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On All Symmetric and Nonsymmetric Exceptional Orthogonal X_1-Polynomials Generated by a Specific Sturm–Liouville Problem

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Abstract: Exceptional orthogonal X_1-polynomials of symmetric and nonsymmetric types can be considered as eigenfunctions of a Sturm–Liouville problem. In this paper, by defining a generic second-order differential equation, a unified classification of all these polynomials is presented, and 10 particular cases of it are then introduced and analyzed.

Keywords: Sturm–Liouville problems; exceptional orthogonal X_1-polynomials; Pearson distributions family; generalized Jacobi, Laguerre and Hermite differential equations

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1. Introduction

Classical orthogonal polynomials can be considered eigenfunctions of a Sturm–Liouville problem [1–3] of the form

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) - \big(\lambda\rho(x) + q(x)\big)y = 0, \tag{1}$$

on an open interval, say (a, b), with the boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) = 0, \\ \alpha_2 y(b) + \beta_2 y'(b) = 0, \end{aligned} \tag{2}$$

in which α_1, α_2 and β_1, β_2 are given constants and the functions $k(x) > 0$, $q(x)$ and $\rho(x) > 0$ in (1) are assumed to be continuous for $x \in [a, b]$. The boundary value problem (1) and (2) is called singular [4] if one of the points a and b is singular, i.e., $k(a) = 0$ or $k(b) = 0$. Sturm–Liouville problems appear in various branches of physics, engineering and biology and are usually studied in three different continuous, discrete and q-discrete spaces; see, for example, [5].

Let $y_n(x)$ and $y_m(x)$ be two solutions of Equation (1). Following the Sturm–Liouville theory [4,6], they are orthogonal with respect to the positive weight function $\rho(x)$ on (a, b) under the given conditions (2), i.e.,

$$\int_a^b \rho(x)y_n(x)y_m(x)\,dx = \left(\int_a^b \rho(x)y_n^2(x)\,dx\right)\delta_{n,m}, \tag{3}$$

where

$$\delta_{n,m} = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases}$$

Many special functions in theoretical and mathematical physics are solutions of a regular or singular Sturm–Liouville problem, satisfying the orthogonality condition (3) [4,7].

There are totally six sequences of real polynomials [5] that are orthogonal with respect to the Pearson distributions family

$$W\left(\begin{array}{c} d^*,\ e^* \\ a,\ b,\ c \end{array} \middle| \, x \right) = \exp\left(\int \frac{d^* x + e^*}{ax^2 + bx + c} \, dx \right) \qquad (a, b, c, d^*, e^* \in \mathbb{R}). \tag{4}$$

Three of them (i.e., Jacobi, Laguerre and Hermite polynomials [3]) are infinitely orthogonal with respect to three special cases of the positive function (4) (i.e., beta, gamma and normal distributions [8]) and three other ones are finitely orthogonal limited to some parametric constraints with respect to F-Fisher, inverse gamma and generalized T-student distributions [8]. Table 1 shows the main properties of these six sequences.

Table 1. Characteristics of six sequences of classical orthogonal polynomials.

Polynomial Notation	Distribution	Weight Function	Kind Interval Parameters Constraint
$P_n^{(\alpha,\beta)}(x)$	Beta	$W\left(\begin{array}{c} -\alpha-\beta,\ -\alpha+\beta \\ -1,\ 0,\ 1 \end{array} \middle| \, x \right) = (1-x)^{\alpha}(1+x)^{\beta}$	Infinite $[-1,1]$ $\forall n, \alpha > -1, \beta > -1$
$L_n^{(\alpha)}(x)$	Gamma	$W\left(\begin{array}{c} -1,\ \alpha \\ 0,\ 1,\ 0 \end{array} \middle| \, x \right) = x^{\alpha}\exp(-x)$	Infinite $[0,\infty)$ $\forall n, \alpha > -1$
$H_n(x)$	Normal	$W\left(\begin{array}{c} -2,\ 0 \\ 0,\ 0,\ 1 \end{array} \middle| \, x \right) = \exp(-x^2)$	Infinite $(-\infty,\infty)$ —
$M_n^{(p,q)}(x)$	Fisher F	$W\left(\begin{array}{c} -p,\ q \\ 1,\ 1,\ 0 \end{array} \middle| \, x \right) = x^{q}(x+1)^{-(p+q)}$	Finite $[0,\infty)$ $\max n < (p-1)/2$ $q > -1$
$N_n^{(p)}(x)$	Inverse Gamma	$W\left(\begin{array}{c} -p,\ 1 \\ 1,\ 0,\ 0 \end{array} \middle| \, x \right) = x^{-p}\exp(-1/x)$	Finite $[0,\infty)$ $\max n < (p-1)/2$
$J_n^{(p,q)}(x)$	Generalized T	$W\left(\begin{array}{c} -2p,\ q \\ 1,\ 0,\ 1 \end{array} \middle| \, x \right) = (1+x^2)^{-p}\exp(q\arctan x)$	Finite $(-\infty,\infty)$ $\max n < p-1/2$

It was shown by S. Bochner [7,9] that if an infinite sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ satisfies a second-order eigenvalue equation of the form

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + r(x)P_n(x) = \lambda_n P_n(x) \qquad n = 0, 1, 2, \ldots,$$

then $\sigma(x)$, $\tau(x)$ and $r(x)$ must be polynomials of degree 2, 1 and 0, respectively. Moreover, if the sequence $\{P_n\}_{n=0}^{\infty}$ is an orthogonal set, then it has to be one of the classical Jacobi, Laguerre or Hermite polynomials, which satisfy a second order differential equation of the form [9–11]

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) - \lambda_n y_n(x) = 0, \tag{5}$$

where

$$\sigma(x) = ax^2 + bx + c \quad \text{and} \quad \tau(x) = dx + e,$$

and

$$\lambda_n = n\big(d + (n-1)a\big),$$

is the eigenvalue depending on $n = 0, 1, 2, \ldots$. However, there are three other sequences of hypergeometric polynomials that are solutions of Equation (5) but finitely orthogonal [12].

It is the presumption in the theory of special functions that any orthogonal polynomial system starts with a polynomial of degree 0. Nevertheless, from the Sturm–Liouville theory point of view, such a restriction is not necessary, and that point gives birth to the so-called 'exceptional orthogonal polynomials'. In this sense, two families of exceptional orthogonal polynomials were recently introduced in [13,14] as solutions of a second-order eigenvalue equation of the form

$$\Big(k_2(x-b)^2 + k_1(x-b) + k_0\Big)y_n''(x) + \frac{ax - ab - 1}{x - b}\Big(k_1(x-b) + 2k_0\Big)y_n'(x) \\ - \Big(\frac{a}{x-b}(k_1(x-b) + 2k_0) + \lambda_n\Big)y_n(x) = 0,$$

for $n \geq 1$, where

$$\lambda_n = (n-1)(nk_2 + ak_1),$$

and $k_0 \neq 0, k_1, k_2$ are real constants. It was also shown in [13] that if a self-adjoint second-order operator has a polynomial eigenfunctions of type $\{P_i(x)\}_{i=1}^{\infty}$, then it can be X_1-Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ with the weight function

$$\hat{W}_{\alpha,\beta}(x) = \Big(x - \frac{\beta+\alpha}{\beta-\alpha}\Big)^{-2}(1-x)^{\alpha}(1+x)^{\beta} \qquad \text{for} \quad x \in (-1, 1), \tag{6}$$

where $\alpha, \beta > -1$, $\alpha \neq \beta$, $\operatorname{sgn}\alpha = \operatorname{sgn}\beta$, and/or X_1-Laguerre polynomials $\hat{L}_n^{(\alpha)}(x)$ with the weight function

$$\hat{W}_{\alpha}(x) = \big(x + \alpha\big)^{-2}x^{\alpha}e^{-x} \qquad \text{for} \quad x \in (0, \infty) \quad \text{and} \quad \alpha > 0. \tag{7}$$

Exceptional orthogonal polynomials were recently of interest due to their important applications in exactly solvable potentials and supersymmetry, Dirac operators minimally coupled to external fields and entropy measures in quantum information theory [15,16].

This paper is organized as follows. In the next section, we consider six sequences of orthogonal X_1-polynomials as particular solutions of a generic differential equation in the form

$$(x-r)\Big(a_2x^2 + a_1x + a_0\Big)y_n''(x) + \Big(b_2x^2 + b_1x + b_0\Big)y_n'(x) \\ - \big(\lambda_n(x-r) + c_0^*\big)y_n(x) = 0, \quad n \geq 1, \tag{8}$$

where r is a real parameter such that $a_2r^2 + a_1r + a_0 \neq 0$ and the roots of $b_2x^2 + b_1x + b_0$ are supposed to be real, see Section 3 for more details. Both infinite and finite types of nonsymmetric exceptional orthogonal X_1-polynomials can be extracted from Equation (8). Although some infinite polynomial sequences were investigated in [17] for particular values of r, the finite cases of nonsymmetric exceptional X_1-polynomials orthogonal on infinite intervals are introduced in this paper for the first time. A key point in this sense is that the weight functions corresponding to these six sequences are exactly a multiplication of the Pearson distributions family introduced in Table 1. Hence, in Section 2, we first have a review on six classical orthogonal polynomials in order to present a unified classification for nonsymmetric exceptional orthogonal X_1-polynomials in Section 3. In Section 4, we study a series of solutions of the generic Equation (8) in order to find some of its polynomial-type solutions. In Section 5, six extended differential equations, as particular cases of the main Equation (8), are introduced, and it is shown that their polynomial solutions are X_1-orthogonal. Finally in Section 6, we apply the generic Equation (8) once again to establish a symmetric Sturm–Liouville equation of the form

$$A(x)\,y_n''(x)+B(x)\,y_n'(x)+\Big(\lambda_n\,C(x)+D(x)+\frac{1-(-1)^n}{2}E(x)\Big)\,y_n(x)=0,$$

and then to introduce four main classes of symmetric orthogonal X_1-polynomials.

2. Classical Orthogonal Polynomials: A Brief Review

It is shown in [5] that the monic polynomial solution of Equation (5) can be represented as

$$y_n(x)=\bar{P}_n\left(\begin{array}{c} d,\quad e \\ a\,,\ b\,,\ c \end{array}\middle|\, x\right)=\sum_{k=0}^{n}\binom{n}{k}G_k^{(n)}(a,b,c,d,e)\,x^k, \tag{9}$$

where

$$G_k^{(n)}=\left(\frac{2a}{b+\sqrt{b^2-4ac}}\right)^{k-n}{}_2F_1\left(\begin{array}{c} k-n,\quad \frac{2ae-bd}{2a\sqrt{b^2-4ac}}+1-\frac{d}{2a}-n \\ -\frac{d}{a}+2-2n \end{array}\middle|\, \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}\right),$$

and

$${}_2F_1\left(\begin{array}{c} a,\quad b \\ c \end{array}\middle|\, x\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},$$

denotes the Gauss hypergeometric function for $(a)_k=a(a+1)\dots(a+k-1)$.

The general Formula (9) is a suitable tool to compute the coefficients of x^k for any fixed degree k and arbitrary a, so that after simplifying it, we obtain

$$\begin{aligned}\bar{P}_n\left(\begin{array}{c} d,\quad e \\ a,\ b,\ c \end{array}\middle|\, x\right)&=x^n+\binom{n}{1}\frac{e+(n-1)b}{d+2(n-1)a}x^{n-1}\\ &+\binom{n}{2}\frac{(e+(n-1)b)(e+(n-2)b)+c(d+2(n-1)a)}{(d+2(n-1)a)(d+(2n-3)a)}x^{n-2}+\dots\\ +\binom{n}{n}\left(\frac{b+\sqrt{b^2-4ac}}{2a}\right)^n&{}_2F_1\left(\begin{array}{c} -n,\quad 1-n-\frac{bd-2ae}{2a\sqrt{b^2-4ac}}-\frac{d}{2a} \\ 2(1-n)-\frac{d}{a} \end{array}\middle|\, \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}\right).\end{aligned}$$

Moreover, by referring to the Nikiforov and Uvarov approach [6] and considering Equation (5) as a self-adjoint form, the Rodrigues representation of the monic polynomials is derived as

$$\bar{P}_n\left(\begin{array}{c} d,\quad e \\ a,\ b,\ c \end{array}\middle|\, x\right)=\frac{1}{\big(\prod_{k=1}^{n}d+(n+k-2)a\big)\,W\left(\begin{array}{c} d,\quad e \\ a,\ b,\ c \end{array}\middle|\, x\right)}\times\frac{d^n\Big((ax^2+bx+c)^nW\left(\begin{array}{c} d,\quad e \\ a,\ b,\ c \end{array}\middle|\, x\right)\Big)}{dx^n}, \tag{10}$$

where

$$W\left(\begin{array}{c} d,\quad e \\ a,\ b,\ c \end{array}\middle|\, x\right)=\exp\Big(\int\frac{(d-2a)x+e-b}{ax^2+bx+c}dx\Big).$$

Using the Formula (9) or (10), we can also obtain a generic three term recurrence equation as [5]

$$\bar{P}_{n+1}(x) = \left(x + \frac{2n(n+1)ab + (d-2a)(e+2nb)}{(d+2na)(d+(2n-2)a)} \right) \bar{P}_n(x)$$
$$+ n(d+(n-2)a) \frac{\left(c(d+(2n-2)a)^2 - nb^2(d+(n-2)a) + (e-b)(a(e+b)-bd) \right)}{(d+(2n-3)a)(d+(2n-2)a)^2(d+(2n-1)a)} \bar{P}_{n-1}(x),$$

in which $\bar{P}_n(x)$ denotes the monic polynomials of (9) with the initial values

$$\bar{P}_0(x) = 1 \;\text{ and }\; \bar{P}_1(x) = x + \frac{e}{d}.$$

Finally, the norm square value of the monic polynomials (9) can be calculated as follows: Let $[L, U]$ be a predetermined orthogonality interval which consists of the zeros of $\sigma(x) = ax^2 + bx + c$ or $\pm\infty$. By noting the Rodrigues representation (10), we have

$$\|\bar{P}_n\|^2 = \int_L^U \bar{P}_n^2\left(\begin{matrix} d, & e \\ a, & b, & c \end{matrix} \middle| x \right) W\left(\begin{matrix} d, & e \\ a, & b, & c \end{matrix} \middle| x \right) dx$$
$$= \frac{1}{\prod_{k=1}^{n} d+(n+k-2)a} \int_L^U \bar{P}_n\left(\begin{matrix} d, & e \\ a, & b, & c \end{matrix} \middle| x \right) \frac{d^n}{dx^n}\left((ax^2+bx+c)^n W\left(\begin{matrix} d, & e \\ a, & b, & c \end{matrix} \middle| x \right) \right) dx. \quad (11)$$

Hence, integrating by parts from the right hand side of (11) eventually yields

$$\|\bar{P}_n\|^2 = \frac{n!\,(-1)^n}{\prod_{k=1}^{n} d+(n+k-2)a} \int_L^U (ax^2+bx+c)^n \left(\exp \int \frac{(d-2a)x+e-b}{ax^2+bx+c} dx \right) dx.$$

Although the Jacobi polynomials

$$\bar{P}_n^{(\alpha,\beta)}(x) = \bar{P}_n\left(\begin{matrix} -\alpha-\beta-2, & \beta-\alpha \\ -1, & 0, & 1 \end{matrix} \middle| x \right),$$

Laguerre polynomials

$$\bar{L}_n^{(\alpha)}(x) = \bar{P}_n\left(\begin{matrix} -1, & \alpha+1 \\ 0, & 1, & 0 \end{matrix} \middle| x \right),$$

and Hermite polynomials

$$\bar{H}_n(x) = \bar{P}_n\left(\begin{matrix} -2, & 0 \\ 0, & 0, & 1 \end{matrix} \middle| x \right),$$

are three polynomial solutions of Equation (5), there are three other sequences of hypergeometric polynomials that are finitely orthogonal with respect to the generalized T, inverse Gamma and F distributions [12] and are solutions of Equation (5). The first finite sequence, i.e.,

$$\bar{M}_n^{(p,q)}(x) = \bar{P}_n\left(\begin{matrix} 2-p, & 1+q \\ 1, & 1, & 0 \end{matrix} \middle| x \right),$$

satisfies the differential equation

$$(x^2+x)y_n''(x) + \big((2-p)x+q+1\big)y_n'(x) - n(n+1-p)y_n(x) = 0,$$

and is finitely orthogonal with respect to the weight function

$$W_1(x; p, q) = x^q(1+x)^{-(p+q)},$$

on $[0, \infty)$ if and only if [12]

$$p > 2\{\max n\} + 1 \quad \text{and} \quad q > -1.$$

The second finite sequence, i.e.,

$$\bar{N}_n^{(p)}(x) = \bar{P}_n\left(\begin{array}{ccc} 2-p, & 1 & \\ 1, & 0, & 0 \end{array} \middle| x \right),$$

satisfies the differential equation

$$x^2 y_n''(x) + \big((2-p)x + 1\big) y_n'(x) - n(n+1-p) y_n(x) = 0,$$

and is finitely orthogonal with respect to the weight function [12]

$$W_2(x;p) = x^{-p} e^{-\frac{1}{x}},$$

on $(0,\infty)$ for $n = 0,1,2,\ldots,N < \frac{p-1}{2}$. The third finite sequence, which is finitely orthogonal with respect to the generalized T-student distribution weight function

$$W_3(x;p,q) = \left(1+x^2\right)^{-p} \exp(q \arctan x),$$

is defined on $(-\infty,\infty)$ as

$$\bar{J}_n^{(p,q)}(x) = \bar{P}_n\left(\begin{array}{ccc} 2-2p, & q & \\ 1, & 0, & 1 \end{array} \middle| x \right),$$

satisfying the equation

$$(1+x^2)\, y_n''(x) + \big(2(1-p)x + q\big)\, y_n'(x) - n(n+1-2p)\, y_n(x) = 0,$$

and the orthogonality property holds if

$$n = 0,1,2,\ldots,N < p - \frac{1}{2} \text{ and } q \in \mathbb{R}.$$

3. A Unified Classification of Nonsymmetric Exceptional Orthogonal X_1-Polynomials

Using identity, which is valid for every real A, B, C, x, r

$$Ax^2 + Bx + C = A(x-r)^2 + (2Ar + B)(x-r) + Ar^2 + Br + C,$$

another form of Equation (8) is as

$$\begin{aligned} (x-r)\Big(a_2(x-r)^2 + (2a_2 r + a_1)(x-r) + a_2 r^2 + a_1 r + a_0\Big) y_n''(x) \\ + \Big(b_2(x-r)^2 + (2b_2 r + b_1)(x-r) + b_2 r^2 + b_1 r + b_0\Big)\, y_n'(x) \\ - \Big(\lambda_n (x-r) + c_0^*\Big)\, y_n(x) = 0, \qquad n \geq 1. \end{aligned} \tag{12}$$

We choose λ_n in (12) so that the relative eigenfunction y_n is a polynomial of degree n. Hence, we first consider a subspace of the whole space of polynomials of degree at most n as

$$\Pi_{n,r,\nu} = \text{span}\Big\{(x-r-\nu), (x-r)^2, \ldots, (x-r)^n\Big\},$$

in which ν is a real constant. By substituting $y_1(x) = x - r - \nu$ and $y_n(x) = (x-r)^n$ for $n \geq 2$ into (12), we respectively obtain

$$\big(b_2 - \lambda_1\big)(x-r)^2 + \big(2\, b_2\, r + b_1 - c_0^* + \nu\, \lambda_1\big)(x-r) + b_2\, r^2 + b_1\, r + b_0 + \nu\, c_0^* = 0, \tag{13}$$

and

$$n(n-1)\Big(a_2(x-r)^2+(2a_2r+a_1)(x-r)+a_2r^2+a_1r+a_0\Big)(x-r)^{n-1}$$
$$+n\Big(b_2(x-r)^2+(2b_2r+b_1)(x-r)+b_2r^2+b_1r+b_0\Big)(x-r)^{n-1}$$
$$-\Big(\lambda_n(x-r)+c_0^*\Big)(x-r)^n=0 \qquad n\geq 2.$$

Therefore,

$$\lambda_n=n\big((n-1)a_2+b_2\big) \qquad \text{for} \qquad n\geq 1,$$

and, using Equation (13) with $n=1$,

$$\begin{cases} 2b_2r+b_1-c_0^*+\nu b_2=0, \\ b_2r^2+b_1r+b_0+\nu c_0^*=0. \end{cases} \tag{14}$$

Solving the system (14) gives

$$\nu=\frac{-b_1\pm\sqrt{b_1^2-4b_0b_2}}{2b_2}-r=\begin{cases} r_1-r, \\ r_2-r, \end{cases}$$

where r_1, r_2 are roots of $b_2x^2+b_1x+b_0$, and

$$c_0^*=c_0^*\Big\{r;b_2,b_1,b_0\Big\}=\frac{1}{2}\Big(2b_2r+b_1\mp\sqrt{{b_1}^2-4b_0b_2}\Big)=\begin{cases} b_2(r-r_2), \\ b_2(r-r_1). \end{cases}$$

Corollary 1. *If we take $b_2x^2+b_1x+b_0=b_2(x-r_1)(x-r_2)$ for $b_2\neq 0$ and*

$$\Pi_{n,r,\nu}=span\{e_k(x)\}_{k=1}^n,$$

then
(i) $e_1(x)=x-r_1$ and $\{e_k(x)\}_{k=2}^\infty=\{(x-r)^k\}_{k=2}^\infty$ lead to $c_0^*=b_2(r-r_2)$.

(ii) $e_1(x)=x-r_2$ and $\{e_k(x)\}_{k=2}^\infty=\{(x-r)^k\}_{k=2}^\infty$ lead to $c_0^*=b_2(r-r_1)$.

Also note that for $b_2=0$, we respectively have $c_0^*=b_1$ and $\nu=-r-\frac{b_0}{b_1}$.

We can now show that the polynomial solutions of Equation (12) in $\Pi_{n,r,\nu}$ are orthogonal on an interval, say $[L,U]$, with respect to a weight function in the form

$$\rho(x)=(x-r)\omega(x),$$

where $\omega(x)$ satisfies the equation

$$\frac{\omega'(x)}{\omega(x)}=\frac{(b_2-3a_2)x^2+(b_1-2a_1+2a_2r)x+b_0-a_0+a_1r}{(x-r)\big(a_2x^2+a_1x+a_0\big)}. \tag{15}$$

To prove the orthogonality, we first consider the self-adjoint form of Equation (12) as

$$\Big(\omega(x)(x-r)\big(a_2x^2+a_1x+a_0\big)y_n'\Big)'=\omega(x)\Big(\lambda_n(x-r)+c_0^*\Big)y_n(x), \tag{16}$$

and for the index m as

$$\Big(\omega(x)(x-r)\big(a_2x^2+a_1x+a_0\big)y_m'\Big)'=\omega(x)\Big(\lambda_m(x-r)+c_0^*\Big)y_m(x). \tag{17}$$

Multiplying by $y_m(x)$ and $y_n(x)$ in relations (16) and (17) respectively, subtracting them and then integrating from both sides yields

$$\Big[\omega(x)(x-r)\big(a_2x^2+a_1x+a_0\big)\big(y_n'(x)y_m(x)-y_m'(x)y_n(x)\big)\Big]_L^U = (\lambda_n-\lambda_m)\int_L^U (x-r)\omega(x)y_n(x)y_m(x)\,dx. \tag{18}$$

Now if the following conditions

$$\omega(L)(L-r)\big(a_2L^2+a_1L+a_0\big)=0,$$
$$\omega(U)(U-r)\big(a_2U^2+a_1U+a_0\big)=0,$$

hold, the left hand side of (18) is equal to zero and therefore

$$\int_L^U (x-r)\omega(x)y_n(x)y_m(x)\,dx=0 \qquad m\neq n,$$

which approves the orthogonality of polynomial sequence $\{y_n(x)\}_{n=1}^{\infty}$ with respect to the weight function $\rho(x)=(x-r)\omega(x)$.

On the other hand, the explicit solution of Equation (15) is as

$$\omega(x)=\exp\left(\int \frac{(b_2-3a_2)x^2+(b_1-2a_1+2a_2r)x+b_0-a_0+a_1r}{(x-r)\big(a_2x^2+a_1x+a_0\big)}\,dx\right). \tag{19}$$

The key point in this relation is that $\omega(x)$ is exactly a multiplication of the Pearson distribution given in (4), because if the integrand function of (19) is written as a sum of two fractions with linear and quadratic denominators in the form

$$\frac{(b_2-3a_2)x^2+(b_1-2a_1+2a_2r)x+b_0-a_0+a_1r}{(x-r)(a_2x^2+a_1x+a_0)}=\frac{\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0}-1}{x-r}$$
$$+\frac{\big(b_2-a_2(2+\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0})\big)x+b_1+b_2r-(\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0})(a_1+a_2r)-a_1}{a_2x^2+a_1x+a_0},$$

then we obtain

$$\omega(x)=(x-r)^{\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0}-1}$$
$$\times\exp\left(\int\frac{\big(b_2-a_2(2+\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0})\big)x+b_1+b_2r-(\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0})(a_1+a_2r)-a_1}{a_2x^2+a_1x+a_0}\,dx\right)$$
$$=(x-r)^{\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0}-1}$$
$$\times W\left(\begin{array}{c|c} b_2-a_2(2+\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0}),\ b_1+b_2r-(\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0})(a_1+a_2r)-a_1 & x \\ a_2\,,\ a_1\,,\ a_0 & \end{array}\right),$$

and accordingly,

$$\rho(x)=(x-r)^{\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0}}$$
$$\times W\left(\begin{array}{c|c} b_2-a_2(2+\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0}),\ b_1+b_2r-(\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0})(a_1+a_2r)-a_1 & x \\ a_2\,,\ a_1\,,\ a_0 & \end{array}\right). \tag{20}$$

Corollary 2. *The polynomial solutions of the generic equation*

$$(x-r)\big(a_2x^2+a_1x+a_0\big)y_n''(x)+\big(b_2x^2+b_1x+b_0\big)y_n'(x) \\ -\Big(n\big(b_2+(n-1)a_2\big)(x-r)+c_0^*(r;b_2,b_1,b_0)\Big)y_n(x)=0, \tag{21}$$

where $(-1)^{\frac{b_2r^2+b_1r+b_0}{a_2r^2+a_1r+a_0}}=1$ *and* $n\geq 1$ *are nonsymmetric exceptional* X_1*-polynomials orthogonal with respect to the weight function* (20).

Now let us assume that the polynomial solution of Equation (21) is symbolically indicated as

$$y_n(x)=Q_{n,r}\left(\begin{array}{c|c} b_2,b_1,b_0 & \\ & x \\ a_2,a_1,a_0 & \end{array}\right). \tag{22}$$

By referring to the Pearson distributions family (4), an inverse process can also be considered as follows.

Suppose that a simplified case of the weight function (20) is given as

$$\rho(x)=(x-r)^{\theta}\,W\left(\begin{array}{c|c} d^*,\ e^* & \\ & x \\ a\ ,\ b\ ,\ c & \end{array}\right), \tag{23}$$

in which $(-1)^{\theta}=1$. Then, by noting Equation (21), the unknown polynomials $p_2(x)$ and $q_2(x)$ of degree 2 in the differential equation

$$(x-r)p_2(x)y_n''(x)+q_2(x)y_n'(x)-\big(\lambda_n(x-r)+c_0^*\big)y_n=0, \tag{24}$$

can be directly derived by computing the logarithmic derivative of the function

$$\frac{\rho(x)}{x-r}=(x-r)^{\theta-1}W\left(\begin{array}{c|c} d^*,\ e^* & \\ & x \\ a\ ,\ b\ ,\ c & \end{array}\right)=(x-r)^{\theta-1}W(x),$$

as

$$\frac{\big((x-r)^{\theta-1}W(x)\big)'}{(x-r)^{\theta-1}W(x)}=\frac{\theta-1}{x-r}+\frac{W'(x)}{W(x)}=\frac{\theta-1}{x-r}+\frac{d^*x+e^*}{ax^2+bx+c} \\ =\frac{\big(d^*+(\theta-1)a\big)x^2+\big(e^*-rd^*+(\theta-1)b\big)x-re^*+(\theta-1)c}{(x-r)(ax^2+bx+c)},$$

and then equating the result with

$$\frac{q_2(x)-\big((x-r)p_2(x)\big)'}{(x-r)p_2(x)},$$

so that we finally obtain

$$p_2(x)=ax^2+bx+c, \tag{25}$$

and

$$q_2(x)=\big(d^*+(\theta+2)a\big)x^2+\big(e^*-r(d^*+2a)+(\theta+1)b\big)x+\theta c-r(e^*+b), \tag{26}$$

provided that the roots of q_2 are real. Relations (25) and (26) show that the polynomial solution of Equation (24) with $\lambda_n = n\big((n+1+\theta)a + d^*\big)$ can be written in terms of the symbol (22) as

$$y_n(x) = Q_{n,r}\left(\begin{array}{c} d^* + (\theta+2)a,\ e^* - r(d^*+2a) + (\theta+1)b,\ \theta c - r(e^*+b) \\ a,\ b,\ c \end{array} \middle|\ x \right).$$

Additionally, according to the Corollary 1, c_0^* in (24) directly depends on the roots of $q_2(x)$ in (26) and is therefore computed as

$$c_0^* = 2\theta(ar^2+br+c)\big(d^*+(\theta+2)a\big)\Big(e^* + rd^* + (\theta+1)(2ra+b) \mp \sqrt{\Big(e^* - r(d^*+2a) + (\theta+1)b\Big)^2 - 4\big(d^*+(\theta+2)a\big)\big(\theta c - r(e^*+b)\big)}\Big)^{-1}.$$

As we observed, $\rho(x)$ was indeed the product of $(x-r)^\theta$ for

$$\theta = \frac{b_2 r^2 + b_1 r + b_0}{a_2 r^2 + a_1 r + a_0},$$

and a special case of the Pearson distributions family. This means that we can classify the nonsymmetric exceptional orthogonal X_1-polynomials into six main sequences.

Corollary 3. *By referring to Table 1 and relation (23), there are, in total, six sequences of nonsymmetric orthogonal X_1-polynomials as follows:*

1. *Infinite X_1-Jacobi polynomials orthogonal with respect to the weight function*

$$\rho_1(x) = (x-r)^\theta (1-x)^\alpha (1+x)^\beta, \qquad (-1 \le x \le 1).$$

2. *Infinite X_1-Laguerre polynomials orthogonal with respect to the weight function*

$$\rho_2(x) = (x-r)^\theta x^\alpha \exp(-x), \qquad (0 \le x < \infty).$$

3. *Infinite X_1-Hermite polynomials orthogonal with respect to the weight function*

$$\rho_3(x) = (x-r)^\theta \exp\left(-x^2\right), \qquad (-\infty < x < \infty).$$

4. *Finite X_1-polynomials orthogonal with respect to the weight function*

$$\rho_4(x) = (x-r)^\theta x^q (x+1)^{-(p+q)}, \qquad (0 \le x < \infty).$$

5. *Finite X_1-polynomials orthogonal with respect to the weight function*

$$\rho_5(x) = (x-r)^\theta x^{-p} \exp\left(-\frac{1}{x}\right), \qquad (0 \le x < \infty).$$

6. *Finite X_1-polynomials orthogonal with respect to the weight function*

$$\rho_6(x) = (x-r)^\theta \left(1+x^2\right)^{-p} \exp(q \arctan x), \qquad (-\infty < x < \infty).$$

In all six above-mentioned cases, $r \in \mathbb{R}$ and θ is a real parameter such that $(-1)^\theta = 1$.

Remark 1. *For $\theta = -2$ in the first and second kind of the above corollary, the weight functions represented in (6) and (7) are retrieved when $r = (\beta+\alpha)/(\beta-\alpha)$ and $r = -\alpha$, respectively.*

4. On the Series Solutions of Equation (12)

Let us reconsider Equation (12) in the form

$$y_n''(x) + \frac{b_2x^2 + b_1x + b_0}{(x-r)\big(a_2x^2 + a_1x + a_0\big)} y_n'(x) - \frac{\lambda_n(x-r) + c_0^*}{(x-r)\big(a_2x^2 + a_1x + a_0\big)} y_n(x) = 0. \tag{27}$$

The indicial equation corresponding to (27) is

$$t^2 + \Big(\frac{b_2r^2 + b_1r + b_0}{a_2r^2 + a_1r + a_0} - 1\Big)t = 0.$$

Hence, using the Frobenius method, we can obtain the series solutions of Equation (12) when

$$t_1 = 1 - \frac{b_2r^2 + b_1r + b_0}{a_2r^2 + a_1r + a_0} = 1 - \theta,$$

for different values of θ.

If $\theta \notin \mathbb{Z}$, the two basic solutions of Equation (27) are, respectively, in the forms

$$y_{n,1}(x) = \sum_{k=0}^{\infty} C_k (x-r)^k, \quad C_0 \neq 0,$$

and

$$y_{n,2}(x) = (x-r)^{1-\theta} \sum_{k=0}^{\infty} d_k (x-r)^k, \quad d_0 \neq 0.$$

If $\theta \in \mathbb{Z}$, three cases can occur for the basis solutions:

(i) If $\theta = 1$, then

$$\begin{cases} y_{n,1}(x) = \sum\limits_{k=0}^{\infty} C_k (x-r)^k, \quad C_0 \neq 0, \\ y_{n,2}(x) = y_{n,1}(x) \ln|x-r| + \sum\limits_{k=1}^{\infty} d_k (x-r)^k. \end{cases}$$

(ii) If $\theta < 1$, then,

$$\begin{cases} y_{n,1}(x) = (x-r)^{1-\theta} \sum\limits_{k=0}^{\infty} C_k (x-r)^k, \quad C_0 \neq 0, \\ y_{n,2}(x) = w y_{n,1}(x) \ln|x-r| + \sum\limits_{k=0}^{\infty} d_k (x-r)^k, \quad d_0 \neq 0, \quad w \in \mathbb{R}. \end{cases}$$

(iii) Finally, if $\theta > 1$, then

$$\begin{cases} y_{n,1}(x) = \sum\limits_{k=0}^{\infty} C_k (x-r)^k, \quad C_0 \neq 0, \\ y_{n,2}(x) = w y_{n,1}(x) \ln|x-r| + |x-r|^{1-\theta} \sum\limits_{k=0}^{\infty} d_k (x-r)^k, \quad d_0 \neq 0, \quad w \in \mathbb{R}. \end{cases}$$

In either case, there is at least one series solution, that it may assume the form

$$y_n(x) = \sum_{k=0}^{\infty} C_k (x-r)^{k-\theta+1}, \qquad (\theta \in \mathbb{Z},\ \theta < 1). \tag{28}$$

Substituting

$$y_n'(x) = \sum_{k=0}^{\infty} (k-\theta+1) C_k (x-r)^{k-\theta},$$
$$y_n''(x) = \sum_{k=0}^{\infty} (k-\theta+1)(k-\theta) C_k (x-r)^{k-\theta-1},$$

in Equation (12) eventually leads to the three-term recurrence relation

$$\begin{aligned}&((k-\theta)\,(a_2(k-\theta-1)+b_2)-\lambda_n)C_{k-1}\\ &\quad+((k-\theta+1)((2a_2r+a_1)(k-\theta)+(2b_2r+b_1))-c_0^*)C_k\\ &\quad+(k-\theta+2)\Big((a_2r^2+a_1r+a_0)(k-\theta+1)+(b_2r^2+b_1r+b_0)\Big)C_{k+1}=0. \end{aligned} \tag{29}$$

Note that, in a similar way, for $\theta \in \mathbb{Z}$ and $\theta \geq 1$, or $\theta \notin \mathbb{Z}$ the assumption

$$y_n(x) = \sum_{k=0}^{\infty} C_k (x-r)^k,$$

eventually leads to the same as recurrence relation (29) for $\theta = 1$.

Some Polynomial Solutions of Equation (21)

According to Corollary 1, the coefficients of the polynomial $B(x) = b_2x^2 + b_1x + b_0$ in (21) have a significant role in determining the value c_0^* in the system (14). In this section, we investigate six special cases of $B(x)$ based on its roots and the real value r, leading to particular cases of Equation (21). First, suppose that $b_2 \neq 0$ and r is a root of $B(x)$. So $b_2r^2 + b_1r + b_0 = 0$, and relations (14) reduce to

$$\begin{cases} 2b_2r + b_1 - c_0^* + \nu b_2 = 0, \\ \nu c_0^* = 0. \end{cases} \tag{30}$$

The equation $\nu\, c_0^* = 0$ in (30) gives three different cases as follows:

- **Case 1.** $\nu = 0$ and $c_0^* = 2b_2r + b_1 = B'(r) \neq 0$,
- **Case 2.** $c_0^* = 0$ and $\nu = -\dfrac{2b_2r + b_1}{b_2} = -\dfrac{B'(r)}{b_2} \neq 0$,
- **Case 3.** $c_0^* = 0$ and $\nu = 0$, leading to $B'(r) = 2b_2r + b_1 = 0$ which means that r is a multiple root of $B(x)$.

  Second, suppose that $b_2 = 0$ and $b_1 \neq 0$. So, relations (14) reduce to

$$\begin{cases} c_0^* = b_1, \\ \nu c_0^* = -(b_1r + b_0). \end{cases}$$

Now, if r is a root of $B(x)$, we have $\nu c_0^* = 0$ leading to

- **Case 4.** $c_0^* = b_1 \neq 0$ and $\nu = 0$, which is indeed a particular case of the first Case 1 for $b_2 = 0$.

  Otherwise, we obtain

- **Case 5.** $c_0^* = b_1 \neq 0$ and $\nu = -\dfrac{b_1r + b_0}{c_0^*} = -\dfrac{B(r)}{b_1} \neq 0$.

Finally, suppose that $b_2 = b_1 = 0$. In this case, relations (14) reduce to

$$\begin{cases} c_0^* = 0, \\ b_0 + \nu c_0^* = 0, \end{cases}$$

which yield $b_0 = 0$ leading to $B(x) \equiv 0$. Therefore, the last case can be considered

- **Case 6.** $c_0^* = 0$ and ν is arbitrary.

Now we consider each of these six cases:

For Case 1. Under the conditions stated in Case 1, the differential Equation (21) reads with $b_0 = -b_1 r - b_2 r^2$, as

$$\begin{aligned}(a_2x^2 + a_1x + a_0)y_n''(x) + \big(b_2x + b_2r + b_1\big)y_n'(x) \\ - \Big(n\big(b_2 + (n-1)a_2\big) + \frac{2b_2r + b_1}{x-r}\Big)y_n(x) = 0,\end{aligned} \tag{31}$$

for $n \geq 1$, whose solutions belong to the space

$$\Pi_{n,r,0} = \operatorname{span}\Big\{(x-r), (x-r)^2, \ldots, (x-r)^n\Big\}. \tag{32}$$

Relation (32) shows that the solution of Equation (31) can be considered as follows:

$$y_n(x) = (x-r)A_{n-1}(x-r) = (x-r)\sum_{k=0}^{n-1} d_k (x-r)^k. \tag{33}$$

Hence, replacing

$$\begin{cases} y_n' = A_{n-1}(x-r) + (x-r)A_{n-1}'(x-r), \\ y_n'' = 2A_{n-1}'(x-r) + (x-r)A_{n-1}''(x-r) \end{cases}$$

in (31) yields

$$\begin{aligned}(x-r)^2(a_2x^2 + a_1x + a_0)A_{n-1}'' + \Big(2(x-r)\big(a_2(x-r)^2 + (2a_2r + a_1)(x-r) + a_2r^2 + a_1r + a_0\big) \\ + (x-r)^2(b_2x + b_2r + b_1)\Big)A_{n-1}' - (n-1)(b_2 + na_2)(x-r)^2 A_{n-1} = 0.\end{aligned} \tag{34}$$

Now, if in (34), we assume that $a_2r^2 + a_1r + a_0 = 0$, which is equivalent to

$$a_0 = -r(a_2r + a_1),$$

then Equation (34) is simplified as

$$\begin{aligned}\big(a_2x^2 + a_1x - r(a_2r + a_1)\big)A_{n-1}'' + \Big((2a_2 + b_2)x + (2a_2 + b_2)r + 2a_1 + b_1\Big)A_{n-1}' \\ - (n-1)(b_2 + na_2)A_{n-1} = 0.\end{aligned} \tag{35}$$

By comparing Equation (35) and Equation (5) and referring to the polynomial solution (9) and also relation (33), we can finally conclude that the polynomial solution of Equation (31) for $a_0 = -r(a_2r + a_1)$ is as

$$y_n(x) = (x-r)P_{n-1}\left(\begin{matrix} 2a_2 + b_2, & (2a_2 + b_2)r + 2a_1 + b_1 \\ a_2, \quad a_1, & -r(a_2r + a_1) \end{matrix}\,\middle|\, x-r\right).$$

In other words, we have

$$\bar{Q}_{n,r}\left(\begin{array}{c} b_2,\quad b_1,\quad -r(b_2r+b_1) \\ a_2,\, a_1,\, -r(a_2r+a_1) \end{array}\middle| x\right)$$
$$= (x-r)\bar{P}_{n-1}\left(\begin{array}{c} 2a_2+b_2,\quad (2a_2+b_2)r+2a_1+b_1 \\ a_2,\quad a_1,\quad -r(a_2r+a_1) \end{array}\middle| x-r\right).$$

For cases 2, 3 and 6: The differential Equation (21) respectively reads as

$$\big(a_2x^2+a_1x+a_0\big)y_n''(x)+\big(b_2x+b_2r+b_1\big)y_n'(x)-n\big(b_2+(n-1)a_2\big)y_n(x)=0,$$
$$\big(a_2x^2+a_1x+a_0\big)y_n''(x)+b_2(x-r)y_n'(x)-n\big(b_2+(n-1)a_2\big)y_n(x)=0,$$
$$\big(a_2x^2+a_1x+a_0\big)y_n''(x)-n(n-1)a_2\,y_n(x)=0,$$

which are all particular cases of the well-known Equation (5). **Finally, for the Case 5,** The differential Equation (21) reduces to

$$\big(a_2x^2+a_1x+a_0\big)y_n''(x)+\left(b_1+\frac{b_1r+b_0}{x-r}\right)y_n'(x)-\left(n(n-1)a_2+\frac{b_1}{x-r}\right)y_n(x)=0, \quad (36)$$

with the polynomial solution space

$$\Pi_{n,r,\nu}=\operatorname{span}\left\{(x+\frac{b_0}{b_1}),(x-r)^2,\ldots,(x-r)^n\right\}.$$

5. On the Differential Equations of Six Nonsymmetric Exceptional Orthogonal X_1-Polynomials

Noting the Corollary 3, in this section, we consider six special cases of the main Equation (21) and study their orthogonal polynomial solutions. For finite cases, we also determine some necessary conditions in order to satisfy the orthogonality relations.

5.1. On the Differential Equation of Exceptional X_1-Jacobi Polynomials

As a generalization of the Jacobi differential equation for $\theta=0$, consider the following equation

$$(x-r)(1-x^2)y_n''(x)+\Big(-(\alpha+\beta+\theta+2)x^2+(\beta-\alpha+r(\alpha+\beta+2))x+\theta-r(\beta-\alpha)\Big)y_n'(x)$$
$$+\Big(n(n+\alpha+\beta+\theta+1)(x-r)-c_0^{(P)}\Big)y_n(x)=0 \qquad n\geq 1, \quad (37)$$

where r,θ,α,β are real parameters such that $\alpha,\beta>-1$, $(-1)^\theta=1$ and

$$c_0^{(P)}=2\theta(1-r^2)(\alpha+\beta+\theta+2)\Big((\alpha+\beta+2\theta+2)r+\alpha-\beta$$
$$\pm\sqrt{\big((\alpha+\beta+2)r+\beta-\alpha\big)^2+4(\alpha+\beta+\theta+2)\big(\theta-r(\beta-\alpha)\big)}\Big)^{-1}.$$

According to Section 3, the polynomial solution of Equation (37), i.e.,

$$y_n(x)=P_{n,r,\theta}^{(\alpha,\beta)}(x)=Q_{n,r}\left(\begin{array}{c} -(\alpha+\beta+\theta+2),\ \beta-\alpha+r(\alpha+\beta+2),\ \theta-r(\beta-\alpha) \\ -1,\ 0,\ 1 \end{array}\middle| x\right),$$

is orthogonal with respect to the weight function

$$\rho_1(x;r,\alpha,\beta,\theta)=(x-r)^\theta\,W\left(\begin{array}{c} -\alpha-\beta,\ \beta-\alpha \\ -1,\ 0,\ 1 \end{array}\middle| x\right)=(x-r)^\theta(1-x)^\alpha(1+x)^\beta,$$

on $[-1,1]$. Additionally, for $\theta = 0$, $r = -1$ or $r = 1$ in (37), $c_0^{(P)} = 0$ and the weight function $\rho_1(x; r, \alpha, \beta, \theta)$ will be a special case of the beta distribution. In fact, in each of these circumstances, Equation (37) is simplified as

$$(1-x^2)\, y_n''(x) + \big(-(\alpha+\beta+2)x + \beta - \alpha \big)\, y_n'(x) + n(n+\alpha+\beta+1)\, y_n(x) = 0,$$

for $\theta = 0$ and

$$(1-x^2)\, y_n''(x) + \big(-(\alpha+\beta+\theta+2)x + \beta + \theta - \alpha \big)\, y_n'(x) + n(n+\alpha+\beta+\theta+1)\, y_n(x) = 0,$$

for $r = -1$ and

$$(1-x^2)\, y_n''(x) + \big(-(\alpha+\beta+\theta+2)x + \beta - \theta - \alpha \big)\, y_n'(x) + n(n+\alpha+\beta+\theta+1)\, y_n(x) = 0,$$

for $r = 1$ with the following Jacobi-type polynomial solutions

$$\begin{aligned} P_{n,r,0}^{(\alpha,\beta)}(x) &= P_n^{(\alpha,\beta)}(x), \\ P_{n,-1,\theta}^{(\alpha,\beta)}(x) &= P_n^{(\alpha,\beta+\theta)}(x), \\ P_{n,1,\theta}^{(\alpha,\beta)}(x) &= P_n^{(\alpha+\theta,\beta)}(x). \end{aligned}$$

5.2. On the Differential Equation of Exceptional X_1-Laguerre Polynomials

As a generalization of Laguerre differential equation for $\theta = 0$, consider the following equation

$$\begin{aligned} x(x-r)y_n''(x) + \Big(-x^2 + (\alpha+r+\theta+1)x - r(\alpha+1) \Big) y_n'(x) & \\ + \big(n(x-r) - c_0^{(L)} \big) y_n(x) = 0 \qquad & n \geq 1, \end{aligned} \tag{38}$$

where r, θ, α are real parameters such that $\alpha > -1$, $(-1)^\theta = 1$ and

$$c_0^{(L)} = 2\, r\, \theta \left(r - \alpha - \theta - 1 \pm \sqrt{(r+\theta)^2 + (\alpha+1)(\alpha+1+2\theta-2r)} \right)^{-1}.$$

According to the Section 3, the polynomial solution of Equation (38), i.e.,

$$y_n(x) = L_{n,r,\theta}^{(\alpha)}(x) = Q_{n,r}\left(\begin{array}{c} -1,\ \alpha+r+\theta+1,\ -r(\alpha+1) \\ 0,\ 1,\ 0 \end{array} \middle|\, x \right),$$

is orthogonal with respect to the weight function

$$\rho_2(x; r, \alpha, \theta) = (x-r)^\theta\, W\left(\begin{array}{c} -1,\ \alpha \\ 0,\ 1,\ 0 \end{array} \middle|\, x \right) = (x-r)^\theta x^\alpha e^{-x},$$

on $[0, \infty)$. Also, for $\theta = 0$ or $r = 0$ in (38), $c_0^{(L)} = 0$ and the weight function $\rho_2(x; r, \alpha, \theta)$ will be a special case of Gamma distribution. In fact, in each of these circumstances Equation (38) is simplified as

$$x\, y_n''(x) + (-x+\alpha+1)\, y_n'(x) + n\, y_n(x) = 0,$$

for $\theta = 0$ and

$$x\, y_n''(x) + (-x+\alpha+\theta+1)\, y_n'(x) + n\, y_n(x) = 0,$$

for $r = 0$ with the following Laguerre-type polynomial solutions

$$L_{n,r,0}^{(\alpha)}(x) = L_n^{(\alpha)}(x),$$

and

$$L_{n,0,\theta}^{(\alpha)}(x) = L_n^{(\alpha+\theta)}(x).$$

5.3. On the Differential Equation of Exceptional X_1-Hermite Polynomials

As a generalization of Hermite differential equation for $\theta = 0$, consider the equation

$$(x-r)y_n''(x) + (-2x^2 + 2rx + \theta)y_n'(x) \\ + \left(2n(x-r) - \frac{2\theta}{r \pm \sqrt{r^2 + 2\theta}}\right)y_n(x) = 0, \quad n \geq 1, \tag{39}$$

where r, θ are real parameters and $(-1)^\theta = 1$. The polynomial solution of Equation (39), i.e.,

$$y_n(x) = H_{n,r,\theta}(x) = Q_{n,r}\left(\begin{array}{c} -2,\ 2r,\ \theta \\ \\ 0,\ 0,\ 1 \end{array} \middle| \ x \right),$$

is orthogonal with respect to the weight function

$$\rho_3(x; r, \theta) = (x-r)^\theta\, W\left(\begin{array}{c} -2,\ 0 \\ 0,\ 0,\ 1 \end{array} \middle| \ x \right) = (x-r)^\theta e^{-x^2},$$

on $(-\infty, \infty)$ and for $\theta = 0$, the solution of Equation (39) is the same as classical Hermite polynomials.

5.4. The First Finite Sequence of Exceptional Orthogonal X_1-Polynomials

Consider the differential equation

$$x(x-r)(x+1)y_n''(x) + \Big((\theta + 2 - p)x^2 + \big(q + \theta + 1 + r(p-2)\big)x - r(q+1)\Big)y_n'(x) \\ - \left(n(n+1+\theta-p)(x-r) + c_0^{(M)}\right)y_n(x) = 0 \qquad n \geq 1, \tag{40}$$

where r, θ are real parameters, $(-1)^\theta = 1$ and

$$c_0^{(M)} = \frac{2\theta r(r+1)(p-\theta-2)}{rp - q - (\theta+1)(2r+1) \pm \left(\big(q+\theta+1+r(p-2)\big)^2 + 4r(q+1)(\theta+2-p)\right)^{\frac{1}{2}}}.$$

Here we show that the polynomial solution of Equation (40), i.e.,

$$y_n(x) = M_{n,r,\theta}^{(p,q)}(x) = Q_{n,r}\left(\begin{array}{c} \theta + 2 - p,\ q + \theta + 1 + r(p-2),\ -r(q+1) \\ \\ 1,\ 1,\ 0 \end{array} \middle| \ x \right),$$

is finitely orthogonal with respect to the weight function

$$\rho_4(x; r, p, q, \theta) = (x-r)^\theta\, W\left(\begin{array}{c} -p,\ q \\ 1,\ 1,\ 0 \end{array} \middle| \ x \right) = (x-r)^\theta x^q (x+1)^{-(p+q)},$$

on $[0, \infty)$ if and only if

$$p > 2\{\max n\} + \theta + 1 \qquad \text{and} \qquad q > -1.$$

In other words, if the self-adjoint form of Equation (40) is written as

$$\Big((x-r)^{\theta}x^{q+1}(x+1)^{1-(p+q)}y_n'(x)\Big)' = (x-r)^{\theta-1}x^{q}(x+1)^{-(p+q)}\Big(n(n+1+\theta-p)(x-r)+c_0^{(M)}\Big)y_n(x), \quad (41)$$

and for the index m as

$$\Big((x-r)^{\theta}x^{q+1}(x+1)^{1-(p+q)}y_m'(x)\Big)' = (x-r)^{\theta-1}x^{q}(x+1)^{-(p+q)}\Big(m(m+1+\theta-p)(x-r)+c_0^{(M)}\Big)y_m(x), \quad (42)$$

then multiplying (41) and (42) by $y_m(x)$ and $y_n(x)$, respectively and subtracting them and finally integrating the resulting equation on the interval $[0,\infty)$ gives

$$\Big[(x-r)^{\theta}x^{q+1}(x+1)^{1-(p+q)}\big(y_n'(x)y_m(x)-y_m'(x)y_n(x)\big)\Big]_0^{\infty} = \big(n(n+1+\theta-p)-m(m+1+\theta-p)\big)\int_0^{\infty}(x-r)^{\theta}x^{q}(x+1)^{-(p+q)}y_n(x)y_m(x)\,dx. \quad (43)$$

Now, since

$$\max\deg\,\{y_n'(x)y_m(x)-y_m'(x)y_n(x)\}=m+n-1,$$

if

$$q>-1 \quad\text{and}\quad p>2N+\theta+1 \quad\text{for}\quad N=\max\{m,n\},$$

the left hand side of (43) tends to zero and for $m,n\geq 1$, we obtain

$$\int_0^{\infty}\frac{(x-r)^{\theta}x^{q}}{(x+1)^{p+q}}\,M_{n,r,\theta}^{(p,q)}(x)\,M_{m,r,\theta}^{(p,q)}(x)\,dx=0 \Leftrightarrow\ m\neq n,\ N=\max\{m,n\}<\frac{p-1-\theta}{2},\ q>-1\ \text{and}\ (-1)^{\theta}=1.$$

Note that for $\theta=0$, $r=-1$ or $r=0$, $\rho_4(x;r,p,q,\theta)$ reduces to a special case of the F-Fisher distribution. Indeed, in each of these circumstances, $c_0^{(M)}=0$ and Equation (40) reads as

$$x(x+1)y_n''(x)+\big((2-p)x+q+1\big)y_n'(x)-n(n+1-p)y_n(x)=0,$$

for $\theta=0$ and

$$x(x+1)y_n''(x)+\big((\theta+2-p)x+q+1\big)y_n'(x)-n(n+1+\theta-p)y_n(x)=0,$$

for $r=-1$ and

$$x(x+1)y_n''(x)+\big((\theta+2-p)x+q+\theta+1\big)y_n'(x)-n(n+1+\theta-p)y_n(x)=0,$$

for $r=0$ with the following polynomial solutions

$$M_{n,r,0}^{(p,q)}(x)=M_n^{(p,q)}(x),$$

$$M_{n,-1,\theta}^{(p,q)}(x)=M_n^{(p-\theta,q)}(x),$$

and

$$M_{n,0,\theta}^{(p,q)}(x)=M_n^{(p-\theta,q+\theta)}(x).$$

5.5. The Second Finite Sequence of Exceptional Orthogonal X_1-Polynomials

Consider the equation

$$(x-r)x^2y_n''(x)+\Big((\theta+2-p)x^2+\big(1+r(p-2)\big)x-r\Big)y_n'(x) \\ -\Big(n(n+1+\theta-p)(x-r)+c_0^{(N)}\Big)y_n(x)=0 \qquad n\geq 1, \quad (44)$$

where r,θ are real parameters, $(-1)^\theta=1$ and

$$c_0^{(N)}=\frac{2\theta r^2(p-\theta-2)}{r\big(p-2(\theta+1)\big)-1\pm\Big(\big(1+r(p-2)\big)^2+4r(\theta+2-p)\Big)^{\frac{1}{2}}}.$$

It can be shown that the polynomial solution of Equation (44), i.e.

$$y_n(x)=N_{n,r,\theta}^{(p)}(x)=Q_{n,r}\left(\begin{array}{c} \theta+2-p,\ 1+r(p-2),\ -r \\ 1,\ 0,\ 0 \end{array}\middle|\ x\right),$$

is finitely orthogonal with respect to the weight function

$$\rho_5(x;\,r,p,\theta)=(x-r)^\theta\, W\left(\begin{array}{c} -p,\ 1 \\ 1,\ 0,\ 0 \end{array}\middle|\ x\right)=(x-r)^\theta x^{-p}e^{-\frac{1}{x}},$$

on $[0,\infty)$ if and only if

$$p>2\{\max n\}+\theta+1,$$

because if the self-adjoint form of Equation (44) is written as

$$\Big((x-r)^\theta x^{-p+2}e^{-\frac{1}{x}}y_n'(x)\Big)'=(x-r)^{\theta-1}x^{-p}e^{-\frac{1}{x}}\Big(n(n+1+\theta-p)(x-r)+c_0^{(N)}\Big)y_n(x), \quad (45)$$

and for the index m as

$$\Big((x-r)^\theta x^{-p+2}e^{-\frac{1}{x}}y_m'(x)\Big)'=(x-r)^{\theta-1}x^{-p}e^{-\frac{1}{x}}\Big(m(m+1+\theta-p)(x-r)+c_0^{(N)}\Big)y_m(x), \quad (46)$$

then multiplying (45) and (46) by $y_m(x)$ and $y_n(x)$, respectively and subtracting them and finally integrating the resulting equation over $[0,\infty)$ gives

$$\Big[(x-r)^\theta x^{-p+2}e^{-\frac{1}{x}}\big(y_n'(x)y_m(x)-y_m'(x)y_n(x)\big)\Big]_0^\infty \\ =\Big(n(n+1+\theta-p)-m(m+1+\theta-p)\Big)\int_0^\infty (x-r)^\theta x^{-p}e^{-\frac{1}{x}}y_n(x)y_m(x)\,dx. \quad (47)$$

Now, if

$$p>2N+\theta+1 \quad \text{for} \quad N=\max\{m,n\},$$

the left-hand side of (47) tends to zero and for $m,n\geq 1$ we obtain

$$\int_0^\infty (x-r)^\theta x^{-p}e^{-\frac{1}{x}}\,N_{n,r,\theta}^{(p)}(x)\,N_{m,r,\theta}^{(p)}(x)\,dx=0 \\ \Leftrightarrow\ m\neq n,\ N=\max\{m,n\}<\frac{p-\theta-1}{2}\ \text{and}\ (-1)^\theta=1.$$

Note that for $\theta=0$ or $r=0$, $\rho_5(x;\,r,p,\theta)$ reduces to a special case of inverse Gamma distribution and in each of these circumstances $c_0^{(N)}=0$ so that Equation (44) changes to

$$x^2y_n''(x)+\big((2-p)x+1\big)y_n'(x)-n(n+1-p)y_n(x)=0,$$

for $\theta = 0$ and

$$x^2 y_n''(x) + ((\theta + 2 - p)x + 1)y_n'(x) - n(n + 1 + \theta - p)y_n(x) = 0,$$

for $r = 0$ with the following polynomial solutions

$$N_{n,r,0}^{(p)}(x) = N_n^{(p)}(x),$$

and

$$N_{n,0,\theta}^{(p)}(x) = N_n^{(p-\theta)}(x).$$

5.6. The Third Finite Sequence of Exceptional Orthogonal X_1-Polynomials

Consider the differential equation

$$\begin{aligned}(x - r)(1 + x^2)y_n''(x) &+ \Big((\theta + 2 - 2p)x^2 + \big(q + 2r(p - 1)\big)x + \theta - rq\Big)y_n'(x)\\ &- \Big(n(n + 1 + \theta - 2p)(x - r) + c_0^{(J)}\Big)y_n(x) = 0 \qquad n \geq 1, \qquad (48)\end{aligned}$$

where r, θ are real parameters, $(-1)^\theta = 1$ and

$$c_0^{(J)} = \frac{2\theta(r^2 + 1)(2p - \theta - 2)}{2r(p - \theta - 1) - q \pm \Big(\big(q + 2r(p - 1)\big)^2 - 4(\theta + 2 - 2p)(\theta - rq)\Big)^{\frac{1}{2}}}.$$

The polynomial solution of Equation (48), i.e.,

$$J_{n,r,\theta}^{(p,q)}(x) = Q_{n,r}\left(\begin{array}{c|c} \theta + 2 - 2p,\ q + 2r(p - 1),\ \theta - rq & x \\ 1,\ 0,\ 1 & \end{array}\right),$$

is finitely orthogonal with respect to the weight function

$$\rho_6(x; r, p, q, \theta) = (x - r)^\theta\, W\left(\begin{array}{c|c} -2p,\ q & x \\ 1,\ 0,\ 1 & \end{array}\right) = (x - r)^\theta (1 + x^2)^{-p} \exp(q \arctan x),$$

on $(-\infty, \infty)$ if and only if

$$p > \{\max n\} + \frac{\theta + 1}{2},$$

because if the self-adjoint form of Equation (48) is written as

$$\begin{aligned}&\Big((x - r)^\theta \Big(1 + x^2\Big)^{1-p} \exp(q \arctan x) y_n'(x)\Big)'\\ &= (x - r)^{\theta - 1}\Big(1 + x^2\Big)^{-p} \exp(q \arctan x)\Big(n(n + 1 + \theta - 2p)(x - r) + c_0^{(J)}\Big)y_n(x), \qquad (49)\end{aligned}$$

and for the index m as

$$\begin{aligned}&\Big((x - r)^\theta \Big(1 + x^2\Big)^{1-p} \exp(q \arctan x) y_m'(x)\Big)'\\ &= (x - r)^{\theta - 1}\Big(1 + x^2\Big)^{-p} \exp(q \arctan x)\Big(m(m + 1 + \theta - 2p)(x - r) + c_0^{(J)}\Big)y_m(x), \qquad (50)\end{aligned}$$

then multiplying (49) and (50) by $y_m(x)$ and $y_n(x)$, respectively and subtracting them and finally integrating from both sides on $(-\infty, \infty)$ gives

$$\left[(x-r)^{\theta}\left(1+x^2\right)^{1-p}\exp(q\arctan x)\left(y_n'(x)y_m(x)-y_m'(x)\,y_n(x)\right)\right]_{-\infty}^{\infty}$$
$$=\left(n(n+1+\theta-2p)-m(m+1+\theta-2p)\right)$$
$$\times\int_{-\infty}^{\infty}(x-r)^{\theta}\left(1+x^2\right)^{-p}\exp(q\arctan x)\,J_{n,r,\theta}^{(p,q)}(x)J_{m,r,\theta}^{(p,q)}(x)\,dx. \quad (51)$$

Now, if

$$p>N+\frac{\theta+1}{2} \qquad \text{for} \qquad N=\max\{m,n\},$$

the left-hand side of (51) tends to zero and for $m,n\geq 1$ we have

$$\int_{-\infty}^{\infty}(x-r)^{\theta}(1+x^2)^{-p}\exp(q\arctan x)\,J_{n,r,\theta}^{(p,q)}(x)J_{m,r,\theta}^{(p,q)}(x)\,dx=0$$
$$\Leftrightarrow \quad m\neq n\,,\,N=\max\{m,n\}<p-\frac{\theta+1}{2}\ \text{and}\ (-1)^{\theta}=1.$$

For $\theta=0$, $\rho_6(x;r,p,q,\theta)$ reduces to the generalized T-Student distribution and Equation (48) reads as

$$(1+x^2)y_n''(x)+\left(2(1-p)x+q\right)y_n'(x)-n(n+1-2p)y_n(x)=0,$$

with the polynomial solution

$$J_{n,r,0}^{(p,q)}(x)=J_n^{(p,q)}(x).$$

6. A Unified Classification for Symmetric Exceptional Orthogonal X_1-Polynomials

Fortunately, most of special functions in theoretical and mathematical physics which are the solutions of Sturm–Liouville problems have the symmetry property, namely

$$\Phi_n(x)=(-1)^n\Phi_n(-x).$$

These functions have usually interesting applications in physics and engineering; see e.g., [4,6] for more details. Hence, if they can be extended when their orthogonality property is preserved, new applications should naturally appear. The following theorem shows this matter.

Theorem 1 ([18])**.** *Let $\Phi_n(x)=(-1)^n\Phi_n(-x)$ be a sequence of independent symmetric functions that satisfy the differential equation*

$$A(x)\,\Phi_n''(x)+B(x)\,\Phi_n'(x)+\left(\lambda_n\,C(x)+D(x)+\frac{1-(-1)^n}{2}E(x)\right)\Phi_n(x)=0, \quad (52)$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $E(x)$ are real functions and $\{\lambda_n\}$ is a sequence of constants. If $A(x)$, $(C(x)>0)$, $D(x)$ and $E(x)$ are even functions and $B(x)$ is odd, then

$$\int_{-v}^{v}W^*(x)\,\Phi_n(x)\,\Phi_m(x)\,dx=\left(\int_{-v}^{v}W^*(x)\,\Phi_n^2(x)\,dx\right)\delta_{n,m},$$

where $W^*(x)$ denotes the corresponding weight function as

$$W^*(x)=C(x)\exp\Big(\int\frac{B(x)-A'(x)}{A(x)}\,dx\Big)=\frac{C(x)}{A(x)}\exp\Big(\int\frac{B(x)}{A(x)}\,dx\Big). \quad (53)$$

Of course, the weight function defined in (53) *must be positive and even on* $[-v, v]$ *and* $x = v$ *must be a root of the function*

$$A(x)K(x) = A(x)\exp\Big(\int \frac{B(x) - A'(x)}{A(x)}\,dx\Big) = \exp\Big(\int \frac{B(x)}{A(x)}\,dx\Big),$$

i.e., $A(v)\,K(v) = 0$. *Notice since* $K(x) = \dfrac{W^*(x)}{C(x)}$ *is an even function, the relation* $A(-v)\,K(-v) = 0$ *follows automatically.*

Based on the above theorem, many symmetric orthogonal functions were recently generalized; see, for example, [19]. In this section, by applying Theorem 1 and the polynomial sequence (22), we establish a class of symmetric orthogonal X_1-polynomials and introduce four special cases of it in the sequel.

For this purpose, let us reconsider the differential Equation (21) for $a_0 = 0$ as

$$\begin{aligned}(x-r)x(a_2x+a_1)y_n''(x) + \big(b_2x^2+b_1x+b_0\big)y_n'(x)\\ -\Big(n\big(b_2+(n-1)a_2\big)(x-r)+c_0^*\Big)y_n(x) = 0,\end{aligned} \tag{54}$$

in which

$$c_0^* = c_0^*\Big\{r; b_2, b_1, b_0\Big\} = b_2r + \frac{b_1 \mp \sqrt{b_1^2 - 4b_0b_2}}{2}. \tag{55}$$

To obtain a symmetric differential equation of type (52), we first substitute

$$\Phi_{2n}(x) = Q_{n,r}\left(\begin{array}{c|c} b_2, b_1, b_0 & \\ & x^2 \\ a_2, a_1, 0 & \end{array}\right),$$

into Equation (54) to obtain

$$\begin{aligned}x^2(a_2x^2+a_1)(x^2-r)\Phi_{2n}''(x) + x\Big((2b_2-a_2)x^4 + (2b_1-a_1+ra_2)x^2 + 2b_0 + ra_1\Big)\Phi_{2n}'(x)\\ -4x^2\Big(n\big(b_2+(n-1)a_2\big)(x^2-r) + c_0^*\Big\{r; b_2, b_1, b_0\Big\}\Big)\Phi_{2n}(x) = 0.\end{aligned} \tag{56}$$

In a similar manner, for

$$\Phi_{2n+1}(x) = x\,Q_{n,r}\left(\begin{array}{c|c} b_2^*, b_1^*, b_0^* & \\ & x^2 \\ a_2^*, a_1^*, 0 & \end{array}\right),$$

we obtain

$$\begin{aligned}&x^2(a_2^*x^2+a_1^*)(x^2-r)\Phi_{2n+1}''(x)\\ &\quad + x\Big((2b_2^*-3a_2^*)x^4 + (2b_1^*-3a_1^*+3ra_2^*)x^2 + 2b_0^* + 3ra_1^*\Big)\Phi_{2n+1}'(x)\\ &+\Big((3a_2^*-2b_2^*)x^4 + (3a_1^*-3ra_2^*-2b_1^*)x^2 - 2b_0^* - 3ra_1^* - 4x^2\big(n(b_2^*+(n-1)a_2^*)(x^2-r)\\ &\qquad + c_0^*\Big\{r; b_2^*, b_1^*, b_0^*\Big\}\big)\Big)\Phi_{2n+1}(x) = 0.\end{aligned} \tag{57}$$

Now, if for simplicity we assume that

$$a_2^* = a_2, \quad a_1^* = a_1,$$

and

$$b_2^* = b_2 + a_2, \quad b_1^* = b_1 + a_1 - ra_2, \quad b_0^* = b_0 - ra_1,$$

the differential Equation (57) changes to

$$\begin{aligned}
&x^2(a_2x^2+a_1)(x^2-r)\Phi''_{2n+1}(x)+x\Big((2b_2-a_2)x^4+(2b_1-a_1+ra_2)x^2+2b_0+ra_1\Big)\Phi'_{2n+1}(x)\\
&\quad+\Big((a_2-2b_2)x^4+(a_1-ra_2-2b_1)x^2-2b_0-ra_1\\
&\quad-4x^2\Big(n(b_2+na_2)(x^2-r)+c_0^*\Big\{r;b_2+a_2,b_1+a_1-ra_2,b_0-ra_1\Big\}\Big)\Big)\Phi_{2n+1}(x)=0,
\end{aligned}\tag{58}$$

with the polynomial solution

$$\Phi_{2n+1}(x)=x\,Q_{n,r}\left(\begin{array}{c}b_2+a_2,\ b_1+a_1-ra_2,\ b_0-ra_1\\ a_2,\ a_1,\ 0\end{array}\,\middle|\,x^2\right).$$

Therefore, by defining the symbol

$$\sigma_n=\frac{1-(-1)^n}{2},$$

and combining both equations (56) and (58) in a unique form, we finally obtain

$$\begin{aligned}
&x^2(a_2x^2+a_1)(x^2-r)\Phi''_n(x)+x\Big((2b_2-a_2)x^4+(2b_1-a_1+ra_2)x^2+2b_0+ra_1\Big)\Phi'_n(x)\\
&\quad+\Big(((a_2-2b_2)x^4+(a_1-ra_2-2b_1\\
&\quad-4c_0^*\Big\{r;b_2+a_2,b_1+a_1-ra_2,b_0-ra_1\Big\}+4c_0^*\Big\{r;b_2,b_1,b_0\Big\})x^2-2b_0-ra_1)\sigma_n\\
&\quad-4x^2((n-\sigma_n)(2b_2+(n+\sigma_n-2)a_2))(x^2-r)+c_0^*\Big\{r;b_2,b_1,b_0\Big\})\Big)\Phi_n(x)=0,
\end{aligned}\tag{59}$$

with the symmetric polynomial solution

$$\Phi_n(x)=x^{\sigma_n}Q_{[\frac{n}{2}],r}\left(\begin{array}{c}b_2+\sigma_na_2,\ b_1+\sigma_n(a_1-ra_2),\ b_0-\sigma_nra_1\\ a_2,\ a_1,\ 0\end{array}\,\middle|\,x^2\right).$$

Once again, if for simplicity we set

$$2b_2-a_2=p_2,\qquad 2b_1-a_1+ra_2=p_1\qquad\text{and}\qquad 2b_0+ra_1=p_0,$$

then Equation (59) is finally simplified as

$$\begin{aligned}
&x^2(a_2x^2+a_1)(x^2-r)\Phi''_n(x)+x(p_2x^4+p_1x^2+p_0)\Phi'_n(x)\\
&\quad-\Big((p_2x^4+(p_1+4c_0^*\{r;\frac{p_2+3a_2}{2},\frac{p_1+3a_1-3ra_2}{2},\frac{p_0-3ra_1}{2}\}\\
&\quad-4c_0^*\{r;\frac{p_2+a_2}{2},\frac{p_1+a_1-ra_2}{2},\frac{p_0-ra_1}{2}\})x^2+p_0)\sigma_n\\
&\quad+4x^2((n-\sigma_n)(p_2+(n+\sigma_n-1)a_2)(x^2-r)\\
&\quad+c_0^*\{r;\frac{a_2+p_2}{2},\frac{a_1+p_1-ra_2}{2},\frac{p_0-ra_1}{2}\})\Big)\Phi_n(x)=0.
\end{aligned}\tag{60}$$

Note in (60) that

$$\begin{aligned}
&c_0^*\Big\{r;\frac{p_2+3a_2}{2},\frac{p_1+3a_1-3ra_2}{2},\frac{p_0-3ra_1}{2}\Big\}\\
&\quad=\frac{1}{4}\Big(3a_1+p_1+3ra_2+2rp_2\mp((3a_1+p_1-3ra_2)^2-4(p_0-3ra_1)(3a_2+p_2))^{\frac{1}{2}}\Big),
\end{aligned}$$

and

$$c_0^*\Big\{r; \frac{p_2+a_2}{2}, \frac{p_1+a_1-ra_2}{2}, \frac{p_0-ra_1}{2}\Big\}$$

$$= \frac{1}{4}\Big(a_1+p_1+ra_2+2rp_2 \mp \big((a_1+p_1-ra_2)^2-4(p_0-ra_1)(a_2+p_2)\big)^{\frac{1}{2}}\Big),$$

are directly computed by referring to (55).

Corollary 4. *If in Theorem 1 we take*

$$\begin{aligned}
A(x) &= x^2(a_2x^2+a_1)(x^2-r),\\
B(x) &= x(p_2x^4+p_1x^2+p_0),\\
C(x) &= x^2(x^2-r),\\
D(x) &= -4x^2c_0^*\Big\{r; \frac{p_2+a_2}{2}, \frac{p_1+a_1-ra_2}{2}, \frac{p_0-ra_1}{2}\Big\},\\
E(x) &= -p_2x^4\Big(p_1+4c_0^*\Big\{r; \frac{p_2+3a_2}{2}, \frac{p_1+3a_1-3ra_2}{2}, \frac{p_0-3ra_1}{2}\Big\}\\
&\quad -4c_0^*\Big\{r; \frac{p_2+a_2}{2}, \frac{p_1+a_1-ra_2}{2}, \frac{p_0-ra_1}{2}\Big\}\Big)x^2-p_0\\
&= -p_2x^4-\Big(p_1+2a_1+2ra_2 \pm \big((a_1+p_1-ra_2)^2-4(p_0-ra_1)(a_2+p_2)\big)^{\frac{1}{2}}\\
&\qquad \mp \big((3a_1+p_1-3ra_2)^2-4(p_0-3ra_1)(3a_2+p_2)\big)^{\frac{1}{2}}\Big)x^2-p_0,
\end{aligned}$$

and

$$\lambda_n = -4(n-\sigma_n)\big(p_2+(n+\sigma_n-1)a_2\big),$$

then its symmetric polynomial solution, i.e.,

$$\Phi_n(x) = x^{\sigma_n} Q_{[\frac{n}{2}],r}\left(\begin{array}{c} \frac{p_2+a_2}{2}+a_2\sigma_n,\ \frac{p_1+a_1-ra_2}{2}+(a_1-ra_2)\sigma_n,\ \frac{p_0-ra_1}{2}-ra_1\sigma_n \\ a_2,\ a_1,\ 0 \end{array}\middle|\ x^2\right),\quad n\geq 1, \tag{61}$$

is orthogonal with respect to the weight function

$$\rho^*(x) = \frac{1}{a_2x^2+a_1}\exp\left(\int \frac{p_2x^4+p_1x^2+p_0}{x(a_2x^2+a_1)(x^2-r)}\,dx\right),$$

which can be simplified as

$$\rho^*(x) = (x^2-r)^{\mu}\exp\left(\int \frac{(p_2-2a_2(\mu+1))x^2-\frac{p_0}{r}}{x(a_2x^2+a_1)}\,dx\right), \tag{62}$$

for

$$\mu = \frac{p_2r^2+p_1r+p_0}{2r(a_1+ra_2)}.$$

Remark 2. *If in (62) we take $\mu = 0$, which is equivalent to $p_0 = -r(p_2r+p_1)$, then we will reach a symmetric class of orthogonal polynomials. In other words, let $p,q,r,s \in \mathbb{R}$ and consider the differential equation*

$$x^2(px^2+q)\,\Phi_n''(x) + x(rx^2+s)\,\Phi_n'(x) - \Big(n(r+(n-1)p)x^2 + \frac{1-(-1)^n}{2}s\Big)\,\Phi_n(x) = 0,$$

whose polynomial solution can be directly represented as [19]

$$\Phi_n(x) = S_n\left(\begin{array}{cc} r, & s \\ p, & q \end{array} \middle| \, x\right) = \sum_{k=0}^{[\frac{n}{2}]} \binom{[\frac{n}{2}]}{k} \left(\prod_{i=0}^{[\frac{n}{2}]-(k+1)} \frac{\left(2i+(-1)^{n+1}+2\left[\frac{n}{2}\right]\right)p+r}{\left(2i+(-1)^{n+1}+2\right)q+s}\right) x^{n-2k}.$$

Additionally, the weight function corresponding to these polynomials is as [19]

$$W^*\left(\begin{array}{cc} r, & s \\ p, & q \end{array} \middle| \, x\right) = \exp\left(\int \frac{(r-2p)x^2+s}{x(px^2+q)}\,dx\right).$$

Now, replacing $\mu = 0$ in (62) gives

$$\rho^*(x) = W^*\left(\begin{array}{cc} p_2, & p_2 r + p_1 \\ a_2, & a_1 \end{array} \middle| \, x\right).$$

Therefore, the symmetric polynomial (61) can be directly represented for $p_0 = -r(p_2 r + p_1)$ as follows

$$x^{\sigma_n} Q_{[\frac{n}{2}],r}\left(\begin{array}{c} \frac{p_2+a_2}{2} + a_2\sigma_n,\ \frac{p_1+a_1-ra_2}{2} + (a_1 - ra_2)\sigma_n,\ -\frac{r}{2}\left(rp_2 + p_1 + (1+2\sigma_n)a_1\right) \\ a_2,\ a_1,\ 0 \end{array} \middle| \, x^2\right)$$
$$= S_n\left(\begin{array}{cc} p_2, & p_2 r + p_1 \\ a_2, & a_1 \end{array} \middle| \, x\right).$$

There are four sequences of symmetric exceptional orthogonal X_1-polynomials as follows.

6.1. First Symmetric Class

Assume in Corollary 4 that

$$(a_2, a_1, p_2, p_1, p_0) = \big(-1, 1, -2(a+b+\mu+1), 2(a+(a+b+1)r-\mu), -2ra\big),$$

with the symmetric polynomial solution

$$\Phi_n(x) = \phi_{n,r,\mu}^{(a,b)}(x) =$$
$$x^{\sigma_n} Q_{[\frac{n}{2}],r}\left(\begin{array}{c} -(a+b+\mu+\frac{3}{2}+\sigma_n),\ (a+b+1)r+a+\mu+(1+r)(\frac{1}{2}+\sigma_n),\ -r(a+\frac{1}{2}+\sigma_n) \\ -1,\ 1,\ 0 \end{array} \middle| \, x^2\right),$$
$$n \geq 1. \qquad (63)$$

According to Theorem 1, the symmetric polynomials (63) are orthogonal with respect to the weight function

$$\rho_1^*(x) = (x^2-r)^{\mu}\, x^{2a} \left(1-x^2\right)^b,$$

on $[-1,1]$ if $(-1)^{\mu} = (-1)^{2a} = 1$, $b > -1$ and $a > -(\frac{1}{2}+\mu)$, (or $a > -\frac{1}{2}$) if $\mu < 0$, (or $\mu \geq 0$).

By noting remark 2, there are three particular cases of the symmetric polynomial $\phi_{n,r,\mu}^{(a,b)}(x)$ for $\mu = 0, r = 0$ and $r = 1$.

If $\mu = 0$, then we have

$$\phi_{n,r,0}^{(a,b)}(x) = S_n\left(\begin{array}{cc} -2(a+b+1), & 2a \\ -1, & 1 \end{array} \middle| \, x\right).$$

If $r = 0$, then

$$\phi^{(a,b)}_{n,0,\mu}(x) = S_n\left(\begin{array}{cc} -2(a+b+\mu+1), & 2(a+\mu) \\ -1, & 1 \end{array}\middle| x\right),$$

and, finally for $r = 1$, the corresponding symmetric polynomial is given by

$$\phi^{(a,b)}_{n,1,\mu}(x) = S_n\left(\begin{array}{cc} -2(a+b+\mu+1), & 2a \\ -1, & 1 \end{array}\middle| x\right).$$

6.2. *Second Symmetric Class*

Assume in Corollary 4 that

$$(a_2, a_1, p_2, p_1, p_0) = \big(0, 1, -2, 2(\mu + a + r), -2ra\big),$$

with the symmetric polynomial solution

$$\begin{aligned} \Phi_n(x) &= \Phi^{(a)}_{n,r,\mu}(x) \\ &= x^{\sigma_n} Q_{[\frac{n}{2}],r}\left(\begin{array}{c} -1,\ \mu + a + r + \frac{1}{2} + \sigma_n,\ -r(a + \frac{1}{2} + \sigma_n) \\ 0,\ 1,\ 0 \end{array}\middle| x^2\right),\ n \geq 1. \end{aligned} \tag{64}$$

According to Theorem 1, the symmetric polynomials (64) are orthogonal with respect to the weight function

$$\rho^*_2(x) = (x^2 - r)^{\mu}\, x^{2a} e^{-x^2},$$

on $(-\infty, \infty)$ if $(-1)^{\mu} = (-1)^{2a} = 1$ and $a > -(\frac{1}{2} + \mu)$, (or $a > -\frac{1}{2}$) if $\mu < 0$, (or $\mu \geq 0$).

By noting remark 2, there are two particular cases of the symmetric polynomial $\Phi^{(a)}_{n,r,\mu}(x)$ for $\mu = 0$ and $r = 0$.
If $\mu = 0$, then we have

$$\Phi^{(a)}_{n,r,0}(x) = S_n\left(\begin{array}{cc} -2, & 2a \\ 0, & 1 \end{array}\middle| x\right).$$

and for $r = 0$, the corresponding symmetric polynomial is given by

$$\Phi^{(a)}_{n,0,\mu}(x) = S_n\left(\begin{array}{cc} -2, & 2(a+\mu) \\ 0, & 1 \end{array}\middle| x\right).$$

6.3. *Third Symmetric Class*

Assume in Corollary 4 that

$$(a_2, a_1, p_2, p_1, p_0) = \Big(1, 1, 2(\mu - a - b + 1), 2\big(\mu - a + r(a + b - 1)\big), 2ra\Big),$$

with the symmetric polynomial solution

$$\begin{aligned} &\Phi_n(x) = \varphi^{(a,b)}_{n,r,\mu}(x) = \\ &x^{\sigma_n} Q_{[\frac{n}{2}],r}\left(\begin{array}{c} \mu - a - b + \frac{3}{2} + \sigma_n,\ \mu - a + r(a + b - 1) + (1 - r)(\frac{1}{2} + \sigma_n),\ r(a - \frac{1}{2} - \sigma_n) \\ 1,\ 1,\ 0 \end{array}\middle| x^2\right), \\ &n \geq 1. \end{aligned} \tag{65}$$

As three particular cases for $\mu = 0$, $r = 0$ and $r = -1$, we respectively have

$$\varphi_{n,r,0}^{(a,b)}(x) = S_n\left(\begin{array}{cc} -2a-2b+2, & -2a \\ 1, & 1 \end{array} \middle| \, x \right),$$

$$\varphi_{n,0,\mu}^{(a,b)}(x) = S_n\left(\begin{array}{cc} -2(a-\mu)-2b+2, & -2(a-\mu) \\ 1, & 1 \end{array} \middle| \, x \right),$$

and

$$\varphi_{n,-1,\mu}^{(a,b)}(x) = S_n\left(\begin{array}{cc} -2a-2b+4, & -2a \\ 1, & 1 \end{array} \middle| \, x \right).$$

According to Theorem 1, the symmetric polynomials (65), $\left\{ \varphi_{n,r,\mu}^{(a,b)} \right\}_{n=1}^{N}$, are finitely orthogonal with respect to the weight function

$$\rho_3^*(x) = (x^2-r)^{\mu}\, x^{-2a} \left(1+x^2\right)^{-b},$$

on $(-\infty, \infty)$ if

$$\begin{aligned} &(-1)^{2a} = (-1)^{\mu} = 1, \\ &\qquad b > 0,\ \ a < \frac{1}{2} + \mu, (\text{or } a < \frac{1}{2}) \text{ if } \mu < 0, (\text{or } \mu \geq 0), \text{and}\ \ N \leq a + b - \mu - \frac{1}{2}. \end{aligned}$$

To observe that why the limitation on N is $a + b - \mu - \frac{1}{2}$, first consider the differential equation

$$\begin{aligned} &x^2(x^2+1)(x^2-r)\Phi_n''(x) + 2x\big((\mu-a-b+1)x^4 + (\mu-a+r(a+b-1))x^2 + ra\big)\Phi_n'(x) \\ &\quad - 2\Big(\big((\mu-a-b+1)x^4 + \big(\mu-a+r(a+b-1)+2d(r;a,b,\mu)\big)x^2 + ra\big)\sigma_n \\ &\qquad + 2x^2\big(n\big(n+\mu-a-b+\frac{1}{2}+\sigma_n\big)(x^2-r) \\ &\qquad + c_0^*\Big\{r; -a-b+\mu+\frac{3}{2}, -a+\mu+\frac{1}{2}+\big(a+b-\frac{3}{2}\big)r, \big(a-\frac{1}{2}\big)r\Big\}\big)\Big)\Phi_n(x) = 0, \end{aligned} \tag{66}$$

in which $c_0^*\{.\}$ and $d(r;a,b,\mu)$ are, respectively, computed as

$$\begin{aligned} &c_0^*\Big\{r; -a-b+\mu+\frac{3}{2}, -a+\mu+\frac{1}{2}+\big(a+b-\frac{3}{2}\big)r, \big(a-\frac{1}{2}\big)r\Big\} \\ &\quad = \frac{1}{2}\Big(\mu - a - 2r(a+b-\mu-\frac{3}{2}) + r(a+b-\frac{3}{2}) + \frac{1}{2} \\ &\qquad\quad \mp \big((\mu-a+r(a+b-\frac{3}{2})+\frac{1}{2})^2 + 4r(a-\frac{1}{2})(a+b-\mu-\frac{3}{2})\big)^{\frac{1}{2}}\Big), \end{aligned}$$

and

$$\begin{aligned} d(r;a,b,\mu) &= c_0^*\Big\{r; -a-b+\mu+\frac{5}{2}, -a+\mu+\frac{3}{2}+\big(a+b-\frac{5}{2}\big)r, \big(a-\frac{3}{2}\big)r\Big\} \\ &\quad - c_0^*\Big\{r; -a-b+\mu+\frac{3}{2}, -a+\mu+\frac{1}{2}+\big(a+b-\frac{3}{2}\big)r, \big(a-\frac{1}{2}\big)r\Big\} \\ &= \frac{1}{2}\Big(r + 1 \pm \big((\mu-a+r(a+b-\frac{3}{2})+\frac{1}{2})^2 + 4r(a-\frac{1}{2})(a+b-\mu-\frac{3}{2})\big)^{\frac{1}{2}} \\ &\qquad \mp \big((\mu-a+r(a+b-\frac{5}{2})+\frac{3}{2})^2 + 4r(a-\frac{3}{2})(a+b-\mu-\frac{5}{2})\big)^{\frac{1}{2}}\Big). \end{aligned}$$

Then write the self-adjoint form of Equation (66) as

$$\begin{aligned}
&\Big((x^2-r)^{\mu}x^{-2a}(x^2+1)^{-b+1}\Phi_n'(x)\Big)'\\
&=2(x^2-r)^{\mu-1}x^{-2a-2}(x^2+1)^{-b}\\
&\times\Big(\big((\mu-a-b+1)x^4+\big(\mu-a+r(a+b-1)+2d(r;a,b,\mu)\big)x^2+ra\big)\sigma_n\\
&+2x^2\big(n\big(n+\mu-a-b+\frac{1}{2}+\sigma_n\big)(x^2-r)\\
&+c_0^*\Big\{r;-a-b+\mu+\frac{3}{2},-a+\mu+\frac{1}{2}+\big(a+b-\frac{3}{2}\big)r,\big(a-\frac{1}{2}\big)r\Big\}\big)\Big)\Phi_n(x),
\end{aligned}\tag{67}$$

and for the index m as

$$\begin{aligned}
&\Big((x^2-r)^{\mu}x^{-2a}(x^2+1)^{-b+1}\Phi_m'(x)\Big)'=2(x^2-r)^{\mu-1}x^{-2a-2}(x^2+1)^{-b}\\
&\times\Big(\big((\mu-a-b+1)x^4+\big(\mu-a+r(a+b-1)+2d(r;a,b,\mu)\big)x^2+ra\big)\sigma_m\\
&+2x^2\big(m\big(m+\mu-a-b+\frac{1}{2}+\sigma_m\big)(x^2-r)\\
&+c_0^*\Big\{r;-a-b+\mu+\frac{3}{2},-a+\mu+\frac{1}{2}+\big(a+b-\frac{3}{2}\big)r,\big(a-\frac{1}{2}\big)r\Big\}\big)\Big)\Phi_m(x).
\end{aligned}\tag{68}$$

Multiplying by $\Phi_m(x)$ and $\Phi_n(x)$ in relations (67) and (68) respectively and subtracting them and finally integrating from both sides on $(-\infty,\infty)$ gives

$$\begin{aligned}
&\Big[(x^2-r)^{\mu}x^{-2a}(x^2+1)^{-b+1}\big(\Phi_n'(x)\Phi_m(x)-\Phi_m'(x)\Phi_n(x)\big)\Big]_{-\infty}^{\infty}\\
&\quad=4\Big((m-n)\big(2(\mu-a-b)+n+m-1\big)-2(\sigma_m-\sigma_n)(\mu-a-b+1)\Big)\\
&\qquad\qquad\times\int_{-\infty}^{\infty}(x^2-r)^{\mu}\,x^{-2a}\big(1+x^2\big)^{-b}\,\Phi_n(x)\Phi_m(x)\,dx.
\end{aligned}\tag{69}$$

Now, since

$$\max\deg\,\{\Phi_n'(x)\Phi_m(x)-\Phi_m'(x)\Phi_n(x)\}=m+n-1,$$

if

$$N\le a+b-\mu-\frac{1}{2}\quad\text{for}\quad N=\max\{m,n\},$$

the left hand side of (69) tends to zero and for $m,n\ge 1$, we obtain

$$\int_{-\infty}^{\infty}(x^2-r)^{\mu}\,x^{-2a}\big(1+x^2\big)^{-b}\,\varphi_{n,r,\mu}^{(a,b)}(x)\,\varphi_{m,r,\mu}^{(a,b)}(x)\,dx=\;0,\qquad(m\neq n).$$

6.4. Fourth Symmetric Class

Assume in Corollary 4 that

$$(a_2,a_1,p_2,p_1,p_0)=\Big(1,0,2(\mu-a+1),2\big(r(a-1)+1\big),-2r\Big),$$

with the symmetric polynomial solution

$$\begin{aligned}
\Phi_n(x)&=\Phi_{n,r,\mu}^{(a)}(x)\\
&=x^{\sigma_n}Q_{[\frac{n}{2}],r}\left(\begin{array}{c|c}\mu-a+\frac{3}{2}+\sigma_n,\ r(a-\frac{3}{2}-\sigma_n)+1,\ -r & \multirow{2}{*}{x^2}\\ 1,\ 0,\ 0 & \end{array}\right),\quad n\ge 1.
\end{aligned}\tag{70}$$

As two particular cases for $\mu = 0$ and $r = 0$, we respectively have

$$\Phi^{(a)}_{n,r,0}(x) = S_n\left(\begin{array}{cc} -2a+2, & 2 \\ 1, & 0 \end{array} \middle| \, x \right),$$

and

$$\Phi^{(a)}_{n,0,\mu}(x) = S_n\left(\begin{array}{cc} -2(a-\mu)+2, & 2 \\ 1, & 0 \end{array} \middle| \, x \right).$$

According to Theorem 1, the symmetric polynomials (70), $\left\{\Phi^{(a)}_{n,r,\mu}\right\}_{n=1}^{N}$, are finitely orthogonal with respect to the weight function

$$\rho_4^*(x) = (x^2-r)^{\mu}\, x^{-2a} e^{-\frac{1}{x^2}},$$

on $(-\infty, \infty)$ if $(-1)^{\mu} = (-1)^{2a} = 1$ and $N \le a - \mu - \frac{1}{2}$. To observe that why the limitation on N is $a - \mu - \frac{1}{2}$, first consider the differential equation

$$\begin{aligned} &x^4(x^2-r)\Phi_n''(x) + 2x\big((\mu-a+1)x^4 + (r(a-1)+1)x^2 - r\big)\Phi_n'(x) \\ &\qquad - 2\Big(\big((\mu-a+1)x^4 + \big(r(a-1)+1+2d(r;a,\mu)\big)x^2 - r\big)\sigma_n \\ &+ 2x^2\big(n(n+\mu-a+\frac{1}{2}+\sigma_n)(x^2-r) + c_0^*\Big\{r; -a+\mu+\frac{3}{2}, 1+(a-\frac{3}{2})r, -r\Big\}\big)\Big)\Phi_n(x) = 0, \end{aligned} \tag{71}$$

in which $c_0^*\{.\}$ and $d(r;a,\mu)$ are respectively computed as

$$\begin{aligned} &c_0^*\Big\{r; -a+\mu+\frac{3}{2}, 1+(a-\frac{3}{2})r, -r\Big\} \\ &\qquad = \frac{1}{2}\Big(2r(\mu-a+\frac{3}{2}) + r(a-\frac{3}{2}) + 1 \mp \big((r(a-\frac{3}{2})+1)^2 + 4r(\mu-a+\frac{3}{2})\big)^{\frac{1}{2}}\Big), \end{aligned}$$

and

$$\begin{aligned} d(r;a,\mu) &= c_0^*\Big\{r; -a+\mu+\frac{5}{2}, 1+(a-\frac{5}{2})r, -r\Big\} - c_0^*\Big\{r; -a+\mu+\frac{3}{2}, 1+(a-\frac{3}{2})r, -r\Big\} \\ &= \frac{1}{2}\Big(r \pm \big((r(a-\frac{3}{2})+1)^2 + 4r(\mu-a+\frac{3}{2})\big)^{\frac{1}{2}} \mp \big((r(a-\frac{5}{2})+1)^2 + 4r(\mu-a+\frac{5}{2})\big)^{\frac{1}{2}}\Big). \end{aligned}$$

Then write the self-adjoint form of Equation (71) as

$$\begin{aligned} &\Big((x^2-r)^{\mu} x^{-2a+2} e^{-\frac{1}{x^2}}\, \Phi_n'(x)\Big)' \\ &\qquad = 2e^{-\frac{1}{x^2}}(x^2-r)^{\mu-1} x^{-2a-2}\Big(\big((\mu-a+1)x^4 + \big(r(a-1)+1+2d(r;a,\mu)\big)x^2 - r\big)\sigma_n \\ &\quad + 2x^2\big(n(n+\mu-a+\frac{1}{2}+\sigma_n)(x^2-r) + c_0^*\Big\{r; -a+\mu+\frac{3}{2}, 1+(a-\frac{3}{2})r, -r\Big\}\big)\Big)\Phi_n(x), \end{aligned} \tag{72}$$

and for the index m as

$$\begin{aligned} &\Big((x^2-r)^{\mu} x^{-2a+2} e^{-\frac{1}{x^2}}\, \Phi_m'(x)\Big)' \\ &\qquad = 2e^{-\frac{1}{x^2}}(x^2-r)^{\mu-1} x^{-2a-2}\Big(\big((\mu-a+1)x^4 + \big(r(a-1)+1+2d(r;a,\mu)\big)x^2 - r\big)\sigma_m \\ &\quad + 2x^2\big(m(m+\mu-a+\frac{1}{2}+\sigma_m)(x^2-r) + c_0^*\Big\{r; -a+\mu+\frac{3}{2}, 1+(a-\frac{3}{2})r, -r\Big\}\big)\Big)\Phi_m(x). \end{aligned} \tag{73}$$

Multiplying by $\Phi_m(x)$ and $\Phi_n(x)$ in relations (72) and (73) respectively and subtracting them and finally integrating from both sides on $(-\infty, \infty)$ gives

$$\begin{aligned}&\Big[(x^2-r)^{\mu}x^{-2a+2}e^{-\frac{1}{x^2}}\left(\Phi_n'(x)\Phi_m(x)-\Phi_m'(x)\Phi_n(x)\right)\Big]_{-\infty}^{\infty}\\&=4\Big((m-n)(2\mu-2a+n+m+1)-2(\sigma_m-\sigma_n)(2\mu-2a+1)\Big)\\&\times\int_{-\infty}^{\infty}(x^2-r)^{\mu}\,x^{-2a}e^{-\frac{1}{x^2}}\,\Phi_n(x)\Phi_m(x)\,dx.\end{aligned} \tag{74}$$

Now, again since

$$\max\deg\,\{\Phi_n'(x)\Phi_m(x)-\Phi_m'(x)\Phi_n(x)\}=m+n-1,$$

if

$$N\le a-\mu-\frac{1}{2}\quad\text{for}\quad N=\max\{m,n\},$$

the left-hand side of (74) tends to zero and for $m,n\ge 1$, we obtain

$$\int_{-\infty}^{\infty}(x^2-r)^{\mu}\,x^{-2a}e^{-\frac{1}{x^2}}\,\Phi_{n,r,\mu}^{(a)}(x)\,\Phi_{m,r,\mu}^{(a)}(x)\,dx=0,\qquad(m\ne n).$$

Corollary 5. *There are, in total, four sequences of symmetric orthogonal X_1-polynomials as follows:*

1. *Infinite X_1 symmetric polynomials orthogonal with respect to the weight function*

$$\rho_1^*(x)=(x^2-r)^{\mu}x^{2a}(1-x^2)^{b},\qquad(-1\le x\le 1).$$

2. *Infinite X_1 symmetric polynomials orthogonal with respect to the weight function*

$$\rho_2^*(x)=(x^2-r)^{\mu}x^{2a}e^{-x^2},\qquad(-\infty<x<\infty).$$

3. *Finite X_1 symmetric polynomials orthogonal with respect to the weight function*

$$\rho_3^*(x)=(x^2-r)^{\mu}x^{-2a}(1+x^2)^{-b},\qquad(-\infty<x<\infty).$$

4. *Finite X_1 symmetric polynomials orthogonal with respect to the weight function*

$$\rho_4^*(x)=(x^2-r)^{\mu}x^{-2a}e^{-\frac{1}{x^2}},\qquad(-\infty<x<\infty).$$

In all four above-mentioned cases, $r\in\mathbb{R}$ and μ is a real parameter such that $(-1)^{\mu}=1$.

7. Conclusions

In this paper, a unified classification of all exceptional orthogonal X_1-polynomials of symmetric and nonsymmetric types is established as a solution of generic second-order differential equations. Ten extended differential equations are introduced, and it is shown that they have polynomial solutions; six of them are X_1-orthogonal and four of them are X_1-symmetric orthogonal. When it comes to the classification nonsymmetric types, the key point is that the weight functions corresponding to the six sequences are exactly a multiplication of Pearson distributions family. Moreover, the finite cases of nonsymmetric exceptional X_1-polynomials orthogonal on infinite intervals and the class of symmetric orthogonal X_1-polynomials are introduced in this paper for the first time. More interesting properties of these polynomials and their applications in theoretical and computational [20] mathematical physics can be investigated in future research.

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