)(b_0+\ell) \\
&\geqq -\frac{\ell^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) - \frac{(\ell-1)^{[-\theta]}}{\Gamma(1-\theta)} \big(\nabla g\big)(b_0+2) - \frac{(\ell-2)^{[-\theta]}}{\Gamma(1-\theta)} \big(\nabla g\big)(b_0+3) \\
&- \frac{(\ell-3)^{[-\theta]}}{\Gamma(1-\theta)} \big(\nabla g\big)(b_0+4) - \underbrace{\sum_{r=b_0+5}^{b_0+\ell-1} \frac{(b_0+\ell-r+1)^{[-\theta]}}{\Gamma(1-\theta)}}_{\geqq 0 \text{ per (5) and our claim}}
\end{aligned}$$

$$\begin{aligned}
&\geqq -\frac{\ell^{[-\theta]}}{\Gamma(1-\theta)}g(b_0+1)-\frac{(\ell-1)^{[-\theta]}}{\Gamma(1-\theta)}(\nabla g)(b_0+2)\\
&-\frac{(\ell-2)^{[-\theta]}}{\Gamma(1-\theta)}(\nabla g)(b_0+3)-\frac{(\ell-3)^{[-\theta]}}{\Gamma(1-\theta)}(\nabla g)(b_0+4)\\
&=\frac{-1}{\Gamma(1-\theta)}\Bigg[\frac{\Gamma(\ell-\theta)}{\Gamma(\ell)}g(b_0+1)+\frac{\Gamma(\ell-\theta-1)}{\Gamma(\ell-1)}(\nabla g)(b_0+2)\\
&+\frac{\Gamma(\ell-\theta-2)}{\Gamma(\ell-2)}(\nabla g)(b_0+3)+\frac{\Gamma(\ell-\theta-3)}{\Gamma(\ell-3)}(\nabla g)(b_0+4)\Bigg]\\
&=\underbrace{\frac{-\Gamma(\ell-\theta-3)}{\Gamma(1-\theta)\Gamma(\ell-3)}}_{>0\text{ per (7)}}\underbrace{\begin{bmatrix} -\frac{\theta(\ell-\theta-2)(\ell-\theta-3)}{(\ell-1)(\ell-2)(\ell-3)}g(b_0+1)\\ -\frac{\theta(\ell-\theta-3)}{(\ell-2)(\ell-3)}g(b_0+2)-\frac{\theta}{\ell-2}g(b_0+3)+g(b_0+4)\end{bmatrix}}_{\geqq 0\text{ per condition (ii)}}\geqq 0,
\end{aligned}$$

which completes the proof. □

Corollary 3. *If the function* $g:\mathbb{J}_{b_0}\longrightarrow\mathbb{R}$ *satisfies*

$$\begin{aligned}
&\text{(i)}\quad \left({}^{RL}_{b_0}\nabla^{\theta}g\right)(z)\leqq 0\quad\text{for each } z\in\mathbb{J}_{b_0+3},\\
&\text{(ii)}\quad g(b_0+4)\geqq\frac{\theta}{\ell-2}g(b_0+3)+\frac{\theta(\ell-\theta-3)}{(\ell-2)(\ell-3)}g(b_0+2)\\
&\qquad\qquad+\frac{\theta(\ell-\theta-2)(\ell-\theta-3)}{(\ell-1)(\ell-2)(\ell-3)}g(b_0+1)\quad\text{for } \ell\in\mathbb{J}_5,
\end{aligned}$$

for $\theta\in(1,2)$*, then,* $(\nabla g)(z)\leqq 0$ *for* $z\in\mathbb{J}_{b_0+4}$.

Proof. This proof follows directly from Theorem 3 applied to the function $h:=-g$. □

3.2. Discrete Nabla Fractional and Integer Differences

Based on Lemma 1, we can now establish a relationship between the discrete nabla fractional and integer differences of the Riemann–Liouville type, and we immediately present the final monotonicity result of this study.

Theorem 4. *Let* g *be defined on* $\mathbb{J}_{b_0}$ *and* $N-1<\theta<N$ *with* $N\in\mathbb{J}_1$*. Then*

$$\begin{aligned}
\left({}^{RL}_{b_0}\nabla^{\theta}g\right)(b_0+N+\ell)&=\sum_{\imath=0}^{N-1}\frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta-\imath)}(\nabla^{\imath}g)(b_0+1)\\
&+\frac{1}{\Gamma(N-\theta)}\sum_{\imath=1}^{\ell-1}(\ell-\imath+1)^{[-\theta+N-1]}\left(\nabla^{N}g\right)(b_0+\imath+N)+\left(\nabla^{N}g\right)(b_0+\ell+N),
\end{aligned}$$

for all $\ell\in\mathbb{J}_0$*. In addition, we have*

$$\begin{aligned}
&\frac{(\ell-\imath+1)^{[-\theta+N-1]}}{\Gamma(N-\theta)}\\
&=\frac{(N-\theta+\ell-\imath-1)(N-\theta+\ell-\imath-2)\cdots(N+1-\theta)(N-\theta)}{(\ell-\imath)!}>0,
\end{aligned}\tag{8}$$

for $\imath=1,2,\ldots,\ell-1$ *with* $\ell\in\mathbb{J}_1$.

Proof. For $N = 1$, from Lemma 1, we find for $z \in \mathbb{J}_{b_0+2}$ that

$$\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) = \frac{(z-b_0)^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) + \sum_{r=b_0+2}^{z} \frac{(z-r+1)^{[-\theta]}}{\Gamma(1-\theta)} (\nabla g)(r). \tag{9}$$

Now, by the same technique used in Lemma 1, for $N = 2$, we have:

$$\begin{aligned}
&\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \\
&= \nabla^2 \left[\sum_{r=b_0+1}^{z} \frac{(z-r+1)^{[1-\theta]}}{\Gamma(2-\theta)} g(r)\right] = \nabla\left[\nabla\left(\sum_{r=b_0+1}^{z} \frac{(z-r+1)^{[1-\theta]}}{\Gamma(2-\theta)} g(r)\right)\right] \\
&= \nabla\left[\frac{(z-b_0)^{[1-\theta]}}{\Gamma(2-\theta)} g(b_0+1) + \sum_{r=b_0+2}^{z} \frac{(z-r+1)^{[1-\theta]}}{\Gamma(2-\theta)} (\nabla g)(r)\right] \\
&= \nabla\left[\frac{(z-b_0)^{[1-\theta]}}{\Gamma(2-\theta)} g(b_0+1)\right] + \nabla\left[\sum_{r=b_0+2}^{z} \frac{(z-r+1)^{[1-\theta]}}{\Gamma(2-\theta)} (\nabla g)(r)\right] \\
&= \frac{(z-b_0)^{[-\theta]}}{\Gamma(1-\theta)} g(b_0+1) + \frac{(z-b_0)^{[1-\theta]}}{\Gamma(2-\theta)} g(b_0+1) \\
&+ \sum_{r=b_0+3}^{z} \frac{(z-r+1)^{[1-\theta]}}{\Gamma(2-\theta)} (\nabla^2 g)(r),
\end{aligned} \tag{10}$$

for $z \in \mathbb{J}_{b_0+3}$, where we have used the following fact

$$\nabla (z-b_0)^{[1-\theta]} = (z-b_0)^{[1-\theta]} - (z-1-b_0)^{[1-\theta]} = (1-\theta)(z-b_0)^{[-\theta]}.$$

We can continue by the same process to obtain

$$\begin{aligned}
\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) = \sum_{\imath=0}^{N-1} \frac{(z-b_0)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)} (\nabla^{\imath} g)(b_0+1) \\
+ \sum_{r=b_0+N+1}^{z} \frac{(z-r+1)^{[N-\theta-1]}}{\Gamma(N-\theta)} (\nabla^N g)(r),
\end{aligned} \tag{11}$$

for $z \in \mathbb{J}_{b_0+N+1}$. Now, we define $z := b_0 + N + \ell$ for $\ell \in \mathbb{J}_1$ to obtain

$$\begin{aligned}
&\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(b_0+N+\ell) \\
&= \sum_{\imath=0}^{N-1} \frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)} (\nabla^{\imath} g)(b_0+1) + \sum_{r=b_0+N+1}^{b_0+N+\ell} \frac{(b_0+N+\ell-r+1)^{[N-\theta-1]}}{\Gamma(N-\theta)} (\nabla^N g)(r) \\
&= \sum_{\imath=0}^{N-1} \frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)} (\nabla^{\imath} g)(b_0+1) + \sum_{\imath=1}^{\ell} \frac{(\ell-\imath+1)^{[N-\theta-1]}}{\Gamma(N-\theta)} (\nabla^N g)(b_0+\imath+N) \\
&= \sum_{\imath=0}^{N-1} \frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)} (\nabla^{\imath} g)(b_0+1) \\
&+ \sum_{\imath=1}^{\ell-1} \frac{(\ell-\imath+1)^{[N-\theta-1]}}{\Gamma(N-\theta)} (\nabla^N g)(b_0+\imath+N) + (\nabla^N g)(b_0+\ell+N),
\end{aligned}$$

which completes the proof of the first part. For the second part of the theorem, we see that

$$\begin{aligned}\frac{(\ell-\imath+1)^{[-\theta+N-1]}}{\Gamma(N-\theta)} &= \frac{\Gamma(\ell-\imath-\theta+N)}{\Gamma(\ell-\imath+1)\Gamma(N-\theta)} \\ &= \frac{(N-\theta+\ell-\imath-1)(N-\theta+\ell-\imath-2)\cdots(N+1-\theta)(N-\theta)}{(\ell-\imath)!} > 0,\end{aligned}$$

for $N-1<\theta<N$ and $\imath=1,2,\ldots,\ell-1$. Hence, the proof is complete. □

Our final result is on the positivity of $\left({}^{RL}_{b_0}\nabla^{\theta}g\right)(z)$, as follows.

Theorem 5. *Let $g:\mathbb{J}_{b_0}\longrightarrow\mathbb{R}$ be a function, $N-1<\theta<N$ with $N\in\mathbb{J}_1$, $(\nabla^N g)(z)\geqq 0$ for $z\in\mathbb{J}_{b_0+1}$, $(-1)^{N-\imath}(\nabla^N g)(b_0+1)\leqq 0$ for $\imath=0,1,\ldots,N-1$. Then $\left({}^{RL}_{b_0}\nabla^{\theta}g\right)(z)\geqq 0$ for each $z\in\mathbb{J}_{b_0+1}$.*

Proof. Let ℓ be a fixed but arbitrary element in $\mathbb{J}_1$. We can then see that

$$\begin{aligned}\frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)} &= \frac{\Gamma(N+\ell-\theta+\imath)}{\Gamma(1-\theta+\imath)\Gamma(N+\ell)} \\ &= \frac{(N+\ell-\theta+\imath-1)(N+\ell-\theta+\imath-2)\cdots(2-\theta+\imath)(1-\theta+\imath)}{(N+\ell-1)!}.\end{aligned}$$

Now, if $N-\imath-1$ is even, then we obtain

$$\frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)} > 0 \quad \text{and} \quad \left(\nabla^N g\right)(b_0+1)\geqq 0,$$

but if $N-\imath-1$ is odd, then we obtain

$$\frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)} < 0 \quad \text{and} \quad \left(\nabla^N g\right)(b_0+1)\leqq 0.$$

These provide that

$$\sum_{\imath=0}^{N-1}\frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)}(\nabla^{\imath}g)(b_0+1)\geqq 0. \tag{12}$$

Hence, according to Theorems 4, (8) and (12) and the assumption that $(\nabla^N g)(z)\geqq 0$, we can deduce

$$\begin{aligned}\left({}^{RL}_{b_0}\nabla^{\theta}g\right)(b_0+N+\ell) &= \sum_{\imath=0}^{N-1}\frac{(N+\ell)^{[-\theta+\imath]}}{\Gamma(1-\theta+\imath)}(\nabla^{\imath}g)(b_0+1) \\ &\quad+\sum_{\imath=1}^{\ell-1}\frac{(\ell-\imath+1)^{[N-\theta-1]}}{\Gamma(N-\theta)}(\nabla^N g)(b_0+\imath+N)+(\nabla^N g)(b_0+\ell+N) \\ &\geqq \sum_{\imath=1}^{\ell-1}\frac{(\ell-\imath+1)^{[N-\theta-1]}}{\Gamma(N-\theta)}(\nabla^N g)(b_0+\imath+N)\geqq 0.\end{aligned}$$

By putting $z:=b_0+N+\ell$ for arbitrary $\ell\in\mathbb{J}_1$, we obtain $\left({}^{RL}_{b_0}\nabla^{\theta}g\right)(z)\geqq 0$ for each $z\in\mathbb{J}_{b_0+1}$ as desired. □

Corollary 4. *Let $g : \mathbb{J}_{b_0} \longrightarrow \mathbb{R}$ be a function, $N-1<\theta<N$ with $N \in \mathbb{J}_1$, $\left(\nabla^N g\right)(z) \leqq 0$ for $z \in \mathbb{J}_{b_0+1}$, $(-1)^{N-\imath}\left(\nabla^N g\right)(b_0+1) \geqq 0$ for $\imath = 0,1,\ldots,N-1$. Then $\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \leqq 0$ for each $z \in \mathbb{J}_{b_0+1}$.*

4. Application: A Specific Example

In this section, we provide a specific example to illustrate our results.

Consider the function

$$g(z) = \left(\frac{8}{3}\right)^z \quad \text{for } z \in \mathbb{J}_{b_0+2}.$$

At first, we will try to show that $\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \geq 0$ for $z \in \{b_0+3, b_0+4\}$, $\theta = \frac{3}{2}$ and $b_0 = 0$. From Definition 3 at $z = b_0+3$, we have

$$\begin{aligned}
&\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(b_0+3) \\
&= \frac{1}{\Gamma(-\theta)} \sum_{r=b_0+1}^{b_0+3} (b_0+3-r+1)^{[-\theta-1]} g(r) \\
&= \frac{1}{\Gamma(-\theta)} \left\{ (3)^{[-\theta-1]} g(b_0+1) + (2)^{[-\theta-1]} g(b_0+2) + (1)^{[-\theta-1]} g(b_0+3) \right\} \\
&= \frac{-\theta(1-\theta)}{2} g(b_0+1) - \theta\, g(b_0+2) + g(b_0+3) \\
&= \frac{251}{27} \geq 0,
\end{aligned}$$

which leads to

$$g(b_0+3) \geq \frac{\theta(1-\theta)}{2} g(b_0+1) + \theta\, g(b_0+2). \tag{13}$$

In addition, Definition 3 at $z = b_0+4$ gives

$$\begin{aligned}
&\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(b_0+4) \\
&= \frac{1}{\Gamma(-\theta)} \sum_{r=b_0+1}^{b_0+4} (b_0+4-r+1)^{[-\theta-1]} g(r) \\
&= \frac{1}{\Gamma(-\theta)} \Big\{ (4)^{[-\theta-1]} g(b_0+1) + (3)^{[-\theta-1]} g(b_0+2) \\
&\qquad + (2)^{[-\theta-1]} g(b_0+3) + (1)^{[-\theta-1]} g(b_0+4) \Big\} \\
&= \frac{-\theta(1-\theta)(2-\theta)}{6} g(b_0+1) - \frac{\theta(1-\theta)}{2} g(b_0+2) - \theta\, g(b_0+3) + g(b_0+4) \\
&= \frac{3179}{162} \geq 0,
\end{aligned}$$

which implies that

$$g(b_0+4) \geq \frac{\theta(1-\theta)(2-\theta)}{6} g(b_0+1) + \frac{\theta(1-\theta)}{2} g(b_0+2) + \theta\, g(b_0+3). \tag{14}$$

On the other hand, we consider the condition:

$$g(b_0+2) \geq \frac{\theta}{\ell-1} g(b_0+1),$$

at $\ell = 3, 4$. At $\ell = 3$, it follows that

$$\left(\frac{8}{3}\right)^2 = g(b_0 + 2) \geq \frac{\theta}{2} g(b_0 + 1) = \frac{3}{4}\left(\frac{8}{3}\right),$$

which means that

$$g(b_0 + 2) \geq \frac{\theta}{2} g(b_0 + 1). \tag{15}$$

In addition, at $\ell = 4$, it follows that

$$\begin{aligned}\left(\frac{8}{3}\right)^2 &= g(b_0 + 2) \\ &\geq \frac{\Gamma(\theta+2)}{\Gamma(\theta)\Gamma(3)} g(b_0 + \theta h) = \frac{\theta}{3} g(b_0 + 1) = \frac{1}{2}\left(\frac{8}{3}\right),\end{aligned}$$

which is equivalent to

$$g(b_0 + 2) \geq \frac{\theta}{3} g(b_0 + 1). \tag{16}$$

Thus, we can conclude from the inequalities (13)–(14) that

$$\begin{aligned}(\nabla g)(b_0 + 3) &= g(b_0 + 3) - g(b_0 + 2) \\ &\geq \frac{\theta(1-\theta)}{2} g(b_0 + 1) + \theta\, g(b_0 + 2) - g(b_0 + 2) \\ &= (\theta - 1)\left\{\frac{-\theta}{2} g(b_0 + 1) + \theta\, g(b_0 + 2)\right\} \geq 0.\end{aligned}$$

Also, from the inequalities (15)–(16) we can conclude that

$$\begin{aligned}(\nabla g)(b_0 + 4) &= g(b_0 + 4) - g(b_0 + 3) \\ &\geq \frac{\theta(1-\theta)(2-\theta)}{6} g(b_0 + 1) + \frac{\theta(1-\theta)}{2} g(b_0 + 2) + \theta\, g(b_0 + 3) - g(b_0 + 3) \\ &= \frac{(\theta-1)(2-\theta)}{2}\left\{\frac{-\theta}{3} g(b_0 + 1) + \theta\, g(b_0 + 2)\right\} \geq 0.\end{aligned}$$

These inequalities imply that g is non-decreasing in the time set $\{b_0 + 3, b_0 + 4\}$.

5. Conclusions and Future Directions

In this paper, we studied the monotonicity analysis for the discrete nabla fractional differences of the Riemann–Liouville type. The first three main results were dedicated to the positivity of $(\nabla g)(z)$ by assuming that $\left({}^{RL}_{b_0}\nabla^{\theta} g\right)(z) \geqq 0$ combined with the condition that $g(b_0 + 2) \geqq \frac{\theta}{\ell-1} g(b_0 + 1)$ for $\ell \in \mathbb{J}_3$ in Theorem 1, $g(b_0 + 3) \geqq \frac{\theta}{\ell-2} g(b_0 + 2) + \frac{\theta(\ell-\theta-2)}{(\ell-1)(\ell-2)} g(b_0 + 1)$ for $\ell \in \mathbb{J}_4$ in Theorem 2, and $g(b_0 + 4) \geqq \frac{\theta}{\ell-2} g(b_0 + 3) + \frac{\theta(\ell-\theta-3)}{(\ell-2)(\ell-3)} g(b_0 + 2) + \frac{\theta(\ell-\theta-2)(\ell-\theta-3)}{(\ell-1)(\ell-2)(\ell-3)} g(b_0 + 1)$ for $\ell \in \mathbb{J}_5$ in Theorem 3.

On the other hand, the relationship between the discrete nabla fractional and integer differences of the Riemann–Liouville type has been made. From which the positivity of the discrete nabla fractional differences of the Riemann–Liouville type has been established. In addition, some particular results have been obtained in the corollaries, which showed the negativity (decreasing) of the function.

There is vast room for monotonicity analysis to be explored in this fertile field of discrete fractional operators, for example, discrete Caputo–Fabrizio and Atangana–Baleanu fractional operators (see [30,32,33] for information about these discrete operators).

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