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Article

Fully Degenerating of Daehee Numbers and Polynomials

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Abstract: In this paper, we consider fully degenerate Daehee numbers and polynomials by using degenerate logarithm function. We investigate some properties of these numbers and polynomials. We also introduce higher-order multiple fully degenerate Daehee polynomials and numbers which can be represented in terms of Riemann integrals on the interval $[0, 1]$. Finally, we derive their summation formulae.

Keywords: degenerate Daehee polynomials; multiple degenerate Daehee numbers; higher-order degenerate Daehee polynomials

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1. Introduction

The generalizations of special polynomials have been one of the most emerging research fields in mathematical analysis and extensively investigated in order to find interesting identities and relations. Applications of the generalized special polynomials arise in problems of number theory, combinatorics, mathematical physics and other subareas of pure and applied mathematics provide motivation for introducing a new class of generalized polynomials. For example, some special polynomials occur in probability as the Edgeworth series; in combinatorics, they arise in the umbral calculus as an example of an Appell sequence which plays an important role in various problems connected with functional equations, interpolation problems, approximation theory, summation methods; in numerical analysis, they play a role in Gaussian quadrature; and in physics, they appear in quantum mechanical and optical beam transport problems. (see [1] for detail).

Recently, many mathematicians as the systematic study of degenerate versions of some special polynomials and numbers (see Kim-Kim [2–4], Kim et al. [5], Khan et al. [6–8], and Sharma et al. [9]) have been established due to Carlitz's degenerate version of Bernoulli polynomials given by (see [10,11])

$$\sum_{\ell=0}^{\infty}\beta_{\ell,\lambda}(\zeta)\frac{\omega^{\ell}}{\ell!}=\frac{\omega}{(1+\lambda\omega)^{\frac{1}{\lambda}}-1}(1+\lambda\omega)^{\frac{\zeta}{\lambda}},\quad (\lambda\in\mathbb{R}-\{0\}),\tag{1}$$

and the degenerate Bernoulli polynomials of higher order are also given by

$$\left(\frac{\omega}{(1+\lambda\omega)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda\omega)^{\frac{\zeta}{\lambda}}=\sum_{\ell=0}^{\infty}\beta_{l,\lambda}^{(r)}(\zeta),\tag{2}$$

with the case

$$\lim_{\lambda\to 0}\beta_{j,\lambda}^{(r)}(\zeta)=B_{j}^{(r)}(\zeta)\quad (j\geq 0),\tag{3}$$

where $B_j^{(r)}(\zeta)$ are the Bernoulli polynomials of higher order.

For any $\lambda \in \mathbb{R}-\{0\}$, degenerate version of the exponential function $e_\lambda^\zeta(\omega)$ is defined as (see [2,3,6–8,10,11])

$$e_\lambda^\zeta(\omega) := (1+\lambda\omega)^{\frac{\zeta}{\lambda}} = \sum_{\ell=0}^{\infty}(\zeta)_{\ell,\lambda}\frac{\omega^\ell}{\ell!}, \tag{4}$$

where $(\zeta)_{0,\lambda} = 1$ and $(\zeta)_{\ell,\lambda} = \zeta(\zeta-\lambda)\cdots(\zeta-(\ell-1)\lambda)$ for $\ell \geq 1$, (see [2–7,10,12]). It is obvious that $\lim_{\lambda\to 0} e_\lambda^\zeta(\omega) = e^{\zeta\omega}$. Additionally, we note that $e_\lambda^1(\omega) := e_\lambda(\omega)$.

The generating functions of the degenerating Stirling numbers of the first and second kinds are defined by

$$\frac{1}{k!}(\log_\lambda(1+\omega))^k = \sum_{j=k}^{\infty} S_{1,\lambda}(j,k)\frac{\omega^j}{j!} \quad \text{and} \quad \frac{1}{k!}(e_\lambda(\omega)-1)^k = \sum_{j=k}^{\infty} S_{2,\lambda}(j,k)\frac{\omega^j}{j!} \tag{5}$$

where

$$\log_\lambda(1+\omega) = \frac{(1+\omega)^\lambda - 1}{\lambda}, \tag{6}$$

which is the inverse of degenerate exponential function (see [4,6]).

Roman [13] defined the Bernoulli polynomials of the second kind given by means of the following generating function:

$$\frac{\omega}{\log(1+\omega)}(1+\omega)^\zeta = \sum_{\ell=0}^{\infty} b_\ell(\zeta)\frac{\omega^\ell}{\ell!}. \tag{7}$$

From (3) and (7), we note that, see [8,14].

$$b_\ell(\zeta) = B_\ell^{(\ell)}(\zeta+1), \quad (\ell \geq 0),$$

It is well known from [2] that

$$\left(\frac{\omega}{\log_\lambda(1+\omega)}\right)^k (1+\omega)^{\zeta-1} = \sum_{\ell=0}^{\infty} B_{\ell,\lambda}^{(\ell-k+1)}(\zeta)\frac{\omega^\ell}{\ell!}, \quad (k \in \mathbb{Z}). \tag{8}$$

where $B_{\ell,\lambda}^{(\alpha)}(\zeta)$ are called λ-analogue of Bernoulli polynomials of higher order given via the following generating function:

$$\left(\frac{\omega}{\lambda e^\omega - 1}\right)^\alpha e^{\zeta\omega} = \sum_{j=0}^{\infty} B_{j,\lambda}^{(\alpha)}(\zeta)\frac{\omega^j}{j!}.$$

In [2], the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{\omega}{\log_\lambda(1+\omega)}(1+\omega)^\zeta = \sum_{\ell=0}^{\infty} b_{\ell,\lambda}(\zeta)\frac{\omega^\ell}{\ell!}. \tag{9}$$

Note that

$$\lim_{\lambda\to 0} b_{\ell,\lambda}(\zeta) = b_\ell(\zeta), \quad (\ell \geq 0).$$

The Daehee polynomials are known as, (see [7–9]).

$$\frac{\log(1+\omega)}{\omega}(1+\omega)^\zeta = \sum_{j=0}^{\infty} D_j(\zeta)\frac{\omega^j}{j!}, \tag{10}$$

In the case when $\zeta = 0$, $D_j = D_j(0)$ are called the Daehee numbers. The Equation (10) will be our main focus to proceed its fully degenerate version with their identities and properties with Section 2. In Section 3, we consider multiple fully degenerate Daehee polynomials of higher order which can be represented in terms of Riemann integrals on the interval $[0,1]$. We derive their identities and properties among some other polynomials which will be mentioned in the next sections.

2. Fully Degenerating Daehee Numbers and Polynomials

Recall from Equation (6) that

$$\begin{aligned}\log_\lambda(1+\omega) &= \frac{(1+\omega)^\lambda-1}{\lambda} \\ &= \sum_{j=1}^{\infty}\lambda^{j-1}(1)_{j,1/\lambda}\frac{\omega^j}{j!},\end{aligned} \tag{11}$$

where

$$(1)_{j,1/\lambda}=(1-\frac{1}{\lambda})(1-\frac{2}{\lambda})\cdots(1-(j-1)\frac{1}{\lambda}).$$

In the case λ approaches to 0, we see that the Equation (11) turns out to be classical one as follows:

$$\lim_{\lambda\to 0}\log_\lambda(1+\omega)=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{\omega^j}{j!}=\log(1+\omega).$$

Note that $e_\lambda(\log_\lambda(\omega))=\log_\lambda(e_\lambda(\omega))=\omega$. By making use of Equation (11), Kim et al. [5] introduced the new type of degenerate Daehee polynomials as follows:

$$\sum_{\ell=0}^{\infty}D_{\ell,\lambda}(\zeta)\frac{\omega^\ell}{\ell!} = \frac{\log_\lambda(1+\omega)}{\omega}(1+\omega)^\zeta \tag{12}$$

$$= \frac{\log(1+\omega)}{\omega}\int_0^1(1+\omega)^{\lambda\eta+\zeta}\mathrm{d}\eta. \tag{13}$$

At the value $\zeta=0$, $D_{j,\lambda}=D_{j,\lambda}(0)$ are called the degenerate Daehee numbers. Motivated by (12) and (13), we give the following definition.

Definition 1. *Let $\lambda\in\mathbb{R}-\{0\}$. The fully degenerating Daehee polynomials are defined by means of the following generating function:*

$$\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}(\zeta)\frac{\omega^j}{j!}=\frac{\log_\lambda(1+\omega)}{\omega}e^{\zeta\log_\lambda(1+\omega)}. \tag{14}$$

Then, from (13), we see that

$$\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}(\zeta)\frac{\omega^j}{j!}=\frac{\log(1+\omega)}{\omega}\int_0^1(1+\omega)^{\lambda\eta}\mathrm{d}\eta\; e^{\zeta\log_\lambda(1+\omega)}. \tag{15}$$

Note that, $\lim_{\lambda\to 0}\tilde{d}_{j,\lambda}(\zeta)=\tilde{d}_j(\zeta),\ \ (j\geq 0)$. We note that $\zeta=0, \tilde{d}_{j,\lambda}:=\tilde{d}_{j,\lambda}(0)$ are called the new type of fully degenerate Daehee numbers. The following identity will be useful for proving next theorem already known in [2]:

$$e^{\zeta\log_\lambda(1+\omega)}=\sum_{\ell=0}^{\infty}\left\{\sum_{k=0}^{\ell}S_{1,\lambda}(\ell,k)\zeta^\ell\right\}\frac{\omega^\ell}{\ell!}. \tag{16}$$

Theorem 1. *Let $j \geq 0$, the following identity holds true:*

$$\tilde{d}_{j,\lambda}(\zeta) = \sum_{r=0}^{j}\sum_{k=0}^{r}\binom{j}{r}\tilde{d}_{j-r,\lambda}S_{1,\lambda}(r,k)\zeta^{r}.$$

Proof. It is proved by using (11) that

$$\begin{aligned}\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}(\zeta)\frac{\omega^{j}}{j!} &= \frac{\log_{\lambda}(1+\omega)}{\omega}e^{\zeta\log_{\lambda}(1+\omega)} \\ &= \sum_{l=0}^{\infty}\tilde{d}_{l,\lambda}\frac{\omega^{l}}{l!}\sum_{r=0}^{\infty}\zeta^{r}\frac{(\log_{\lambda}(1+\omega))^{r}}{r!} \\ &= \sum_{l=0}^{\infty}\tilde{d}_{l,\lambda}\frac{\omega^{l}}{l!}\sum_{r=0}^{\infty}\sum_{k=0}^{r}S_{1,\lambda}(r,k)\zeta^{r}\frac{\omega^{r}}{r!} \\ &= \sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\sum_{k=0}^{r}\binom{j}{r}\tilde{d}_{j-r,\lambda}S_{1,\lambda}(r,k)\zeta^{r}\right)\frac{\omega^{j}}{j!}.\end{aligned}$$

By comparing the coefficients of $\frac{\omega^{j}}{j!}$ on the above, we get the proof of this theorem. □

Theorem 2. *For $j \geq 0$, we have*

$$\tilde{d}_{j,\lambda}(\zeta) = \sum_{i=0}^{j}\frac{1}{j-i+1}\sum_{r=1}^{j-i+1}\sum_{k=0}^{i}\binom{j}{i}\lambda^{r-1}S_{1}(j-i+1,r)\zeta^{k}S_{1,\lambda}(i,k). \tag{17}$$

Proof. By using (11), we get

$$\begin{aligned}\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}(\zeta)\frac{\omega^{j}}{j!} &= \frac{\log(1+\omega)}{\omega}\int_{0}^{1}(1+\omega)^{\lambda\eta}\mathrm{d}\eta e^{\zeta\log_{\lambda}(1+\omega)} \\ &= \frac{\log(1+\omega)}{\omega}\sum_{r=0}^{\infty}\lambda^{r}\frac{(\log(1+\omega))^{r}}{(r+1)!}\sum_{k=0}^{\infty}\zeta^{k}\frac{(\log_{\lambda}(1+\omega))^{k}}{k!} \\ &= \frac{1}{\omega}\left(\sum_{r=1}^{\infty}\lambda^{r-1}\frac{(\log(1+\omega))^{r}}{r!}\right)\left(\sum_{k=0}^{\infty}\zeta^{k}\frac{(\log_{\lambda}(1+\omega))^{k}}{k!}\right) \\ &= \frac{1}{\omega}\left(\sum_{j=1}^{\infty}\sum_{r=1}^{j}\lambda^{r-1}S_{1}(j,r)\frac{\omega^{j}}{j!}\right)\left(\sum_{i=0}^{\infty}\sum_{k=0}^{i}\zeta^{k}S_{1,\lambda}(i,k)\frac{\omega^{i}}{i!}\right) \\ &= \left(\sum_{j=0}^{\infty}\frac{1}{j+1}\sum_{r=1}^{j+1}\lambda^{r-1}S_{1}(j+1,r)\frac{\omega^{j}}{j!}\right)\left(\sum_{i=0}^{\infty}\sum_{k=0}^{i}\zeta^{k}S_{1,\lambda}(i,k)\frac{\omega^{i}}{i!}\right) \\ &= \sum_{j=0}^{\infty}\left(\frac{1}{j-i+1}\sum_{i=0}^{j}\sum_{r=1}^{j-i+1}\sum_{k=0}^{i}\binom{j}{i}\lambda^{r-1}S_{1}(j-i+1,r)\zeta^{k}S_{1,\lambda}(i,k)\right)\frac{\omega^{j}}{j!}\end{aligned}$$

Thus, we get the result. □

From (1), we note that

$$\left(\sum_{j=0}^{\infty}\beta_{j,\lambda}\frac{\omega^{j}}{j!}\right)\left(\sum_{r=0}^{\infty}\zeta^{r}\frac{\omega^{r}}{r!}\right) = \frac{\omega}{e_{\lambda}(\omega)-1}e^{\zeta\omega} = \sum_{j=0}^{\infty}\left(\sum_{r=0}^{j}\binom{j}{r}\beta_{j-r,\lambda}\zeta^{r}\right)\frac{\omega^{j}}{j!} \tag{18}$$

Theorem 3. *For $j \geq 0$, we have*

$$\tilde{d}_{j,\lambda}(\zeta) = \sum_{i=0}^{j} \sum_{r=0}^{i} \binom{i}{r} \beta_{i-r,\lambda} \zeta^r S_{1,\lambda}(j,i). \tag{19}$$

Proof. By replacing ω by $\log_\lambda(1+\omega)$ in (18), we get

$$\begin{aligned} \frac{\log_\lambda(1+\omega)}{\omega} e^{x \log_\lambda(1+\omega)} &= \sum_{i=0}^{\infty} \sum_{r=0}^{i} \binom{i}{r} \beta_{i-r,\lambda} \zeta^r \frac{(\log_\lambda(1+\omega))^i}{i!} \\ &= \sum_{i=0}^{\infty} \sum_{r=0}^{i} \binom{i}{r} \beta_{i-r,\lambda} \zeta^r \sum_{j=i}^{\infty} S_{1,\lambda}(j,i) \frac{\omega^j}{j!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \sum_{r=0}^{i} \binom{i}{r} \beta_{i-r,\lambda} \zeta^r S_{1,\lambda}(j,i) \right) \frac{\omega^j}{j!}. \end{aligned} \tag{20}$$

On the other hand,

$$\frac{\log_\lambda(1+\omega)}{\omega} e^{\zeta \log_\lambda(1+\omega)} = \sum_{r=0}^{\infty} \tilde{d}_{j,\lambda}(\zeta) \frac{\omega^j}{j!}. \tag{21}$$

Therefore, by (20) and (21), we obtain the required result. □

Theorem 4. *For $j \geq 0$, we have*

$$\beta_{j,\lambda} = \sum_{r=0}^{j} \tilde{d}_{r,\lambda}(\zeta) S_{2,\lambda}(j,r).$$

Proof. By replacing ω by $e_\lambda(\omega) - 1$ in (11), we get

$$\begin{aligned} \frac{\omega}{e_\lambda(\omega) - 1} e^{\zeta\omega} &= \sum_{r=0}^{\infty} \tilde{d}_{r,\lambda}(\zeta) \frac{(e_\lambda(\omega) - 1)^r}{r!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{r=0}^{j} \tilde{d}_{r,\lambda}(\zeta) S_{2,\lambda}(j,r) \right) \frac{\omega^j}{j!}. \end{aligned} \tag{22}$$

By using (18) and (22), we conclude the proof. □

Theorem 5. *For $j \geq 0$, we have*

$$\sum_{r=0}^{j} \binom{j}{r} b_{j-r} \tilde{d}_{r,\lambda}(\zeta) = \sum_{i=0}^{j} \binom{j}{i} \sum_{r=0}^{j-i} \sum_{k=0}^{i} \frac{\lambda^r}{r+1} \zeta^k S_{j-i,r} S_{1,\lambda}(i,k).$$

Proof. By using (11), we note that

$$\begin{aligned} \int_0^1 (1+\omega)^{\lambda\eta} \mathrm{d}\eta \; e^{\zeta \log_\lambda(1+\omega)} &= \frac{\omega}{\log(1+\omega)} \frac{\log_\lambda(1+\omega)}{\omega} e^{\zeta \log_\lambda(1+\omega)} \\ &= \left(\sum_{j=0}^{\infty} b_j \frac{\omega^j}{j!} \right) \left(\sum_{r=0}^{\infty} \tilde{d}_{r,\lambda}(\zeta) \frac{\omega^r}{r!} \right) \\ &= \sum_{j=0}^{\infty} \left(\sum_{r=0}^{j} \binom{j}{r} b_{j-r} \tilde{d}_{r,\lambda}(\zeta) \right) \frac{\omega^j}{j!}. \end{aligned} \tag{23}$$

On the other hand, we have

$$\begin{aligned}\int_0^1 (1+\omega)^{\lambda\eta} d\eta \, e^{\zeta \log_\lambda(1+\omega)} &= \sum_{r=0}^{\infty} \frac{\lambda^r (\log(1+\omega))^r}{(r+1)!} \sum_{k=0}^{\infty} \zeta^k \frac{(\log_\lambda(1+\omega))^k}{k!} \\ &= \sum_{r=0}^{\infty} \frac{\lambda^r}{r+1} \sum_{j=r}^{\infty} S_1(j,r) \frac{\omega^j}{j!} \sum_{k=0}^{\infty} \zeta^k \sum_{i=k}^{\infty} S_{1,\lambda}(i,k) \frac{\omega^i}{i!} \\ &= \sum_{j=0}^{\infty} \frac{\lambda^r}{r+1} \sum_{r=0}^{j} S_1(j,r) \frac{\omega^j}{j!} \sum_{i=0}^{\infty} \zeta^k \sum_{k=0}^{i} S_{1,\lambda}(i,k) \frac{\omega^i}{i!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \binom{j}{i} \sum_{r=0}^{j-i} \sum_{k=0}^{i} \frac{\lambda^r}{r+1} \sigma^k S_{j-i,r} S_{1,\lambda}(i,k) \right) \frac{\omega^j}{j!}. \end{aligned} \tag{24}$$

Thus, by (23) and (24), we arrive at the required the proof. □

3. New Type of Higher-Order Fully Degenerating Daehee Numbers and Polynomials

Let us define the new type of fully degenerate Daehee polynomials of order r ($\in \mathbb{N}$) by the following multiple Riemann integral on the interval $\underbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}_{\text{k-times}}$:

$$\begin{aligned} &\left(\frac{\log(1+\omega)}{\omega}\right)^r \int_0^1 \cdots \int_0^1 (1+\omega)^{\lambda(\zeta_1+\cdots+\zeta_r)} d\zeta_1 \cdots d\zeta_r \, e^{\zeta \log_\lambda(1+\omega)} \\ &= \left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta \log_\lambda(1+\omega)} = \sum_{j=0}^{\infty} \tilde{d}_{j,\lambda}^{(r)}(\zeta) \frac{\omega^j}{j!} \end{aligned} \tag{25}$$

In the case when $\zeta = 0, \tilde{d}_{j,\lambda}^{(r)} = \tilde{d}_{j,\lambda}^{(r)}(0)$ are called the new type of fully degenerate Daehee numbers of the order r.

Theorem 6. *For $j \geq 0$, we have*

$$\tilde{d}_{j,\lambda}^{(r)}(\zeta) = \sum_{k=0}^{j} \binom{j}{k} \sum_{i=0}^{k} \tilde{d}_{j-k,\lambda}^{(r)} \zeta^i S_{1,\lambda}(k,i).$$

Proof. From (25), we note that

$$\begin{aligned} &\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta \log_\lambda(1+\omega)} \\ &= \sum_{j=0}^{\infty} \tilde{d}_{j,\lambda}^{(r)} \frac{\omega^j}{j!} \sum_{i=0}^{\infty} \frac{1}{i!} \zeta^i (\log_\lambda(1+\omega))^i = \sum_{j=0}^{\infty} \tilde{d}_{j,\lambda}^{(r)} \frac{\omega^j}{j!} \sum_{i=0}^{\infty} \zeta^i \sum_{k=i}^{\infty} S_{1,\lambda}(k,i) \frac{\omega^k}{k!} \\ &= \sum_{j=0}^{\infty} \tilde{d}_{j,\lambda}^{(r)} \frac{\omega^j}{j!} \sum_{k=0}^{\infty} \zeta^i \sum_{i=0}^{k} S_{1,\lambda}(k,i) \frac{\omega^k}{k!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \binom{j}{k} \sum_{i=0}^{k} \zeta^i S_{1,\lambda}(k,i) \tilde{d}_{j-k,\lambda}^{(r)} S_{1,\lambda}(j,i) \right) \frac{\omega^j}{j!}. \end{aligned}$$

Thus, by comparing the coefficients of $\frac{\omega^j}{j!}$ on the above, we obtain the result. □

Theorem 7. *For $n \geq 0$, we have*

$$\tilde{d}_{j,\lambda}^{(r)}(\zeta) = \sum_{i=0}^{j} \sum_{k=0}^{j-i} \binom{j}{i} \frac{S_{1,\lambda}(i+r,r)}{\binom{i+r}{i}} \zeta^k S_{1,\lambda}(j-i,k).$$

Proof. Using (25), we have

$$\begin{aligned}
\sum_{j=0}^{\infty} \tilde{d}_{j,\lambda}^{(r)}(\zeta) \frac{\omega^j}{j!} &= \frac{r!}{\omega^r} \frac{1}{r!} (\log_\lambda(1+\omega))^r e^{\zeta \log_\lambda(1+\omega)} \\
&= \frac{r!}{\omega^r} \left(\sum_{i=r}^{\infty} S_{1,\lambda}(i,r) \frac{\omega^i}{i!} \right) \left(\sum_{k=0}^{\infty} \zeta^k \frac{(\log_\lambda(1+\omega))^k}{k!} \right) \\
&= \left(\sum_{i=0}^{\infty} \frac{S_{1,\lambda}(i+r,r)}{\binom{i+r}{i}} \frac{\omega^i}{i!} \right) \left(\sum_{j=0}^{\infty} \sum_{k=0}^{j} \zeta^k S_{1,\lambda}(j,k) \frac{\omega^j}{j!} \right) \\
&= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \sum_{k=0}^{j-i} \binom{j}{i} \frac{S_{1,\lambda}(i+r,r)}{\binom{i+r}{i}} \zeta^k S_{1,\lambda}(j-i,k) \right) \frac{\omega^j}{j!}.
\end{aligned}$$

Therefore, by comparing the coefficients of $\frac{\omega^j}{j!}$ on the above, we arrive at the desired result. □

Theorem 8. *For $j \geq 0$, we have*

$$\sum_{i=0}^{j} \binom{j}{i} \beta_{j-i,\lambda} \zeta^i = \sum_{\ell=0}^{j} \tilde{d}_{\ell,\lambda}^{(r)}(\zeta) S_{2,\lambda}(j,\ell).$$

Proof. By replacing ω by $e_\lambda(\omega) - 1$ in (25), we get

$$\begin{aligned}
\sum_{i=0}^{\infty} \tilde{d}_{i,\lambda}^{(r)}(\zeta) \frac{(e_\lambda(\omega)-1)^i}{i!} &= \left(\frac{\omega}{e_\lambda(\omega)-1} \right)^r e^{\zeta\omega} \\
&= \sum_{j=0}^{\infty} \beta_{j,\lambda} \frac{\omega^j}{j!} \sum_{i=0}^{\infty} \zeta^i \frac{\omega^i}{i!} \\
&= \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \binom{j}{i} \beta_{j-i,\lambda} \zeta^i \right) \frac{\omega^j}{j!}. \qquad (26)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{i=0}^{\infty} \tilde{d}_{i,\lambda}^{(r)}(\zeta) \frac{[e_\lambda(\omega)-1]^i}{i!} &= \sum_{i=0}^{\infty} \tilde{d}_{i,\lambda}^{(r)}(\zeta) \sum_{j=i}^{\infty} S_{2,\lambda}(j,i) \frac{\omega^j}{j!} \\
&= \sum_{j=0}^{\infty} \left(\sum_{\ell=0}^{j} \tilde{d}_{\ell,\lambda}^{(r)}(\zeta) S_{2,\lambda}(j,\ell) \right) \frac{\omega^j}{j!}. \qquad (27)
\end{aligned}$$

By (26) and (27), we complete the proof. □

From (2), we get

$$\left(\frac{\omega}{e_\lambda(\omega)-1} \right)^r e^{\zeta\omega} = \sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \binom{j}{i} \beta_{j-i,\lambda} \zeta^i \right) \frac{\omega^j}{j!} \qquad (28)$$

Theorem 9. *For $j \geq 0$, we have*

$$\tilde{d}^{(r)}_{j,\lambda}(\zeta) = \sum_{p=0}^{j}\sum_{q=0}^{p}\binom{p}{q}\beta_{p-q,\lambda}\zeta^q S_{1,\lambda}(j,p).$$

Proof. By changing t to $\log_\lambda(1+t)$ in (28), we get

$$\begin{aligned}\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta\log_\lambda(1+\omega)} &= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}\beta_{p-q,\lambda}\zeta^q\frac{[\log_\lambda(1+\omega)]^p}{p!}\\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p}{q}\beta_{p-q,\lambda}\zeta^q\sum_{j=p}^{\infty}S_{1,\lambda}(j,p)\frac{\omega^j}{j!}\\ &= \sum_{j=0}^{\infty}\left(\sum_{p=0}^{j}\sum_{q=0}^{p}\binom{p}{q}\beta_{p-q,\lambda}\zeta^q S_{1,\lambda}(j,p)\right)\frac{\omega^j}{j!}. \end{aligned} \tag{29}$$

In view of (25) and (29), we get the result. □

Theorem 10. *For $j \geq 0$, we have*

$$\begin{aligned}&\sum_{k=0}^{j}\binom{j}{k}B_k^{(k-r+1)}\tilde{d}^{(r)}_{j-k,\lambda}(\zeta)\\ &= \sum_{i=0}^{j}\binom{j}{i}\sum_{p=0}^{i}\lambda^p\sum_{q_1+\cdots+q_r=p}\binom{p}{q_1,\cdots,q_r}\frac{1}{(q_1+1)\cdots(q_r+1)}S_1(i,p)\\ &\quad\times\sum_{k=0}^{j-i}\zeta^k S_{1,\lambda}(j-i,k).\end{aligned}$$

Proof. From (25), we see that

$$\begin{aligned}&\int_0^1\cdots\int_0^1(1+\omega)^{\lambda(\zeta_1+\cdots+\zeta_r)}d\zeta_1\cdots d\zeta_r e^{\zeta\log_\lambda(1+\omega)}\\ &= \left(\frac{\omega}{\log(1+\omega)}\right)^r\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta\log_\lambda(1+\omega)} = \sum_{k=0}^{\infty}B_k^{(k-r+1)}\frac{\omega^k}{k!}\sum_{j=0}^{\infty}\tilde{d}^{(r)}_{j,\lambda}(\zeta)\frac{\omega^j}{j!}\\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}B_k^{(k-r+1)}\tilde{d}^{(r)}_{j-k,\lambda}(\zeta)\right)\frac{\omega^j}{j!}.\end{aligned} \tag{30}$$

On the other hand, we have

$$\begin{aligned}&\int_0^1\cdots\int_0^1(1+\omega)^{\lambda(\zeta_1+\cdots+\zeta_r)}d\zeta_1\cdots d\zeta_r e^{\zeta\log_\lambda(1+\omega)}\\ &= \sum_{p=0}^{\infty}\lambda^p\frac{(\log(1+\omega))}{p!}\int_0^1\cdots\int_0^1(\zeta_1+\cdots+\zeta_r)^p d\zeta_1\cdots d\zeta_r\sum_{k=0}^{\infty}\zeta^k\frac{(\log_\lambda(1+\omega))^k}{k!}\\ &= \sum_{p=0}^{\infty}\lambda^p\sum_{q_1+\cdots+q_r=p}\binom{p}{q_1,\cdots,q_r}\frac{1}{(q_1+1)\cdots(q_r+1)}\frac{(\log(1+\omega))^p}{p!}\\ &\quad\times\sum_{k=0}^{\infty}\zeta^k\frac{[\log_\lambda(1+\omega)]^k}{k!}\end{aligned}$$

$$= \sum_{p=0}^{\infty} \lambda^p \sum_{q_1+\cdots+q_r=p} \binom{p}{q_1,\cdots,q_r} \frac{1}{(q_1+1)\cdots(q_r+1)} \sum_{i=p}^{\infty} S_1(i,p)\frac{\omega^i}{i!}$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{j} \zeta^k S_{1,\lambda}(j,k)\frac{\omega^j}{j!}$$

$$= \sum_{i=0}^{\infty}\sum_{p=0}^{i} \lambda^p \sum_{q_1+\cdots+q_r=p} \binom{p}{q_1,\cdots,q_r} \frac{1}{(q_1+1)\cdots(q_r+1)} \sum_{p=0}^{i} S_1(i,p)\frac{\zeta^i}{i!}$$

$$\times \sum_{j=0}^{\infty}\sum_{k=0}^{j} \zeta^k S_{1,\lambda}(j,k)\frac{\omega^j}{j!}$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{j} \binom{j}{i} \sum_{p=0}^{i} \lambda^p \sum_{q_1+\cdots+q_r=p} \binom{p}{q_1,\cdots,q_r} \frac{1}{(q_1+1)\cdots(q_r+1)} S_1(i,p)$$

$$\times \sum_{k=0}^{j-i} \zeta^k S_{1,\lambda}(j-i,k)\frac{\omega^j}{j!}. \tag{31}$$

Therefore, by (30) and (31), we get the result. □

Theorem 11. *The following relationship holds true*

$$\sum_{p=0}^{j}\sum_{k=0}^{j-p} \binom{j}{p} S_{1,\lambda}(q,k)\zeta^k \int_0^1 (\lambda(\zeta_1+\cdots+\zeta_r))_p d\zeta_1\cdots d\zeta_r$$

$$= \sum_{k=0}^{j} \binom{j}{k} B_k^{(k-r+1)} \tilde{d}_{j-k,\lambda}^{(r)}(\zeta).$$

Proof. From (25), we see that

$$\int_0^1 \cdots \int_0^1 (1+\omega)^{\lambda(\zeta_1+\cdots+\zeta_r)} d\zeta_1\cdots d\zeta_r e^{\zeta \log_\lambda(1+\omega)}$$

$$= \left(\frac{\omega}{\log(1+\omega)}\right)^r \left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta\log_\lambda(1+\omega)} = \sum_{k=0}^{\infty} B_k^{(k-r+1)}\frac{\omega^k}{k!}\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}^{(r)}(\zeta)\frac{\omega^j}{j!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k} B_k^{(k-r+1)} \tilde{d}_{j-k,\lambda}^{(r)}(\zeta)\right)\frac{\omega^j}{j!}. \tag{32}$$

On the other hand, we have

$$\int_0^1 \cdots \int_0^1 (1+\omega)^{\lambda(\zeta_1+\cdots+\zeta_r)} d\zeta_1\cdots d\zeta_r e^{\zeta \log_\lambda(1+\omega)}$$

$$= \sum_{p=0}^{\infty}\int_0^1\cdots\int_0^1 (\lambda(\zeta_1+\cdots+\zeta_r))_p d\zeta_1\cdots d\zeta_r \frac{\omega^p}{p!}\sum_{q=0}^{\infty}\sum_{k=0}^{q} S_{1,\lambda}(q,k)\zeta^k\frac{\omega^q}{q!}$$

$$= \sum_{j=0}^{\infty}\sum_{p=0}^{j}\sum_{k=0}^{j-p}\binom{j}{p} S_{1,\lambda}(q,k)\zeta^k \int_0^1 (\lambda(\zeta_1+\cdots+\zeta_r))_p d\zeta_1\cdots d\zeta_r \frac{\omega^j}{j!}. \tag{33}$$

Thus, by (32) and (33), we complete the proof. □

Theorem 12. *For $j \geq 0$, we have*

$$\tilde{d}^{(r)}_{j,\lambda}(\zeta) = \sum_{k=0}^{j}\binom{j}{k}\sum_{p=0}^{j-k}\sum_{q_1+\cdots+q_r=k}\binom{k}{q_1,\cdots,q_r}\tilde{d}^{(r)}_{q_1,\lambda}\cdots\tilde{d}^{(r)}_{q_r,\lambda}\zeta^p S_{1,\lambda}(j-k,p).$$

Proof. From (25), we note that

$$\sum_{j=0}^{\infty}\tilde{d}^{(r)}_{j,\lambda}(\zeta)\frac{\omega^j}{j!} = \left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta\log_\lambda(1+\omega)}$$

$$= \underbrace{\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)\times\cdots\times\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)}_{\text{r-times}}e^{\zeta\log_\lambda(1+\omega)}$$

$$= \left(\sum_{k=0}^{\infty}\sum_{q_1+\cdots+q_r=k}\binom{k}{q_1,\cdots,q_r}\tilde{d}^{(r)}_{q_1,\lambda}\cdots\tilde{d}^{(r)}_{q_r,\lambda}\frac{\omega^k}{k!}\right)$$

$$\times\left(\sum_{j=0}^{\infty}\sum_{p=0}^{j}\zeta^p S_{1,\lambda}(j,p)\frac{\omega^j}{j!}\right)$$

$$= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}\sum_{p=0}^{j-k}\sum_{q_1+\cdots+q_r=k}\binom{k}{q_1,\cdots,q_r}\tilde{d}^{(r)}_{q_1,\lambda}\cdots\tilde{d}^{(r)}_{q_r,\lambda}\zeta^p S_{1,\lambda}(j-k,p)\right)\frac{\omega^j}{j!}. \tag{34}$$

In view of (34), we complete the proof. □

Theorem 13. *For $n \geq 0$, we have*

$$\tilde{d}^{(r)}_{j,\lambda}(\zeta) = \sum_{i=0}^{j}\binom{j}{i}\sum_{p=0}^{i}\lambda^p\sum_{q_1+\cdots+q_r=p}\frac{\binom{i}{q_1,\cdots,q_r}}{(q_1+1)\cdots(q_r+1)}S_1(i+r,p+r)\frac{\binom{p+r}{r}}{\binom{i+r}{r}}$$

$$\times\sum_{k=0}^{j-i}\zeta^k S_{1,\lambda}(j-i,k).$$

Proof. By (25), we note that

$$\sum_{j=0}^{\infty}\tilde{d}^{(r)}_{j,\lambda}(\zeta)\frac{\omega^j}{j!} = \left(\frac{\log(1+\omega)}{\omega}\right)^r\int_0^1\cdots\int_0^1(1+\omega)^{\lambda(\zeta_1+\cdots+\zeta_r)}d\zeta_1\cdots d\zeta_r e^{\zeta\log_\lambda(1+\omega)}$$

$$= \left(\frac{\log(1+\omega)}{\omega}\right)^r\sum_{p=0}^{\infty}\lambda^p\frac{(\log(1+\omega))}{p!}\int_0^1\cdots\int_0^1(\zeta_1+\cdots+\zeta_r)^p d\zeta_1\cdots d\zeta_r\sum_{k=0}^{\infty}\zeta^k\frac{[\log_\lambda(1+\omega)]^k}{k!}$$

$$= \frac{1}{\omega^r}\sum_{p=0}^{\infty}\lambda^p\sum_{q_1+\cdots+q_r=p}\binom{p}{q_1,\cdots,q_r}\frac{1}{(q_1+1)\cdots(q_r+1)}\frac{(\log(1+\omega))^{p+r}}{m!}$$

$$\times\sum_{k=0}^{\infty}\zeta^k\frac{[\log_\lambda(1+\omega)]^k}{k!}$$

$$= \frac{1}{\omega^r}\sum_{p=0}^{\infty}\lambda^p\sum_{q_1+\cdots+q_r=p}\binom{p}{q_1,\cdots,q_r}\frac{1}{(q_1+1)\cdots(q_r+1)}\frac{(p+r)!}{p!}\sum_{i=p+r}^{\infty}S_1(i,p+r)\frac{\omega^i}{i!}$$

$$\times\sum_{j=0}^{\infty}\sum_{k=0}^{j}\zeta^k S_{1,\lambda}(j,k)\frac{\omega^j}{j!}$$

$$=\sum_{p=0}^{\infty}\sum_{q_1+\cdots+q_r=p}\binom{p}{q_1,\cdots,q_r}\frac{1}{(q_1+1)\cdots(q_r+1)}\frac{(p+r)!}{m!}\sum_{i=p}^{\infty}S_1(i+r,p+r)\frac{\omega^i}{(i+r)!}$$

$$\times\sum_{j=0}^{\infty}\sum_{k=0}^{j}\zeta^k S_{1,\lambda}(j,k)\frac{\omega^j}{j!}$$

$$=\sum_{i=0}^{\infty}\sum_{p=0}^{i}\lambda^p\sum_{q_1+\cdots+q_r=p}\frac{\binom{i}{q_1,\cdots,q_r}}{(q_1+1)\cdots(q_r+1)}S_1(i+r,p+r)\frac{\binom{p+r}{r}}{\binom{i+r}{r}}\frac{\omega^i}{i!}$$

$$\times\sum_{j=0}^{\infty}\sum_{k=0}^{j}\zeta^k S_{1,\lambda}(j,k)\frac{\omega^j}{j!}$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}\sum_{p=0}^{i}\lambda^p\sum_{q_1+\cdots+q_r=p}\frac{\binom{i}{q_1,\cdots,q_r}}{(q_1+1)\cdots(q_r+1)}S_1(i+r,p+r)\frac{\binom{p+r}{r}}{\binom{i+r}{r}}$$

$$\times\sum_{k=0}^{j-i}\zeta^k S_{1,\lambda}(j-i,k)\frac{\omega^j}{j!}. \tag{35}$$

Thus we get what we want. □

Theorem 14. *For $j\geq 0$, we have*

$$\tilde{d}^{(-r)}_{j,\lambda}(\zeta)=\sum_{i=0}^{j}\sum_{k=0}^{i}\binom{j}{i}d^{(r)}_{j-i,\lambda}\zeta^k S_{1,\lambda}(i,k).$$

Proof. From 25, we note that

$$\sum_{j=0}^{\infty}\tilde{d}^{(-r)}_{j,\lambda}(\zeta)\frac{\omega^j}{j!}=\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta\log_\lambda(1+\omega)}$$

$$=\left(\sum_{j=0}^{\infty}d^{(r)}_{j,\lambda}\frac{\omega^j}{j!}\right)\left(\sum_{i=0}^{\infty}\sum_{k=0}^{i}\zeta^k S_{1,\lambda}(i,k)\frac{\omega^i}{i!}\right)$$

$$=\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\sum_{k=0}^{i}\binom{j}{i}d^{(r)}_{j-i,\lambda}\zeta^k S_{1,\lambda}(i,k)\right)\frac{\omega^j}{j!}. \tag{36}$$

Therefore, by (25) and (36), we complete the proof. □

Theorem 15. *For $r,k\in\mathbb{N}$, with $r>k$, we have*

$$\tilde{d}^{(r)}_{j,\lambda}(\zeta)=\sum_{i=0}^{j}\binom{j}{i}\tilde{d}^{(r-k)}_{i,\lambda}\tilde{d}^{(k)}_{j-i,\lambda}(\zeta),(j\geq 0).$$

Proof. Since

$$\sum_{j=0}^{\infty}\tilde{d}^{(r)}_{j,\lambda}(\zeta)\frac{\omega^j}{j!}=\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta\log_\lambda(1+\omega)},$$

we have

$$=\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^{r-k}\left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^k e^{\zeta\log_\lambda(1+\omega)}$$

$$=\left(\sum_{i=0}^{\infty}\tilde{d}^{(r-k)}_{i,\lambda}\frac{\omega^i}{i!}\right)\left(\sum_{p=0}^{\infty}\tilde{d}^{(k)}_{m,\lambda}(x)\frac{\omega^p}{p!}\right)$$

$$= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\binom{j}{i}\tilde{d}_{i,\lambda}^{(r-k)}\tilde{d}_{j-i,\lambda}^{(k)}(\zeta)\right)\frac{\omega^j}{j!}. \tag{37}$$

Therefore, by (25) and (37), we obtain the result. □

Theorem 16. *For $j \geq 0$, we have*

$$\tilde{d}_{j,\lambda}^{(r)}(\zeta+\eta) = \sum_{k=0}^{j}\sum_{p=0}^{k}\binom{j}{k}\tilde{d}_{j-k,\lambda}^{(r)}(\zeta)S_{1,\lambda}(k,p)\eta^p.$$

Proof. Now, we observe that

$$\begin{aligned}\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}^{(r)}(\zeta+\eta)\frac{\omega^j}{j!} &= \left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{(\zeta+\eta)\log_\lambda(1+\omega)} \\ &= \left(\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}^{(r)}(\zeta)\frac{\omega^j}{j!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{p=0}^{k}S_{1,\lambda}(k,p)\eta^p\right)\frac{t^k}{k!}\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{p=0}^{k}\binom{j}{k}\tilde{d}_{j-k,\lambda}^{(r)}(\zeta)S_{1,\lambda}(k,p)\eta^p\right)\frac{\omega^j}{j!}.\end{aligned}$$

Equating the coefficients of $\frac{\omega^j}{j!}$ on both sides of the above, we get the result. □

4. Conclusions and Observation

Motivated by [2], we have defined fully degenerating Daehee polynomials, which turn out to be classical ones in the special cases, as follows:

$$\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}(\zeta)\frac{\omega^j}{j!} = \frac{\log(1+\omega)}{\omega}\int_0^1(1+\omega)^{\lambda\eta}d\eta\, e^{\zeta\log_\lambda(1+\omega)}.$$

By making use of this generating function, we derived some new explicit expressions and identities. Later, we have considered multiple point of view this generating function as follows:

$$\begin{aligned}\sum_{j=0}^{\infty}\tilde{d}_{j,\lambda}^{(r)}(\zeta)\frac{\omega^j}{j!} &= \left(\frac{\log(1+\omega)}{\omega}\right)^r\int_0^1\cdots\int_0^1(1+\omega)^{\lambda(\zeta_1+\cdots+\zeta_r)}d\zeta_1\cdots d\zeta_r\, e^{\zeta\log_\lambda(1+\omega)} \\ &= \left(\frac{\log_\lambda(1+\omega)}{\omega}\right)^r e^{\zeta\log_\lambda(1+\omega)},\end{aligned}$$

by virtue of which we obtained some new identities, equalities and properties. Seemingly, the new generalizations, methods and applications for special polynomials involve some known special polynomials, such as Bernoulli polynomials, Euler polynomials, Genocchi polynomials, Frobenius–Euler polynomials, Daehee polynomials, and Changee polynomials, and these will be continued because they have interesting relations in mathematical physics and statistics.

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