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Some Generalized Versions of Chevet–Saphar Tensor Norms

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Abstract: The paper is concerned with some generalized versions $g_E$ and $w_E$ of classical tensor norms. We find a Banach space $E$ for which $g_E$ and $w_E$ are finitely generated tensor norms, and show that $g_E$ and $w_E$ are associated with the ideals of some $E$-nuclear operators. We also initiate the study of some theories of our tensor norms.

Keywords: Schauder basis; vector-valued sequence; tensor norm; operator ideal

MSC: 46B45; 47L20

1. Introduction

One of the important theories in the study of Banach spaces is the theory of tensor norms (see Section 2 for the definition of tensor norm). It provides not only new examples of Banach spaces but also a powerful tool in the study of Banach operator ideals. One may refer to [1–5] and the references therein for various information and content about tensor norms. Throughout this paper, Banach spaces will be denoted by $X$ and $Y$ over $\mathbb{R}$ or $\mathbb{C}$, with dual spaces $X^*$ and $Y^*$, and the closed unit ball of $X$ will be denoted by $B_X$. We will denote by $X \otimes Y$ the algebraic tensor product of $X$ and $Y$. The most classical two tensor norms are the injective norm $\varepsilon$ and the projective norm $\pi$, which were systematically investigated by Grothendieck [6,7]. For $u \in X \otimes Y$,

$$\varepsilon(u; X, Y) := \sup \left\{ \left\| \sum_{n=1}^{l} x_n^*(y_n) \right\| : x_n^* \in B_{X^*}, y_n^* \in B_{Y^*} \right\},$$

where $\sum_{n=1}^{l} x_n \otimes y_n$ is any representation of $u$, and

$$\pi(u; X, Y) := \inf \left\{ \left\| \sum_{n=1}^{l} x_n \right\| \left\| y_n \right\| : u = \sum_{n=1}^{l} x_n \otimes y_n, l \in \mathbb{N} \right\}.$$

More recently, the author [8] introduced a tensor norm related with the injective norm. Lapresté [9] introduced the most generalized version $a_{p,q}$ of the projective norm, and its some related topics were studied by Díaz, López-Molina, Rivera [10] and the author [11]. Many of the interesting tensor norms can be obtained from the tensor norm $a_{p,q}$ ($1 \leq p, q \leq \infty, 1/p + 1/q \geq 1$), which is defined as follows. Let $1 \leq r \leq \infty$ with $1/r = 1/p + 1/q - 1$. For $u \in X \otimes Y$, let

$$a_{p,q}(u) := \inf \left\{ \left\| \left( \lambda_n \right)_{n=1}^{l} \right\|_r \sup_{x_n^* \in B_{X^*}} \left\| (x_n^*) \right\|_{n=1}^{l} \sup_{y_n^* \in B_{Y^*}} \left\| (y_n^*) \right\|_{n=1}^{l} \| p^* : u = \sum_{n=1}^{l} \lambda_n x_n \otimes y_n, l \in \mathbb{N} \right\},$$

where $p^*$ is the conjugate index of $p$ and $\| \cdot \|_p$ means the $\ell_p$-norm. Then, we see that

$$g_p(u) := \inf \left\{ \left\| \left( \| x_n \| \right)_{n=1}^{l} \right\|_p \sup_{y_n^* \in B_{Y^*}} \left\| (y_n^*) \right\|_{n=1}^{l} \| p^* : u = \sum_{n=1}^{l} x_n \otimes y_n, l \in \mathbb{N} \right\} = a_{p,1}(u),$$
\[ w_p(u) := \inf \left\{ \sup_{x^* \in B_{X^*}} \left| (x^*(x_n))_n \right|_p \sup_{y^* \in B_{Y^*}} \left| (y^*(y_n))_n \right|_{p'} : u = \sum_{n=1}^\infty x_n \otimes y_n, l \in \mathbb{N} \right\} = a_{p,p'}(u) \]

and \( \pi(u) = a_{1,1}(u) \). The tensor norms \( g_p \) and \( w_p \) were introduced and studied by Chevet and Saphar [12,13]; see [10,11,14–24] and the references therein for the investigation on related topics.

In this paper, we consider another generalization of \( g_p \) and \( w_p \). These tensor norms are somehow determined by the Banach space \( \ell_p \). Naturally, one may extend these notions by replacing \( \ell_p \) by a general Banach space with a Schauder basis. Throughout this paper, \( E \) is a Banach space having the 1-unconditional Schauder basis \( (e_n)_n \) is the sequence of coordinate functionals for \( (e_n)_n \) and \( E_1 := \overline{\text{span}} \{e_n\}_{n=1}^\infty \). For a finite subset \( F \) of \( \mathbb{N} \) and \( \{x_n\}_{n \in F} \subset X \), let

\[ \|(x_n)_{n \in F}\|_{E(X)} := \left\{ \sum_{n \in F} \|x_n\|_{E} \right\} \quad \text{and} \quad \|(x_n)_{n \in F}\|_{E^\pi(X)} := \sup_{x^* \in B_{X^*}} \left\{ \sum_{n \in F} \|x^*(x_n)e_n\|_{E^\pi} \right\} \].

We are now ready to introduce the main notion in this paper.

**Definition 1.** For \( u \in X \otimes Y \), let

\[ g_E(u; X, Y) := \inf \left\{ \|(x_n)_{n \in F}\|_{E(X)} \left\| (y_n)_{n \in F} \right\|_{E^\pi(Y)} : u = \sum_{n \in F} x_n \otimes y_n, F \subset \mathbb{N} \right\} \],

\[ w_E(u; X, Y) := \inf \left\{ \|(x_n)_{n \in F}\|_{E^\pi(X)} \left\| (y_n)_{n \in F} \right\|_{E^\pi(Y)} : u = \sum_{n \in F} x_n \otimes y_n, F \subset \mathbb{N} \right\} \].

For instance, \( g_{\ell_p} = g_p \) and \( w_{\ell_p} = w_p \) (1 \( \leq p < \infty \)), and \( g_{c_0} = w_{c_0} = g_{\infty} = w_{\infty} \).

Tensor norms are closely related with normed operator ideals. Actually, in view of the monograph of Defant and Floret [2], there is a one-to-one correspondence between maximal Banach operator ideals and finitely generated tensor norms. A tensor norm \( \alpha \) is said to be associated with a normed operator ideal \([A, \| \cdot \|_A]\) if the canonical map from \( A(M, N) \) to \( M^* \otimes_\alpha N \) equipped with the norm \( \alpha \) is an isometry for every finite-dimensional normed spaces \( M \) and \( N \). It is well known that \( g_p \) is associated with the ideal of \( p \)-nuclear operators. The starting point of this paper comes from [25], where the \( E \)-nuclear operators (see Section 2 for the definition of \( E \)-nuclear operators) were defined and replaced \( \ell_p \) by \( E \) in the notion of \( p \)-nuclear operators. The main goal of this paper is to find a Banach space \( E \) for which \( g_E \) and \( w_E \) are tensor norms, and show that \( g_E \) and \( w_E \) are associated with the ideals of \( E \)-nuclear operators. Obtaining some results for \( g_E \) and \( w_E \), we provide a base for further investigations of the \( g_E \)- and \( w_E \)-tensor norms and \( E \)-operator ideals. We focus on the Banach space \( E = (\sum_1^p \ell_q) \) of infinite \( \ell_p \) direct sum of \( \ell_q \)-spaces, which is a generalization of \( \ell_p \). For this case, we extend some well known results for \( g_E \) and \( w_E \) as follows.

In Section 2, for \( E = (\sum_1^p \ell_q) \) (1 \( \leq p, q \leq \infty \)), we prove that \( g_E \) and \( w_E \) are finitely generated tensor norms, and it is demonstrated that \( g_E \) and \( w_E \) are associated with the ideals of \( E \)-nuclear operators. In Section 3, we prove that \( g_E \) is left injective and for every Banach space \( X \), the injective tensor product \( X \otimes E \) is isometric to \( X \otimes_{w_E} E \); furthermore, if \( (e_n)_n \) is shrinking, then \( E^\pi \otimes_E X \) is isometric to \( E^\pi \otimes_{w_E} X \). Additionally, we establish the completions of our \( E \)-tensor norms for \( E = (\sum_1^p \ell_q) \), and as an application, we represent \( E \)-nuclear operators acting on dual spaces. We refer to the book of Defant and Floret [2] as a reference to the main notions and formulas in the theory of tensor norms and (quasi) normed operator ideals.

2. The \( g_E \)- and \( w_E \)-Tensor Norms and Their Associated Operator Ideals

Let us recall that a tensor norm \( \alpha \) is a norm on \( X \otimes Y \) for each pair of Banach spaces \( X \) and \( Y \) such that
(TN1) $\epsilon \leq \kappa \leq \pi$;
(TN2) for all operators $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$,
\[ \|T_1 \otimes T_2 : X_1 \otimes_\kappa X_2 \to Y_1 \otimes_\kappa Y_2\| \leq \|T_1\|\|T_2\|. \]

A tensor norm $\alpha$ is said to be finitely generated if
\[ \alpha(u; X, Y) = \inf\{\alpha(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty\} \]
for every $u \in X \otimes Y$.

Proposition 1. Suppose that $(e_n)_{n}$ is normalized. If $g_E$ and $w_E$ satisfy the triangle inequality, then they are finitely generated tensor norms.

Proof. We only consider $g_E$. Let $X$ and $Y$ be Banach spaces. Let $c \in \mathbb{C}$ and let $u = \sum_{n \in F} x_n \otimes y_n$ be an arbitrary representation in $X \otimes Y$. Then
\[ g_E(cu; X, Y) \leq |c|g_E(u; X, Y). \]
Thus $g_E(cu; X, Y) \leq |c|g_E(u; X, Y)$. Since $g_E(u; X, Y) = g_E((1/c)(cu); X, Y) \leq (1/|c|)g_E(cu; X, Y)$, $g_E(cu; X, Y) \leq |c|g_E(u; X, Y)$.

(TN1): Let $u = \sum_{n \in F} x_n \otimes y_n$ be an arbitrary representation in $X \otimes Y$. Let $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$. Then
\[ \left| \sum_{n \in F} x^*(x_n)y^*(y_n) \right| = \left| \left( \sum_{k \in F} y^*(y_k)\right) \left( \sum_{n \in F} x^*(x_n)e_n \right) \right| \leq \left\| \sum_{n \in F} x_n \otimes y_n \right\| \left\| \sum_{n \in F} y_n \otimes x_n \right\|. \]
and
\[ g_E(u; X, Y) \leq \sum_{n \in F} g_E(x_n \otimes y_n) \leq \sum_{n \in F} \|x_n\|\|y_n\|. \]
It follows that $\epsilon(u; X, Y) \leq g_E(u; X, Y) \leq \pi(u; X, Y)$, and so
\[ g_E(u; X, Y) = 0 \iff u = 0 \]
for $u \in X \otimes Y$.

(TN2): Let $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ be operators. Let $u \in X_1 \otimes X_2$ and let $u = \sum_{n \in F} x_n^1 \otimes x_n^2$ be an arbitrary representation. Then
\[ g_E((T_1 \otimes T_2)(u); Y_1, Y_2) = g_E\left( \sum_{n \in F} T_1 x_n^1 \otimes T_2 x_n^2, Y_1, Y_2 \right) \leq \|T_1\|\|T_2\|\|c\|_{E(Y_1)}\|c\|_{E(Y_2)} \]
\[ \leq \|T_1\|\|T_2\|\|c\|_{E(Y_1)}\|c\|_{E(Y_2)} \]
\[ \leq \|T_1\|\|T_2\|\|c\|_{E(Y_1)}\|c\|_{E(Y_2)}. \]
Hence
\[ g_E((T_1 \otimes T_2)(u); Y_1, Y_2) \leq \|T_1\|\|T_2\|g_E(u; X_1, X_2). \]

To show that $g_E$ is finitely generated, let $u \in X \otimes Y$ and let $u = \sum_{n \in F} x_n \otimes y_n$ be an arbitrary representation. Let $M_0 := \text{span}\{x_n\}_{n \in F}$ and $N_0 := \text{span}\{y_n\}_{n \in F}$. Using the Hahn–Banach extension theorem, we have
We can find representations

\[ \inf \{ g_E(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty \} \]
\[ \leq g_E(u; M_0, N_0) \]
\[ \leq \| (x_n)_{n \in F} \|_{E(M_0)} \sup_{z^* \in B_{M_0}} \| \sum_{n \in F} z^*(y_n) e_n^* \|_E, \]
\[ = \| (x_n)_{n \in F} \|_{E(X)} \| (y_n)_{n \in F} \|_{E^p(Y)}. \]

Hence,
\[ \inf \{ g_E(u; M, N) : u \in M \otimes N, \dim M, \dim N < \infty \} \leq g_E(u; X, Y). \]

We can now prove:

**Theorem 1.** If \( E = (\sum \ell_q)_p \) (1 \( \leq \) p, q \( < \infty \)), E = (\( \sum \ell_q \))_p (1 \( \leq \) p \( < \infty \)) or E = (\( \sum \ell_q \))_c (1 \( \leq \) q \( < \infty \)), then \( g_E \) and \( \inf \) are finitely generated tensor norms.

**Proof.** We only consider \( g_E \). Let X and Y be Banach spaces. By Proposition 1, we only need to show the triangle inequality of \( g_E \).

For the he case \( E = (\sum \ell_q)_p \) (1 \( < \) p, q \( < \infty \)), let \( u, v \in X \otimes Y \) and let \( \delta > 0 \) be given. We can find representations
\[ u = \sum_{n=1}^{l} \sum_{k=1}^{l} x_{nk}^1 \otimes y_{nk}^1 \]
\[ \text{and} \ v = \sum_{n=1}^{l} \sum_{k=1}^{l} x_{nk}^2 \otimes y_{nk}^2 \]
such that
\[ \| (x_{nk}^1)_{k=1}^{l} \|_{E(X)} \| (y_{nk}^1)_{k=1}^{l} \|_{E^p(Y)} \leq (1 + \delta) g_E(u; X, Y), \]
\[ \| (x_{nk}^2)_{k=1}^{l} \|_{E(X)} \| (y_{nk}^2)_{k=1}^{l} \|_{E^p(Y)} \leq (1 + \delta) g_E(v; X, Y). \]

We may assume that
\[ \| (x_{nk}^1)_{k=1}^{l} \|_{E(X)} \leq ((1 + \delta) g_E(u; X, Y))^{1/p}, \]
\[ \| (y_{nk}^1)_{k=1}^{l} \|_{E^p(Y)} \leq ((1 + \delta) g_E(u; X, Y))^{1/p}, \]
\[ \| (x_{nk}^2)_{k=1}^{l} \|_{E(X)} \leq ((1 + \delta) g_E(v; X, Y))^{1/p}, \]
\[ \| (y_{nk}^2)_{k=1}^{l} \|_{E^p(Y)} \leq ((1 + \delta) g_E(v; X, Y))^{1/p}. \]

Since
\[ u + v = \sum_{i=1}^{2} \sum_{n=1}^{l} \sum_{k=1}^{l} x_{nk}^i \otimes y_{nk}^i, \]
\[ g_E(u; X, Y) \leq \left( \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{p/q} \right)^{1/p} \sup_{y^* \in B^*_r} \left( \sum_{i=1}^{l} \sum_{k=1}^{l} |y^*(y_{n_k}^i)|^{p^*/q^*} \right)^{1/p^*} \]

\[ \leq \left( \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{p/q} \right)^{1/p} \sup_{y^* \in B^*_r} \left( \sum_{i=1}^{l} \sum_{k=1}^{l} |y^*(y_{n_k}^i)|^{p^*/q^*} \right)^{1/p^*} \]

\[ \leq \left( \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{p/q} + \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{p/q} \right)^{1/p} \]

\[ \left( \sup_{y^* \in B^*_r} \left( \sum_{i=1}^{l} \sum_{k=1}^{l} |y^*(y_{n_k}^i)|^{p^*/q^*} \right)^{1/p^*} \right) \]

\[ \leq ((1 + \delta)(g_E(u; X, Y) + g_E(v; X, Y)))^{1/p} ((1 + \delta)(g_E(u; X, Y) + g_E(v; X, Y)))^{1/p^*} \]

\[ = (1 + \delta)(g_E(u; X, Y) + g_E(v; X, Y)). \]

Since \( \delta > 0 \) was arbitrary,

\[ g_E(u; v; X, Y) \leq g_E(u; X, Y) + g_E(v; X, Y). \]

For the case \( E = (\sum \ell_1)_p \ (1 < p < \infty) \):

\[ g_E(u; v; X, Y) \]

\[ \leq \left( \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{p/q} \right)^{1/p} \sup_{y^* \in B^*_r} \left( \sum_{i=1}^{l} \sum_{k=1}^{l} |y^*(y_{n_k}^i)|^{p^*/q^*} \right)^{1/p^*} \]

\[ \leq \left( \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{p/q} + \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{p/q} \right)^{1/p} \]

\[ \left( \sup_{y^* \in B^*_r} \left( \sum_{i=1}^{l} \sum_{k=1}^{l} |y^*(y_{n_k}^i)|^{p^*/q^*} \right)^{1/p^*} \right) \]

\[ \leq ((1 + \delta)(g_E(u; X, Y) + g_E(v; X, Y)))^{1/p} ((1 + \delta)(g_E(u; X, Y) + g_E(v; X, Y)))^{1/p^*} \]

\[ = (1 + \delta)(g_E(u; X, Y) + g_E(v; X, Y)). \]

For the case \( E = (\sum \ell_q)_1 \ (1 < q < \infty) \): We may assume that

\[ \|(x_{n_k}^i)_{k=1}^{l}\|_{E_1} \leq (1 + \delta)g_E(u; X, Y), \|(y_{n_k}^i)_{k=1}^{l}\|_{E_1} \leq 1, \]

\[ \|(x_{n_k}^2)_{k=1}^{l}\|_{E_1} \leq (1 + \delta)g_E(v; X, Y), \|(y_{n_k}^2)_{k=1}^{l}\|_{E_1} \leq 1. \]

Then

\[ g_E(u; v; X, Y) \]

\[ \leq \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{1/q} \sup_{y^* \in B^*_r} \left( \sum_{i=1}^{l} \sum_{k=1}^{l} |y^*(y_{n_k}^i)|^{p^*/q^*} \right)^{1/p^*} \]

\[ \leq \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{1/q} + \sum_{i=1}^{l} \left( \sum_{k=1}^{l} \|x_{n_k}^i\|^q \right)^{1/q} \]

\[ \leq (1 + \delta)(g_E(u; X, Y) + g_E(v; X, Y)). \]

For the case \( E = (\sum c_0)_p \ (1 < p < \infty) \):
\[ g_E(u + v; X, Y) \]
\[ \leq \left( \sum_{i=1}^{l} \left( \sup_{1 \leq k \leq l} \| x_{nk}^l \|^p \right) + \sup_{1 \leq k \leq l} \left( \sum_{n=1}^{l} \| y^r(y_{nk}^l) \|^p \right) \right)^{1/p} \]
\[ \leq \left( \sum_{n=1}^{l} \left( \sup_{1 \leq k \leq l} \| x_{nk}^l \|^p \right) + \sup_{1 \leq k \leq l} \left( \sum_{n=1}^{l} \| y^r(y_{nk}^l) \|^p \right) \right)^{1/p} \]
\[ \leq \left( 1 + \delta \right) \left( g_E(u; X, Y) + g_E(v; X, Y) \right) \]
\[ = \left( 1 + \delta \right) \left( g_E(u; X, Y) + g_E(v; X, Y) \right). \]

For the case \( E = (\sum \ell_q)_0 \) \((1 < q < \infty)\): We may assume that
\[ \|(x_{nk}^l \|_{E(E(X)} \leq 1, \|(y_{nk}^l \|_{E(Y)} \leq 1 + \delta) g_E(u; X, Y), \]
\[ \|(x_{nk}^l \|_{E(E(X)} \leq 1, \|(y_{nk}^l \|_{E(Y)} \leq 1 + \delta) g_E(v; X, Y). \]

Then
\[ g_E(u + v; X, Y) \]
\[ \leq \sup_{i=1,2,1 \leq n \leq l} \left( \sum_{k=1}^{l} \| x_{nk}^l \|^q \right)^{1/q} \sup_{y^r \in B_Y} \left( \sum_{n=1}^{l} \| y^r(y_{nk}^l) \|^q \right)^{1/q} \]
\[ \leq \sup_{y^r \in B_Y} \left( \sum_{n=1}^{l} \| y^r(y_{nk}^l) \|^q \right)^{1/q} + \sup_{y^r \in B_Y} \left( \sum_{n=1}^{l} \| y^r(y_{nk}^l) \|^q \right)^{1/q} \]
\[ \leq \left( 1 + \delta \right) \left( g_E(u; X, Y) + g_E(v; X, Y) \right). \]

The cases \( E = (\sum \ell_1)_0 \) and \( E = (\sum \ell_0)_1 \) also follow from similar proofs. \( \square \)

Throughout the remainder of this paper, we will assume that \( g_E \) and \( w_E \) are finitely generated tensor norms. For a Banach space \( X \), let us consider the Banach spaces
\[ E(X) := \left\{ (x_n)_n \in X : \sum_{n=1}^{\infty} \| x_n \| e_n \text{ converges in } E \right\} \]
equipped with the norm \( \|(x_n)_n\|_{E(X)} := \sum_{n=1}^{\infty} \| x_n \| e_n \|_E \)
\[ E^w(X) := \left\{ (x_n)_n \in X : \sum_{n=1}^{\infty} x^*(x_n) e_n \text{ converges in } E \text{ for each } x^* \in X^* \right\} \]
equipped with the norm \( \|(x_n)_n\|_{E^w(X)} := \sup_{x^* \in B_{X^*}} \| \sum_{n=1}^{\infty} x^*(x_n) e_n \|_E \)
\[ E^u(X) := \left\{ (x_n)_n \in X : \lim_{l \to \infty} \sup_{n \geq l} \| \sum_{n=l}^{\infty} x^*(x_n) e_n \|_E = 0 \right\} \]
equipped with the norm \( \|(x_n)_n\|_{E^u(X)} \).

Let \( X \) and \( Y \) be Banach spaces, and let \( T : X \to Y \) be an operator such that
\[ T = \sum_{n=1}^{\infty} x_n^* \otimes e_n, \]
where \( x_n^* \otimes e_n(x) = x_n^*(x) e_n \). The following operators were introduced in [25]. We say that \( T \) is \( E\text{-nuclear} \) (respectively, dual \( E\text{-nuclear} \)) if \((x_n)_n \in E(X^*) \) (respectively, \( E^w(X^*) \)) and \((y_n)_n \in E^u(Y) \) (respectively, \( E(Y) \)). The collection of all \( E\text{-nuclear} \) (respectively, dual
E-nuclear) operators from $X$ to $Y$ is denoted by $\mathcal{N}^E_E(X, Y)$ (respectively, $\mathcal{N}^E_E(X, Y)$), and for $T \in \mathcal{N}^E_E(X, Y)$ (respectively, $\mathcal{N}^E_E(X, Y)$), let

$$
\| T \|_{\mathcal{N}^E_E} := \inf \{ (x_n^*)_n \| E(X^*) \| (y_n)_n \| E(Y) \}
$$

(respectively, $\| T \|_{\mathcal{N}^E_E} := \inf \{ (x_n^*)_n \| E^*(X) \| (y_n)_n \| E(Y) \}$),

where the infimum is taken over all such representations. We say that $T$ is uniform E-nuclear (respectively, dual uniform E-nuclear) if $(x_n^*)_n \in E^u(X^*)$ (respectively, $E^u(X^*)$) and $(y_n)_n \in E^u(Y)$ (respectively, $E^u(Y)$). The collection of all uniform E-nuclear (respectively, dual uniform E-nuclear) operators from $X$ to $Y$ is denoted by $\mathcal{N}^E(X, Y)$ (respectively, $\mathcal{N}^E(X, Y)$) and for $T \in \mathcal{N}^E(X, Y)$ (respectively, $\mathcal{N}^E(X, Y)$), let

$$
\| T \|_{\mathcal{N}^E} := \inf \{ (x_n^*)_n \| E^u(X^*) \| (y_n)_n \| E^u(Y) \}
$$

(respectively, $\| T \|_{\mathcal{N}^E} := \inf \{ (x_n^*)_n \| E^u(X) \| (y_n)_n \| E^u(Y) \}$),

where the infimum is taken over all such representations. For instance, $\mathcal{N}_{E_p}$ is the ideal of $p$-nuclear operators, and $\mathcal{N}_{E_p}$ is the ideal of $p$-compact operators (cf. [2,15]).

Let $\mathcal{F}$ be the ideal of finite rank operators and let $X$ and $Y$ be Banach spaces. For $T \in \mathcal{F}(X, Y)$, let

$$
\| T \|_{\mathcal{F}} := \inf \left\{ \left( \| (x_n^*)_n \| E(X^*) \| (y_n)_n \| E(Y) \right) : T = \sum_{n \in F} x_n^* \otimes y_n, \text{ finite } F \subset \mathbb{N} \right\},
$$

$$
\| T \|_{\mathcal{F}_E} := \inf \left\{ \left( \| (x_n^*)_n \| E^*(X^*) \| (y_n)_n \| E(Y) \right) : T = \sum_{n \in F} x_n^* \otimes y_n, \text{ finite } F \subset \mathbb{N} \right\},
$$

$$
\| T \|_{\mathcal{N}^0_E} := \inf \left\{ \left( \| (x_n^*)_n \| E^u(X^*) \| (y_n)_n \| E(Y) \right) : T = \sum_{n \in F} x_n^* \otimes y_n, \text{ finite } F \subset \mathbb{N} \right\},
$$

$$
\| T \|_{\mathcal{N}^E} := \inf \left\{ \left( \| (x_n^*)_n \| E^u(X) \| (y_n)_n \| E(Y) \right) : T = \sum_{n \in F} x_n^* \otimes y_n, \text{ finite } F \subset \mathbb{N} \right\}.
$$

Let $\alpha$ be the transposed tensor norm (see [2]) of a tensor norm $\alpha$. Let $X$ and $Y$ be Banach spaces. For $T = \sum_{n \in F} x_n^* \otimes y_n \in \mathcal{F}(X, Y)$, let $u_T := \sum_{n \in F} x_n^* \otimes y_n \in X^* \otimes Y$. Then we see that

$$
\| T \|_{\alpha} = g_E(u_T; X^*, Y), \| T \|_{\alpha_0} = w_E(u_T; X^*, Y),
$$

$$
\| T \|_{\alpha^E} = g^E(u_T; X^*, Y), \| T \|_{\alpha_0^E} = w^E(u_T; X^*, Y).
$$

**Proposition 2.** If $X$ or $Y$ is a finite-dimensional normed space, then

$$
\| T \|_{\alpha^E} = \| T \|_{\mathcal{N}^E}, \| T \|_{\alpha^E} = \| T \|_{\mathcal{N}^E}, \| T \|_{\alpha_0^E} = \| T \|_{\mathcal{N}^E}, \| T \|_{\alpha_0^E} = \| T \|_{\mathcal{N}^E}
$$

for every operator $T$ from $X$ to $Y$.

**Proof.** We only consider $\mathcal{N}^E_0$. Let $T : X \to Y$ be an operator, and let $\delta > 0$ be given. Let

$$
T = \sum_{n=1}^{\infty} x_n^* \otimes y_n
$$

be a dual E-nuclear representation such that

$$
\| (x_n^*)_n \| E^u(X^*) \| (y_n)_n \| E(Y) \leq (1 + \delta) \| T \|_{\mathcal{N}^E}.
$$
If \( X \) is finite-dimensional, then there exists an \( l \in \mathbb{N} \) such that
\[
\left\| \sum_{n \geq l+1} x_n^* \otimes y_n \right\| \leq \sup_{x \in X} \sum_{n \geq l+1} |x_n^*(x)| \|y_n\| \\
= \sup_{x \in X} \left( \sum_{n \geq l+1} |x_n^*(x)| \right) \left( \sum_{n \geq l+1} \|y_n\| \|e_n\| \right) \\
\leq \left\| (x_n^*_n)_{n \in \mathbb{N}} \right\|_{E^p(X')} \| y_{l+1} \|_{L(E)} \\
\leq \delta \| T \|_{\mathcal{L}(X)} \| \text{id}_X \|_{\mathcal{L}(X)}.
\]
where \( \text{id}_X \) is the identity map on \( X \). We have
\[
\| T \|_{\mathcal{L}(X)} \leq \left( \sum_{n=1}^l \| x_n^*_n \|_{L(E)} \right) + \left( \sum_{n \geq l+1} \| x_n^*_n \|_{L(E)} \right) \\
\leq \left\| (x_n^*_n)_{n \in \mathbb{N}} \right\|_{E^p(X')} \| y_{l+1} \|_{L(E)} + \left( \sum_{n \geq l+1} \| x_n^*_n \|_{L(E)} \right) \| \text{id}_X \|_{\mathcal{L}(X)} \\
\leq (1 + 2\delta) \| T \|_{\mathcal{L}(X)}.
\]
If \( Y \) is finite-dimensional, then \( \text{id}_Y \) can be replaced by \( \text{id}_Y \) in the above proof. \( \square \)

From Proposition 2, we have:

**Corollary 1.** The tensor norms \( s_E, s^l_E, w_E \) and \( w^l_E \), respectively, are associated with \( [\mathcal{N}_E, \| \cdot \|_{\mathcal{N}_E}], [\mathcal{N}^E, \| \cdot \|_{\mathcal{N}^E}], \| u_{\mathcal{N}_E} \| \| u_{\mathcal{N}_E} \| \) and \( [u_{\mathcal{N}^E}, \| \cdot \|_{u_{\mathcal{N}^E}}] \).

### 3. Some Results of the \( g_E \) and \( w_E \)-Tensor Norms

A tensor norm \( \alpha \) is called left-projective if, for every quotient operator \( q : Z \to X \), the operator
\[
q \otimes \text{id}_Y : Z \otimes \alpha Y \to X \otimes \alpha Y
\]
is a quotient operator for all Banach spaces \( X, Y \) and \( Z \). If the transposed \( \alpha^t \) of \( \alpha \) is left-projective, then \( \alpha \) is called right-projective.

**Proposition 3.** The tensor norm \( g_E \) is left-projective.

**Proof.** Let \( q : Z \to X \) be a quotient operator. To show that the map
\[
q \otimes \text{id}_Y : Z \otimes \alpha Y \to X \otimes \alpha Y
\]
is a quotient operator, let \( u = \sum_{n \in F} x_n \otimes y_n \in X \otimes \alpha Y \). We should show that
\[
g_E(u; X, Y) \geq \inf \{ g_E(v; Z, Y) : v \in Z \otimes \alpha Y, q \otimes \text{id}_Y(v) = u \}.
\]
Let \( \delta > 0 \) be given. Since \( q \) is a quotient operator, there exists \( \{z_n\}_{n \in F} \subset Z \) such that
\[
qz_n = x_n, \|z_n\| \leq (1 + \delta) \|x_n\|
\]
for every $n \in F$. Then we have
\[
\inf\{g_{E}(v; Z, Y) : v \in Z \otimes_{g_{E}} Y, q \otimes id_{Y}(v) = u\}
\leq g_{E}\left( \sum_{n \in F} z_{n} \otimes y_{n}; Z, Y \right)
\leq \|\sum_{n \in F} (z_{n})_{n \in F}\|_{(E_{s})^{w}(Y)}
= \left\| \sum_{n \in F} \|z_{n}\|_{E} \right\|_{E} \|\sum_{n \in F} (y_{n})_{n \in F}\|_{(E_{s})^{w}(Y)}
\leq (1 + \delta) \left\| \sum_{n \in F} \|x_{n}\|_{E} \right\|_{E} \|\sum_{n \in F} (y_{n})_{n \in F}\|_{(E_{s})^{w}(Y)}.
\]
Since $u = \sum_{n \in F} x_{n} \otimes y_{n}$ was an arbitrary representation,
\[
\inf\{g_{E}(v; Z, Y) : v \in Z \otimes_{g_{E}} Y, q \otimes id_{Y}(v) = u\} \leq (1 + \delta) g_{E}(u; X, Y).
\]
Since $\delta > 0$ was also arbitrary, we complete the proof. \(\square\)

For a tensor norm $\alpha$, we will denote by $X \hat{\otimes}_{\alpha} Y$ the completion of the normed space $X \otimes_{\alpha} Y$.

**Lemma 1** ([2], Proposition 21.7(1)). For a finitely generated tensor norm $\alpha$, if a Banach space $X$ has the approximation property, then for every Banach space $Y$, the natural map
\[
I_{\alpha} : Y \hat{\otimes}_{\alpha} X \rightarrow Y \hat{\otimes}_{\epsilon} X
\]
is injective.

**Theorem 2.** For every Banach space $X$,
\[
X \otimes_{\epsilon} E = X \otimes_{w_{E}} E
\]
holds isometrically, and if $(e_{n})_{n}$ is shrinking, then
\[
E^{*} \otimes_{\epsilon} X = E^{*} \otimes_{w_{E}} X
\]
holds isometrically.

**Proof.** In order to prove the first statement, let $u \in X \otimes E$, and let $U : X^{*} \rightarrow E$ be the corresponding finite rank operator for $u$. Then, $U^{*}(E^{*})$ can be viewed with a subset of $X$. Thus, for every $x^{*} \in X^{*}$,
\[
Ux^{*} = \sum_{i=1}^{\infty} (e_{i}^{*} Ux^{*}) e_{i} = \sum_{i=1}^{\infty} x^{*}(U^{*}e_{i}^{*}) e_{i}.
\]
Since $U(B_{X^{*}})$ is a relatively compact subset of $E$,
\[
\lim_{l \rightarrow \infty} \epsilon\left( \sum_{i=1}^{l} U^{*} e_{i}^{*} \otimes e_{i} - u; X, E \right) = \lim_{l \rightarrow \infty} \left\| \sum_{i=1}^{l} U^{*} e_{i}^{*} \otimes e_{i} - U \right\|
= \lim_{l \rightarrow \infty} \sup_{x^{*} \in B_{X^{*}}} \left\| \sum_{i=1}^{l} (e_{i}^{*} Ux^{*}) e_{i} - Ux^{*} \right\|_{E} = 0.
\]
Consequently,
\[
u = \sum_{i=1}^{\infty} U^{*} e_{i}^{*} \otimes e_{i}.
converges in $X \hat{\otimes}_k E$.

To show that the above series unconditionally converges in $X \hat{\otimes}_w E$, let $\delta > 0$ be given. Let $\{Ux_k^*\}_{k=1}^m$ be a $\delta/2$-net for $U(B_{X^*})$. Choose an $I_0 \in \mathbb{N}$ so that

$$\left\| \sum_{i \geq I_0} (e_i^* Ux_k^*) e_i \right\|_E \leq \frac{\delta}{2}$$

for every $k = 1, \ldots, m$. Now, let $G$ be an arbitrary finite subset of $\mathbb{N}$ with $\min G > I_0$. Let $x^* \in B_{X^*}$ and $e^* \in B_{E^*}$. Let $k_0 \in \{1, \ldots, m\}$ be such that

$$\|Ux^* - Ux_{k_0}^*\|_E \leq \frac{\delta}{2}.$$

Then we have

$$\left\| \sum_{i \in G} x^* (U^* e_i^*) e_i \right\|_E \leq \left\| \sum_{i \in G} (e^* e_i) e_i \right\|_{E^*} = \left\| \sum_{i \in G} (e_i^* Ux^*) e_i \right\|_E \leq \sup_{\alpha \in \Sigma \alpha \in G} \left\| \sum_{i \in G} e_i^*(\alpha e_i) \right\|_E,$$

$$\leq \left\| \sum_{i \in G} (e_i^* U(x^* - x_{k_0}^*)) e_i \right\|_E + \left\| \sum_{i \in G} (e_i^* Ux_{k_0}^*) e_i \right\|_E,$$

$$\leq \left\| \sum_{i \in G} (e_i^* U(x^* - x_{k_0}^*)) e_i \right\|_E + \left\| \sum_{i \geq I_0} (e_i^* Ux_{k_0}^*) e_i \right\|_E,$$

$$\leq \|Ux^* - Ux_{k_0}^*\|_E + \frac{\delta}{2} \leq \delta.$$

Consequently,

$$w_E\left( \sum_{i \in G} U^* e_i^* \otimes e_i; X, E \right) \leq \| (U^* e_i^*)_{i \in G} \|_{E^w(X)} \| (e_i)_{i \in G} \|_{E^*_w(E)} \leq \delta$$

and so

$$v := \sum_{i=1}^{\infty} U^* e_i^* \otimes e_i$$

unconditionally converges in $X \hat{\otimes}_w E$. Since a Banach space with a basis has the approximation property, by Lemma 1, $u = v$ in $X \hat{\otimes}_w E$. Then, since for every $l \in \mathbb{N}$,

$$w_E\left( \sum_{i=1}^{l} U^* e_i^* \otimes e_i; X, E \right) \leq \| (U^* e_i^*)_{i=1}^{l} \|_{E^w(X)} \| (e_i)_{i=1}^{l} \|_{E^*_w(E)}$$

$$\leq \sup_{x^* \in B_{X^*}} \left\| \sum_{i=1}^{l} (e_i^* Ux^*) e_i \right\|_{E^*},$$

$$w_E(u; X, E) \leq \|U\| = \epsilon(u; X, E).$$

In order to prove the second statement, let $v \in E^* \otimes X$ and let $V : E \to X$ be the corresponding finite rank operator for $v$. For every $e = \sum a_i e_i \in E$,

$$Ve = \sum_{i=1}^{\infty} a_i Ve_i = \sum_{i=1}^{\infty} (e_i^* e) Ve_i.$$
Since \((e^*_i)_i\) is a basis for \(E^*\), and \(V^*(B_{X^*})\) is a relatively compact subset of \(E^*\),

\[
\lim_{l \to \infty} \epsilon \left( \sum_{i=1}^{l} e^*_i \otimes V e_i - V^*, X \right) = \lim_{l \to \infty} \left\| \sum_{i=1}^{l} e^*_i \otimes V e_i - V^* \right\|
\]

\[
= \lim_{l \to \infty} \left\| \sum_{i=1}^{l} V e_i \otimes e^*_i - V^* \right\|
\]

\[
= \lim_{l \to \infty} \sup_{x \in B_{X^*}} \left\| \sum_{i=1}^{l} (V^* x^*)(e_i) e^*_i - V^* x^* \right\|_{E^*} = 0.
\]

Consequently,

\[
v = \sum_{i=1}^{\infty} e^*_i \otimes V e_i
\]

converges in \(E^* \otimes_{\varepsilon} X\).

To show that the above series unconditionally converges in \(E^* \otimes_{\omega_E} X\), let \(\delta > 0\) be given. Let \(\{V^* x^*_k\}_{k=1}^{m} \) be a \(\delta/2\)-net for \(V^*(B_{X^*})\). Choose an \(l_g \in \mathbb{N}\) so that

\[
\left\| \sum_{l \geq l_g} V^* x^*_k (e_i) e^*_i \right\|_{E^*} \leq \frac{\delta}{2}
\]

for every \(k = 1, ..., m\). Now, let \(G\) be an arbitrary finite subset of \(\mathbb{N}\) with \(\min G > l_g\). Let \(x^* \in B_{X^*}\) and \(e^{**} \in B_{E^{**}}\). Let \(k_0 \in \{1, ..., m\}\) be such that

\[
\| V^* x^* - V^* x^*_{k_0} \|_{E^*} \leq \delta.
\]

Then, we have

\[
\left\| \sum_{i \in G} e^{**} (e^*_i) e_i \right\|_{E^*} \left\| \sum_{i \in G} (x^* V e_i) e^*_i \right\|_{E^*} = \sup_{\sum_k a_k e^*_k \in \mathbb{R} e^*} \left\| \sum_{i \in G} e^{**} (a_i e^*_i) \right\| \left\| \sum_{i \in G} V^* x^*(e_i) e^*_i \right\|_{E^*}
\]

\[
\leq \left\| \sum_{i \in G} V^* x^*(e_i) e^*_i \right\|_{E^*} + \left\| \sum_{i \in G} V^* x^*_{k_0} (e_i) e^*_i \right\|_{E^*}
\]

\[
\leq \left\| \sum_{i=1}^{\infty} V^* (x^* - x^*_{k_0}) (e_i) e^*_i \right\|_{E^*} + \left\| \sum_{i \geq l_g} V^* x^*_{k_0} (e_i) e^*_i \right\|_{E^*}
\]

\[
\leq \| V^* (x^* - x^*_{k_0}) \|_{E^*} + \frac{\delta}{2} \leq \delta.
\]

Consequently,

\[
\omega_E \left( \sum_{i \in G} e^*_i \otimes V e_i; E^*, X \right) \leq \| (e^*_i)_{i \in G} \|_{E^* (E^*)} \| (V e_i)_{i \in G} \|_{\omega_E (X)} \leq \delta
\]

and so

\[
u := \sum_{i=1}^{\infty} e^*_i \otimes V e_i
\]
unconditionally converges in $E^* \hat{\otimes}_{w_Y} X$. By Lemma 1, since $v^l = u^l$ in $X \hat{\otimes} E^*$, $v^l = u^l$ in $X \hat{\otimes}_{w_E} E^*$, and so $v = u$ in $E^* \hat{\otimes}_{w_E} X$. Since for every $l \in \mathbb{N}$,

$$w_E \left( \sum_{j=1}^{l} e_j^* \otimes V e_j^* , X \right) \leq \| (e_j^*)_{j=1}^{l} \|_{E^* (E^*)} \| (V e_j)_{j=1}^{l} \|_{E^* (X)}$$

$$\leq \sup_{x \in \mathbb{B}_{c_X}} \left| \sum_{j=1}^{l} V^* x^* (e_j^*) \right|_{E^*}$$

$$w_E (v; E^*, X) \leq \| V \| = e (v; E^*, X).$$

Now, we consider the completions of our tensor norms. The following lemma is well known.

**Lemma 2.** Let $(Z, \| \cdot \|)$ be a normed space, and let $(\hat{Z}, \| \cdot \|)$ be its completion. If $z \in \hat{Z}$, then for every $\delta > 0$, there exists a sequence $(z_n)_n$ in $Z$ such that

$$\sum_{n=1}^{\infty} \| z_n \| \leq (1 + \delta) \| z \|$$

and $z = \sum_{n=1}^{\infty} z_n$ converges in $\hat{Z}$.

**Proposition 4.** Suppose that $E = (\sum \ell_q)_p (1 \leq p, q < \infty)$, $E = (\sum c_q)_p (1 \leq p < \infty)$ or $E = (\sum \ell_q)_c (1 \leq q < \infty)$. If $u \in X \hat{\otimes}_{w_E} Y$, then there exist $(x_n)_n \in E^w (X)$ and $(y_n)_n \in E^w (Y)$ such that

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

unconditionally converges in $X \hat{\otimes}_{w_E} Y$ and

$$w_E (u; X, Y) = \inf \left\{ \| (x_n)_n \|_{E^w (X)} \| (y_n)_n \|_{E^w (Y)} : u = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}.$$

**Proof.** Let $u \in X \hat{\otimes}_{w_E} Y$, and let $\delta > 0$ be given. Then, by Lemma 2, there exists a sequence $(u_n)_n$ in $X \otimes Y$ such that

$$\sum_{n=1}^{\infty} w_E (u_n; X, Y) \leq (1 + \delta) w_E (u; X, Y)$$

and $u = \sum_{n=1}^{\infty} u_n$ converges in $X \hat{\otimes}_{w_E} Y$.

We only consider the case $E = (\sum \ell_q)_p (1 < p, q < \infty)$. The proofs of the other cases are similar. For every $n \in \mathbb{N}$, let

$$u_n = \sum_{i=1}^{m_n} \sum_{j=1}^{m_n} x_{ij}^n \otimes y_{ij}^n$$

be such that

$$\left\| (x_{ij}^n)_{j=1}^{m_n} \right\|_{E^w (X)} \left\| (y_{ij}^n)_{j=1}^{m_n} \right\|_{E^w (Y)} \leq (1 + \delta) w_E (u_n; X, Y).$$

We may assume that

$$\left\| (x_{ij}^n)_{j=1}^{m_n} \right\|_{E^w (X)} \leq ((1 + \delta) w_E (u_n; X, Y))^{1/p},$$
\[
\|((y_{ij}^n)_{i=1}^m)_{n=1}^m\|_{E^p(Y)} \leq ((1 + \delta)w_E(u_n; X, Y))^{1/p}.
\]

In order to show that \( u = \sum_{n=1}^\infty \sum_{i=1}^m x_i^n \otimes y_i^n \) unconditionally converges in \( X \hat{\otimes} w_2 Y \) and \(((x_{ij}^n)_{i=1}^m)_{j=1}^m) \in E^u(X) \) and \(((y_{ij}^n)_{j=1}^m)_{i=1}^m) \in E^u(Y)\), let \( \gamma > 0 \) be given. Choose an \( N_\gamma \in \mathbb{N} \) so that for all \( l \geq N_\gamma \),

\[
w_E\left(u - \sum_{n=1}^l u_n; X, Y\right) \leq \gamma \quad \text{and} \quad \sum_{n \geq l} w_E(u_n; X, Y) \leq \gamma.
\]

Then, for all \( l \geq N_\gamma \) and \( 1 \leq a, b \leq m_{l+1} \),

\[
w_E\left(u - \left( \sum_{n=1}^l u_n + \sum_{i=1}^a \sum_{j=1}^{m_{l+1}} x_i^{l+1} \otimes y_i^{l+1} + \sum_{j=1}^b x_{(a+1)j}^{l+1} \otimes y_{(a+1)j}^{l+1} \right); X, Y\right)
\]

\[
\leq \gamma + w_E\left( \sum_{i=1}^a \sum_{j=1}^{m_{l+1}} x_i^{l+1} \otimes y_i^{l+1} + \sum_{j=1}^b x_{(a+1)j}^{l+1} \otimes y_{(a+1)j}^{l+1}; X, Y\right)
\]

\[
\leq \gamma + \|((x_{ij}^{l+1})_{i=1}^m)_{j=1}^m\|_{E^u(X)} \|((y_{ij}^{l+1})_{j=1}^m)_{i=1}^m\|_{E^u(Y)}
\]

\[
\leq \gamma + (1 + \delta)w_E(u_{l+1}; X, Y)
\]

\[
\leq \gamma + (1 + \delta)\gamma.
\]

This shows that

\[
u = \sum_{n=1}^\infty \sum_{i=1}^m x_i^n \otimes y_i^n
\]

converges in \( X \hat{\otimes} w_2 Y \). To show that the above series converges unconditionally, let \( F \) be an arbitrary finite subset of \( \mathbb{N} \) with \( \min F > \sum_{n=1}^{N_\gamma} m_n^2 \) and let \( \{s_k \otimes t_k\}_{k \in F} \) be the set of corresponding tensors. Then, there exists \( I_1, I_2 > N_\gamma \) such that \( \{s_k \otimes t_k\}_{k \in F} \subset \{(x_i^n \otimes y_i^n)_{i,j=1}^{l_2} \}_{n=l_1}^{l_2} \). We have

\[
w_E\left( \sum_{k \in F} s_k \otimes t_k; X, Y\right) \leq \sum_{n=l_1}^{l_2} \|((x_{ij}^n)_{i=1}^m)_{j=1}^m\|_{E^u(X)} \|((y_{ij}^n)_{j=1}^m)_{i=1}^m\|_{E^u(Y)}
\]

\[
\leq \sum_{n=l_1}^{l_2} (1 + \delta)w_E(u_n; X, Y)
\]

\[
\leq (1 + \delta)\gamma.
\]

Since for all \( l \geq N_\gamma \) and \( 1 \leq a, b \leq m_l \),

\[
\sup_{x^n \in B_{x^n}} \left( \sum_{j=b}^{m_l} |x^n(x_{ij})|^q \right)^{p/q} + \sum_{j=1}^{m_l} \left( \sum_{i=a+1}^{m_l} |x^n(x_{ij})|^q \right)^{p/q} + \sum_{n \geq l+1} m_{n-1} \left( \sum_{i=1}^{m_n} |x^n(x_{ij})|^q \right)^{p/q} \right)^{1/p}
\]

\[
\leq \sup_{x^n \in B_{x^n}} \left( \sum_{n \geq l+1} m_{n-1} \left( \sum_{j=1}^{m_n} |x^n(x_{ij})|^q \right)^{p/q} \right)^{1/p}
\]

\[
\leq \left( \sum_{n \geq l+1} (1 + \delta)w_E(u_n; X, Y) \right)^{1/p}
\]

\[
\leq ((1 + \delta)\gamma)^{1/p},
\]

\(((x_{ij}^n)_{i=1}^m)_{j=1}^m) \in E^u(X) \) and we see that

\[
\|((x_{ij}^n)_{i=1}^m)_{j=1}^m\|_{E^u(X)} \leq \left( (1 + \delta) \sum_{n=1}^\infty w_E(u_n; X, Y) \right)^{1/p}.
\]
Similarly,
\[(\sum_{i=1}^{m_n} y_{i,j})_{n} \in E^u(Y) \text{ and } \|((y_{i,j})_{n})_{i=1}^{m_n}\|_{E^u(Y)} \leq \left((1 + \delta) \sum_{n=1}^{\infty} w_E(u_n, X, Y)\right)^{1/p^*}.
\]

Consequently, the infimum
\[\inf\{\cdot\} \leq \|((x_{i,j})_{n})_{i=1}^{m_n}\|_{E^0(X)} \|((y_{i,j})_{n})_{i=1}^{m_n}\|_{E^u(Y)} \leq (1 + \delta)^2 w_E(u; X, Y).
\]

Since \(\delta > 0\) was arbitrary, \(\inf\{\cdot\} \leq w_E(u; X, Y)\).

For every such representation
\[u = \sum_{n=1}^{\infty} x_n \otimes y_n\]
unconditionally converging in \(X \hat{\otimes} E^u Y\),
\[w_E(u; X, Y) = \lim_{l \to \infty} w_E\left(\sum_{n=1}^{l} x_n \otimes y_n\right) \leq \lim_{l \to \infty} \|\sum_{n=1}^{l} x_n \otimes y_n\|_{E^u(Y)} = \sum_{n=1}^{\infty} \|x_n \otimes y_n\|_{E^u(Y)} = \inf\{\cdot\}.
\]

Thus, \(w_E(u; X, Y) \leq \inf\{\cdot\}\).

As in the proof of Proposition 4, we have:

**Proposition 5.** Suppose that \(E = (\sum_{k} \ell_{k})_{p} (1 \leq p, q < \infty)\), \(E = (\sum_{k} \ell_{k})_{p} (1 \leq p < \infty)\) or \(E = (\sum_{k} \ell_{k})_{c_0} (1 \leq q < \infty)\). If \(u \in X \hat{\otimes} E^u Y\), then there exist \((x_n)_{n} \in E(X)\) and \((y_n)_{n} \in E^u(Y)\) such that
\[u = \sum_{n=1}^{\infty} x_n \otimes y_n\]
unconditionally converges in \(X \hat{\otimes} E^u Y\) and
\[g_E(u; X, Y) = \inf\left\{\|x_n\|_{E(X)} \|y_n\|_{E^u(Y)} : u = \sum_{n=1}^{\infty} x_n \otimes y_n\right\}.
\]

Let \(\lambda\) be a finitely generated tensor norm. Let \(L(X, Y)\) be the Banach space of all operators from \(X\) to \(Y\). The operator \(j_\lambda : X^* \hat{\otimes} \lambda Y \to L(X, Y)\) is defined by \(j_\lambda(\sum_{n=1}^{m} x_n^* \otimes y_n) = \sum_{n=1}^{m} x_n^* \otimes y_n\), and let
\[j_\lambda : X^* \hat{\otimes} \lambda Y \to L(X, Y),
\]
be the continuous extension of \(j_\lambda\). We equip \(J_\lambda(X^* \hat{\otimes} \lambda Y)\) with the quotient norm of \(X^* \hat{\otimes} \lambda Y / \ker j_\lambda\), which will be denoted by \(\|\cdot\|_{J_\lambda}\). According to a well-known result of Grothendieck [16] (cf. [10], Proposition 1.5.4), if \(X^*\) or \(Y\) has the approximation property (AP), then \(J_\lambda\) is injective; hence, \(X^* \hat{\otimes} \lambda Y\) is isometric to \((J_\lambda(X^* \hat{\otimes} \lambda Y), \|\cdot\|_{J_\lambda})\).

**Lemma 3 ([21], Theorem 2.4).** Assume that \(X^{***}\) or \(Y\) has the AP.
If \(T \in J_\lambda(X^{***} \hat{\otimes} \lambda Y) \subset L(X^*, Y)\) and \(T^*(Y^*) \subset X\), then \(T \in \overline{I_\lambda(X^{***} \hat{\otimes} \lambda Y)}\).

The prototype of the following theorem is described in [21] (Theorem 3.1).

**Theorem 3.** Suppose that \(E = (\sum_{k} \ell_{k})_{p} (1 \leq p, q < \infty)\), \(E = (\sum_{k} \ell_{k})_{p} (1 \leq p < \infty)\) or \(E = (\sum_{k} \ell_{k})_{c_0} (1 \leq q < \infty)\). Assume that \(X^{***}\) or \(Y\) has the AP. If \(T \in \overline{N_\lambda(X^*, Y)}\) (respectively,
\( uN_E(X^*, Y) \) and \( T^*(Y^*) \subset X \), then there exist \((x_n)_n \in E(X)\) (respectively, \(E^u(X)\)) and \((y_n)_n \in E^u_*(Y)\) such that

\[
T = \sum_{n=1}^{\infty} x_n \otimes y_n
\]

unconditionally converges in \( N_E(X^*, Y) \) (respectively, \( uN_E(X^*, Y) \)).

**Proof.** We only consider \( N_E \). The proof of the case \( uN_E \) is similar. First, we show that \( (J_{SE}(X^{**} \otimes_{SE} Y), \| \cdot \|_{J_{SE}}) = (N_E(X^*, Y), \| \cdot \|_{N_E}) \). Let \( J_{SE}(u) \in J_{SE}(X^{**} \otimes_{SE} Y) \). Let \( u = \sum_{n=1}^{\infty} x_n^{**} \otimes y_n \) be an arbitrary representation in Proposition 5. Then

\[
J_{SE}(u) = \sum_{n=1}^{\infty} x_n^{**} \otimes y_n \in N_E(X^*, Y)
\]

and \( \| J_{SE}(u) \|_{N_E} \leq \| (x_n^{**})_n \|_{E(X)} \| (y_n)_n \|_{E^u_*(Y)} \). Since the representation of \( u \) was arbitrary,

\[
\| J_{SE}(u) \|_{N_E} \leq g_E(u; X^{**}, Y) = \| J_{SE}(u) \|_{J_{SE}}.
\]

Let \( T \in N_E(X^*, Y) \) and let \( \delta > 0 \) be given. Let \( T = \sum_{n=1}^{\infty} x_n^{**} \otimes y_n \) be an arbitrary \( N_E \)-representation. Since

\[
g_E \left( \sum_{n=m}^{l} x_n^{**} \otimes y_n; X^*, Y \right) \leq \| (x_n^{**})_n \|_{E(X)} \| (y_n)_n \|_{E^u_*(Y)} \leq \| (y_n)_n \|_{E^u_*(Y)} \sum_{n=m}^{l} \| x_n^{**} \|_{E_1} \|
\]

\( \sum_{n=1}^{\infty} x_n^{**} \otimes y_n \) converges in \( X^{**} \otimes_{SE} Y \). Thus,

\[
T = J_{SE} \left( \sum_{n=1}^{\infty} x_n^{**} \otimes y_n \right) \in J_{SE}(X^{**} \otimes_{SE} Y).
\]

Choose an \( l \in \mathbb{N} \) so that \( g_E(\sum_{n>l} x_n^{**} \otimes y_n; X^{**}, Y) \leq \delta \). Then, we have

\[
\| T \|_{J_{SE}} = g_E \left( \sum_{n=1}^{\infty} x_n^{**} \otimes y_n; X^{**}, Y \right) \leq g_E \left( \sum_{n=1}^{l} x_n^{**} \otimes y_n; X^{**}, Y \right) + \delta \leq \| (x_n^{**})_n \|_{E(X)} \| (y_n)_n \|_{E^u_*(Y)} + \delta.
\]

Since the representation of \( T \) was arbitrary, \( \| T \|_{J_{SE}} \leq \| T \|_{N_E} \).

Now, let \( T \in N_E(X^*, Y) \). Choose \( u \in X^{**} \otimes_{SE} Y \) so that \( T = J_{SE}(u) \). By Lemma 3, \( J_{SE}(u) \in J_{SE}(X \otimes Y) \|_{J_{SE}} \). Since \( J_{SE} \) is an isometry and \( X \otimes_{SE} Y \) is isometrically embeded in \( X^{**} \otimes_{SE} Y \) (cf. [3], Proposition 6.4), we see that \( u \in X \otimes_{SE} Y \). By Proposition 5, there exist \((x_n)_n \in E(X)\) and \((y_n)_n \in E^u_*(Y)\) such that \( u = \sum_{n=1}^{\infty} x_n \otimes y_n \) unconditionally converges in \( X \otimes_{SE} Y \). Hence,

\[
T = J_{SE}(u) = \sum_{n=1}^{\infty} x_n \otimes y_n
\]

unconditionally converges in \( N_E(X^*, Y) \). \( \square \)

**4. Discussion**

This work is the general and natural extension of some results about the tensor norms \( g_p \) and \( w_p \). There have been many more investigations about \( g_p \) and \( w_p \) since their introduction. We expect that several more results on \( g_p \) and \( w_p \), and the ideals of \( p \)-nuclear and \( p \)-compact operators, can be developed. For instance, for a finitely generated tensor
norm $\alpha$, a Banach space $X$ is said to have the $\alpha$-approximation property ($\alpha$-AP) if for every Banach space $Y$, the natural map

$$J_{\alpha} : Y \hat{\otimes} X \longrightarrow Y \otimes \varepsilon X$$

is injective (cf. [2]), Section 21.7. The $g_p$-AP and the $w_p$-AP were well studied, and the $g_p$-AP (respectively, $w_p$-AP) is closely related with an approximation property of the ideal of $p$-summing operators (respectively, ideal of $p$-dominated operators) (cf. [11]). We can consider the $g_E$-AP and the $w_E$-AP as the following subjects:

1. An investigation of the ideals of $E$-summing operators and $E$-dominated operators;
2. Some relationships of the ideals of $E$-summing operators and $E$-dominated operators, respectively, between the $g_E$-AP and the $w_E$-AP, respectively.

**Funding:** This work was supported by the National Research Foundation of Korea (NRF-2021R1F1A1047322).

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**