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**Abstract:** We give a family of counterexamples of a theorem on a new upper bound for the α-indices of graphs in the paper that is mentioned in the title. We also provide a new upper bound for corrigendum.

**Keywords:** nonnegative matrix; graph; index; α-index

**MSC:** 05C50; 15A42

**The Statement, Counterexamples, and Corrigendum**

Let $C$ be an $n \times n$ real symmetric matrix. The index of $C$, denoted by $\rho(C)$, is the largest eigenvalue of $C$. Let $G = (V, E)$ be a connected graph of order $n = |V|$ and size $m = |E|$ with adjacency matrix $A(G)$ and diagonal matrix $D(G)$ of degree sequence. Nikiforov [1] proposed the following matrix:

$$A_{\alpha}(G) = \alpha D(G) + (1 - \alpha) A(G),$$

where $0 \leq \alpha \leq 1$. The α-index of $G$, denoted by $\rho_{\alpha}(G)$, is the index of $A_{\alpha}(G)$. E. Lenes, E. Mallea-Zepeda, and J. Rodríguez [2] (Theorem 4) gave the following upper bound for $\rho_{\alpha}(G)$.

$$\rho_{\alpha}(G) \leq \delta - 1 + \alpha + \sqrt{(\delta + 1 - \alpha)^2 + 4(2m - \delta)(1 - \alpha)},$$

where $\delta$ is the minimum degree of $G$.

The upper bound of $\rho_{\alpha}(G)$ in (1) is not true by the following family of counterexamples.

**Example 1.** It was shown in [1] that the α-index of the star graph $K_{1, n-1}$ is

$$\rho_{\alpha}(K_{1, n-1}) = \frac{1}{2} \left( \alpha n + \sqrt{\alpha^2 n^2 + 4(n - 1)(1 - 2\alpha)} \right).$$

Suppose $\alpha \neq 0$. Then

$$\lim_{n \to \infty} \frac{\rho_{\alpha}(K_{1, n-1})}{an} = 1.$$  \hspace{1cm} (2)

Applying $\delta = 1$ for $K_{1, n-1}$ with $n \geq 2$ in (1), we find

$$\rho_{\alpha}(K_{1, n-1}) \leq \frac{1}{2} \left( \alpha + \sqrt{(2 - \alpha)^2 + 4(n - 2)(1 - \alpha)} \right) \sim n^{\frac{1}{2}}.$$

Hence

$$\lim_{n \to \infty} \frac{\rho_{\alpha}(K_{1, n-1})}{an} = 0.$$
a contradiction to (2).

We follow the idea of the proof of the inequality (1) in [2] and give the following corrected version.

**Theorem 1.** Let $G$ be a connected graph of order $n$ and size $m$ with maximum degree $\Delta$ and minimum degree $\delta$. Then

$$\rho_\alpha(G) \leq \frac{\alpha\Delta + (1-\alpha)(\delta - 1) + \sqrt{(\alpha\Delta + (1-\alpha)(\delta - 1))^2 + 4(1-\alpha)(2m - (n-1)\delta)}}{2}$$

for $0 \leq \alpha < 1$. Equality holds if and only if $G$ is regular, or $\alpha = 0$ and every vertex in $G$ has degree $n-1$ or $\delta$.

**Proof.** Let $G$ have the degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, and $r_i(C)$ denote the $i$-th row sum of an $n \times n$ matrix $C$. Note that $r_i(A_\alpha(G)) = ad_i + (1-\alpha)d_i = d_i$, and

$$r_i(A_\alpha(G)^2) = ad_i^2 + (1-\alpha) \sum_{j \in E} d_j = ad_i^2 + (1-\alpha)(2m - d_i - \sum_{j \notin E \setminus i} d_j)$$

$$\leq \alpha\Delta d_i + (1-\alpha)(2m - d_i - (n-d_i-1)\delta)$$

$$= (\alpha\Delta + (1-\alpha)(\delta - 1))d_i + (1-\alpha)(2m - (n-1)\delta).$$

Therefore, for $1 \leq i \leq n$,

$$r_i(A_\alpha(G)^2) - (\alpha\Delta + (1-\alpha)(\delta - 1))A_\alpha(G) \leq (1-\alpha)(2m - (n-1)\delta).$$

(4)

Note that $A_\alpha^2(G) - (\alpha\Delta + (1-\alpha)(\delta - 1))A_\alpha(G)$ has eigenvalue $\rho_\alpha^2(G) - (\alpha\Delta + (1-\alpha)(\delta - 1))\rho_\alpha(G)$ associated with a nonnegative eigenvector which is also a $\rho_\alpha(G)$ eigenvector of $A_\alpha(G)$. By [3],

$$\rho_\alpha^2(G) - (\alpha\Delta + (1-\alpha)(\delta - 1))\rho_\alpha(G) \leq (1-\alpha)(2m - (n-1)\delta),$$

with equality if, and only if, the equality in (4) (or equivalently in (3)) holds for every $1 \leq i \leq n$. Solving the above quadratic inequality of $\rho_\alpha(G)$ and studying the equality, the theorem follows. $\square$

Theorem 1 is a generalization of a result of Hong, Shu, and Fang [4]. It is worth mentioning that many different upper bounds of $\rho_\alpha(G)$ are already given in [5–7].

**Remark 1.** If we give an additional assumption in Theorem 1

$$t := \min_{i \in V} \sum_{j \notin E \setminus i} (d_j - \delta),$$

then with little modification of the proof in line (3), we have

$$\rho_\alpha(G) \leq \frac{\alpha\Delta + (1-\alpha)(\delta - 1) + \sqrt{(\alpha\Delta + (1-\alpha)(\delta - 1))^2 + 4(1-\alpha)(2m - (n-1)\delta - 1)}}{2}.$$  

The above equality holds if, and only if, (i) $G$ is regular, or (ii) $\alpha = 0$ and

$$t = \sum_{j \notin E \setminus i} (d_j - \delta)$$

for $i \in V$.  

(5)

Theorem 1 is a special case of Remark 1 with $t = 0$. The following is a graph that satisfies (5) with $\delta = 2$ and $t = 1$. It is of independent interest to find all graphs that satisfy (5).
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