A New Type-3 Fuzzy Logic Approach for Chaotic Systems: Robust Learning Algorithm

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Abstract: The chaotic systems have extensive applications in various branches of engineering problems such as financial problems, image processing, secure communications, and medical problems, among many others. In most applications, a synchronization needs to be made with another favorite chaotic system, or output trajectories track the desired signal. The dynamics of these systems are complicated, they are very sensitive to the initial conditions, and they exhibit a stochastic unpredictable behavior. In this study, a new robust type-3 fuzzy logic control (T3-FLC) is designed that can be applied for a large case of chaotic systems under faulty actuators and unknown perturbed dynamics. The dynamic uncertainties are estimated by the online learned type-3 fuzzy logic systems (T3-FLSs). The rules of T3-FLS are optimized by the Lyapunov theorem. The actuator nonlinearities are identified by a new method. The effects of approximation error (AE), dynamic perturbations and unknown time-varying control gains are tackled by the designed adaptive compensator. The designed compensator is constructed by online estimation of the upper bound of AE. By several simulations and comparison with the new FLS-based controllers, the better performance of the designed T3-FLC is shown. In addition, the performance of the designed controller is examined in a secure communication system.

Keywords: type-3 fuzzy control; faulty actuator; online-learning; chaotic systems; perturbed dynamics

MSC: 37D45; 37N35; 93C42

1. Introduction

The chaotic systems have many potential applications in various science and technologies, such as optimization algorithms [1], encryption [2], feature selection [3], chaotic maps [4], control systems [5], watermarking [6], secure communications [7], and cryptography techniques [8], among many others. Due to the aforementioned extensive applications of chaotic systems, the study in this field has been one of the interesting topics recently.

The control of chaotic systems is a challenging control problem, because these systems exhibit a stochastic behavior and they are strongly sensitive to the initial conditions. Some aspects of chaotic systems have been studied, and some control techniques for various applications have been proposed. The studies in the field of control can be categorized in two classes: classic and FLS-based controllers [9,10].

The first category has been more studied. For example, in [11], the sliding mode control (SMC) is designed, and by the Lyapunov–Krasovskii approach, the stability is investigated. In [12], the response chaotic system is identified by the derive system, and then
by the Lyapunov approach, a control signal is designed. In [13], the active control scheme is developed for the control of a new 5D hyperchaotic system, and the applicability of the suggested chaotic system in cryptography problems is studied. In [14], the event-triggered control system is designed, and by simulation on the Lorenz system, the accuracy of the suggested control scheme is compared with the uniform impulsive synchronization approach. In [15], a feedback control system is suggested for the synchronization and control of the 3D chaotic plants, and its capability in both continuous and discrete time systems is investigated. In [16], the observer-based SMC is developed for chaotic systems, and by the Lyapunov–Krasovskii technique, the convergence of the synchronization error is investigated. In [17], the terminal SMC is developed, and the effect of dynamic perturbations is studied by applying on Lorenz and Chen systems. In [18], the exponential synchronization is studied, a feedback control system is designed and its proficiency is investigated considering cellular neural networks. In [19], the $H_{\infty}$ approach is proposed and the states of the response and derive systems are estimated, and by Wirtinger-based inequality, the stability is studied. The lag projective control and synchronization approach are studied in [20], and time-delay is analytically investigated. An adaptive control system by the use of the linear matrix inequalities (LMI) technique is suggested in [21], and its application in an image encryption problem is studied. In [22], the integral SMC is developed for control and synchronization under uncertainties, and it is examined on Hindmarsh–Rose and FitzHugh–Nagumo systems. In [23], the terminal SMC is designed, and various issues in this controller such as the singularity and fluctuation in the control signal is investigated. In [24], $l_2$ and $H_{\infty}$ techniques are combined to design a robust control scheme, and the effect of time-delay is studied. In [25], by the quasi-projective approach, the synchronization and control region are estimated, and the effects of input saturation are analyzed. In [26], a feedback impulsive controller is designed, and considering the Lorenz system, the effect of external disturbances is investigated. In [27], some necessary conditions are obtained for a passivity-based control scheme, and by an Arduino microcontroller, its applicability is investigated.

For the second category, FLS-based control systems have been presented. The FLSs are widely used in various problems. The FLSs are used to estimate the nonlinearities and uncertainties [28,29]. In the problem of control of chaotic systems, the dynamics of chaotic systems are estimated by FLSs [30]. For example, in [31], the dynamics are modeled by a simple FLS, and by the LMI method and bifurcation trajectories, the stability is studied. In [32], the SMC is developed using FLSs, and the reducing of the chattering problem in conventional SMC is studied. The backstepping approach is designed in [33], by the use of FLSs, and the effectiveness of using FLSs is studied by considering Rössler and Arneodo systems. In [34], based on the dynamics of the synchronization error, an FLS-based control system is introduced and the rules of FLS are adjusted to construct a desired control signal. In [35], the problem of secure communication is considered, and an $H_{\infty}$-based control technique is designed using FLSs for both synchronization and control objects. In [36], the effectiveness of FLS to deal with time-delay is studied, and an adaptive synchronization scheme is designed. In [37], a feedback controller is designed by the model estimation using FLSs, and by Chua’s system, its efficiency is examined. In [38], an FLS controller is designed for the Duffing system, and its applicability for energy management in nano air vehicles is studied. In [39], the problem of FLS-based exponential synchronization is studied, and by the LMI technique, some adjustable rules are developed. In [40], the performance improvement of a conventional terminal SMC is examined by FLSs, and by applying on a Newton–Leipnik system, its stability is investigated. In [41], the event-triggered technique is combined with $H_{\infty}$ criteria and FLSs to construct a robust chaotic controller, and its performance is examined on Chua’s system. In [42], the effectiveness of FLS-based control systems is shown by applying on Liu and Chen systems. In [43], an adaptive control is designed by an immersion and invariance scheme, and then, the stability is guaranteed by an FLS, and it is employed to control a brushless DC motor. In [44], an SMC is developed by FLSs, and its improvement versus a conventional SMC is examined on Duffing–Holmes and Jerk systems. In [45], the event-triggered scheme is improved by FLS, and some
conditions are derived to deal with faulty synchronization situations. In [46], an observer is designed for the Lorenz system by the use of FLSs and the SMC scheme, and solving of the chattering problem in conventional SMC is studied.

The high-order FLSs, specially type-2 (T2) FLSs and T3-FLSs, results in better control performance in term of accuracy and robustness against uncertainties [47,48]. However, the control of chaotic systems with T2-FLS/T3-FLS has been rarely studied. For instance, in [49], a T2-FLS is designed to estimated the uncertainties. By an unscented Kalman filter, some tuning rules are obtained, and by Lorenz and Chen systems, the superiority of T2-FLS is shown. In [50], an immersion and invariance control system is developed using T2-FLSs, and the tracking performance improvement is shown by several simulations.

In [51], an active controller is improved by the use of T2-FLS, and its better performance against T1-FLS is shown. In [52], a T2-FLS is optimized by the grey wolf technique to construct a controller for chaotic satellite systems, and its stability is studied. In [53,54], a T2-FLS-based controller is proposed by the use of $H_\infty$ criteria, and its better robustness against uncertainties in contrast to T1-FLSs is shown by applying on Duffing systems.

In [55], a T2-FLS-based controller is designed such that the parameters of T2-FLS are learned by the gradient technique, and the adaptation law is derived by the particle swarm optimization. In [56], the backstepping scheme is developed using T2-FLSs, and by considering the Duffing system, it is shown that the robustness of the conventional backstepping is improved. In [57], the synchronization and control accuracy improvement of SMC is studied using T2-FLSs and a PI controller, and by simulation on chaotic gyro systems, the effectiveness of T2-FLSs is verified. In [58], a hybrid control system is presented by SMC and T2-FLSs, and the better tracking outcome in comparison with T1-FLS is shown.

In [59], projection synchronization is developed, and the performance improvement in terms of accuracy and better control signal is investigated. In [60–62], deep learning techniques and neural networks are developed to handle the uncertainties.

In most of the aforementioned studies, only some nonlinear functions are considered to be uncertain, and the effect of actuator faulty behavior is neglected. In addition, the asymptotic stability is rarely studied. Motivated by the above discussion, in the current study, a new T3-FLS-based control scheme is presented such that in addition to the unknown dynamics, the actuator faulty behavior is also taken into account. Furthermore, unlike in existence studies, the upper bound of uncertainties is considered to be unknown, and it is approximated in an online scheme, and consequently, an adaptive compensator is constructed. By several simulations, the sufficient control and synchronization performance is shown under unknown dynamics, actuator faulty behavior and dynamic perturbations.

The basic advantages and contributions are as follows:

- The effect of the faulty actuator is taken to account.
- Unlike existence studies, the upper bound of AE is assumed to be unknown, and a new adaptation law is derived for online estimation.
- The robustness and stability of the suggested control system is proved under faulty actuators and unknown dynamics.
- The better performance of the introduced scheme is shown by comparison with new FLS-based controllers and simulation on chaotic systems.

The paper is organized as follows. In Section 2, a general overview is presented. In Section 3, the suggested T3-FLS is illustrated. In Section 4, the main results are presented. The simulations are provided in Section 5, and finally, the conclusions are given in Section 6.

2. General View

In this study, the following chaotic systems are taken into account as:

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\vdots \\
\dot{x}_n &= f(x) + \Delta f(x) + \varphi(u)
\end{align*}$$
where $x = [x_1, x_2, ..., x_n]^T$, $f(x)$ is an unknown function, $\Delta f(x)$ is a dynamic perturbation, $u$ is a controller, and $\varphi(u)$ represents the faulty actuator. General block diagrams of problem and control systems are given in Figures 1 and 2. The dynamic uncertainties are identified by the suggested adaptive T3-FLS. The actuator nonlinearities are modeled by the suggested simple adaptive models, and finally, the approximation errors (AEs) and perturbation effects are handled by the designed adaptive compensator.

![Figure 1. A general block diagram of the problem.](image1)

![Figure 2. A general block diagram for the control system.](image2)

3. Type-3 Fuzzy System

The output of the T3FLS is obtained as follows (see Figure 3).
(1) The inputs of \( r \)-th T3FLS are \( x_i, i = 1, ..., n \).

(2) The upper/lower memberships of \( \tilde{\vartheta}_j^i \) at slice \( u_s = q_k \) and \( u_s = \bar{q}_k \) are obtained as [63]:

\[
\tilde{\rho}_{\tilde{\vartheta}_j^i | u_s = \bar{q}_k} = \begin{cases} 
1 - \left( \frac{x_i - c_{\tilde{\vartheta}_j^i}}{\tilde{d}_{\tilde{\vartheta}_j^i}} \right) & \text{if } c_{\tilde{\vartheta}_j^i} - \tilde{d}_{\tilde{\vartheta}_j^i} < x_i \leq c_{\tilde{\vartheta}_j^i} \\
1 - \left( \frac{x_i - c_{\tilde{\vartheta}_j^i}}{\bar{d}_{\tilde{\vartheta}_j^i}} \right) & \text{if } c_{\tilde{\vartheta}_j^i} < x_i \leq c_{\tilde{\vartheta}_j^i} + \bar{d}_{\tilde{\vartheta}_j^i} \\
0 & \text{if } x_i > c_{\tilde{\vartheta}_j^i} + \bar{d}_{\tilde{\vartheta}_j^i} \text{ or } x_i \leq c_{\tilde{\vartheta}_j^i} - \bar{d}_{\tilde{\vartheta}_j^i}
\end{cases}
\]

(3)

\[
\bar{\rho}_{\tilde{\vartheta}_j^i | u_s = \bar{q}_k} = \begin{cases} 
1 - \left( \frac{x_i - c_{\tilde{\vartheta}_j^i}}{\tilde{d}_{\tilde{\vartheta}_j^i}} \right) & \text{if } c_{\tilde{\vartheta}_j^i} - \tilde{d}_{\tilde{\vartheta}_j^i} < x_i \leq c_{\tilde{\vartheta}_j^i} \\
1 - \left( \frac{x_i - c_{\tilde{\vartheta}_j^i}}{\bar{d}_{\tilde{\vartheta}_j^i}} \right) & \text{if } c_{\tilde{\vartheta}_j^i} < x_i \leq c_{\tilde{\vartheta}_j^i} + \bar{d}_{\tilde{\vartheta}_j^i} \\
0 & \text{if } x_i > c_{\tilde{\vartheta}_j^i} + \bar{d}_{\tilde{\vartheta}_j^i} \text{ or } x_i \leq c_{\tilde{\vartheta}_j^i} - \bar{d}_{\tilde{\vartheta}_j^i}
\end{cases}
\]

(4)
(3) The upper/lower rule memberships at slice levels $u_s = q_k$ and $u_s = q_k$ are obtained as:

$$v^l_{q_k} = p_{\tilde{\vartheta}^l_{|u_s=q_k}} \cdot p_{\tilde{\vartheta}^l_{|u_s=q_k}} \cdots p_{\tilde{\vartheta}^l_{|u_s=q_k}}$$  (6)
$$v^l_{q_k} = p_{\tilde{\vartheta}^l_{|u_s=q_k}} \cdot p_{\tilde{\vartheta}^l_{|u_s=q_k}} \cdots p_{\tilde{\vartheta}^l_{|u_s=q_k}}$$  (7)
$$v^l_{q_k} = p_{\tilde{\vartheta}^l_{|u_s=q_k}} \cdot p_{\tilde{\vartheta}^l_{|u_s=q_k}} \cdots p_{\tilde{\vartheta}^l_{|u_s=q_k}}$$  (8)
$$v^l_{q_k} = p_{\tilde{\vartheta}^l_{|u_s=q_k}} \cdot p_{\tilde{\vartheta}^l_{|u_s=q_k}} \cdots p_{\tilde{\vartheta}^l_{|u_s=q_k}}$$  (9)

where the $l$-th rule is:

- $l$-th Rule:
  if $x_1$ is $\tilde{\vartheta}^l_{p_1}$ and $x_2$ is $\tilde{\vartheta}^l_{p_2}$ and $\cdots$ $x_n$ is $\tilde{\vartheta}^l_{p_n}$
  Then, $y \in [\tilde{\omega}_l, \tilde{\omega}_l], l = 1, ..., M$

where $\tilde{\omega}_l$ and $\tilde{\omega}_l$ are rule parameters.

(4) The output is written as:

$$\hat{y} = \sum_{k=1}^{K} \left( q_k \tilde{z}_k + q_k \tilde{z}_k \right) / \sum_{k=1}^{K} (q_k + q_k)$$  (11)
where

\[
\bar{z}_k = \frac{1}{M} \left( \sum_{l=1}^{M} \left( \bar{\varphi}_{q_k}^l \bar{w}_l + \bar{\varphi}_{\bar{q}_k}^l \bar{\varphi}_{\bar{w}_l}^l \right) \right)
\]

\[
\bar{z}_k = \frac{1}{M} \left( \sum_{l=1}^{M} \left( \bar{\varphi}_{\bar{q}_k}^l \bar{w}_l + \bar{\varphi}_{\bar{q}_k}^l \bar{\varphi}_{\bar{w}_l}^l \right) \right)
\]  

(12)

\[
\dot{\hat{F}} \text{ in (11) can be written as:}
\]

\[
\dot{\hat{F}} = \bar{w}^T \psi(x)
\]

where

\[
\bar{w}^T = [\bar{w}_1, \ldots, \bar{w}_M, \bar{\varphi}_{\bar{q}_k}^l, \ldots, \bar{\varphi}_{\bar{q}_k}^l]
\]

\[
\bar{\mu}(x) = [\bar{\mu}_1, \ldots, \bar{\mu}_M, \bar{\varphi}_{\bar{q}_k}^l, \ldots, \bar{\varphi}_{\bar{q}_k}^l]
\]

(14)

4. Main Results

The main results, control signal and tuning laws are described in the following theorem.

**Theorem 1.** The system (1) is asymptotically stable if the control signal is considered as Equation (19), the compensator is considered as Equation (20) and the tuning laws are considered as Equations (21)–(25):

\[
u = \frac{1}{\pi} \left[ -KS - \hat{f}(x|w) + r^{(n)} \right] - \hat{p} \dot{u} - \hat{q} \int_0^1 u(\tau)d\tau - \lambda_0 e^{(n-1)} - \cdots - \lambda_1 \dot{e}
\]

\[
\dot{u}_1 = -\tanh(S)
\]

\[
u_c = -S \varphi
\]

\[
\dot{\hat{\varphi}} = \eta \tanh(S) \varphi(x)
\]

\[
\dot{\hat{\varphi}} = \eta \tanh(S) u
\]

\[
\dot{\hat{\varphi}} = \eta \tanh(S) \dot{u}
\]

\[
\dot{\hat{\varphi}} = \eta \tanh(S) \int_0^1 u(\tau)d\tau
\]

\[
\dot{\hat{\varphi}} = \eta \tanh(S)|\varphi(x)|
\]

(19)
where $\lambda_i$ and $K$ are positive constants, $\eta$ is between 0 and 1, the variables $a$, $\beta$ and $\gamma$ are described in (27), $S$ is defined in (29), and $\dot{v}$ is the estimated upper bound of AE that is defined in (41). Considering the suggested T3-FLS, the dynamics of (1) can be rewritten as:

$$
\dot{x}_1 = x_2 \\
\vdots \\
\dot{x}_n = \hat{f}(x|w) + \phi(u)
$$

(26)

where $\hat{f}(x|w)$ is the suggested T3-FLS and without losing generality, the unknown actuator is modeled as follows:

$$
\phi(u) = au + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau + \epsilon
$$

(27)

where $\gamma$, $\beta$ and $a$ are unknown parameters and $\epsilon$ indicates the approximation error (AE). $a$, $\beta$ and $\gamma$ are estimated as $\hat{a}$, $\hat{\beta}$ and $\hat{\gamma}$, respectively. Then, Equation (27) can be rewritten as

$$
\hat{\phi}(u) = \hat{a}u + \hat{\beta}\dot{u} + \hat{\gamma}\int_0^t u(\tau)d\tau + \epsilon
$$

(28)

Consider the following definition:

$$
S = e^{(n-1)} + \lambda_{n-1}e^{(n-2)} + \cdots + \lambda_1 e
$$

(29)

where $e = y - r$, $y$ is the system output and $r$ represents the reference signal. $\lambda_i$, $i = 1, ..., n$ are positive constants. The time-derivative of $S$ in (29) yields:

$$
\dot{S} = e^{(n)} + \lambda_{n}e^{(n-1)} + \cdots + \lambda_1 \dot{e}
$$

(30)

Considering (26), the dynamics of $e^{(n)}$ can be obtained as:

$$
e^{(n)} = y^{(n)} - r^{(n)}
$$

(31)

by replacing $\phi(u)$ from (27) into (31), $e^{(n)}$ becomes:

$$
e^{(n)} = f(x) + \Delta f(x) + au + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau + \epsilon - r^{(n)}
$$

(32)

Substituting $e^{(n)}$ form (32) into (30) results in:

$$
\dot{S} = f(x) + \Delta f(x) + \dot{\alpha}u + \hat{\beta}\dot{u} + \hat{\gamma}\int_0^t u(\tau)d\tau + \epsilon - r^{(n)} + \lambda_{n}e^{(n-1)} + \cdots + \lambda_1 \dot{e}
$$

(33)

By adding and subtracting $\hat{\alpha}u + \hat{\beta}\dot{u} + \hat{\gamma}\int_0^t u(\tau)d\tau$ into (33), one has:

$$
\dot{S} = f(x) + \Delta f(x) + \dot{\alpha}u + \hat{\beta}\dot{u} + \hat{\gamma}\int_0^t u(\tau)d\tau
$$

$$
- \hat{\alpha}u + \hat{\beta}\dot{u} + \hat{\gamma}\int_0^t u(\tau)d\tau + \epsilon - r^{(n)} + \lambda_{n}e^{(n-1)} + \cdots + \lambda_1 \dot{e}
$$

(34)

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are defined as:

$$
\hat{\alpha} = \alpha - \hat{\alpha}
$$

$$
\hat{\beta} = \beta - \hat{\beta}
$$

$$
\hat{\gamma} = \gamma - \hat{\gamma}
$$

(35)
Similarly, adding/subtracting $\dot{f}(x|w)$ into (34), one has:

\[
\dot{S} = \dot{f}(x|w) + \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau \\
- r^{(n)} + \lambda_1 e^{(n-1)} + \cdots + \lambda_1 \dot{e} \\
+ \left[ f(x) + \Delta f(x) - \dot{f}(x|w) \right] \\
+ \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau + \epsilon
\]  

(36)

Now, considering the T3-FLS with optimal parameters $\hat{f}^*(x|w^*)$, $\dot{S}$ becomes:

\[
\dot{S} = \dot{f}(x|w) + \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau \\
- r^{(n)} + \lambda_1 e^{(n-1)} + \cdots + \lambda_1 \dot{e} \\
+ \hat{f}^*(x|w^*) - \dot{f}(x|w) \\
+ \left[ f(x) + \Delta f(x) - \hat{f}^*(x|w^*) \right] \\
+ \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau + \epsilon
\]  

(37)

Considering Equation (14), the term $\hat{f}^*(x|w^*) - \dot{f}(x|w)$ in (37) can be written as:

\[
\hat{f}^*(x|w^*) - \dot{f}(x|w) = \tilde{w}^T \psi(x)
\]  

(38)

where $\tilde{w}$ is defined as

\[
\tilde{w} = w^* - w
\]  

(39)

Then, form (38)–(39), $\dot{S}$ in (37) becomes:

\[
\dot{S} = \dot{f}(x|w) + \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau \\
- r^{(n)} + \lambda_1 e^{(n-1)} + \cdots + \lambda_1 \dot{e} + \tilde{w}^T \psi(x) \\
+ \left[ f(x) + \Delta f(x) - \hat{f}^*(x|w^*) \right] \\
+ \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau + \epsilon
\]  

(40)

Consider the general AE as:

\[
v = \left[ f(x) + \Delta f(x) - \hat{f}^*(x|w^*) \right] \\
+ \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau + \epsilon
\]  

(41)

Then, (40) is simplified as:

\[
\dot{S} = \dot{f}(x|w) + \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau \\
- r^{(n)} + \lambda_1 e^{(n-1)} + \cdots + \lambda_1 \dot{e} + \tilde{w}^T \psi(x) + v \\
+ \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau
\]  

(42)

By applying the control signal (19), Equation (42) becomes:

\[
\dot{S} = -KS + \tilde{w}^T \psi(x) + v \\
+ \dot{\alpha} u + \beta \dot{u} + \gamma \int_0^t u(\tau)d\tau \\
- |S| \tanh(S) + u_1 + u_c
\]  

(43)

Consider the Lyapunov function (44),

\[
V = \frac{1}{2\eta} \tilde{w}^T \tilde{w} + \frac{1}{2\eta} \lambda_1^2 + \frac{1}{2\eta} \beta^2 + \frac{1}{2\eta} \gamma^2 \\
+ \int_0^t \tanh(\tau) + \frac{1}{2} u_1^2 + \frac{1}{2} \dot{u}_1^2
\]  

(44)
where $\eta$ is a positive constant, $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are defined in (35), $\hat{w}$ is defined in (39) and $\hat{v}$ is defined as:

$$\hat{v} = v - \hat{v}$$  \hspace{1cm} (45)

where $\hat{v}$ is the estimation of $v$, and $v$ represents the upper bound of $AE$ $v$ that is defined in (41). It is assumed that:

$$\nu \geq |v|$$  \hspace{1cm} (46)

The time derivative of $V$ in (44), yields:

$$V = -\frac{1}{\eta} \hat{w}^T \hat{w} - \frac{1}{\eta} \hat{\alpha} \hat{\alpha} - \frac{1}{\eta} \hat{\beta} \hat{\beta} - \frac{1}{\eta} \hat{\gamma} \hat{\gamma} + \tanh(S) \hat{S} + u_1 \hat{u}_1 - \frac{1}{\eta} \hat{\nu} \hat{\nu}$$  \hspace{1cm} (47)

Substituting $\hat{S}$ from (43) into (47) results in:

$$V = -\frac{1}{\eta} \hat{w}^T \hat{w} - \frac{1}{\eta} \hat{\alpha} \hat{\alpha} - \frac{1}{\eta} \hat{\beta} \hat{\beta} - \frac{1}{\eta} \hat{\gamma} \hat{\gamma} + \tanh(S) \hat{S} + u_1 \hat{u}_1 - \frac{1}{\eta} \hat{\nu} \hat{\nu} + \frac{\alpha}{\eta} \nu + \frac{\beta}{\eta} \nu + \frac{\gamma}{\eta} \int_0^\tau u(\tau) d\tau - |S| \tanh(S) + u_1 + u_c$$  \hspace{1cm} (48)

Equation (48) is rewritten as:

$$V = -KS \tanh(S) + \hat{w}^T \left[ \tanh(S) \psi(x) - \frac{1}{\eta} \hat{w} \right] + \frac{\alpha}{\eta} \nu + \frac{\beta}{\eta} \nu + \frac{\gamma}{\eta} \int_0^\tau u(\tau) d\tau - |S| \tanh(S) + u_1 + u_c \tanh(S) + u_1 + u_c$$  \hspace{1cm} (49)

Substituting the tuning laws (21–24) into (49) results in:

$$V = -KS \tanh(S) - \tanh^2(S) |S| + \tanh(S) u_c + \tanh(S) v - \frac{1}{\eta} \hat{\nu} \hat{\nu}$$  \hspace{1cm} (50)

Equation (50) can be rewritten as:

$$V \leq -K |S| - \tanh^2(S) |S| + \tanh(S) u_c + |\tanh(S)| |v| - \frac{1}{\eta} \hat{\nu} \hat{\nu}$$  \hspace{1cm} (51)

From (46) and (51), one has:

$$V \leq -K |S| - \tanh^2(S) |S| + \tanh(S) u_c + |\tanh(S)| |v| - \frac{1}{\eta} \hat{\nu} \hat{\nu}$$  \hspace{1cm} (52)

By adding and subtracting $|\tanh(S)| \hat{\nu}$ into (52) and considering $\hat{\nu}$ from (45), one has:

$$V \leq -K |S| - \tanh^2(S) |S| + \tanh(S) u_c + \hat{\nu} \left[ |\tanh(S)| - \frac{1}{\eta} \nu \right] + |\tanh(S)| \hat{\nu}$$  \hspace{1cm} (53)

Substituting the tuning rule form (25) results in:

$$V \leq -K |S| - \tanh^2(S) |S| + \tanh(S) u_c + |\tanh(S)| \hat{\nu}$$  \hspace{1cm} (54)
From the compensator (20), one has:

\[ \dot{V} \leq -K|S| - \tanh^2(S)|S| \quad (55) \]

From (55) and considering the fact that:

\[ - \int_0^t \dot{V}(\tau)d\tau = V(0) - V(t) < \infty \quad (56) \]

It can be written:

\[ \int_0^t K|S| + \tanh^2(S)|S| < \infty \quad (57) \]

From (57), it can be derived that \( \int_0^t |S|^2 < \infty \). Then, \( S \in \ell^2 \) and considering the Barbalat’s lemma, the proving of asymptotic stability is completed.

5. Simulations

In this section, some of the most useful chaotic systems are considered to investigate the control performance of the schemed approach. For each example, we have the following main figures: tracking performance, tracking error, control signal, and finally, phase portrait. In the figure of tracking performance, both reference and output signals are depicted to show how the output signal tracks the reference signal. In the figure of tracking error, the error signal is depicted to show the performance of tracking and also to show the stability. In the figure of the control signal, the control signal is depicted to show the feasibility of implementation. Finally, in the figure of the phase portrait, all states are depicted simultaneously in one figure to have a general overview of the tracking.

Example 1. For the first example, the following Chua’s system is considered:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \Delta f + \frac{14}{1805} x_1 - \frac{168}{9025} x_2 + \frac{1}{58} x_3 \\
&
\quad - \frac{2}{53} \left( \frac{23}{501} x_1 + \frac{7}{58} x_2 + x_3 \right)^3 + \varphi(u)
\end{align*}
\tag{58}
\]

where the simulation parameters are given in Table 1. The dynamic perturbation is considered to be:

\[ \Delta f = 0.01 \sin(7t) + 0.03 \cos(11t) \tag{59} \]

The behavior of the faulty actuator is considered to be:

\[ \varphi(u) = \begin{cases} 
(u - 2)(1 - 0.3 \sin(u)) & u > 2 \\
0 & -2 < u < 2 \\
(u + 2)(0.9 - 0.4 \sin(u)) & u < -2 
\end{cases} \tag{60} \]

The rules are written as:

\[
\begin{align*}
\text{if } x_1 &\text{ is } \bar{A}_1^1 \text{ and } x_2 &\text{ is } \bar{A}_2^1 \text{ and } x_3 &\text{ is } \bar{A}_3^1, \text{ Then } y &\in [\bar{w}_1, \bar{w}_1] \\
\text{if } x_1 &\text{ is } \bar{A}_1^2 \text{ and } x_2 &\text{ is } \bar{A}_2^2 \text{ and } x_3 &\text{ is } \bar{A}_3^2, \text{ Then } y &\in [\bar{w}_2, \bar{w}_2] \\
\text{if } x_1 &\text{ is } \bar{A}_1^3 \text{ and } x_2 &\text{ is } \bar{A}_2^3 \text{ and } x_3 &\text{ is } \bar{A}_3^3, \text{ Then } y &\in [\bar{w}_3, \bar{w}_3] \\
\text{if } x_1 &\text{ is } \bar{A}_1^4 \text{ and } x_2 &\text{ is } \bar{A}_2^4 \text{ and } x_3 &\text{ is } \bar{A}_3^4, \text{ Then } y &\in [\bar{w}_4, \bar{w}_4] \\
\text{if } x_1 &\text{ is } \bar{A}_1^5 \text{ and } x_2 &\text{ is } \bar{A}_2^5 \text{ and } x_3 &\text{ is } \bar{A}_3^5, \text{ Then } y &\in [\bar{w}_5, \bar{w}_5] \\
\text{if } x_1 &\text{ is } \bar{A}_1^6 \text{ and } x_2 &\text{ is } \bar{A}_2^6 \text{ and } x_3 &\text{ is } \bar{A}_3^6, \text{ Then } y &\in [\bar{w}_6, \bar{w}_6] \\
\text{if } x_1 &\text{ is } \bar{A}_1^7 \text{ and } x_2 &\text{ is } \bar{A}_2^7 \text{ and } x_3 &\text{ is } \bar{A}_3^7, \text{ Then } y &\in [\bar{w}_7, \bar{w}_7] \\
\text{if } x_1 &\text{ is } \bar{A}_1^8 \text{ and } x_2 &\text{ is } \bar{A}_2^8 \text{ and } x_3 &\text{ is } \bar{A}_3^8, \text{ Then } y &\in [\bar{w}_8, \bar{w}_8] 
\end{align*}
\tag{61} \]
Table 1. Example 1: the simulation parameters.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(0)$</td>
<td>0.2, 0.5, 0.3</td>
</tr>
<tr>
<td>$c_\tilde{\varphi}$</td>
<td>−100, 0, 100</td>
</tr>
<tr>
<td>$d_\tilde{\varphi}$</td>
<td>100, 100, 100</td>
</tr>
<tr>
<td>$d_\tilde{\varphi}$</td>
<td>50, 50, 50</td>
</tr>
<tr>
<td>$K$</td>
<td>100</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>30, 300, 1000</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

The trajectory of the output signal $y$ is shown in Figure 5. We see that the output $y$ is converged to the target signal $r$ in a desired finite time. The tracking error $e$ is depicted in Figure 6, and the control signal is given in Figure 7. One can see a bounded control signal with no un-implementable fluctuations. The phase trajectory is depicted in Figure 8. Figure 8 shows that the output signals move along the target path with a desired accuracy.

Figure 5. Example 1: the tracking performance.

Figure 6. Example 1: tracking error.

Figure 7. Example 1: control signal.
Example 2. For the second example, the following Genesio system is considered:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \Delta f - 6x_1 - 2.9x_2 - 1.2x_3 + x_1^2 + \varphi(u)
\end{align*}
\]  

(62)

where the simulation condition and control parameters are the same as in Example 1. The initial conditions are considered as \(x_1(0) = 3.9, x_2(0) = -2.9\) and \(x_3(0) = 2.9\). The reference signal is considered to be the output of the Genesio system with different initial conditions as \(x_1(0) = 0.4, x_2(0) = 0.6\) and \(x_3(0) = 0.3\). The dynamic perturbation is considered as (59). The actuator fault is considered to be a Gaussian noise by variance 0.15 and zero mean that is added on control signal \(u\). The output trajectory is given in Figure 9, the error \(e\) is depicted in Figure 10 and the control signal is given in Figure 11. The phase trajectory is shown in Figure 12. It is known that the tracking error trajectory approaches the zero level at a very finite time, and a good synchronization is achieved between the control plant and target system. It should be noted that the Genesio system is a chaotic system. It is very sensitive to the initial condition. By changing the initial condition, the behavior of the output signal is completely changed. As seen from Figure 12, the trajectories of the output of the Genesio system with different initial conditions are completely different. Figure 12 shows that the suggested control system could establish a strong synchronization in spite of perturbed dynamics and considering the reference system as a chaotic system.

Figure 8. Example 1: phase trajectory, after and before control.

Figure 9. Example 2: The tracking performance.
The synchronization problem is widely used in secure communication application. As shown in Figure 13, the information signal in the transmitter side is encrypted by a chaotic system and in the receiver side, by an accurate synchronization, the information is extracted. For further examination, the synchronization performance is examined in a secure communication application. The input message signal, encrypted signal and the extracted signal in the receiver side are depicted in Figure 14. We see that the input message is well hidden in the chaotic system and the output signal is extracted with desirable accuracy.
Example 3. For the third examination, the following system is considered:

\[
\begin{align*}
x_1 &= x_2 \\
x_2 &= \Delta f - 0.5x_1 - 0.12x_2 - \sin(x_1)
\end{align*}
\]

(63)

where \( \Delta f \) is:

\[
\Delta f = -0.0187 \cos(0.75t) - 0.027 \sin(0.75t) + \sin(0.3 \cos(0.75t) - 3) - 1.5
\]

(64)

The other simulation conditions are the same as in Example 1. The initial conditions are considered as \( x_1(0) = 3, x_2(0) = -4 \) and \( x_3(0) = 2 \). The reference signal is considered to be the output of the Genesio system with different initial conditions as \( x_1(0) = 0.3, x_2(0) = 0.5 \) and \( x_3(0) = 0.2 \). The actuator fault is the same as in Example 1. The output control result is shown in Figure 15, the error \( e \) is depicted in Figure 16, and the control signal is given in Figure 17. Figures 15 and 16 show that under unknown dynamics, actuator fault and dynamic perturbation, a strong tracking performance is achieved. It is seen that by the suggested controller, the output signal \( y \) well converged to the reference signal \( r \) at a finite time. The phase trajectory in two cases, after and before control, is depicted in Figure 18. One can see that the under control system accurately follows the target path. It should be noted that the target path is a chaotic path.

Figure 13. Example 2: Secure communication application.

Figure 14. Example 2: Synchronization performance in a secure communication application.

Figure 15.
Example 4. A comparison with new FLCs is given to better show the superior efficiency of the introduced T3-FLC. The tracking capability of T3-FLC is compared with type-1 FLC (T1-FLC) [64], EKF-neural network combined by T2-FLs (EN-T2-FLC) [65], feedback linearization by T2-FLS (FL-T2-FLS) [66] and SMC by T2-FLS (T2-SMC) [67]. The comparison results in terms of root-mean-square-errors (RMSEs) are given in Table 2. It can be observed that the accuracy of the suggested T3-FLC is better than other fuzzy-based controllers.
Table 2. Example 4: Comparison of RMSE.

<table>
<thead>
<tr>
<th>Controller</th>
<th>Example 1</th>
<th>Example 1</th>
<th>Example 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed</td>
<td>0.0101</td>
<td>0.1190</td>
<td>0.0744</td>
</tr>
<tr>
<td>T1-FLC</td>
<td>0.1701</td>
<td>0.6343</td>
<td>0.2442</td>
</tr>
<tr>
<td>T2-SMC</td>
<td>0.02314</td>
<td>0.2124</td>
<td>0.1031</td>
</tr>
<tr>
<td>EN-T2-FLC</td>
<td>0.0541</td>
<td>0.3347</td>
<td>0.1571</td>
</tr>
<tr>
<td>FL-T2-FLS</td>
<td>0.0813</td>
<td>0.3742</td>
<td>0.1801</td>
</tr>
</tbody>
</table>

The RMSE is computed as:

$$\text{RMSE} = \sqrt{\frac{1}{N} \sum_{k=1}^{N} e^2(k)}$$

where $e$ is the tracking error, and $N$ is the number of samples.

6. Conclusions

In this study, a case of chaotic systems is considered and a new T3-FLC is designed. A new T3-MF is presented with uncertain FOU. Unlike the existing studies, the actuator behavior is not ideal, but it is considered to be faulty. In other words, an unknown time-varying control gain is considered. In addition, the system dynamics are fully unknown and are disturbed by bounded perturbations. A new adaptive compensator is proposed to tackle the effects of AE, actuator fault and dynamic perturbations. In four examples, the performance is examined. In the first example, the suggested method is applied on Chua’s system. In the second example, the Genesio system is considered as the under control plant, and the actuator fault is considered to be a noise by variance 0.15 and zero mean. In example 3, two non-identical chaotic systems are considered as master–slave systems. Finally, in the last example, the performance of the suggested technique is compared with other similar methods. It is shown that by T3-FLC, the under control chaotic system accurately tracks the target plant even if the target plant exhibits stochastic chaotic behavior. In addition to unknown dynamics and disturbances, a white noise, bounded sinusoidal perturbation and dead-zone perturbation are also added to the control signal as the actuator fault. It is shown that the suggested controller tackles the effects of perturbations and unknown dynamics well. Finally, the high capability of T3-FLC is examined by a comparison with some new fuzzy-based controllers such as type-1 FLC, EKF-neural network combined by T2-FLs, feedback linearization by T2-FLS and SMC by T2-FLS. The main limitation is that the MFs of T3-FLC are not optimized. For future studies, in addition to rules, the MFs of T3-FLC can also be optimized by an appropriate approach.


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