Sums of Pell/Lucas Polynomials and Fibonacci/Lucas Numbers

Dongwei Guo and Wenchang Chu

Abstract: Seven infinite series involving two free variables and central binomial coefficients (in denominators) are explicitly evaluated in closed form. Several identities regarding Pell/Lucas polynomials and Fibonacci/Lucas numbers are presented as consequences.

Keywords: Gaussian hypergeometric function; arcsin-function; central binomial coefficient; Pell/Lucas polynomial; Fibonacci/Lucas numbers

MSC: 11B39; 05A10

1. Introduction and Motivation

As polynomial extensions of the Fibonacci/Lucas numbers, the Pell/Lucas polynomials have many remarkable properties and wide applications in mathematics, physics, and computer sciences (see Koshy [1,2] and Grimaldi [3]). There exist numerous summation formulae involving Fibonacci/Lucas numbers [4–8] and the Pell/Lucas polynomials [9–15]. Motivated by a beautiful formula (see Corollary 3) due to Ohtsuka [16], we shall investigate in this paper further summation formulae involving inverse central binomial coefficients and Pell/Lucas polynomials, as well as their applications to identities concerning Fibonacci/Lucas numbers.

By means of the linearization method, several \( _2F_1 \)-series with a free variable were evaluated in closed form by the second author [17]. Some of these will be utilized in this paper to show seven algebraic identities (Theorems 1–7) that express infinite series containing two free variables and central binomial coefficients (in denominators) in terms of arcsin-function. As applications, a number of summation formulae involving Pell/Lucas polynomials and Fibonacci/Lucas numbers will be derived consequently. To the best of authors’ knowledge, most of the results presented in the propositions and corollaries of this paper are new, except for the inspiration formula (see Corollary 1) found recently by Ohtsuka [16].

In order to facilitate the presentation, it is necessary to review briefly Pell/Lucas polynomials and Fibonacci/Lucas numbers (cf. Koshy [1,2]). The Pell/Lucas polynomials were introduced by Horadam and Mahon [18] that are determined by the same recurrence relations:

\[
P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),
\]
\[
Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x);
\]

but with different initial conditions:

\[
P_0(x) = 0 \quad \text{and} \quad P_1(x) = 1,
\]
\[
Q_0(x) = 2 \quad \text{and} \quad Q_1(x) = 2x.
\]
Their Binet form formulae are given by
\[
P_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad Q_n(x) = \alpha^n + \beta^n,
\]
where
\[
\alpha := \alpha(x) = x + \sqrt{x^2 + 1} \quad \text{and} \quad \beta := \beta(x) = x - \sqrt{x^2 + 1}.
\]
When \( x = \frac{1}{2} \), they become the Fibonacci numbers \( F_n = P_n(\frac{1}{2}) \) and Lucas numbers \( L_n = Q_n(\frac{1}{2}) \), respectively. Let \( \phi, \psi = \frac{1 \pm \sqrt{5}}{2} \) be two algebraic numbers, with the former being the golden ratio. Then, there are the Binet form expressions:
\[
F_n = \frac{\phi^n - \psi^n}{\phi - \psi} \quad \text{and} \quad L_n = \phi^n + \psi^n,
\]
which are valid for all the integers \( n \in \mathbb{Z} \).

Throughout the paper, we shall utilize the following notation for the Gaussian hypergeometric series (cf. Bailey [19], §1.1)
\[
_{2}F_{1}\left[ a, \ b \mid c \right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n,
\]
where the shifted factorial is defined, for an indeterminate \( x \), by
\[
(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = x(x+1) \cdots (x+n-1) \quad \text{for} \ n \in \mathbb{N}.
\]

2. The First Algebraic Identity and Applications
We begin with the following infinite series identity. An equivalent form of it can be found in Adegoke–Frontczak–Goy [4], Lehmer [20], and Zucker [21].

**Theorem 1** (Algebraic identity: \( u > 0 \) and \( v > 0 \)).
\[
\sum_{n=0}^{\infty} \frac{4^n}{2n+1} \frac{u^n + v^n}{(u+v)^n} \left( \begin{array}{c} 2n \\ n \end{array} \right) = \frac{\pi (u+v)}{2 \sqrt{uv}}.
\]

**Proof.** Recalling the summation formula (cf. Chu [17], Corollary 11)
\[
_{2}F_{1}\left[ 1 + x, \ 1 - \frac{x}{2} \mid y^2 \right] = \frac{\sin(2x \arcsin y)}{2xy \sqrt{1 - y^2}},
\]
we can easily deduce the following limit as \( x \to 0 \)
\[
\sum_{n=0}^{\infty} \frac{(2y)^{2n}}{2n+1} \left( \begin{array}{c} 2n \\ n \end{array} \right) = \frac{\arcsin y}{y \sqrt{1 - y^2}}.
\]
Making the replacement \( y \to \sqrt{\frac{u}{u+v}} \) yields the expression below
\[
\sum_{n=0}^{\infty} \frac{u^n}{2n+1} \left( \frac{4}{u+v} \right)^n \left( \begin{array}{c} 2n \\ n \end{array} \right) = \frac{u+v}{\sqrt{uv}} \arcsin \sqrt{\frac{u}{u+v}}.
\]
Consequently, by unifying the two sums
\[
\sum_{n=0}^{\infty} \frac{u^n + v^n}{2n+1} \left( \frac{4}{u+v} \right)^n \left( \begin{array}{c} 2n \\ n \end{array} \right) = \frac{u+v}{\sqrt{uv}} \left\{ \arcsin \sqrt{\frac{u}{u+v}} + \arcsin \sqrt{\frac{v}{u+v}} \right\}.
\]
and then making use of the following identity for the inverse trigonometric function:

\[
\arcsin x + \arcsin \sqrt{1 - x^2} = \frac{\pi}{2}, \quad \text{where} \quad x \geq 0,
\]

we find the summation formula in the theorem. \(\square\)

For an even number \(m \in \mathbb{N}\), letting \(u = \alpha^m\) and \(v = \beta^m\), we deduce from Theorem 1 the following identity about Lucas polynomials.

**Proposition 1** (Even \(m \in \mathbb{N}\)).

\[
\sum_{n=0}^{\infty} \frac{Q_{mn}(x)}{2n+1} \left( \frac{4}{Q_m(x)} \right)^n \left( \frac{2n}{n} \right)^{-1} = \frac{\pi}{2} Q_m(x).
\]

In particular, when \(x = \frac{1}{2}\), we confirm the following identity about Lucas numbers, which was proposed recently by Ohtsuka [16] as a problem in “Fibonacci Quarterly”. More identities of a similar type were derived recently by Adegoke–Frontczak–Goy [4].

**Corollary 1** (Even \(m \in \mathbb{N}\)).

\[
\sum_{n=0}^{\infty} \frac{L_{mn}}{2n+1} \left( \frac{4}{L_m} \right)^n \left( \frac{2n}{n} \right)^{-1} = \frac{\pi}{2} L_m.
\]

3. The Second Algebraic Identity and Applications

Now, we examine the identity (cf. Chu [22,23] and Slater [24], §1.5)

\[
_{2}F_{1}\left[ x, -\frac{1}{2} \left| y^2 \right. \right] = \cos(2x \arcsin y).
\]

By shifting the initial term of the \(_{2}F_{1}\)-series to another side and then dividing across by \(x^2\), we get the equality

\[
\sum_{n=1}^{\infty} \frac{(1+x)^{n-1}(1-x)^{n-1}(2y)^{2n}}{(2n)!} = \frac{1 - \cos(2x \arcsin y)}{x^2}.
\]

Now, letting \(x \to 0\) gives the identity

\[
\sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n^2(2n)!} = 2 \arcsin^2 y.
\]

Keeping in mind that

\[
\arcsin \sqrt{\frac{u}{u+v}} - \arcsin \sqrt{\frac{v}{u+v}} = \arcsin \frac{u-v}{u+v},
\]

we derive the following summation formula, that its equivalent forms can be located in Lehmer [20], Zucker [21], and Elsner [25].

**Theorem 2** (Algebraic identity: \(u > 0\) and \(v > 0\)).

\[
\sum_{n=1}^{\infty} \frac{4^n(u^n - v^n)}{n^2(u+v)^n} \left( \frac{2n}{n} \right)^{-1} = \pi \arcsin \frac{u-v}{u+v}.
\]

For an even number \(m \in \mathbb{N}\), letting \(u = \alpha^m\) and \(v = \beta^m\), we obtain from Theorem 2 the identity below concerning Pell/Lucas polynomials.
Proposition 2 (Even \( m \in \mathbb{N} \)).
\[
\sum_{n=1}^{\infty} \frac{P_{mn}(x)}{n^2} \left( \frac{4}{Q_m(x)} \right)^n \left( \frac{2n}{n} \right)^{-1} = \frac{\pi}{2\sqrt{1 + x^2}} \arcsin \frac{2\sqrt{1 + x^2}P_m(x)}{Q_m(x)}.
\]

For the special case corresponding to \( x = \frac{1}{2} \), we have the following identity involving Fibonacci/Lucas numbers.

Corollary 2 (Even \( m \in \mathbb{N} \)).
\[
\sum_{n=1}^{\infty} \frac{F_{mn}}{n^2} \left( \frac{4}{L_m} \right)^n \left( \frac{2n}{n} \right)^{-1} = \frac{\pi}{\sqrt{5}} \arcsin \frac{\sqrt{5}F_m}{L_m}.
\]

4. The Third Algebraic Identity and Applications

By reindexing \( n \rightarrow n - 1 \), we can rewrite (1) as
\[
\sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n} \left( \frac{2n}{n} \right)^{-1} = \frac{2y \arcsin y}{\sqrt{1 - y^2}}. \tag{2}
\]
Subtracting twice (1) from the above equation yields
\[
\sum_{n=1}^{\infty} \frac{4^n y^{2n+2}}{n(2n+1)(u+v)^n} \left( \frac{2n}{n} \right)^{-1} = 2y^2 - 2y \sqrt{1 - y^2} \arcsin y. \tag{3}
\]

Keeping in mind that
\[
\arcsin \sqrt{\frac{u}{u+v}} - \arcsin \sqrt{\frac{v}{u+v}} = \arcsin \frac{u-v}{u+v}
\]
we derive the following summation formulae, which is equivalent to those recorded by Lehmer [20], and Zucker [21].

Theorem 3 (Algebraic identity: \( u > 0 \) and \( v > 0 \)).
\[
\sum_{n=1}^{\infty} \frac{4^n (u^{n+1} \pm v^{n+1})}{n(2n+1)(u+v)^n} \left( \frac{2n}{n} \right)^{-1} = 2(u \pm v) - 2\sqrt{uv} \times \begin{cases} \frac{\pi}{2}, & u \pm v \geq 0; \\ \arcsin \frac{u-v}{u+v}, & u \pm v < 0. \end{cases}
\]

For an even number \( m \in \mathbb{N} \), letting \( u = \alpha^m \) and \( v = \beta^m \), we deduce from Theorem 3 two formulae as in the following proposition.

Proposition 3 (Even \( m \in \mathbb{N} \)).
\[
\sum_{n=1}^{\infty} \frac{Q_{mn+m}(x)}{n(2n+1)(u+v)^n} \left( \frac{4}{Q_m(x)} \right)^n \left( \frac{2n}{n} \right)^{-1} = 2Q_m(x) - \pi; \\
\sum_{n=1}^{\infty} \frac{P_{mn+m}(x)}{n(2n+1)(u+v)^n} \left( \frac{4}{Q_m(x)} \right)^n \left( \frac{2n}{n} \right)^{-1} = 2P_m(x) - \frac{1}{\sqrt{1 + x^2}} \arcsin \frac{2\sqrt{1 + x^2}P_m(x)}{Q_m(x)}.
\]

Letting further \( x = \frac{1}{2} \) result in two series below about Fibonacci/Lucas numbers.
Corollary 3 (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=1}^{\infty} \frac{L_{mn+m}}{n(2n+1)} \left( \frac{4}{L_m} \right)^n \binom{2n}{n}^{-1} = 2L_m - \pi;
\]

\[
\sum_{n=1}^{\infty} \frac{F_{mn+m}}{n(2n+1)} \left( \frac{4}{L_m} \right)^n \binom{2n}{n}^{-1} = 2F_m - \frac{2}{\sqrt{5}} \arcsin \sqrt{\frac{5}{F_m}} L_m.
\]

5. The Fourth Algebraic Identity and Applications

In view of the summation formula (cf. Chu [17], Corollary 12)

\[
\begin{align*}
2F_1 \left[ 1 + x, 1 - \frac{x}{2}, 1 \right] & = \frac{\cos(2\arcsin y)}{1 - y^2} + \frac{y \sin(2\arcsin y)}{2x \sqrt{1 - y^2}^3}, \\
\sum_{n=0}^{\infty} 4^n y^{2n-4} \binom{2n}{n} & = \frac{1}{y^4(1 - y^2)^2} + \frac{\arcsin y}{y^3 \sqrt{1 - y^2}}^3,
\end{align*}
\]

we can easily deduce the following limit as \( x \to 0 \)

\[
\sum_{n=0}^{\infty} 4^n y^{2n-4} \binom{2n}{n} = \frac{1}{y^4(1 - y^2)^2} + \frac{\arcsin y}{y^3 \sqrt{1 - y^2}}^3,
\]

where we have divided the equation by \( y^4 \). Now, by letting \( y \to \sqrt{\frac{u}{u+v}} \) and \( y \to \sqrt{\frac{v}{u+v}} \), and then by adding the two resulting equations, we derive the following formula. Its equivalent forms can be found in two different papers by Adegoke–Frontczak–Goy [4], and Lehmer [20].

Theorem 4 (Algebraic identity: \( u > 0 \) and \( v > 0 \)).

\[
\sum_{n=0}^{\infty} 4^n \frac{u^{n-2} + v^{n-2}}{(u+v)^n} \binom{2n}{n}^{-1} = \frac{(u+v)^2}{u^2 v^2} + \frac{\pi(u+v)}{2 \sqrt{uv}}.
\]

For an even number \( m \in \mathbb{N} \), letting \( u = \alpha^m \) and \( v = \beta^m \), we deduce from Theorem 4 the infinite series below concerning Lucas polynomials.

Proposition 4 (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=0}^{\infty} \frac{Q_{mn-2m}(x)}{\binom{2n}{n}} \left( \frac{4}{Q_m(x)} \right)^n = Q_m^2(x) + \frac{\pi}{2} Q_m(x).
\]

Especially, letting \( x = \frac{1}{2} \), we have the following series involving Lucas numbers.

Corollary 4 (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=0}^{\infty} L_{mn-2m} \left( \frac{4}{L_m} \right)^n \binom{2n}{n} = L_m^2 + \frac{\pi}{2} L_m.
\]

6. The Fifth Algebraic Identity and Applications

Recall the formula (cf. Chu [17], Corollary 16), we can derive

\[
2F_1 \left[ 1 + x, 1 - \frac{x}{2}, 1 \right] = \frac{3\cos(2\arcsin y)}{4(1 - x^2)^2 y^2(1 - y^2)^2} - \frac{3(1 - 2y^2) \sin(2\arcsin y)}{8x(1 - x^2) y^3 \sqrt{1 - y^2}^3}.
\]
which reduces as \( x \to 0 \), to the following formula

\[
\sum_{n=0}^{\infty} \frac{n + 1}{n + 2} \frac{(2y)^{2n}}{(2n + 3)} = \frac{1}{8y^2 (1 - y^2)} - \frac{(1 - 2y^2) \arcsin y}{8y^3 \sqrt{(1 - y^2)^3}}.
\]

Then, we can analogously examine the series

\[
\sum_{n=0}^{\infty} \frac{n + 1}{n + 2} \frac{(4u)^n}{(u + v)^n} \left( \frac{2n + 3}{n + 1} \right)^{-1} = \left( \frac{u + v)^2}{8uv} + \frac{(u + v)^2 (u - v) \arcsin \sqrt{\frac{u}{u + v}}}{8 \sqrt{(uv)^3}}.
\]

By symmetry, we can evaluate further the sum in the theorem below.

**Theorem 5** (Algebraic identity: \( u > 0 \) and \( v > 0 \)).

\[
\sum_{n=1}^{\infty} 4^n (u^n - v^n) \left( \frac{2n + 3}{n + 1} \right)^{-1} = \frac{\pi (u + v)^2 (u - v)}{16 \sqrt{(uv)^3}}.
\]

For an even number \( m \in \mathbb{N} \), letting \( u = \alpha^m \) and \( v = \beta^m \), we deduce from Theorem 5 the infinite series below concerning Pell/Lucas polynomials.

**Proposition 5** (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=1}^{\infty} \frac{n + 1}{n + 2} \frac{P_{mn}(x)}{(2n + 3)} \left( \frac{4}{Q_m(x)} \right)^n = \frac{\pi}{16} Q_m(x) P_{2m}(x).
\]

Letting \( x = \frac{1}{2} \) further, we have the series below regarding Fibonacci/Lucas numbers.

**Corollary 5** (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=1}^{\infty} \frac{n + 1}{n + 2} \frac{F_{mn}(x)}{(2n + 3)} \left( \frac{4}{L_m} \right)^n = \frac{\pi}{16} L_m F_{2m}.
\]

7. The Sixth Algebraic Identity and Applications

By employing Theorems 5 and 8 in [17], we can derive the following closed formula for \( \Omega_{6,3}(x, y) \):

\[
_{2}F_{1}\left[ \frac{3 + x, 3 - x}{2} ; \frac{y^2}{2} \right] = \frac{15}{128x(1 + x)(1 - x)(2 + x)(2 - x)y^5 \sqrt{(1 - y^2)^5}}
\]

\[
\times \left\{ \frac{2(2 + x)(2 - x) \sin(2x \arcsin y)}{2(2 + x)(2 - x) \sin(2x \arcsin y)} \right\}.
\]

Its limiting case as \( x \to 0 \) gives that

\[
\sum_{n=0}^{\infty} \frac{(n + 1)(n + 2)}{n + 3} \frac{(2y)^{2n}}{(2n + 3)} = \frac{\arcsin y}{32y^5 \sqrt{(1 - y^2)^5}} + \frac{\arcsin y \cos(4 \arcsin y)}{64y^8 \sqrt{(1 - y^2)^9}} - \frac{3 \sin(4 \arcsin y)}{256y^8 \sqrt{(1 - y^2)^9}}.
\]

By checking, for \( x \geq 0 \), the two equalities below

\[
\sin(4 \arcsin x) + \sin(4 \arcsin \sqrt{1 - x^2}) = 0,
\]

\[
\arcsin x \cos(4 \arcsin x) + \arcsin \sqrt{1 - x^2} \cos(4 \arcsin \sqrt{1 - x^2}) = \frac{\pi}{2} (1 - 8x^2 + 8x^4);
\]

we find the following summation formula.
Theorem 6 (Algebraic identity: \( u > 0 \) and \( v > 0 \)).

\[
\sum_{n=0}^{\infty} 4^n \frac{(n+1)(n+2)}{n+3} \frac{u^n + v^n}{(u+v)^n} \left( \frac{2n+5}{n+2} \right)^{-1} = \frac{\pi(u+v)^3(3u^2 - 2uv + 3v^2)}{128 \sqrt{(uv)^5}}.
\]

For an even number \( m \in \mathbb{N} \), letting \( u = \alpha^m \) and \( v = \beta^m \), we deduce from Theorem 6 the infinite series below concerning Lucas polynomials.

Proposition 6 (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{n+3} \frac{Q_{mn}(x)}{(2n+5)} \left( \frac{4}{Q_m(x)} \right)^n = \frac{\pi}{128} Q_m^2(x) \left\{ 3Q_{2m}(x) - 2 \right\}.
\]

In particular, letting \( x = \frac{1}{2} \), we have the following infinite series identity involving Lucas numbers.

Corollary 6 (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{n+3} \frac{L_{mn}}{(2n+5)} \left( \frac{4}{L_m} \right)^n = \frac{\pi}{128} L_m^3 \left\{ 3L_{2m} - 2 \right\}.
\]

8. The Seventh Algebraic Identity and Applications

Finally, by utilizing again Theorems 5 and 8 in [17], we can derive further the closed formula for \( \Omega_{3,4}(x,y) \):

\[
\begin{align*}
2F1 & \left[ \frac{4 + x}{2}, \frac{4 - x}{2} \mid \frac{y^2}{2} \right] = \\
& \frac{105}{128 x(1 + x)(2 + x)(3 + x)(1 - x)(2 - x)(3 - x)y^2 \sqrt{(1 - y^2)^3}} \\
& \times \left\{ 2y \sqrt{1 - y^2}(15 - 44y^2 - 4x^2y^2 + 44y^4 + 4x^2y^4) \cos(2x \arcsin y) \right\} \\
& - 3(1 - 2y^2)(5 - 8y^2 - 8x^2y^2 + 8y^4 + 8x^2y^4) \sin(2x \arcsin y) \right\}.
\end{align*}
\]

Its limiting case as \( x \to 0 \)

\[
\sum_{n=0}^{\infty} \frac{(n+1)_3}{n+4} \frac{(2y)^{2n}}{(2n+7)} = \frac{15 - 44y^2 + 44y^4}{512y^6(1 - y^2)^3} - \frac{3(1 - 18y^2 + 24y^4 - 16y^6)}{512y^7 \sqrt{(1 - y^2)^7}} \arcsin y
\]

leads us to the following companion formula.

Theorem 7 (Algebraic identity: \( u > 0 \) and \( v > 0 \)).

\[
\sum_{n=0}^{\infty} 4^n \frac{(n+1)_3}{n+4} \frac{u^n - v^n}{(u+v)^n} \left( \frac{2n+7}{n+3} \right)^{-1} = \frac{3\pi(u+v)^4(u-v)(5u^2 + 2uv + 5v^2)}{1024 \sqrt{(uv)^7}}.
\]

For an even number \( m \in \mathbb{N} \), letting \( u = \alpha^m \) and \( v = \beta^m \), we deduce from Theorem 7 the infinite series below concerning Pell/Lucas polynomials.

Proposition 7 (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=0}^{\infty} \frac{(n+1)_3}{n+4} \frac{P_{mn}(x)}{(2n+7)} \left( \frac{4}{Q_m(x)} \right)^n = \frac{3\pi}{1024} P_m(x)Q^4_m(x) \left\{ 2 + 5Q_{2m}(x) \right\}.
\]

Letting \( x = \frac{1}{2} \) in particular, we have the following infinite series involving Fibonacci/Lucas numbers.
Corollary 7 (Even \( m \in \mathbb{N} \)).

\[
\sum_{n=0}^{\infty} \frac{(n+1)}{(n+4)^{2m+1}} \binom{4}{L_m} \mathcal{F}_{mn} \left( \frac{4}{L_m} \right)^n = \frac{3\pi}{1024} F_m L_m^4 \left\{ 2 + 5L_{2m} \right\}.
\]

According to Theorems 5 and 8 in [17], the same procedure can be carried out for producing additional infinite series identities. However, the resulting expressions will be more complicated and less elegant. Hence, we shall not pursue further along that direction.

9. Further Identities on Fibonacci/Lucas Numbers

By assigning different values to \( u \) and \( v \) in the theorems shown in the previous sections, numerous interesting identities concerning Fibonacci/Lucas numbers can further be derived. We limit ourselves to presenting one representative example for each theorem.

Letting \( u = \frac{5F_m}{F_2} \) and \( v = \frac{L_m}{L_2} \) in Theorem 1 and then making use of

\[
5F_m L_m = 2L_{2m},
\]

we obtain the following summation formula.

**Corollary 8** (\( m \in \mathbb{N} \)).

\[
\sum_{n=0}^{\infty} \frac{2n}{n+1} \frac{5^n F_{2n} + L_{2n}}{L_{2n}} \binom{2n}{n}^{-1} = \frac{\pi L_{2m}}{\sqrt{5F_2}}.
\]

Letting \( u = F_{m+1} \) and \( v = F_{m-1} \) in Theorem 2 and then making use of

\[
F_{m+1} + F_{m-1} = L_m,
\]

we get the following summation formula.

**Corollary 9** (\( m \in \mathbb{N} \)).

\[
\sum_{n=1}^{\infty} \frac{4^n F_m}{n^2} \frac{F_{m+1} - F_{m-1}}{L_m} \binom{2n}{n}^{-1} = \pi \arcsin \left( \frac{F_m}{L_m} \right).
\]

Letting \( u = L_{m+1} \) and \( v = L_{m-1} \) in Theorem 3 and then making use of

\[
L_{m+1} + L_{m-1} = 5F_m,
\]

we deduce the following summation formula.

**Corollary 10** (\( m \in \mathbb{N} \)).

\[
\sum_{n=1}^{\infty} \frac{(4/5)^n}{n(2n+1)} \frac{L_{m+1}^{n+1} + L_{m-1}^{n+1}}{F_m} \binom{2n}{n}^{-1} = \begin{cases} 10F_m - \pi \sqrt{L_{2m} - 3(-1)^m}, & \text{“+”}, \\ 2L_m - 2\sqrt{L_{2m} - 3(-1)^m} \arcsin \left( \frac{L_m}{5F_m} \right), & \text{“-”} \end{cases}
\]

Letting \( u = 5F_m L_m \) and \( v = L_m L_{2m} \) in Theorem 4 and then making use of

\[
5F_m L_m + L_m L_{2m} = 2L_{3m} \quad \text{and} \quad 5F_m L_m \times L_m L_{2m} = 5F_m L_{4m},
\]

we can prove the following summation formula.
we can prove the following summation formula.

Corollary 11 \((m \in \mathbb{N})\).

\[
\sum_{n=0}^{\infty} 2^n \left(5F_mF_{2m}\right)^{n-2} \left(L_mL_{2m}\right)^{n-2} \frac{\binom{2n}{n}}{\binom{n}{n}} = \frac{4L_{3m}^2}{25F_{2m}^2 F_{4m}} + \frac{\pi L_{3m}}{(5F_{2m}F_{4m})^{3/2}}.
\]

Letting \(u = F_{m+1}^2\) and \(v = F_m^2\) in Theorem 5 and then making use of
\[
P_{m+1}^2 + P_m^2 = F_{2m+1},
\]
we can establish the following summation formula.

Corollary 12 \((m \in \mathbb{N})\).

\[
\sum_{n=1}^{\infty} 4^n \frac{n+1}{n+2} \frac{F_{m+1}^2 - F_m^2}{F_{2m+1}} \left(\binom{2n+3}{n+1}\right)^{-1} = \frac{\pi F_{2m+1}^2 F_{m+1}F_{m+2}}{16F_m^3 F_{m+1}}.
\]

Letting \(u = 5F_m\) and \(v = L_m\) in Theorem 6 and then making use of
\[
5F_m + L_m = 2L_{m+1},
\]
we can prove the following summation formula.

Corollary 13 \((m \in \mathbb{N})\).

\[
\sum_{n=0}^{\infty} 2^n \left(n+1\right)\left(n+2\right) \frac{5^n F_m^2 + L_m^2}{n+2} \frac{\binom{2n+5}{n+2}}{\binom{n+2}{n+2}} = \frac{\pi L_{m+1}^3}{16 \left(5F_{2m}\right)^{5/2}} \left\{75F_m^2 - 10F_{2m} + 3L_m^2\right\}.
\]

Finally letting \(u = F_m\) and \(v = L_m\) in Theorem 7 and then making use of
\[
F_m + L_m = 2F_{m+1},
\]
we can prove the following summation formula.

Corollary 14 \((m \in \mathbb{N})\).

\[
\sum_{n=0}^{\infty} 2^n \left(n+1\right)\left(n+2\right) \frac{3^n F_m^2 - L_m^2}{n+2} \frac{\binom{2n+7}{n+3}}{\binom{n+3}{n+3}} = \frac{3\pi F_{m+1}^4}{64F_{2m}^2} \left(F_m - L_m\right) \left\{5F_m^2 + 2F_{2m} + 5L_m^2\right\}.
\]

Up to now, we have restricted ourselves to the value \(x = 1/2\) to exhibit corollaries for applications. Instead, for \(x = 1\), we have the Pell/Lucas numbers \(P_n = P_n(1)\) and \(Q_n = Q_n(1)\). The corresponding Binet form expressions are given by:

\[
P_n = \frac{\left(1 + \sqrt{2}\right)^n - \left(1 - \sqrt{2}\right)^n}{2\sqrt{2}} \quad \text{and} \quad Q_n = \left(1 + \sqrt{2}\right)^n + \left(1 - \sqrt{2}\right)^n.
\]

Therefore for each Proposition, one can directly write down the corresponding formula for Pell/Lucas numbers.

In addition, this paper can be considered as a continuation of the authors’ recent work [9], where more “bridges” were constructed for passing from Pell/Lucas polynomials to Fibonacci/Lucas numbers (see [9], Equations 2–5). For example, by letting \(x = \sqrt{\frac{5}{2}} F_{2\lambda}\), we have
\[
P_{2m} \left(\sqrt{\frac{5}{2}} F_{2\lambda}\right) = \frac{F_{4m\lambda}}{L_{2\lambda}} \sqrt{5}, \quad \text{and} \quad Q_{2m} \left(\sqrt{\frac{5}{2}} F_{2\lambda}\right) = L_{4m\lambda}.
\]

Then, applying these relations to Propositions 1, 3 (the first identity), 4 and 5, we can show the following four elegant formulae. Further identities can be deduced analogously.
Corollary 15 (Even $m \in \mathbb{N}$).

\[
\sum_{n=0}^{\infty} \frac{L_{2mn\lambda}}{2n+1} \left( \frac{4}{L_{2m\lambda}} \right)^n \binom{2n}{n}^{-1} = \frac{\pi}{2} L_{2m\lambda},
\]

\[
\sum_{n=1}^{\infty} \frac{L_{2mn\lambda+2m\lambda}}{n(2n+1)} \left( \frac{4}{L_{2m\lambda}} \right)^n \binom{2n}{n}^{-1} = 2L_{2m\lambda} - \pi,
\]

\[
\sum_{n=0}^{\infty} \frac{L_{2mn\lambda-4m\lambda}}{\binom{2n}{n}} \left( \frac{4}{L_{2m\lambda}} \right)^n = L_{2m\lambda}^2 + \frac{\pi}{2} L_{2m\lambda},
\]

\[
\sum_{n=1}^{\infty} \frac{n+1}{n+2} \frac{F_{2mn\lambda}}{\binom{2n+3}{n+1}} \left( \frac{4}{L_{2m\lambda}} \right)^n = \frac{\pi}{16} L_{2m\lambda} F_{4m\lambda}.
\]

Author Contributions: Writing—original draft & supervision, W.C.; Writing—review & editing, D.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

15. Trojovský, P. On terms of generalized Fibonacci sequences which are powers of their indexes. *Mathematics* **2019**, 7, 700. [CrossRef]
21. Zucker, I.J. On the series $\sum_{k=1}^{\infty} \left( \frac{2k}{k} \right)^{-1} k^{-n}$. *J. Number Theory* **1985**, 20, 92–102. [CrossRef]