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New Z-Eigenvalue Localization Set for Tensor and Its Application in Entanglement of Multipartite Quantum States

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Abstract: This study focuses on tensor Z-eigenvalue localization and its application in the geometric measure of entanglement for multipartite quantum states. A new Z-eigenvalue localization theorem and the bounds for the Z-spectral radius are derived, which are more precise than some of the existing results. On the other hand, we present theoretical bounds of the geometric measure of entanglement for a weakly symmetric multipartite quantum state with non-negative amplitudes by virtue of different distance measures. Numerical examples show that these conclusions are superior to the existing results in quantum physics in some cases.

Keywords: Z-eigenvalue; non-negative tensors; spectral radius; geometric measure of entanglement

MSC: 15A18; 15A69; 15A21



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1. Introduction

Tensors, namely multidimensional arrays, have become more and more important in many different fields of applied mathematics and computational mathematics, and they have promoted the development of numerical multilinear algebra. They have a very rich diversity in practical applications, especially in the positive definiteness of even-order multivariate forms in automatical control [1], higher-order statistics [2,3] and multiple dimensional data analysis [4,5].

Tensor eigenvalues are widely used in a large amount of scientific and engineering problems. However, the calculation of the Z-eigenvalue of a higher-order tensor is usually NP hard, which is different from the case of matrices. Nevertheless, there are some algorithms for calculating one or more eigenvalues of tensors, such as [6–12]. Unfortunately, these methods do not work well in larger-sized tensors, even on a medium scale. In this situation, the eigenvalue localization methods can capture all eigenvalues of a high-order tensor in a certain interval. For example, Geršgorin and Brauer-type tensor eigenvalue inclusion sets are introduced in [13]. Therefore, eigenvalue localization is one of the important methods to investigate the spectral radius of higher-order tensors.

Entanglements in composite systems are a basic and important feature of quantum physics and the core resource of the field of quantum information science [14], but it has been proved difficult to quantify. It makes all the difference that we know whether a quantum state is entangled or not in many practical applications [15]. There are many elegant entanglement criteria, such as the Bell inequality [16], entanglement witness [17] and the positive partial transposition (PPT) criterion [18,19]. However, in the case of multipartite systems, the situation is substantially more complicated. The geometric measure of entanglement has become the most basic method for measuring the entanglement of multipartite systems, which is proposed to Shimony [20] for bipartite systems and extended to multipartite systems by Wei and Goldbart [21]. Despite its significance, the explicit value of the

geometric measures of entanglement can be derived for only a few entangled states, such as generalized W states [22], Dicke states and m -qubit GHZ states [21]. Since the definition involves the optimization process of all separable state sets, it is still impossible to obtain geometric measures for most of the multipartite states, which represents a formidable task in the general case even with numerical approaches.

The geometric measure of entanglement can be attributed to the spectral radius of the normalized tensors [11] from a mathematical perspective. Recently, the authors in [23] indicate that the maximal overlap of the state Ψ with a pure separable state is equivalent to the Z -spectral radius of symmetric nonnegative tensor \mathcal{A}_Ψ ; that is, the geometric measure of entanglement is derived. Moreover, these conclusions are generalized to the weakly symmetric non-negative tensor case [24]. On the other hand, based on Bures distance, the authors in [25] propose an upper bound for a maximally geometric measure of entanglement for an m -partite system composed of subsystems of dimensions d_1, \dots, d_m .

Highly entangled multipartite states are very important in the fields of quantum information processing, quantum error correction and quantum communication, especially in the exponential acceleration of quantum algorithms; for details, see [26–28]. However, the authors in [29] argue that the entanglement in symmetric case is much smaller than in the general case, and most symmetric quantum states are close to being maximally entangled. They also present the upper bound of the maximal possible geometric measure of entanglement for Boson quantum states. On this basis, an upper bound for the entanglement is derived in [27].

In this literature, we focus on the Z -eigenvalue localization set for a tensor and its application in the geometric measure of entanglement for multipartite quantum states, which is beneficial to the cross development of tensor theory and quantum information. A new Z -eigenvalue localization theorem and the bounds for the Z -spectral radius are derived, which is more precise than some of the existing results. As applications, we are devoted to the geometric measure of entanglement on the ground of tensor Z -eigenvalue theory. Based on different distance measures, we present theoretical bounds of the geometric measure of entanglement for a weakly symmetric pure state with non-negative amplitudes. Numerical examples show that our bounds are more precise than some existing conclusions in quantum physics.

2. Preliminaries

2.1. Preliminaries for Tensors

For a positive integer $n \in \mathbb{N}$, we denote $[n]$ by the set of positive integers $\{1, \dots, n\}$. An m th-order n -dimensional real tensor denoted by

$$\mathcal{A} := (a_{i_1 \dots i_m}) \in \mathbb{R}^{n_1 \times \dots \times n_m}$$

is a multidimensional array consisting of n^m numbers $a_{i_1 \dots i_m} \in \mathbb{R}$ for all $i_j \in [n_j]$ and $j \in [m]$. A symmetric tensor is a square tensor, that is $i_k = n, k = 1, \dots, m$, if its entries $a_{i_1 \dots i_m}$ are invariant under any permutation of m indices (i_1, i_2, \dots, i_m) , which are denoted as $S^m(\mathbb{R}^n)$. We use $\mathbb{R}^{[m \times n]} (\mathbb{C}^{[m \times n]})$ to represent the set of all m -order n -dimensional real (complex) tensors. For a tensor $\mathcal{A} \in \mathbb{R}^{[m \times n]}$, \mathcal{A} is non-negative (positive) if every entry $a_{i_1 i_2 \dots i_m} \geq (>) 0$. $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{[m \times n]}$ is weakly symmetric [30] if the associated homogeneous polynomial

$$\mathcal{A}x^m = \sum_{i_1, i_2, \dots, i_m=1}^n a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}$$

satisfies $\nabla \mathcal{A}x^m = m \mathcal{A}x^{m-1}$, where ∇ denotes the gradient of the associated multivariable function and $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, where $\mathcal{A}x^{m-1}$ is an n dimension vector in \mathbb{C}^n , whose i th component is

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, i_3, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m}.$$

It is worth noting that a symmetric tensor must be a weakly symmetric tensor but not vice versa. In a word, some conclusions for weakly symmetric tensors are applicable for symmetric tensors.

Definition 1 ([31,32]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be an m -order n -dimensional real tensor. If there is a real number λ and a nonzero real vector x such that

$$\mathcal{A}x^{m-1} = \lambda x, \quad x^\top x = 1,$$

where $\mathcal{A}x^{m-1}$ is an n -dimension vector in \mathbb{R}^n , whose i th component is

$$(\mathcal{A}x^{m-1})_i := \sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m}, \quad \forall i \in [n].$$

Then, we say that λ is an Z -eigenvalue of \mathcal{A} and x is an Z -eigenvector of \mathcal{A} associated with λ .

We denote the Z -spectrum of tensor \mathcal{A} by $\sigma(\mathcal{A})$: that is the set of all Z -eigenvalues of \mathcal{A} . The Z -spectrum radius of \mathcal{A} is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda|; \lambda \in \sigma(\mathcal{A})\}.$$

For non-negative tensor \mathcal{A} , the authors in [30] imply that the Z -spectrum radius $\rho(\mathcal{A})$ is a Z -eigenvalue of \mathcal{A} if \mathcal{A} is weakly symmetric.

Geršgorin and Brauer-type tensor eigenvalues inclusion sets are introduced in [13].

Theorem 1 ([13]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{C}^{[m \times n]}$ be an m -order n -dimension tensor. It follows that

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i \in N} \mathcal{K}_i(\mathcal{A}),$$

where $\mathcal{K}_i(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq R_i(\mathcal{A})\}$, $R_i(\mathcal{A}) = \sum_{i_2, \dots, i_m} |a_{ii_2 \dots i_m}|$.

Theorem 2 ([13]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{C}^{[m \times n]}$ be an m -order n -dimension tensor. It follows that

$$\sigma(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) = \bigcup_{i, j \in N, i \neq j} \mathcal{N}_{i,j}(\mathcal{A}),$$

where

$$\mathcal{N}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : [|z| - (R_i(\mathcal{A}) - P_i^j(\mathcal{A}))] |z| \leq P_i^j(\mathcal{A}) R_j(\mathcal{A}), \quad P_i^j(\mathcal{A}) = \sum_{\substack{i_2 \dots i_m \in N, \\ i \notin \{i_2, \dots, i_m\}}} |a_{ii_2 \dots i_m}|.$$

2.2. Tensor Representation of Quantum States

For a composite m -partite quantum system, an m -partite pure state $|\Psi\rangle$ can be interpreted as a normalized element in a tensor product Hilbert space $\mathcal{H} = \otimes_{k=1}^m \mathcal{H}_k = \otimes_{k=1}^m \mathbb{R}^{n_k}$, where the dimension of \mathcal{H}_k is n_k . We suppose that $\left\{ \left| e_{i_k}^{(k)} \right\rangle : i_k \in [n_k] \right\}$ is an orthogonal basis of \mathbb{R}^{n_k} , which yields

$$\left\{ \left| e_{i_1}^{(1)} e_{i_2}^{(2)} \cdots e_{i_m}^{(m)} \right\rangle : i_k \in [n_k]; k \in [m] \right\},$$

that is also an orthogonal basis of $\otimes_{k=1}^m \mathbb{R}^{n_k}$. In this expression, $|\Psi\rangle$ can be regarded as

$$|\Psi\rangle := \sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} a_{i_1 \dots i_m} |e_{i_1}^{(1)} e_{i_2}^{(2)} \dots e_{i_m}^{(m)}\rangle,$$

where $a_{i_1 \dots i_m} \in \mathbb{R}$. Under the orthogonal basis, the quantum state $|\Psi\rangle$ has a corresponding tensor representation denoted by

$$\mathcal{A}_{|\Psi\rangle} := (a_{i_1 \dots i_m}).$$

In this sense, a weakly symmetric pure state always has a corresponding weakly symmetric tensor. A separable m -partite pure state can be considered as a product state

$$|\phi\rangle := \otimes_{k=1}^m |\phi^{(k)}\rangle, \quad |\phi^{(k)}\rangle = \sum u_{i_k}^{(k)} |e_{i_k}^{(k)}\rangle \in \mathcal{H}_k, \quad \|\phi^{(k)}\| = 1, \quad k = 1, \dots, m.$$

We denote the set of all separable pure states in \mathcal{H} by $Separ(\mathcal{H})$. We call the state an entangled state if it is inseparable.

The Hilbert–Schmidt distance is the Hilbert–Schmidt norm, such as trace operators and Hilbert–Schmidt operators ($\|\cdot\|_2$). Based on Hilbert–Schmidt distance, the geometric measure of entanglement for multipartite pure states $|\Psi\rangle$ is defined as

$$GME_{\Psi} \triangleq \min\{\|\Psi - |\Phi\rangle\| : |\Phi\rangle = \otimes_{k=1}^m |\phi^{(k)}\rangle \in Separ(\mathcal{H})\}. \tag{1}$$

The minimization of GME_{Ψ} always has a solution because the minimization in (1) is taken with a continuous function on a compact set $Separ(\mathcal{H})$ in a finite dimensional space \mathcal{H} . It is evident that the nearest separable state $|\Phi_{\Psi}\rangle$ can be chosen as a symmetric one.

Based on Bures distance, the authors in [25] propose an upper bound for a maximally geometric measure of entanglement for an m -partite system composed of subsystems of dimensions d_1, \dots, d_m .

Theorem 3 ([25]). For any normalized pure state $|\Psi\rangle \in \mathcal{H}$, we have

$$GME_{\Psi} \leq \sqrt{2 - 2/\sqrt{d_1 \dots d_{m-1}}}.$$

On the basis of von Neumann entropy, a commonly used entropy form of geometric measure [27] can be defined as:

$$E_G(|\Psi\rangle) = -\log_2 \max_{|\Phi\rangle = \otimes_{k=1}^m |\phi^{(k)}\rangle \in \mathcal{H}} |\langle \Psi | \Phi \rangle|^2 = -\log_2 G_{\Psi}^2. \tag{2}$$

The following upper bound for the entanglement is derived in [27].

Theorem 4 ([27]). For all unit length tensors $\mathcal{A}_{\Psi} \in \mathbb{R}^{[m \times n]}$, one has

$$E_G(|\Psi\rangle) \leq (m - 1) \log_2(n).$$

3. New Z-Eigenvalue Localization Set and the Bounds for Z-Spectral Radius

In this section, we present a new Z -eigenvalue inclusion theorem of tensors and show that our localization set is tighter than some existing localization sets. On this basis, lower and upper bounds for the Z -spectral radius of weakly symmetric non-negative tensors are available.

For a tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{C}^{[m \times n]}$, we denote

$$P_i^j(\mathcal{A}) = \sum_{\substack{i_2 \dots i_m \in N, \\ i \notin \{i_2, \dots, i_m\}}} |a_{i i_2 \dots i_m}|, \quad r_i(\mathcal{A}) = \sum_{i_2, \dots, i_m \in N} |a_{i i_2 \dots i_m}|.$$

Theorem 5. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{C}^{[m \times n]}$, $n \geq 2$, there is the following Z-eigenvalue localization sets.

$$\sigma(\mathcal{A}) \subseteq Y(\mathcal{A}) = \left(\bigcup_{i,j \in N, i \neq j} \hat{Y}_{i,j}(\mathcal{A}) \right) \cup \left(\bigcup_{i,j \in N, i \neq j} \tilde{Y}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A}) \right),$$

where

$$\hat{Y}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq r_i(\mathcal{A}) - P_i^j(\mathcal{A}), \quad |z| \leq r_j(\mathcal{A}) - |a_{ji \dots i}|\},$$

$$\tilde{Y}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : \left(|z| - (r_i(\mathcal{A}) - P_i^j(\mathcal{A})) \right) \left(|z| - (r_j(\mathcal{A}) - |a_{ji \dots i}|) \right) \leq P_i^j(\mathcal{A}) |a_{ji \dots i}|\}.$$

Proof. Let λ be a Z-eigenvalue of \mathcal{A} with the corresponding eigenvector x , then

$$\mathcal{A}x^{m-1} = \lambda x, \quad x^\top x = 1. \tag{3}$$

We assume that $|x_p| \geq |x_q| \geq \max_{i \in N, i \neq p,q} |x_i|$, then $0 < |x_p|^m \leq |x_p| \leq 1$. Based on (3), it follows that

$$\lambda x_p = \sum_{\substack{i_2 \dots i_m \in N, \\ p \in \{i_2, \dots, i_m\}}} a_{p i_2 \dots i_m} x_{i_2} \dots x_{i_m} + \sum_{\substack{i_2 \dots i_m \in N, \\ p \notin \{i_2, \dots, i_m\}}} a_{p i_2 \dots i_m} x_{i_2} \dots x_{i_m}.$$

There are the following inequalities by virtue of the absolute value and the triangle inequality:

$$|\lambda| \leq \sum_{\substack{i_2 \dots i_m \in N, \\ p \in \{i_2, \dots, i_m\}}} |a_{p i_2 \dots i_m}| + \sum_{\substack{i_2 \dots i_m \in N, \\ p \notin \{i_2, \dots, i_m\}}} |a_{p i_2 \dots i_m}| \frac{|x_q|}{|x_p|},$$

which is equivalent to

$$|\lambda| \leq (r_p(\mathcal{A}) - P_p^p(\mathcal{A})) + P_p^p(\mathcal{A}) \frac{|x_q|}{|x_p|}. \tag{4}$$

From inequalities (4), it is obvious that $\lambda \in \mathcal{K}_p(\mathcal{A})$.

If $|x_q| = 0$, it yields $|\lambda| - (r_p(\mathcal{A}) - P_p^p(\mathcal{A})) \leq 0$ on the ground of $|x_p| > 0$. If $|\lambda| \geq r_q(\mathcal{A}) - |a_{qp \dots p}|$, there is

$$\left(|\lambda| - (r_p(\mathcal{A}) - P_p^p(\mathcal{A})) \right) \left(|\lambda| - (r_q(\mathcal{A}) - |a_{qp \dots p}|) \right) \leq 0 \leq P_p^p(\mathcal{A}) |a_{qp \dots p}|,$$

which indicates $\lambda \in \tilde{Y}_{p,q}(\mathcal{A})$. Otherwise, $|\lambda| < r_q(\mathcal{A}) - |a_{qp \dots p}|$, we have $\lambda \in \hat{Y}_{p,q}(\mathcal{A}) \subseteq Y(\mathcal{A})$.

If $|x_q| > 0$, we can derive the following inequalities in a similar way

$$|\lambda| \leq (r_q(\mathcal{A}) - |a_{qp \dots p}|) + |a_{qp \dots p}| \frac{|x_p|}{|x_q|}. \tag{5}$$

Multiplying (4) and (5) yields

$$\left(|\lambda| - (r_p(\mathcal{A}) - P_p^p(\mathcal{A})) \right) \left(|\lambda| - (r_q(\mathcal{A}) - |a_{qp \dots p}|) \right) \leq P_p^p(\mathcal{A}) |a_{qp \dots p}|,$$

which indicates that $\lambda \in \tilde{Y}_{p,q}(\mathcal{A}) \subseteq Y(\mathcal{A})$.

If $|\lambda| \leq r_p(\mathcal{A}) - P_p^p(\mathcal{A})$ and $|z| \leq r_q(\mathcal{A}) - |a_{qp\dots p}|$, then $\lambda \in \hat{Y}_{p,q}(\mathcal{A}) \subseteq Y(\mathcal{A})$. Therefore, the conclusion is proved. \square

In order to further compare Theorems 2 and 5, we introduce the following Lemma.

Lemma 1 ([33]). *Let $a, b, c \geq 0$ and $d > 0$. If $\frac{a}{b+c+d} \geq 1$, then*

$$\frac{a - (b + c)}{d} \geq \frac{a - b}{c + d} \geq \frac{a}{b + c + d}.$$

Theorem 6. *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{C}^{[m \times n]}$, it follows that*

$$\sigma(\mathcal{A}) \subseteq Y(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

Proof. The authors in reference [13] have shown that $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$. Therefore, the proof of $Y(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A})$ is only needed. For any $z \in Y(\mathcal{A})$, there are $i, j \in N, j \neq i$ such that $z \in \hat{Y}_{i,j}(\mathcal{A})$ or $z \in (\check{Y}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A}))$. In this situation, we prove our result from two cases.

In the case of $z \in \hat{Y}_{i,j}(\mathcal{A})$, there are

$$|z| \leq R_i(\mathcal{A}) - P_i^i(\mathcal{A}), \quad |z| \leq r_j(\mathcal{A}) - |a_{ji\dots i}|,$$

These indicate that $z \in \mathcal{N}_{i,j}(\mathcal{A})$.

In the case of $z \in (\check{Y}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A}))$, there are $|z| \leq R_i(\mathcal{A})$

$$[|z| - (R_i(\mathcal{A}) - P_i^i(\mathcal{A}))][|z| - (R_j(\mathcal{A}) - |a_{ji\dots i}|)] \leq P_i^i(\mathcal{A})|a_{ji\dots i}|. \tag{6}$$

If $P_i^i(\mathcal{A})|a_{ji\dots i}| = 0$, it yields

$$R_i(\mathcal{A}) - P_i^i(\mathcal{A}) \leq |z| \leq R_j(\mathcal{A}) - |a_{ji\dots i}|, \tag{7}$$

or

$$R_j(\mathcal{A}) - |a_{ji\dots i}| \leq |z| \leq R_i(\mathcal{A}) - P_i^i(\mathcal{A}). \tag{8}$$

When inequalities (7) hold, there are

$$R_i(\mathcal{A}) - P_i^i(\mathcal{A}) \leq |z| \leq R_i(\mathcal{A}), \quad |z| \leq R_j(\mathcal{A}).$$

This implies $z \in \mathcal{N}_{i,j}(\mathcal{A})$.

When inequalities (8) hold, it follows that

$$[|z| - (R_i(\mathcal{A}) - P_i^i(\mathcal{A}))]|z| \leq 0 \leq P_i^i(\mathcal{A})R_j(\mathcal{A}),$$

which also implies $z \in \mathcal{N}_{i,j}(\mathcal{A})$.

If $P_i^i(\mathcal{A})|a_{ji\dots i}| > 0$, then inequalities (6) show

$$\frac{|z| - (R_i(\mathcal{A}) - P_i^i(\mathcal{A}))}{P_i^i(\mathcal{A})} \frac{|z| - (R_j(\mathcal{A}) - |a_{ji\dots i}|)}{|a_{ji\dots i}|} \leq 1.$$

When $\frac{|z| - (R_j(\mathcal{A}) - |a_{ji\dots i}|)}{|a_{ji\dots i}|} \leq 1$, there is $|z| \leq R_j(\mathcal{A})$, that is to say, $z \in \mathcal{N}_{i,j}(\mathcal{A})$.

When $\frac{|z| - (R_j(\mathcal{A}) - |a_{ji\dots i}|)}{|a_{ji\dots i}|} \geq 1$, according to Lemma 1, one has

$$\frac{|z| - (R_i(\mathcal{A}) - P_i^i(\mathcal{A}))}{P_i^i(\mathcal{A})} \frac{|z|}{R_j(\mathcal{A})} \leq \frac{|z| - (R_i(\mathcal{A}) - P_i^i(\mathcal{A}))}{P_i^i(\mathcal{A})} \frac{|z| - (R_j(\mathcal{A}) - |a_{ji\dots i}|)}{|a_{ji\dots i}|} \leq 1.$$

This indicates that $z \in \mathcal{N}_{i,j}(\mathcal{A})$. In summary, we have completed the proof that $Y(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A})$. \square

The following simple numerical example can verify the superiority of our conclusion in the bounds of the tensor spectrum.

Example 1. Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$ with 10 nonzero elements defined as follows;

$$\mathcal{A}(:, :, 1) = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}(:, :, 2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}(:, :, 3) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

It follows from Theorem 1 that

$$\mathcal{K} = (\mathcal{A}) = \bigcup_{i \in N} \mathcal{K}_i(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 5\}.$$

It follows from Theorem 2 that

$$\mathcal{N}(\mathcal{A}) = \bigcup_{i,j \in N, i \neq j} \mathcal{N}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : |z| \leq 2 + 2\sqrt{2} \doteq 4.8184\}.$$

However, it follows from Theorem 5 that

$$\sigma(\mathcal{A}) \subseteq Y(\mathcal{A}) = \bigcup_{i,j \in N, j \neq i} \left(\hat{Y}_{i,j}(\mathcal{A}) \cup (\check{Y}_{i,j}(\mathcal{A}) \cap \mathcal{K}_i(\mathcal{A})) \right) = \{z \in \mathbb{C} : |z| \leq \frac{7 + \sqrt{5}}{2} \doteq 4.618\}.$$

Lemma 2 ([30]). For a weakly symmetric non-negative tensor \mathcal{A} , there is

$$\rho(\mathcal{A}) = \lambda^*,$$

where λ^* denotes the largest Z-eigenvalue of tensor \mathcal{A} .

Theorem 7 ([13]). Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{C}^{[m \times n]}$ be a weakly symmetric non-negative tensor, it follows that

$$\rho(\mathcal{A}) \leq \bar{\omega} = \max_{i,j \in N, i \neq j} \frac{1}{2} \left\{ R_i(\mathcal{A}) - P_i^j(\mathcal{A}) + \sqrt{(R_i(\mathcal{A}) - P_i^j(\mathcal{A}))^2 + 4P_i^j(\mathcal{A})R_j(\mathcal{A})} \right\}.$$

In a similar manner, based on Theorem 5 and Lemma 2, we derive the following low and upper bounds for a weakly symmetric non-negative tensor, which is tighter than bound in Theorem 7.

Theorem 8. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{C}^{[m \times n]}$ be a weakly symmetric non-negative tensor; it follows that

$$\min_{i,j \in N, j \neq i} \underline{\varphi}_{i,j}(\mathcal{A}) = \underline{\varphi} \leq \rho(\mathcal{A}) \leq \bar{\varphi} = \max_{i,j \in N, j \neq i} \bar{\varphi}_{i,j}(\mathcal{A}), \tag{9}$$

where

$$\begin{aligned} \underline{\varphi}_{i,j}(\mathcal{A}) &= \max \left\{ \frac{1}{2} [R_i(\mathcal{A}) - P_i^j(\mathcal{A}) + R_j(\mathcal{A}) - |a_{ji \dots i}| - \hat{\varphi}_{i,j}(\mathcal{A})], 0 \right\}, \\ \bar{\varphi}_{i,j}(\mathcal{A}) &= \max \left\{ \max \{ R_i(\mathcal{A}) - P_i^j(\mathcal{A}), R_j(\mathcal{A}) - |a_{ji \dots i}| \}, \right. \\ &\quad \left. \min \{ R_i(\mathcal{A}), \frac{1}{2} [R_i(\mathcal{A}) - P_i^j(\mathcal{A}) + R_j(\mathcal{A}) - |a_{ji \dots i}| + \hat{\varphi}_{i,j}(\mathcal{A})] \} \right\}, \end{aligned}$$

and

$$\hat{\phi}_{i,j}(\mathcal{A}) = \sqrt{(R_i(\mathcal{A}) - P_i^i(\mathcal{A}) - R_j(\mathcal{A}) + |a_{ji\dots i}|)^2 + 4P_i^i(\mathcal{A})|a_{ji\dots i}|}.$$

Moreover, it follows that

$$0 \leq \underline{\varphi} \leq \rho(\mathcal{A}) \leq \bar{\varphi} \leq \bar{\omega} \leq \max_{i \in N} R_i(\mathcal{A}).$$

4. The Geometric Measure of Entanglement of Multipartite Pure States

This section is devoted to the geometric measure of entanglement on the ground of tensor Z -eigenvalue localization theory. Theoretical bounds of the geometric measure of entanglement for a weakly symmetric pure state with non-negative amplitudes are proposed. It is worth noting that the geometric measures derived based on different distance measures will be slightly different, such as Hilbert–Schmidt distance, Bures distance and trace distance.

As we know, a multipartite quantum state $|\Psi\rangle$ has a corresponding tensor representation $\mathcal{A}_{|\Psi\rangle} := (a_{i_1\dots i_m})$ under the orthogonal basis. We define the product of the tensor and vector as follows:

$$\mathcal{A}_{\Psi} u^{(1)} \cdots u^{(m)} \triangleq \sum a_{i_1 i_2 \dots i_m} u_{i_1}^{(1)} \cdots u_{i_m}^{(m)}.$$

In other words, the inner product between the entangled state $|\Psi\rangle$ and separable states $|\Phi\rangle$ can be regarded as

$$\langle \Psi | \Phi \rangle = \mathcal{A}_{\Psi} u^{(1)} \cdots u^{(m)}.$$

In this situation, the spectral radius of the tensor \mathcal{A}_{Ψ} is denoted as

$$\rho(\mathcal{A}_{\Psi}) \triangleq \max_{\|\phi^{(k)}\|=1, k=1, \dots, m} |\mathcal{A}_{\Psi} u^{(1)} \cdots u^{(m)}|.$$

In general, we consider as follows instead of solving (1) directly:

$$\text{GME}_{\Psi}^2 \triangleq \min\{\|\Psi - |\Phi\rangle\|^2 : |\Phi\rangle \in \text{Separ}(\mathcal{H})\},$$

which yields

$$\|\Psi - |\Phi\rangle\|^2 = 2 - \langle \Psi | \Phi \rangle - \langle \Phi | \Psi \rangle.$$

In other words, the minimization problem in (1) transforms into the maximization problem as follows:

$$\max_{\|\phi^{(k)}\|=1, k=1, \dots, m} \{\langle \Psi | \otimes_{j=1}^m |\phi^{(j)}\rangle + \otimes_{j=1}^m \langle \phi^{(j)} | \Psi \rangle\}. \tag{10}$$

By introducing Lagrange multipliers λ and applying complex differentiation, we show that the maximization problem in (10) is regarded as the largest entanglement eigenvalue λ , satisfying

$$\begin{cases} \langle \Psi | \left(\otimes_{j \neq k} |\phi^{(j)}\rangle \right) = \lambda \langle \phi^{(k)} |, \\ \left(\otimes_{j \neq k} \langle \phi^{(j)} | \right) \Psi = \lambda |\phi^{(k)}\rangle, \\ \|\phi^{(k)}\| = 1, k = 1, \dots, m. \end{cases} \tag{11}$$

According to (11), it follows that

$$\lambda = \langle \Psi | \Phi \rangle = \langle \Phi | \Psi \rangle,$$

is a real number in the interval $[-1, 1]$. For an m -partite pure state $|\Psi\rangle \in \mathcal{H}$, we denote the maximal overlap by

$$G_{\Psi} \triangleq \max_{|\Phi\rangle = \otimes_{k=1}^m |\phi^{(k)}\rangle \in \mathcal{H}} |\langle \Psi | \Phi \rangle|. \tag{12}$$

where $|\Phi\rangle = \otimes_{k=1}^m |\phi^{(k)}\rangle$ is the closest separable state to $|\Psi\rangle$.

Based on Bures distance, the geometric measure of entanglement for a multipartite pure state $|\Psi\rangle$ is defined as

$$GME_{\Psi} = \sqrt{2 - 2G_{\Psi}}. \tag{13}$$

In other words, the geometric measure of entanglement for a multipartite pure state $|\Psi\rangle$, expressed in (13), becomes

$$GME_{\Psi} = \sqrt{2 - 2\rho(\mathcal{A}_{\Psi})}. \tag{14}$$

In a similar way, (2) can be regarded as:

$$E_G(|\Psi\rangle) = -2\log_2 \rho(\mathcal{A}_{\Psi}). \tag{15}$$

When $\mathcal{H}_1 = \dots = \mathcal{H}_m$, \mathcal{A}_{Ψ} is symmetric if and only if $|\Psi\rangle$ is permutation symmetric. The geometric measure of symmetric states attracted much attention recently [23,34–37]. When $|\Psi\rangle$ is symmetric, (12) becomes

$$G_{\Psi} = \max_{|\Phi\rangle=|\phi\rangle^{\otimes m} \in \mathcal{H}} |\langle \Psi | \Phi \rangle|.$$

Therefore, the nearest separable state can be chosen as a symmetric one; for details, see [34,35].

In [24], the authors show that the maximal overlap for the geometric measure of entanglement for $|\Psi\rangle \in \mathcal{H}$ is equivalent to the Z -spectral radius of the corresponding tensor \mathcal{A}_{Ψ} in a weakly symmetric non-negative case.

We know that a weakly symmetric m -partite pure state $|\Psi\rangle \in \mathcal{H}$ with non-negative amplitude corresponding always has a corresponding m -order weakly symmetric non-negative tensor $\mathcal{A}_{|\Psi\rangle} := (a_{i_1 \dots i_m})$. Thus, we consider the lower and upper bounds for the geometric measure of entanglement for $|\Psi\rangle$ by virtue of Bures distance. It follows from Theorem 8 and (14) that the desired lower and upper bounds can be obtained.

Theorem 9. *For a weakly symmetric pure state with non-negative amplitudes $|\Psi\rangle \in \mathcal{H}$, there are the following lower and upper bounds for the geometric measure of entanglement for $|\Psi\rangle$:*

$$\sqrt{2 - 2\bar{\varphi}} \leq GME_{\Psi} \leq \sqrt{2 - 2\underline{\varphi}},$$

where $\bar{\varphi}$ and $\underline{\varphi}$ are as in Equation (9).

On the other side, we consider the bounds for the geometric measure of entanglement for $|\Psi\rangle$ on the grounds of von Neumann entropy; there are the following conclusions on the basis of Theorem 8 and (15).

Theorem 10. *For a weakly symmetric pure state with non-negative amplitudes $|\Psi\rangle \in \mathcal{H}$, according to von Neumann entropy, there are the following bounds for the geometric measure of entanglement for $|\Psi\rangle$:*

$$-2\log_2 \bar{\varphi} \leq E_G(|\Psi\rangle) \leq -2\log_2 \underline{\varphi},$$

where $\bar{\varphi}$ and $\underline{\varphi}$ are as in Equation (9).

On the one hand, an m -order n -dimensional real tensor \mathcal{A} has n^m independent entries, and a symmetric m -order n -dimensional real tensor \mathcal{A} has

$$C_{m+n-1}^m = \binom{m+n-1}{m}$$

independent entries [38]. However, a weakly symmetric tensor has at least

$$n^m - n \binom{m+n-2}{m} + n$$

independent entries [39], and the number of independent elements still increases exponentially.

It is worth noting that our conclusion in Theorems 9 and 10 depends on the characteristics of elements of the tensor \mathcal{A}_Ψ corresponding to the m -partite quantum state $|\Psi\rangle \in \mathcal{H}$, while Theorems 3 and 4 only depending on the dimension and order of the tensor. Therefore, we have numerical advantages in most cases, regardless of the size of the tensor.

Example 2. We consider a simple three-partite state with a three-level system, such as 3-qutrit GHZ state (3-qutrit system), as follows:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle).$$

From Theorem 3, the upper bound is

$$\text{GME}_\Psi \leq \sqrt{2 - 2/\sqrt{d_1 \cdots d_{m-1}}} = 1.1547.$$

However, by virtue of Theorem 9, the lower and upper bounds are

$$0.9194 = \sqrt{2 - 2\bar{\varphi}} \leq \text{GME}_\Psi \leq \sqrt{2 - 2\underline{\varphi}} = 0.9194.$$

In fact, the GME_Ψ of the 3-qutrit GHZ state is 0.9194 with the closest product state $|\phi_\Psi\rangle = |000\rangle$.

In addition, it follows from Theorem 4 that

$$E_G(|\Psi\rangle) \leq (m - 1) \log_2(n) = 3.1699.$$

However, based on Theorem 10, the lower and upper bounds are

$$1.5850 = -2 \log_2 \bar{\varphi} \leq E_G(|\Psi\rangle) \leq -2 \log_2 \underline{\varphi} = 1.5850.$$

which is less than the upper bound in Theorem 4.

Example 3. We consider the following more general 3-qutrit weakly symmetric state in a three-level with non-negative amplitudes

$$\begin{aligned} |\Psi\rangle = & 0.6805|000\rangle + 0.4990|111\rangle + 0.4768|222\rangle \\ & + 0.0500(|001\rangle + |010\rangle + |100\rangle + |200\rangle + |020\rangle + |002\rangle \\ & + |022\rangle + |202\rangle + |220\rangle + |011\rangle + |101\rangle + |110\rangle) \\ & + 0.0150(|122\rangle + |212\rangle + |221\rangle) \\ & + 0.1000(|112\rangle + |121\rangle + |211\rangle). \end{aligned}$$

It is easy to verify

$$\|\mathcal{A}_\Psi\|_F = 1.0000, \quad \rho(\mathcal{A}_\Psi) = 0.6940 \quad \text{GME}_\Psi = \sqrt{2 - 2\rho(\mathcal{A}_\Psi)} = 0.7823.$$

According to Theorem 3 in [25], there is

$$\text{GME}_\Psi \leq \sqrt{2 - 2/3} = 1.1547.$$

However, it follows from Theorem 9 that

$$0.3728 = \sqrt{2 - 2\bar{\varphi}} \leq \text{GME}_\Psi \leq \sqrt{2 - 2\underline{\varphi}} = 0.8713.$$

On the other side, the upper bound from Theorem 4 is

$$E_G(|\Psi\rangle) \leq (m-1) \log_2(n) = 3.1699.$$

It follows from Theorem 10 that

$$0.4156 = -2 \log_2 \bar{\varphi} \leq E_G(|\Psi\rangle) \leq -2 \log_2 \underline{\varphi} = 2.7550.$$

In fact, we can verify that

$$E_G(|\Psi\rangle) = -\log_2 \rho(\mathcal{A}_\Psi) = 1.4168.$$

Therefore, it is evident that Theorem 10 not only obtains a smaller upper bound compared with Theorem 4 but also a lower bound. This lower bound plays a significant role in the geometric measure of entanglement and other quantum information topic.

5. Conclusions

In this paper, we concentrate on the tensor Z -eigenvalue inclusion theorem and its application in the geometric measure of entanglement for multipartite quantum states. Firstly, we propose a new Z -eigenvalue localization theorem and bounds for the Z -spectral radius of non-negative tensors, which prove to be tighter than existing results. As applications, on the basis of the connection between the geometric measure of entanglement and the Z -spectral radius for a weakly symmetric non-negative tensor, we present theoretical bounds of the geometric measure of entanglement for a weakly symmetric multipartite quantum state with non-negative amplitudes by virtue of different distance measures. Numerical examples show that our bounds are tighter than the existing results in quantum physics in some cases. We believe that our results may be beneficial to the development of the intersection between tensor theory and quantum information.

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