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Probabilistic Interpretation of Number Operator Acting on Bernoulli Functionals

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Abstract: Let $N$ be the number operator in the space $H$ of real-valued square-integrable Bernoulli functionals. In this paper, we further pursue properties of $N$ from a probabilistic perspective. We first construct a nuclear space $G$, which is also a dense linear subspace of $H$, and then by taking its dual $G^*$, we obtain a real Gel'fand triple $G \subseteq H \subseteq G^*$. Using the well-known Minlos theorem, we prove that there exists a unique Gauss measure $\gamma_N$ on $G^*$ such that its covariance operator coincides with $N$. We examine the properties of $\gamma_N$, and, among others, we show that $\gamma_N$ can be represented as a convolution of a sequence of Borel probability measures on $G^*$. Some other results are also obtained.

Keywords: Bernoulli functionals; number operator; Gel'fand triple; Gauss measure; convolution of measures

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1. Introduction

Bernoulli functionals (also known as Bernoulli noise functionals) are measurable functions defined on the Bernoulli space and can be regarded as functionals of a Bernoulli process. Much attention has been paid to Bernoulli functionals and their application in the past two decades (for example, [1–5] and the references therein).

Let $N$ be the number operator in the space $L^2(Z)$ of complex-valued square-integrable Bernoulli functionals. Then, $N$ is a self-adjoint unbounded operator with a spectrum consisting of all non-negative integers, and, moreover, $N$ has a countably infinite family of eigenvectors that form an orthonormal basis for $L^2(Z)$. Recent years have seen much attention paid to $N$ and its application. Chen [6,7] showed that $N$ can play an important role in the study of quantum Markov semigroups. Ren et al. [8] applied $N$ to stochastic Schrödinger equations of the exclusion type, while Han [9] used $N$ and other related operators to compute quantum entropy. There are other works on $N$ and its application (see, for example, [10–12] and the references therein).

Let $H$ be the space of real-valued square-integrable Bernoulli functionals. Then, $H \subseteq L^2(Z)$ and $N$ leaves $H$ invariant, which implies that the restriction of $N$ to $H$ is an operator in $H$. In fact, it can be shown that the restriction of $N$ to $H$ is exactly the number operator in $H$. Just like the number operator $N$ in $L^2(Z)$, the number operator in $H$ is also a self-adjoint unbounded operator with a spectrum consisting of all non-negative integers. In this paper, we would like to further pursue the properties of the number operator in $H$ from a probabilistic perspective.

It is known that Gel'fand triples are appropriate frameworks for the study of probability measures in infinite dimensions. Roughly speaking, a Gel'fand triple consists of a Hilbert space, a nuclear space and its dual, where the nuclear space is continuously embedded into the Hilbert space. By the well-known Minlos theorem [13], there is a correspondence between characteristic functions on the nuclear space of a Gel'fand triple and
probability measures on its dual. There are many works showing excellent applications of Gel’fand triples (see, for example, [14–17] and references therein).

Essentially, this paper would like to present an application of Gel’fand triples for probabilistically pursuing properties of the number operator in $H$. Our main work is as follows. We first construct a nuclear space $\mathcal{G}$, which is also a dense linear subspace of $H$, and then by taking its dual $\mathcal{G}^*$, we obtain a real Gel’fand triple as:

$$\mathcal{G} \subset H \subset \mathcal{G}^*.$$  \hspace{1cm} (1)

We show that $\mathcal{G} \subset \text{Dom} \, N$ and $\text{Dom} \, N|_{\mathcal{G}} : \mathcal{G} \to H$ is continuous, where $N$ denotes the number operator in $H$. Based on these properties, we further prove that there exists a unique Gauss measure $\gamma_N$ on $\mathcal{G}^*$ such that its covariance operator coincides with $N$. Finally, we examine the properties of $\gamma_N$, and, among others, we show that $\gamma_N$ can be represented as a convolution of a sequence of Borel probability measures on $\mathcal{G}^*$.

The paper is organized as follows. In Section 2, we recall some necessary notions and facts about the Bernoulli space, real-valued Bernoulli functionals and the number operator acting on these functionals. In Section 3, we show the Gel’fand triple mentioned above. Our main work then lies in Section 4, where we first prove that there exists a unique Gauss measure $\gamma_N$ on $\mathcal{G}^*$ such that $N$ is the covariant operator of $\gamma_N$, and then we show that $\gamma_N$ can be represented as a convolution of a sequence of Borel probability measures on $\mathcal{G}^*$. Some other interesting results are also presented in that section.

Throughout this paper, $\mathbb{N}$ always denotes the set of all non-negative integers, while $\Gamma$ means the finite power set of $\mathbb{N}$, namely

$$\Gamma = \{ \sigma \mid \sigma \subset \mathbb{N}, \#(\sigma) < \infty \},$$  \hspace{1cm} (2)

where $\#(\sigma)$ means the cardinality of $\sigma$. Unless otherwise specified, we use letters such as $j$, $k$ and $n$ to mean non-negative integers, namely, the element of $\mathbb{N}$.

2. Bernoulli Functionals and Number Operator

Let $\Sigma$ be the set of all mappings $\omega : \mathbb{N} \mapsto \{-1, 1\}$. For each $n \geq 0$, we denote by $\xi_n$ the canonical projection associated with $n$, which is the mapping on $\Sigma$ defined by

$$\xi_n(\omega) = \omega(n), \quad \omega \in \Sigma.$$  \hspace{1cm} (3)

We write $\mathcal{A} = \sigma(\xi_n, n \geq 0)$, namely, $\mathcal{A}$ is the $\sigma$-field generated by the sequence $(\xi_n)_{n \geq 0}$ over $\Sigma$. Let $(\theta_n)_{n \geq 0}$ be a given sequence of real numbers with the property that $0 < \theta_n < 1$ for all $n \geq 0$. Then, there exists a unique probability measure $\mu$ on $\mathcal{A}$ such that

$$\mu \circ \left( \xi_{n_1}, \xi_{n_2}, \cdots, \xi_{n_k} \right)^{-1}\{ (\epsilon_1, \epsilon_2, \cdots, \epsilon_k) \} = \prod_{j=1}^{k} \theta_{n_j}^{1-\epsilon_j} (1 - \theta_{n_j})^{1-\epsilon_j},$$  \hspace{1cm} (4)

where $k$ is any positive integer, $\{n_j \mid 1 \leq j \leq k \} \subset \mathbb{N}$ with $n_i \neq n_j$ (if $i \neq j$) and $\epsilon_j \in \{-1, 1\}$, $1 \leq j \leq k$. Thus, we come to a probability measure space $(\Sigma, \mathcal{A}, \mu)$, which is called the Bernoulli space. Measurable functions on $(\Sigma, \mathcal{A}, \mu)$ are known as Bernoulli functionals. In particular, square-integrable functions on $(\Sigma, \mathcal{A}, \mu)$ are referred to as square-integrable Bernoulli functionals.

Consider the usual Hilbert space $L^2(\Sigma, \mathcal{A}, \mu; \mathbb{R})$ of real-valued square-integrable Bernoulli functionals. Let $Z = (Z_n)_{n \geq 0}$ be the standardized sequence of the canonical projection sequence $(\xi_n)_{n \geq 0}$, namely

$$Z_n = \frac{\xi_n + 1 - 2\theta_n}{2\sqrt{\theta_n(1 - \theta_n)}}, \quad n \geq 0.$$  \hspace{1cm} (5)
Then, the system \( \{ Z_\sigma \mid \sigma \in \Gamma \} \) forms an orthonormal basis for \( L^2(\Sigma, \mathscr{A}, \mu; \mathbb{R}) \), where \( Z_\emptyset = 1 \) and
\[
Z_\sigma = \prod_{j \in \sigma} Z_j, \quad \sigma \in \Gamma, \; \sigma \neq \emptyset,
\]
where, as indicated in (2), \( \Gamma \) denotes the finite power set of \( \mathbb{N} \). From the viewpoint of probability, \( Z = (Z_n)_{n \geq 0} \) is a Bernoulli process on the Bernoulli space \( (\Sigma, \mathscr{A}, \mu) \). Thus, elements of \( L^2(\Sigma, \mathscr{A}, \mu; \mathbb{R}) \) can be viewed as functionals of the Bernoulli process \( Z = (Z_n)_{n \geq 0} \).

In what follows, we just simply write \( \mathcal{H} \) for the Hilbert space \( L^2(\Sigma, \mathcal{A}, \mu; \mathbb{R}) \) of real-valued square-integrable Bernoulli functionals, namely, we set
\[
\mathcal{H} \equiv L^2(\Sigma, \mathcal{A}, \mu; \mathbb{R}).
\]

We use \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) to mean the inner product and norm in \( \mathcal{H} \), respectively. We call \( \{ Z_\sigma \mid \sigma \in \Gamma \} \) the canonical orthonormal basis (ONB) for \( \mathcal{H} \). For \( \sigma \in \Gamma \) and \( n \geq 0 \), we simply use \( \sigma \setminus n \) to mean \( \sigma \setminus \{ n \} \), and similarly we use \( \sigma \cup n \).

For each non-negative integer \( k \geq 0 \), it can be shown that there corresponds to a bounded operator \( \partial_k \) on \( \mathcal{H} \) such that
\[
\partial_k Z_\sigma = 1_\sigma(k) Z_{\sigma \setminus \{ k \}}, \quad \partial_k^* Z_\sigma = (1 - 1_\sigma(k)) Z_{\sigma \cup \{ k \}},
\]
where \( \partial_k^* \) denotes the adjoint of \( \partial_k \), and \( 1_\sigma \) means the indicator of \( \sigma \) as a subset of \( \mathbb{N} \). Usually, \( \partial_k \) and \( \partial_k^* \) are known as the annihilation and creation operators, respectively.

**Definition 1.** The number operator \( N \) in \( \mathcal{H} \) is the positive self-adjoint operator defined by
\[
N = \sum_{\sigma \in \Gamma} \#(\sigma) |Z_\sigma\rangle \langle Z_\sigma|,
\]
where \( |Z_\sigma\rangle \langle Z_\sigma| \) denotes the Dirac operator associated with the basis vector \( Z_\sigma \).

Denote by \( \text{Dom} N \) the domain of \( N \). Then, \( \text{Dom} N \) consists of vectors \( \xi \) in \( \mathcal{H} \) satisfying
\[
\sum_{\sigma \in \Gamma} [\#(\sigma)]^2 \langle Z_\sigma, \xi \rangle^2 < \infty,
\]
and, for \( \xi \in \text{Dom} N \), one has \( N\xi = \sum_{\sigma \in \Gamma} \#(\sigma) \langle Z_\sigma, \xi \rangle Z_\sigma \), where the series is actually an orthogonal vector series in \( \mathcal{H} \) and convergent in norm. It can be verified that \( \{ Z_\sigma \mid \sigma \in \Gamma \} \subset \text{Dom} N \) and \( N Z_\sigma = \#(\sigma) Z_\sigma \) for each \( \sigma \in \Gamma \), which means that \( \#(\sigma) \) is an eigenvalue of \( N \), while \( Z_\sigma \) is the corresponding eigenvector. However, \( N \) is an unbounded operator. In fact, one can show that
\[
\| N \| := \sup \left\{ \| N\xi \| \mid \xi \in \text{Dom} N, \; \| \xi \| = 1 \right\} = \infty.
\]

**Remark 1.** Let \( L^2(\mathbb{Z}) := L^2(\Sigma, \mathcal{A}, \mu; \mathbb{C}) \) be the space of complex-valued square integrable Bernoulli functionals. Then, one can show that \( L^2(\mathbb{Z}) \) and \( \mathcal{H} \) share a common ONB \( \{ Z_\sigma \mid \sigma \in \Gamma \} \) although \( \mathcal{H} \subset L^2(\mathbb{Z}) \). Based on this fact, one can further show that the number operator in \( \mathcal{H} \) coincides with the restriction of the number operator in \( L^2(\mathbb{Z}) \) to \( \mathcal{H} \).

### 3. The Gel'fand Triple

This section describes the Gel'fand triple mentioned in Section 1 and presents some necessary results concerning this triple.

We follow the notation introduced in the previous section. Consider the operator \( A \) in \( \mathcal{H} \) defined by
\[
A = \sum_{\sigma \in \Gamma} \lambda_\sigma |Z_\sigma\rangle \langle Z_\sigma|,
\]
where \( \lambda_{\emptyset} = 1 \) and \( \lambda_{\sigma} = \prod_{k \in \sigma} (1 + k) \) for \( \sigma \in \Gamma \) with \( \sigma \neq \emptyset \). Clearly, \( A \) is a positive self-adjoint operator and \( \{ Z_{\sigma} \mid \sigma \in \Gamma \} \subset \text{Dom } A^p \), in particular \( Z_{\sigma} \) is an eigenvector of \( A \).

For a nonnegative integer \( p \geq 0 \), we set \( \mathcal{G}_p = \text{Dom } A^p \), the domain of \( A^p \), which is given by

\[
\mathcal{G}_p = \{ \xi \in \mathcal{H} \mid \sum_{\sigma \in \Gamma} \lambda_{\sigma}^{2p} \langle Z_{\sigma}, \xi \rangle^2 < \infty \},
\]

and endow it with the inner product \( \langle \cdot, \cdot \rangle_p \equiv \langle A^p \cdot, A^p \cdot \rangle \). It can be shown that \( (\mathcal{G}_p, \langle \cdot, \cdot \rangle_p) \) is a real Hilbert space and has an ONB like \( \{ \lambda_{\sigma}^{-p} Z_{\sigma} \mid \sigma \in \Gamma \} \). In the sequel, we abbreviate \( (\mathcal{G}_p, \langle \cdot, \cdot \rangle_p) \) as \( \mathcal{G}_p \) and denote by \( \| \cdot \|_p \) the norm induced by \( \langle \cdot, \cdot \rangle_p \).

For \( q \geq p \geq 0 \), one obviously has \( \mathcal{G}_q \subset \mathcal{G}_p \). Moreover, if \( p \geq 0 \) and \( q > p + \frac{1}{2} \), then the inclusion mapping \( i_{pq} : \mathcal{G}_q \to \mathcal{G}_p \) satisfies that

\[
\| i_{pq} \|^2_{HS} = \sum_{\sigma \in \Gamma} \| i_{pq}(\lambda_\sigma^{-q} Z_{\sigma}) \|_p^2 = \sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-p)} \leq \exp \left[ \sum_{n=1}^{\infty} n^{-(q-p)} \right] < \infty,
\]

which means that \( i_{pq} \) is of Hilbert–Schmidt class. This justifies the next lemma.

**Lemma 1.** Let \( \mathcal{G} = \bigcap_{p \geq 1} \mathcal{G}_p \) and endow it with the topology generated by the norm sequence \( \| \cdot \|_p, p \geq 0 \). Then \( \mathcal{G} \) is a real nuclear space.

For each non-negative integer \( p \geq 0 \), we denote by \( \mathcal{G}_p^* \) the dual of \( \mathcal{G}_p \) and by \( \| \cdot \|_{-p} \) the norm in \( \mathcal{G}_p^* \). Then, for \( 0 \leq p \leq q \), one has \( \| \cdot \|_{-p} \geq \| \cdot \|_{-q} \) and \( \mathcal{G}_p^* \subset \mathcal{G}_q^* \). Denote by \( \mathcal{G}^* \) the dual of the nuclear space \( \mathcal{G} \) and endow it with the strong topology. Then, as an immediate consequence of the general theory of countably Hilbert nuclear space (see, for example, \([13,18]\)) , one has

\[
\mathcal{G}^* = \bigcup_{p=0}^{\infty} \mathcal{G}_p^*,
\]

and moreover the strong topology over \( \mathcal{G}^* \) coincides with the inductive limit topology given by space sequence \( \{ \mathcal{G}_p^* \}_{p \geq 1} \). By identifying \( \mathcal{H} \) with its dual, we come to a real Gel’fand triple

\[
\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*,
\]

which is the framework where we will work. By convention, elements of \( \mathcal{G} \) are called (real-valued) Bernoulli testing functionals, while elements of \( \mathcal{G}^* \) are called (real-valued) Bernoulli generalized functionals. In the following, we denote by \( \langle \langle \cdot, \cdot \rangle \rangle \) the canonical bilinear form on \( \mathcal{G}^* \times \mathcal{G} \), which is defined as

\[
\langle \langle x, \xi \rangle \rangle = x(\xi), \quad (x, \xi) \in \mathcal{G}^* \times \mathcal{G},
\]

where \( x(\xi) \) means the value of functional \( x \) at \( \xi \). From the definition of \( \mathcal{G} \) as well as the properties of \( \mathcal{G}_p \), one can see that \( \{ Z_\sigma \mid \sigma \in \Gamma \} \subset \mathcal{G} \), which implies that \( \mathcal{G} \) is dense in \( \mathcal{H} \). Additionally, it can be verified that

\[
\xi = \sum_{\sigma \in \Gamma} \langle Z_\sigma, \xi \rangle Z_\sigma, \quad \forall \xi \in \mathcal{G},
\]

where the series on the right-hand side converges in the topology of \( \mathcal{G} \).

For Bernoulli generalized functionals \( x \in \mathcal{G}^* \), its Fock transform is the function \( \tilde{x} \) on \( \Gamma \) given by

\[
\tilde{x}(\sigma) = \langle \langle x, Z_\sigma \rangle \rangle, \quad \sigma \in \Gamma,
\]

where \( \langle \langle \cdot, \cdot \rangle \rangle \) is the canonical bilinear form on \( \mathcal{G}^* \times \mathcal{G} \). It can be shown that a Bernoulli generalized functional \( x \in \mathcal{G}^* \) is completely determined by its Fock transform \( \tilde{x} \). Additionally, using the same methods as in \([4]\) , one can establish the next useful characterization lemma for Bernoulli generalized functionals.
Lemma 2. Let $F$ be a real-valued function on $\Gamma$. Then $F$ is the Fock transform of an element $x \in G^*$ if and only if there exist some constants $C \geq 0$ and $p \geq 0$ such that

$$|F(\sigma)| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma. \quad (18)$$

In that case, for any $q > p + \frac{1}{2}$, one has

$$\|x\|_{-q} \leq C\left(\sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-p)}\right)^{\frac{1}{2}}, \quad (19)$$

and in particular $x \in G_q^*$.

4. Probabilistic Interpretation

In the final section, we pursue properties of the number operator $N$ from a probabilistic perspective. We continue to use the notions and notation introduced in previous sections.

We denote by $\mathcal{B}(G^*)$ the Borel $\sigma$-field over $G^*$. According to the general theory of Gel’fand triples, $\mathcal{B}(G^*)$ coincides with the $\sigma$-field generated by all cylindrical sets in $G^*$. This suggests that two probability measures on $\mathcal{B}(G^*)$ are equal if and only if they have the same value at every cylindrical set in $G^*$. By convention, a probability measure on $\mathcal{B}(G^*)$ is also called a probability measure on $G^*$.

Recall that a probability measure $\nu$ on $G^*$ is called a Gauss measure if, for all $\xi \in G$, the function $x \mapsto \langle\langle x, \xi \rangle\rangle$ is a Gauss random variable on the probability space $(G^*, \mathcal{B}(G^*), \nu)$, where $\langle\langle , \rangle\rangle$ denotes the bilinear form on $G^* \times G$. By the well-known Minlos theorem [13], there exists a unique Gauss measure $\gamma$ on $G^*$ such that

$$\int_{G^*} e^{\langle\langle x, \xi \rangle\rangle} d\gamma(x) = e^{-\frac{1}{4}||\xi||^2}, \quad \xi \in G, \quad (20)$$

which we call the standard Gauss measure on $G^*$ below.

Recall again that the number operator $N$ is densely defined in $H$, namely $\text{Dom } N$ is a dense linear subspace of $H$. On the other hand, by the inclusion relation indicated in (14), one has $G \subset H$. The next proposition shows the relation between $\text{Dom } N$ and $G$.

**Proposition 1.** It holds that $G \subset \text{Dom } N$. Moreover, the restriction of $N$ to $G$ is a continuous linear operator from $G$ to $H$.

**Proof.** Let $\xi \in G$. Then, for all $p \geq 0$, we have $\xi \in G_p$, especially $\xi \in G_1$, which together with the inequality $\#(\sigma) \leq \lambda_\sigma$ implies that

$$\sum_{\sigma \in \Gamma} [\#(\sigma)]^2 (Z_\sigma, \xi)^2 \leq \sum_{\sigma \in \Gamma} \lambda_\sigma^2 (Z_\sigma, \xi)^2 = ||\xi||^2_1 < \infty,$$

which means that $\xi \in \text{Dom } N$. Thus $G \subset \text{Dom } N$. On the other hand, for all $\xi \in G$, careful calculations and estimations give

$$\|N\xi\| = \left\{\sum_{\sigma \in \Gamma} [\#(\sigma)]^2 (Z_\sigma, \xi)^2\right\}^{\frac{1}{2}} \leq \left\{\sum_{\sigma \in \Gamma} \lambda_\sigma^2 (Z_\sigma, \xi)^2\right\}^{\frac{1}{2}} = ||A\xi|| = ||\xi||_1,$$

which, together with the fact that the topology over $G$ is generated by the norm sequence $||\cdot||_p$, $p \geq 0$, implies that $N|_G : G \to H$ is continuous. \[\square\]

**Proposition 2.** The function $\xi \mapsto e^{-\frac{1}{4}||\xi||^2}$ is positive definite on $G$, where $\langle\langle , \rangle\rangle$ is the inner product in $H$.

**Proof.** Consider the function $\varphi$ on $G$ defined by $\varphi(\xi) = e^{-\frac{1}{4}||\xi||^2}$, $\xi \in H$, where $||\cdot||$ is the norm in $H$. According to the Minlos theorem and (20), $\varphi$ is positive definite on $G$. 


which, together with the denseness of $\mathcal{G}$ in $\mathcal{H}$, implies that $\varphi$ is also positive definite on $\mathcal{H}$. On the other hand, from the definition of $\mathcal{N}$, we know that $\text{Dom} \, \mathcal{N} \subset \text{Dom} \, \mathcal{N}^\sharp$ and $\mathcal{N}^\sharp(\text{Dom} \, \mathcal{N}) \subset \text{Dom} \, \mathcal{N}^\sharp$. Moreover, it holds that

$$\langle \xi, \mathcal{N} \xi \rangle = \langle \xi, \mathcal{N}^\sharp (\mathcal{N}^\sharp \xi) \rangle = \| \mathcal{N}^\sharp \xi \|^2, \quad \forall \xi \in \text{Dom} \, \mathcal{N}.$$  

Thus, by Proposition 1, we have

$$e^{-\frac{1}{2} \langle \xi, \mathcal{N} \xi \rangle} = e^{-\frac{1}{2} \| \mathcal{N}^\sharp \xi \|^2} = \varphi(\mathcal{N}^\sharp \xi), \quad \xi \in \mathcal{G},$$

which together with the positive definiteness of $\varphi$ on $\mathcal{H}$ implies that the function $\xi \mapsto e^{-\frac{1}{2} \langle \xi, \mathcal{N} \xi \rangle}$ is positive definite on $\mathcal{G}$. \qed

**Theorem 1.** There exists a unique probability measure $\gamma_\mathcal{N}$ on $\mathcal{G}^\ast$ such that

$$\int_{\mathcal{G}^\ast} e^{i \langle x, \xi \rangle} \, d\gamma_\mathcal{N}(x) = e^{-\frac{1}{2} \langle \xi, \mathcal{N} \xi \rangle}, \quad \xi \in \mathcal{G},$$  \hspace{1cm} (21)

where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}$.

**Proof.** Consider the function $\psi_\mathcal{N}(\xi) = e^{-\frac{1}{2} \langle \xi, \mathcal{N} \xi \rangle}, \xi \in \mathcal{G}$. Clearly, $\psi_\mathcal{N}(0) = 1$ and, by Proposition 1, $\psi_\mathcal{N}$ is continuous on $\mathcal{G}$. Additionally, by Proposition 2, $\psi_\mathcal{N}$ is also positive definite on $\mathcal{G}$. Therefore, by the well-known Minlos theorem, there exists a probability measure $\gamma_\mathcal{N}$ on $\mathcal{G}^\ast$ such that

$$\int_{\mathcal{G}^\ast} e^{i \langle x, \xi \rangle} \, d\gamma_\mathcal{N}(x) = \psi_\mathcal{N}(\xi), \quad \xi \in \mathcal{G},$$

which is the same as (21). Using the monotone class method in probability theory as well as the fact that $\mathcal{G}^\ast$ is generated by all cylindrical sets in $\mathcal{G}^\ast$, we can easily come to the uniqueness of $\gamma_\mathcal{N}$. \qed

**Proposition 3.** $\gamma_\mathcal{N}$ is a Gauss measure and its covariant operator coincides with $\mathcal{N}$, namely, it holds that

$$\int_{\mathcal{G}^\ast} \langle x, \eta \rangle \langle x, \eta \rangle \, d\gamma_\mathcal{N}(x) = \langle \xi, \mathcal{N} \eta \rangle, \quad \xi, \eta \in \mathcal{G}.$$  \hspace{1cm} (22)

**Proof.** For each $\xi \in \mathcal{G}$, it is easy to see that the random variable $X_\xi(\cdot) = \langle \cdot, \xi \rangle$ is a Gauss random variable on the probability space $(\mathcal{G}^\ast, \mathfrak{B}(\mathcal{G}^\ast), \gamma_\mathcal{N})$ with mean 0 and variance $\langle \xi, \mathcal{N} \xi \rangle$. Thus, $\gamma_\mathcal{N}$ is a Gauss measure and

$$\int_{\mathcal{G}^\ast} \langle x, \xi \rangle \langle x, \xi \rangle \, d\gamma_\mathcal{N}(x) = \int_{\mathcal{G}^\ast} |X_\xi(x)|^2 \, d\gamma_\mathcal{N}(x) = \langle \xi, \mathcal{N} \xi \rangle, \quad x \in \mathcal{G}.$$  

This together with the polarization formula implies (22). \qed

**Remark 2.** In view of Proposition 3, we may call $\gamma_\mathcal{N}$ the Gauss measure with covariant $\mathcal{N}$.

**Proposition 4.** There exists a continuous linear operator $m : \mathcal{G}^\ast \rightarrow \mathcal{G}^\ast$ such that

$$\overline{m}(\sigma) = \sqrt{\#(\sigma)} \, \hat{x}(\sigma), \quad \sigma \in \Gamma, \quad x \in \mathcal{G}^\ast,$$  \hspace{1cm} (23)

where $\hat{x}$ means the Fock transform of a Bernoulli generalized functional $y \in \mathcal{G}^\ast$ and $\#(\sigma)$ denotes the cardinality of $\sigma$.

**Proof.** Let $x \in \mathcal{G}^\ast$. Then, by Lemma 2, there exist some constants $C \geq 0$ and $p \geq 0$ such that

$$|\hat{x}(\sigma)| \leq C \lambda^p, \quad \sigma \in \Gamma,$$
which, together with the inequality \( \#(\sigma) \leq \lambda_{\sigma} \), gives
\[
|\sqrt{\#(\sigma)} \hat{x}(\sigma)| \leq C\lambda_{\sigma}^{p+1} \leq C\lambda_{\sigma}^{p+1}, \quad \sigma \in \Gamma.
\]

This, together with Lemma 2, implies that there exists a unique \( y_x \in G^* \) such that
\[
g_x(\sigma) = \sqrt{\#(\sigma)} \hat{x}(\sigma), \quad \sigma \in \Gamma.
\]

Now, by defining \( mx = y_x, x \in G^* \), we obtain an operator \( m : G^* \rightarrow G^* \). It is easy to see that \( m \) is linear and satisfies (23). Next, we prove its continuity.

Recall that the strong topology over \( G^* \) coincides with the inductive limit topology generated by the space sequence \( G_{p*}^p, p \geq 0 \). In view of this, it suffices to prove that \( m \circ j_p : G_p^* \rightarrow G^* \) is continuous for all \( p \geq 0 \), where \( j_p : G_p^* \rightarrow G^* \) denotes the inclusion mapping.

Let \( p \geq 0 \). Then, for each \( x \in G_p^* \), we have
\[
\sum_{\sigma \in \Gamma} \lambda_{\sigma}^{-2(p+1)}|m \circ j_p(x)(\sigma)|^2 = \sum_{\sigma \in \Gamma} \lambda_{\sigma}^{-2(p+1)}|\hat{m}(\sigma)|^2
\]
\[
= \sum_{\sigma \in \Gamma} \lambda_{\sigma}^{-2(p+1)}\#(\sigma)|\hat{x}(\sigma)|^2
\]
\[
\leq \sum_{\sigma \in \Gamma} \lambda_{\sigma}^{-2p}|\hat{x}(\sigma)|^2,
\]
which implies that \( m \circ j_p(x) \in G_{p+1}^* \) and \( \|m \circ j_p(x)\|_{(p+1)} \leq \|x\|_{-p} \). Thus, \( m \circ j_p \) is a bounded linear operator from \( G_p^* \) to \( G_{p+1}^* \) (equivalently, a continuous linear operator from \( G_p^* \) to \( G_{p+1}^* \)), which implies that \( m \circ j_p \) is also a continuous linear operator from \( G_p^* \) to \( G^* \).

It then follows from the arbitrariness \( p \geq 0 \) that \( m \circ j_p : G_p^* \rightarrow G^* \) is continuous for all \( p \geq 0 \). \( \Box \)

**Theorem 2.** \( \gamma_N \) coincides with the measure induced by the standard Gauss measures \( \gamma \) and the operator \( m \), namely
\[
\gamma_N = \gamma \circ m^{-1}.
\]

**Proof.** Consider the positive self-adjoint operator \( \sqrt{N} = N^{\frac{1}{2}} \) in \( \mathcal{H} \). It can be shown that \( G \subset \text{Dom} \sqrt{N} \) and \( \sqrt{N} \) leaves \( G \) invariant. Additionally, it can also be verified that \( \sqrt{N}|_G : G \rightarrow G \) is continuous, where \( \sqrt{N}|_G \) means the restriction to \( G \) of the operator \( \sqrt{N} \).

Let \( x \in G^* \) and \( \xi \in G \) be given. Then, for any \( \sigma \in \Gamma \), a careful calculation gives
\[
\hat{m}(\sigma) = \sqrt{\#(\sigma)} \hat{x}(\sigma) = \langle x, \sqrt{\#(\sigma)} Z_{\sigma} \rangle = \langle x, \sqrt{N} Z_{\sigma} \rangle = x \circ \sqrt{N}|_G(\sigma),
\]
which implies that \( mx = x \circ \sqrt{N}|_G \), hence \( \langle mx, \xi \rangle = \langle x, \sqrt{N} \xi \rangle \).

Finally, for any \( \xi \in G \), by using the above formula, the formula of the change of variables in the theory of integration, as well as properties of the standard Gauss measure \( \gamma \), we obtain
\[
\int_{G^*} e^{\langle x, \xi \rangle} d\gamma \circ m^{-1}(x) = \int_{G^*} e^{\langle mx, \xi \rangle} d\gamma(x) = \int_{G^*} e^{\langle x, \sqrt{N} \xi \rangle} d\gamma(x) = e^{-\frac{1}{2}\|\sqrt{N} \xi\|^2},
\]
which, together with the equality \( \|\sqrt{N} \xi\|^2 = \langle \xi, N\xi \rangle \) and Theorem 1, yields
\[
\int_{G^*} e^{\langle x, \xi \rangle} d\gamma \circ m^{-1}(x) = \int_{G^*} e^{\langle x, \xi \rangle} d\gamma_N(x).
\]

Therefore, by the arbitrariness of \( \xi \in G \), we come to the equality \( \gamma_N = \gamma \circ m^{-1} \). \( \Box \)
**Definition 2.** A sequence \((\kappa_n)_{n \geq 1}\) of probability measures on \((G^*, \mathcal{B}(G^*))\) is said to converge characteristically to a probability measure \(\kappa\) on \((G^*, \mathcal{B}(G^*))\) if

\[
\lim_{n \to \infty} \int_{G^*} e^{i\langle x, \xi \rangle} d\kappa_n(x) = \int_{G^*} e^{i\langle x, \xi \rangle} d\kappa(x), \quad \forall \xi \in G.
\]

In that case, we simply write \(\kappa_n \xrightarrow{\ast} \kappa\).

Recall that, for a non-negative integer \(k \geq 0\), the annihilation operator \(\partial_k\) is a bounded linear operator defined on the whole space \(H\). Let \(k \geq 0\) be given. Then, it is easy to show that \(G\) is an invariant subspace of \(\partial_k\), and moreover the restriction of \(\partial_k\) to \(G\) is a continuous linear operator from \(G\) to itself. Thus, by the Minlos theorem, there exists a probability measure \(\nu^{(k)}\) on \(G^*\) such that

\[
\int_{G^*} e^{i\langle x, \xi \rangle} d\nu^{(k)}(x) = e^{-\frac{1}{2}||\partial_k\xi||^2}, \quad \xi \in G.
\]

In what follows, we call \(\nu^{(k)}\) the probability measure associated with the annihilation operator \(\partial_k\). It is easy to see that \(\nu^{(k)}\) is also a Gauss measure and its covariant operator is just \(\partial_k^2\), which is a projection operator on \(H\).

Let \(\kappa_1, \kappa_2, \ldots, \kappa_n\) be probability measures on the measurable space \((G^*, \mathcal{B}(G^*))\). Then, their convolution \(\kappa_1 \ast \kappa_2 \ast \cdots \ast \kappa_n\) is the probability measure on \((G^*, \mathcal{B}(G^*))\) defined by

\[
\kappa_1 \ast \kappa_2 \ast \cdots \ast \kappa_n = (\kappa_1 \times \kappa_2 \times \cdots \times \kappa_n) \circ S^{-1},
\]
where \(\kappa_1 \times \kappa_2 \times \cdots \times \kappa_n\) means the product measure, while \(S\) is the mapping from \(G^{\ast n}\) to \(G^*\) given by

\[
S(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n, \quad (x_1, x_2, \ldots, x_n) \in G^{\ast n},
\]
where \(G^{\ast n}\) is the \(n\)-fold product space of \(G^*\). The convolution \(\kappa_1 \ast \kappa_2 \ast \cdots \ast \kappa_n\) admits the following property

\[
\int_{G^*} e^{i\langle x, \xi \rangle} d(\kappa_1 \ast \kappa_2 \ast \cdots \ast \kappa_n)(x) = \prod_{j=1}^{n} \int_{G^*} e^{i\langle x, \xi \rangle} d\kappa_j(x), \quad \xi \in G, \quad (26)
\]
which is not hard to verify.

**Theorem 3.** As \(n \to \infty\), one has \(\nu^{(0)} \ast \nu^{(1)} \ast \cdots \ast \nu^{(n)} \xrightarrow{\ast} \gamma_N\), where \(\nu^{(k)}\) is the probability measure associated with the annihilation operator \(\partial_k\).

**Proof.** We first prove that \(\lim_{n \to \infty} \sum_{k=0}^{n} ||\partial_k^2 \kappa_n \xi||^2 = \langle \xi, N\xi \rangle\) for all \(\xi \in G\). Let \(\xi \in G\) be given. Then, using the canonical ONB \(\{Z_{\sigma} \mid \sigma \in \Gamma\}\) for \(H\), we have

\[
\xi = \sum_{\sigma \in \Gamma} \langle Z_{\sigma}, \xi \rangle Z_{\sigma},
\]
which, together with the properties of the annihilation and creation operators (see (8)), gives

\[
\partial_k^2 \kappa_n \xi = \sum_{\sigma \in \Gamma} \mathbf{1}_{\sigma}(k) \langle Z_{\sigma}, \xi \rangle Z_{\sigma}, \quad k \geq 0.
\]

Thus, for \(n \geq 0\), it holds that

\[
\sum_{k=0}^{n} ||\partial_k^2 \kappa_n \xi||^2 = \sum_{k=0}^{n} \sum_{\sigma \in \Gamma} \mathbf{1}_{\sigma}(k) \langle Z_{\sigma}, \xi \rangle^2 = \sum_{\sigma \in \Gamma} \left[ \sum_{k=0}^{n} \mathbf{1}_{\sigma}(k) \right] \langle Z_{\sigma}, \xi \rangle^2.
\]
For each $\sigma \in \Gamma$, we can easily see that $\sum_{k=0}^n 1_{\sigma}(k) \nearrow \#(\sigma)$ as $n \to \infty$, which implies that

$$\left[ \sum_{k=0}^n 1_{\sigma}(k) \right] \langle Z_{\sigma}, \xi \rangle^2 \nearrow \#(\sigma) \langle Z_{\sigma}, \xi \rangle^2 \quad (\text{as } n \to \infty).$$

Thus, by the monotone convergence theorem, we obtain

$$\lim_{n \to \infty} \sum_{k=0}^n \| \partial_k^* \partial_k \xi \|_2^2 = \lim_{n \to \infty} \sum_{\sigma \in \Gamma} \left[ \sum_{k=0}^n 1_{\sigma}(k) \right] \langle Z_{\sigma}, \xi \rangle^2 = \sum_{\sigma \in \Gamma} \#(\sigma) \langle Z_{\sigma}, \xi \rangle^2.$$

On the other hand, by using (9), we find $\langle \xi, N\xi \rangle = \sum_{\sigma \in \Gamma} \#(\sigma) \langle Z_{\sigma}, \xi \rangle^2$. Therefore,

$$\lim_{n \to \infty} \sum_{k=0}^n \| \partial^*_k \partial_k \xi \|_2^2 = \langle \xi, N\xi \rangle.$$

Next, we show that $\nu^{(0)} * \nu^{(1)} * \cdots * \nu^{(n)} \xrightarrow{\text{d}} \gamma_N$. Let $\xi \in \mathcal{G}$ be given. Then, for each $n \geq 0$, by using properties of the convolution as well as (25), we have

$$\int_{\mathcal{G}^*} e^{i\langle (x, \xi) \rangle} d(\nu^{(0)} * \nu^{(1)} * \cdots * \nu^{(n)})(x) = \prod_{k=0}^n \int_{\mathcal{G}^*} e^{i\langle (x, \xi) \rangle} d\nu^{(k)}(x) = e^{-\frac{1}{2} \sum_{k=0}^n \| \partial_k^* \partial_k \xi \|_2^2},$$

which, together with the formula $\| \partial_k^* \partial_k \xi \|_2^2 = \| \partial_k^* \partial_k \xi \|_2^2, k \geq 0$, yields

$$\lim_{n \to \infty} \int_{\mathcal{G}^*} e^{i\langle (x, \xi) \rangle} d(\nu^{(0)} * \nu^{(1)} * \cdots * \nu^{(n)})(x) = \lim_{n \to \infty} e^{-\frac{1}{2} \sum_{k=0}^n \| \partial_k^* \partial_k \xi \|_2^2} = e^{-\frac{1}{2} \langle \xi, N\xi \rangle},$$

which together with Theorem 1 implies that

$$\lim_{n \to \infty} \int_{\mathcal{G}^*} e^{i\langle (x, \xi) \rangle} d(\nu^{(0)} * \nu^{(1)} * \cdots * \nu^{(n)})(x) = \int_{\mathcal{G}^*} e^{i\langle (x, \xi) \rangle} d\gamma_N(x).$$

Therefore, $\nu^{(0)} * \nu^{(1)} * \cdots * \nu^{(n)} \xrightarrow{\text{d}} \gamma_N$. This completes the proof. \qed

As is seen, Theorem 3 actually shows that the Gauss measure $\gamma_N$ can be represented as a convolution of the form $\gamma_N = \nu^{(0)} * \nu^{(1)} * \cdots * \nu^{(n)} * \cdots$. The next proposition, together with Theorem 3, further suggests that $\gamma_N$ can also be viewed as a limit of Gauss measures with small fluctuations.

**Proposition 5.** For each $n \geq 0$, the convolution $\nu^{(0)} * \nu^{(1)} * \cdots * \nu^{(n)}$ is a Gauss measure on $\mathcal{G}^*$ and its covariant operator is $\sum_{k=0}^n \partial_k^* \partial_k$.

**Proof.** Write $\kappa_n = \nu^{(0)} * \nu^{(1)} * \cdots * \nu^{(n)}$ and $S_n = \sum_{k=0}^n \partial_k^* \partial_k$. Then, from the proof of Theorem 3 we know that

$$\int_{\mathcal{G}^*} e^{i\langle (x, \xi) \rangle} d\kappa_n(x) = e^{-\frac{1}{2} \sum_{k=0}^n \| \partial_k^* \partial_k \xi \|_2^2}, \quad \xi \in \mathcal{G},$$

which together with $\sum_{k=0}^n \| \partial_k^* \partial_k \xi \|_2^2 = \langle \xi, S_n\xi \rangle$ gives

$$\int_{\mathcal{G}^*} e^{i\langle (x, \xi) \rangle} d\kappa_n(x) = e^{-\frac{1}{2} \langle \xi, S_n\xi \rangle}, \quad \xi \in \mathcal{G},$$

which implies that $\kappa_n$ is a Gauss measure and its covariant operator is $S_n$. \qed
5. Conclusions Remarks

Consider the family \( X = \{ X_\sigma \mid \sigma \in \Gamma \} \) of random variables defined on the probability space \((\mathcal{G}^*, \mathcal{B}(\mathcal{G}^*), \gamma_N)\) in the following manner

\[
X_\sigma(\omega) = \langle \langle \omega, Z_\sigma \rangle \rangle, \quad \omega \in \mathcal{G}^*,
\]

where \( Z_\sigma \) is the basis vector of the canonical ONB for \( \mathcal{H} \) and \( \langle \langle \cdot, \cdot \rangle \rangle \) is the canonical bilinear form on \( \mathcal{G}^* \times \mathcal{G} \). Using Theorem 1, one can find that \( X_\sigma \) and \( X_\tau \) are independent for \( \sigma, \tau \in \Gamma \) with \( \sigma \neq \tau \). Moreover, for each \( \sigma \in \Gamma \), one has \( \mathbb{E}X_\sigma = 0 \) and \( \text{Var}X_\sigma = \#(\sigma) \). In particular, by putting \( \psi_k = X\{k\}, k \geq 0 \), one gets a sequence \( \{ \psi_k \mid k \geq 0 \} \) of independent Gaussian random variables of mean 0 and variance 1. Such a sequence is usually known as a discrete-time Gaussian white noise, and can have potential application in time series analysis as well as in mathematical physics.

As is seen in Section 4, we refer to the Minlos theorem several times. Here, for reader’s benefits, we give some reviews on the theorem. Let \( \mathcal{E} \subset \mathcal{X} \subset \mathcal{E}^* \) be a real Gelf’and triple.

For a probability measure \( \nu \) on the Borel \( \sigma \)-field \( \mathcal{B}(\mathcal{E}^*) \), its Fourier transform \( \hat{\nu} \) is defined as

\[
\hat{\nu}(f) = \int_{\mathcal{E}^*} e^{i \langle \langle x, f \rangle \rangle} d\nu, \quad f \in \mathcal{E},
\]

where \( \langle \langle \cdot, \cdot \rangle \rangle \) denotes the canonical linear form on \( \mathcal{E}^* \times \mathcal{E} \). The Minlos theorem then reads as follows.

A complex-valued function \( C(\cdot) \) on \( \mathcal{E} \) is the Fourier transform of a unique probability measure on \( \mathcal{B}(\mathcal{E}^*) \) if and only if it satisfies the following:

1. \( C(0) = 1 \);
2. \( C(\cdot) \) is continuous;
3. \( C(\cdot) \) is positive definite, namely, for any positive integer \( n \), any \( z_1, z_2, \ldots, z_n \in \mathbb{C} \) and any \( u_1, u_2, \ldots, u_n \in \mathcal{E} \), it holds that

\[
\sum_{j,k=1}^n z_j\overline{z}_k C(u_j - u_k) \geq 0.
\]

In that case, the function \( C(\cdot) \) is called a characteristic function.

The Minlos theorem establishes a correspondence between characteristic functions on a nuclear space and probability measures on the Borel \( \sigma \)-field of its dual. As a generalization of the Bochner theorem to the infinite-dimensional case, the Minlos theorem is a powerful tool for dealing with probability measures on an infinite-dimensional space.

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