Exponential Synchronization of Hyperbolic Complex Spatio-Temporal Networks with Multi-Weights

Hongkun Ma 1 and Chengdong Yang 2,*

1 School of Economics, Shandong Normal University, Jinan 250358, China; mahongkun@sdnu.edu.cn
2 School of Information Science and Technology, Linyi University, Linyi 276005, China
* Correspondence: yangchengdong@lyu.edu.cn

Abstract: This paper deals with the leader-following synchronization of first-order, semi-linear, complex spatio-temporal networks. Firstly, two sorts of complex spatio-temporal networks based on hyperbolic partial differential equations (CSTNHPDEs) are built: one with a single weight and the other with multi-weights. Then, a new distributed controller is designed to address CSTNHPDE with a single weight. Sufficient conditions for the synchronization and exponential synchronization of CSTNHPDE are presented by showing the gain ranges. Thirdly, the proposed distributed controller addresses of CSTNHPDE with multi-weights, and gain ranges are obtained for synchronization and exponential synchronization, respectively. Finally, two examples show the effectiveness and good performance of the control methods.

Keywords: synchronization; complex networks; multi-weights; hyperbolic partial differential equations

MSC: 05C82

1. Introduction

The synchronization of complex networks, a group dynamical behavior, aims to drive nodes to perform a designated task synchronously. It has been applied to many engineering aspects, such as intelligent traffic [1,2], circuit systems [3], image processing [4–6], smart grids [7], secure communication [8,9], multi-agent systems [10], rumor propagation [11], data security [12], biological systems [13], etc.

A number of important works discuss the synchronization of complex networks [14–18]. This literature shows node dynamics depending only on time. In practice, the dynamics of all processes are spatio-temporal [19–21]. As a consequence, it is necessary to study complex spatio-temporal networks (CSTNs), which is with spatio-temporal characteristics [22]. Wu et al. studied the synchronization of CSTNs with space-independent coefficients and space-dependent coefficients, with or without spatio-temporal disturbance [23]. Huang et al. proposed a fuzzy synchronization method for nonlinear CSTNs with reaction—diffusion terms [24]. Luo et al. studied event-triggered control for the finite-time synchronization of reaction–diffusion CSTNs [25]. Yang et al. studied the boundary control of fractional-order CSTNs [26]. Zheng et al. researched synchronization analysis for fractional-order CSTNs with time delays [27]. Shen et al. studied the $H_{\infty}$ synchronization of Markov jump CSTNs using an observer-based method [28]. Kabalan et al. studied boundary control for the synchronization of leader-follower CSTNs with in-domain coupling [29].

Most references are modeled by parabolic PDEs, while there are few methods studying hyperbolic PDEs. There are many hyperbolic PDEs systems in practice, including shallow-water systems [30], epidemic models [31], district heating networks [32], heat exchangers [33], and reactor models [34]. Therefore, it is important to study the synchronization of hyperbolic PDEs-based CSTNs (HPDECSTNs).
Chueshov presented invariant manifolds and nonlinear master–slave synchronization hyperbolic and parabolic CSTNs [35]. Li studied the synchronization, exact synchronization and approximate synchronization of HPDECSTNs [36]. Li and Lu researched exact-boundary synchronization for a kind of first-order hyperbolic system [37]. Lu proposed a local exact-boundary synchronization for a kind of first-order, quasi-linear hyperbolic system [38]. However, technical difficulties remain regarding the synchronization of a semi-linear, first-order HPDECSTNs when the convection coefficient is symmetric semi-negative definite or semi-positive definite, which motivate this paper. Multi-weights exist in many physical networks [39–43]. As a result, HPDECSTN with multi-weights is important and remains challenging.

This paper mainly studies the leader-following synchronization control of a semi-linear HPDECSTN with two sorts of boundary conditions in a one-dimensional space. This paper’s contributions are as follows: (1) Two sorts of HPDECSTN models are built, one with a single weight and the other with multi-weights. (2) A new distributed controller is designed to address CSTNHPDE with a single weight. Sufficient conditions for the synchronization and exponential synchronization of CSTNHPDE are presented by providing the gain ranges. (3) The proposed distributed controller addresses CSTNHPDE with multi-weights and gain ranges, obtained for synchronization and exponential synchronization, respectively. (4) Two examples show the effectiveness and good performance of the control methods.

Notations: Let $I_N$ denote the identity matrix with Nth order, $P > 0 \ (P < 0)$ denote symmetric positive definite (negative definite), and $\lambda_{\text{max(min)}}(\cdot)$ denote the maximum (minimum) eigenvalue.

2. Problem Formulation

This paper first studies a class of leader-following, semi-linear, hyperbolic PDE-based, complex spatio-temporal networks (HPDECSTNs) with a single weight. The following node is assumed to be

$$
\begin{cases}
\frac{\partial z_i(\xi, t)}{\partial t} = \frac{\partial z_i(\xi, t)}{\partial \xi} + Az_i(\xi, t) + B f(z_i(\xi, t)) + c \sum_{j=1}^{N} g_{ij} \Gamma z_j(\xi, t) + u_i(\xi, t), \\
z_i(L, t) = 0, \\
z_i(\xi, 0) = z_i^0(\xi), i \in \{1, 2, \cdots, N\},
\end{cases}
$$

(1)

where $(\xi, t) \in [0, L] \times [0, \infty)$ are space and time, respectively. $z_i(\xi, t)$ and $u_i(\xi, t) \in \mathbb{R}^n$ are the state and control input, respectively. $0 < L \in \mathbb{R}$ is a constant. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$, and $\Gamma \in \mathbb{R}^{n \times n}$ are constant matrices. $f(\cdot)$ is a nonlinear function. The coupling strength $c > 0$ is a constant. $G = (g_{ij})_{N \times N}$ satisfies $g_{ii} = -\sum_{j=1, j \neq i}^{N} g_{ij}$.

The leader node is assumed to be

$$
\begin{cases}
\frac{\partial s(\xi, t)}{\partial t} = \frac{\partial s(\xi, t)}{\partial \xi} + As(\xi, t) + B f(s(\xi, t)), \\
s(L, t) = 0, \\
s(\xi, 0) = s^0(\xi),
\end{cases}
$$

(2)

where $s(\xi, t) \in \mathbb{R}^n$ is the state.

This paper aims to study a distributed controller $u_i(\xi, t)$, driving HPDECSTN (1) to the leader node (2), designed as

$$
u_i(\xi, t) = d_i(s(\xi, t) - z_i(\xi, t)),
$$

(3)

where $d_i$ are the control gains to be determined.
Definition 1. HPDECSTN (1) reaches synchronization, if
\[
\lim_{t \to \infty} ||z_i(\xi, t) - s(\xi, t)|| = 0, i \in \{1, 2, \cdots, N\}. \tag{4}
\]

Definition 2. Given \(\rho > 0\), HPDECSTN (1) reaches exponential synchronization, if there is a real number \(\sigma > 0\) such that
\[
||z_i(\xi, t) - s(\xi, t)|| \leq \sigma \exp(-2\rho t)||z_i^0(\xi) - s^0(\xi)||, i \in \{1, 2, \cdots, N\}. \tag{5}
\]

Assumption 1. For any \(\xi_1, \xi_2 \in \mathbb{R}\), then \(0 < \mathcal{X} \in \mathbb{R}\), satisfying
\[
|f(\xi_1) - f(\xi_2)| \leq \mathcal{X}|\xi_1 - \xi_2|. \tag{6}
\]

3. Synchronization of HPDECSTNs with a Single Weight

Let the synchronization error be \(e_i(\xi, t) \triangleq z_i(\xi, t) - s(\xi, t)\). The error system of between HPDECSTN (1) and (2) yields
\[
\begin{align*}
\frac{\partial e_i(\xi, t)}{\partial t} &= \frac{\partial e_i(\xi, t)}{\partial \xi} + (I_N \otimes A)e_i(\xi, t) + (I_N \otimes B)F(e_i(\xi, t)) + (G \otimes \Gamma)e_i(\xi, t) + u_i(\xi, t), \\
e(L, t) &= 0, \\
e(\xi, 0) &= e_i^0(\xi),
\end{align*}
\tag{7}
\]

where \(e_i^0(\xi) \triangleq z_i^0(\xi) - s^0(\xi)\), \(u_i(\xi) \triangleq [u_i^T(\xi), u_i^2(\xi), \cdots, u_i^N(\xi)]^T\), \(e(\xi, t) \triangleq [e_1^T(\xi, t), e_2^T(\xi, t), \cdots, e_N^T(\xi, t)]^T\), \(F(e_i(\xi, t)) \triangleq f(z_i(\xi, t)) - f(s(\xi, t))\), and
\[
F(e(\xi, t)) \triangleq [F^T(e_1(\xi, t)), F^T(e_2(\xi, t)), \cdots, F^T(e_N(\xi, t))]^T.
\]

Theorem 1. Suppose Assumption 1 holds. HPDECSTN (1) reaches synchronization under the controller (2), if
\[
d_i > \lambda_{\text{max}}(\Psi), \tag{8}
\]

where \(\Psi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5I_N \otimes BB^T + 0.5\mathcal{X}^2I_N + 0.5c(G \otimes \Gamma + G^T \otimes \Gamma^T)\).

Proof. Choose the Lyapunov functional candidate as follows:
\[
V(t) = 0.5 \int_0^L e^T(\xi, t)e(\xi, t)d\xi. \tag{9}
\]

One has
\[
\begin{align*}
V(t) &= \int_0^L e^T(\xi, t)\frac{\partial e_i(\xi, t)}{\partial t}d\xi \\
&= \int_0^L e^T(\xi, t)\frac{\partial e_i(\xi, t)}{\partial \xi}d\xi \\
&+ \int_0^L e^T(\xi, t)(I_N \otimes A + cG \otimes \Gamma)e(\xi, t)d\xi \\
&+ \int_0^L e^T(\xi, t)F(e(\xi, t))d\xi - \int_0^L e^T(\xi, t)(D \otimes I_n)e(\xi, t)d\xi,
\end{align*}
\tag{10}
\]

where \(D \triangleq \text{diag}\{d_1, d_2, \cdots, d_N\}\).
By integrating by parts,
\[
\int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial \zeta} d\zeta = e^T(\zeta, t)e(\zeta, t) \Big|_{\zeta=0}^{\zeta=L} \quad - \int_0^L \frac{\partial e^T(\zeta, t)}{\partial \zeta} e(\zeta, t) d\zeta = -e^T(0, t)e(0, t) - \int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial \zeta} d\zeta \quad \leq -\int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial \zeta} d\zeta,
\]
which implies
\[
\int_0^L e^T(\zeta, t) \frac{\partial e(\zeta, t)}{\partial \zeta} d\zeta \leq -0.5e^T(0, t)e(0, t).
\]
Under Assumption 1,
\[
\int_0^L e^T(\zeta, t)BF(e(\zeta, t))d\zeta \leq 0.5\int_0^L e^T(\zeta, t)BB^T e(\zeta, t)d\zeta + 0.5\int_0^L F^T(\zeta, t)F(\zeta, t)d\zeta \quad \leq \int_0^L e^T(\zeta, t)(0.5I_N \otimes BB^T + 0.5\chi^2 I_{Nn})e(\zeta, t)d\zeta.
\]
The substitution of (11)–(13) into (10) yields,
\[
V(t) \leq \int_0^L e^T(\zeta, t)(\Psi - D \otimes I_n)e(\zeta, t)d\zeta,
\]
where \( \Psi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5I_N \otimes BB^T + 0.5\chi^2 I_{Nn} + cG \otimes \Gamma \) and \( D = \text{diag}\{d_1, d_2, \ldots, d_N\} \).
It is obvious that (8) implies
\[
\Psi - D \otimes I_n < 0.
\]
The substitution of (15) into (14) yields, \( \dot{V}(t) \leq -\lambda_{\min}(D \otimes I_n - \Psi)||e(\cdot, t)||, \) for all non-zero \( e(\zeta, t), \) implying synchronization of HPDECSTN (1).

**Theorem 2.** Suppose Assumption 1 holds. Given \( \rho > 0, \) HPDECSTN (1) reaches exponential synchronization under the controller (2), if
\[
d_i > \lambda_{\max}(\Psi + \rho I_{Nn}),
\]
where \( \Psi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5I_N \otimes BB^T + 0.5\chi^2 I_{Nn} + 0.5c(G \otimes \Gamma + G^T \otimes \Gamma^T). \)

**Proof.**
\[
\dot{V}(t) + 2\rho V(t) \leq \int_0^L e^T(\zeta, t)(\Psi + \rho I_{Nn} - D \otimes I_n)e(\zeta, t)d\zeta \quad \leq 0,
\]
which implies
\[
V(t) \leq V(0) \exp(-2\rho t).
\]
It follows from (18) that
\[
||e_i(\zeta, t)||_2^2 \leq \sigma \exp(-2\rho t),
\]
where \(\sigma = ||e_i^0(\zeta)||_2^2\). Therefore, exponential synchronization is obtained. \(\square\)

4. Synchronization of HPDECSTNs with Multi-Weights

This section studies a class of semi-linear HPDECSTNs with multi-weights, where the following node is as follows:

\[
\begin{align*}
\frac{\partial z_i(\zeta, t)}{\partial t} &= \frac{\partial z_i(\zeta, t)}{\partial \zeta} + Az_i(\zeta, t) + Bf(z_i(\zeta, t)) + c_1 \sum_{j=1}^{N} g_{ij}^1 \Gamma_1 z_j(\zeta, t) \\
&\quad + c_2 \sum_{j=1}^{N} g_{ij}^2 \Gamma_2 z_j(\zeta, t) + \cdots + c_l \sum_{j=1}^{N} g_{ij}^l \Gamma_l z_j(\zeta, t) + u_i(\zeta, t), \\
\end{align*}
\]

where \(\Gamma_1 \in \mathbb{R}^{n \times n}, \Gamma_2 \in \mathbb{R}^{n \times n}, \ldots, \Gamma_l \in \mathbb{R}^{n \times n}\) are constant matrices. \(G_k = (g_{ij}^k)_{N \times N}\) satisfies
\[
g_{ii}^k = -\sum_{j=1, j \neq i}^{N} g_{ij}^k.
\]

The error system of between HPDECSTN (20) and (2) with multi-weights can be obtained as

\[
\begin{align*}
\frac{\partial e(\zeta, t)}{\partial t} &= \Theta \frac{\partial e(\zeta, t)}{\partial \zeta} + (I_N \otimes A)e(\zeta, t) + F(e(\zeta, t)) + c_1(G_1 \otimes \Gamma_1)e(\zeta, t) \\
&\quad + c_2(G_2 \otimes \Gamma_2)e(\zeta, t) + \cdots + c_l(G_l \otimes \Gamma_l)e(\zeta, t) + u_i(\zeta, t), \\
\frac{\partial e(0, t)}{\partial \zeta} &= 0, \\
e(\zeta, 0) &= e^0(\zeta).
\end{align*}
\]

Theorem 3. Suppose that Assumption 1 holds. HPDECSTN (20) reaches synchronization under the controller (2), if
\[
d_i > \lambda_{\text{max}}(\Xi),
\]
where \(\Xi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5I_N \otimes BB^T + 0.5\chi^2 I_{Nn} + 0.5c_1(G_1 \otimes \Gamma_1 + G_1^T \otimes \Gamma_1^T) + 0.5c_2(G_2 \otimes \Gamma_2 + G_2^T \otimes \Gamma_2^T) + \cdots + 0.5c_l(G_l \otimes \Gamma_l + G_l^T \otimes \Gamma_l^T)\).

Proof. The proof is similar to that of Theorem 1, and so it is omitted. \(\square\)

Theorem 4. Suppose that Assumption 1 holds. Given \(\rho > 0\), HPDECSTN (20) reaches exponential synchronization under the controller (2), if
\[
d_i > \lambda_{\text{max}}(\Xi + \rho I_{Nn}),
\]
where \(\Xi \triangleq I_N \otimes \frac{A + A^T}{2} + 0.5I_N \otimes BB^T + 0.5\chi^2 I_{Nn} + 0.5c_1(G_1 \otimes \Gamma_1 + G_1^T \otimes \Gamma_1^T) + 0.5c_2(G_2 \otimes \Gamma_2 + G_2^T \otimes \Gamma_2^T) + \cdots + 0.5c_l(G_l \otimes \Gamma_l + G_l^T \otimes \Gamma_l^T)\).

Proof. The proof is similar to that of Theorem 2, and so it is omitted. \(\square\)

Remark 1. This paper addresses not only the synchronization of HPDECSTNs, but also the exponential synchronization. Moreover, this paper addresses HPDECSTNs not only with a single weight, but also with multi-weights.
Remark 2. Compared with the results modeled by ordinary differential equations with multi-weights [39–43], this paper addresses spatio-temporal models with multi-weights.

Remark 3. Different from the control design for synchronization of parabolic PDEs-based CSTNs [44,45], this paper deals with the synchronization of hyperbolic PDEs-based CSTNs.

Remark 4. Only a few important results discussed the synchronization, exact synchronization and approximate synchronization of HPDECSTNs [36–38]. Different from those with a single weight, this paper addresses the case with multi-weights.

5. Numerical Simulation

Example 1. Consider a single weighted HPDECSTN (1) with random initial conditions and

\[
A = \begin{bmatrix} 5.1 & 2.7 \\ -1.1 & 4.2 \end{bmatrix}, B = \begin{bmatrix} 0.5 & -0.2 \\ 0.2 & 1.5 \end{bmatrix}, \Gamma = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, L = 1, c = 0.2, f(\cdot) = \tanh(\cdot).
\]

The single weight takes

\[
G = \begin{bmatrix} -5 & 1 & 2 & 2 \\ -1 & 4 & 3 & 0 \\ 1 & 1 & -3 & 1 \\ -3 & -2 & -3 & 8 \end{bmatrix}.
\]

Figure 1 shows that HPDECSTN (1) cannot reach synchronization without control. It is obvious that \(\chi = 1\). With Theorem 1, solve (16) by Matlab, the feedback gains \(d_i = 12.04\) are obtained. Figure 2 shows that HPDECSTN (1) reaches exponential synchronization under the controller (2) with \(d_i = 12.04\). The controller (2) with the feedback gains \(d_i = 12.04\) is shown in Figure 3.

![Figure 1. \(e(\zeta, t)\) of HPDECSTN (1) without control.](image-url)
Figure 2. $e(\zeta, t)$ of HPDECSTN (1) with control.

Figure 3. The control input of HPDECSTN (1).

Example 2. Consider multi-weighted HPDECSTN (20) with random initial conditions and the same parameters as those of Example 1, except:

$$c_1 = 0.8, c_2 = 0.3, c_3 = 0.4, c_4 = 0.5$$

(26)
The weights take

\[
G_1 = \begin{bmatrix}
-5 & 1 & 2 & 2 \\
-1 & 4 & 3 & 0 \\
1 & 1 & -3 & 1 \\
-3 & -2 & -3 & 8
\end{bmatrix}, \quad G_2 = \begin{bmatrix}
6 & -1 & -2 & -3 \\
-2 & 4 & -3 & 1 \\
1 & 2 & -3 & 0 \\
-1 & -3 & -3 & 7
\end{bmatrix},
\]

(27)

\[
G_3 = \begin{bmatrix}
-2 & 4 & 1 & -3 \\
2 & -1 & 3 & -4 \\
-2 & -1 & -2 & 5 \\
6 & 2 & -3 & -5
\end{bmatrix}, \quad G_4 = \begin{bmatrix}
-5 & 1 & 2 & 2 \\
1 & 3 & -2 & -2 \\
-7 & -2 & 3 & 6 \\
-3 & 1 & -3 & 5
\end{bmatrix}.
\]

(28)

Figure 4 shows that HPDECSTN (20) cannot reach synchronization without control. With Theorem 4, solving (23) using Matlab, the feedback gains \( d_i = 26.21 \) are obtained. Figure 5 shows that HPDECSTN (20) reaches exponential synchronization under controller (2) with \( d_i = 26.21 \). The controller (2) with the feedback gains \( d_i = 26.21 \) is shown in Figure 6.

Figure 4. \( e(\zeta, t) \) of HPDECSTN (1) without control.
6. Conclusions

This paper has dealt with the leader-following synchronization control of two classes of semi-linear HPDECSTNs: one HPDECSTN with a single weight, and the other with multi-weights. To drive HPDECSTNs to synchronization, one new distributed controller was constructed. Dealing with HPDECSTNs with a single weight, sufficient conditions for synchronization and exponential synchronization of CSTNHPDE were presented by providing gain ranges. Furthermore, the proposed distributed controller was used to address...
CSTNHPE with multi-weights and gain ranges, which were obtained for synchronization and exponential synchronization, respectively. Two examples illustrated the effectiveness of the developed theoretical results. In future work, the event-triggered control and pinning control of HPDECSTNs will be studied.

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