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Acyclic Chromatic Index of 1-Planar Graphs

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Abstract: The acyclic chromatic index $\chi'_a(G)$ of a graph G is the smallest k for which G is a proper edge colorable using k colors. A 1-planar graph is a graph that can be drawn in plane such that every edge is crossed by at most one other edge. In this paper, we prove that every 1-planar graph G has $\chi'_a(G) \leq \Delta + 36$, where Δ denotes the maximum degree of G. This strengthens a result that if G is a triangle-free 1-planar graph, then $\chi'_a(G) \leq \Delta + 16$.

Keywords: 1-planar graph; acyclic edge coloring; acyclic chromatic index; discharging

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1. Introduction

Suppose that G is a simple graph with vertex set $V(G)$, edge set $E(G)$, minimum degree $\delta(G)$, and maximum degree $\Delta(G)$ (for short, Δ). A *proper edge-k-coloring* of G is a function ϕ from $E(G)$ to the color set $\{1, 2, \ldots, k\}$ such that any two adjacent edges receive distinct colors. We say that ϕ is *acyclic* if there do not exist bichromatic cycles in G. Let $\chi'(G)$ and $\chi'_a(G)$ denote the *chromatic index* and *acyclic chromatic index* of G, respectively, which are defined as the smallest k such that G is edge-k-colorable and acyclic edge-k-colorable.

On the one hand, it is evident that $\chi'_a(G) \geq \chi'(G) \geq \Delta$ for any graph G. On the other hand, Alon et al. [1] raised the following conjecture.

Conjecture 1. *For any graph G, $\chi'_a(G) \leq \Delta + 2$.*

Alon, McDiarmid, and Reed [2] used probabilistic analysis to show that $\chi'_a(G) \leq 64\Delta$ for any graph G. After then, this bound has been improved to that $\chi'_a(G) \leq 16\Delta$ in [3], that $\chi'_a(G) \leq \lceil 9.62(\Delta - 1) \rceil$ in [4], that $\chi'_a(G) \leq 4\Delta - 4$ in [5], and that $\chi'_a(G) \leq \lceil 3.74(\Delta - 1) \rceil + 1$ in [6]. The acyclic edge coloring of some classical classes of graphs has been extensively investigated, including subcubic graphs [7–9], outerplanar graphs [10,11], and 2-degenerate graphs [12].

Suppose that G is a planar graph. Let C_n denote a cycle of length n. In 2011, Basavaraju et al. [13] showed that $\chi'_a(G) \leq \Delta + 12$. This bound was later improved to that $\chi'_a(G) \leq \Delta + 7$ by Wang, Shu, and Wang [14] and furthermore to that $\chi'_a(G) \leq \Delta + 6$ by Wang and Zhang [15].

A graph G is called *1-planar* if it admits a drawing in the plane such that every edge has at most one crossing. Ringle [16] first introduced this concept in the study of the problem of simultaneously coloring the vertices and faces of plane graphs. Zhang, Liu, and Wu [17] proved that if G is a 1-planar graph, then $\chi'_a(G) \leq \max\{2\Delta - 2, \Delta + 83\}$, which implies a upper bound that $\chi'_a(G) \leq 2\Delta - 2$ for any 1-planar graph G with $\Delta \geq 85$. Song and Miao [18] showed that every C_3-free 1-planar graph G has $\chi'_a(G) \leq \Delta + 22$. Chen, Wang, and Zhang [19] improved this result by replacing the constant 22 with 16.

In this paper, we will prove that every 1-planar graph G has $\chi'_a(G) \leq \Delta + 36$, which improved the previously stated results on the acyclic edge coloring of 1-planar graphs.

2. Structure of 1-Planar Graphs

Given a plane graph, we denote its face set by $F(G)$. For a face $f \in F(G)$, we use $\partial(f)$ to denote the boundary of f, and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $\partial(f)$. Note that $\partial(f)$ forms a cycle once G is 2-connected. A vertex v of degree k (at most k, at least k, respectively) is called a *k-vertex* (*k^--vertex*, *k^+-vertex*, respectively). The similar definition can be given for a face f. A cycle C in a plane graph G is *separating* if there exists at least one vertex in its interior and exterior, respectively. Let $V_{\text{int}}(C)$ denote the set of vertices in G that lie interior to C.

To investigate the $L(p,q)$-labeling of planar graphs, Heuvel and McGuinness [20] established the following structural theorem on planar graphs.

Theorem 1 (Heuvel and McGuinness [20])**.** *Every planar graph G contains a k-vertex v with neighbors $v_1, v_2, \ldots, v_k$ with $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$ such that one of the following holds:*

(A1) $k \leq 2$;
(A2) $k = 3$ with $d_G(v_1) \leq 11$;
(A3) $k = 4$ with $d_G(v_1) \leq 7$ and $d_G(v_2) \leq 11$;
(A4) $k = 5$ with $d_G(v_1) \leq 6$, $d_G(v_2) \leq 7$, and $d_G(v_3) \leq 11$.

Let G be a 1-planar graph. Now we shall define a new graph $G^\times$ as follows. Let $Z(G)$ denote the set of crossings in G. For each crossing $z \in Z(G)$, there is a unique pair of crossing edges $x_1x_2, y_1y_2 \in E(G)$. Let

$$V(G^\times) = V(G) \cup Z(G), \qquad E(G^\times) = E_0(G) \cup E_1(G),$$

where $E_0(G)$ is the set of non-crossed edges in G and

$$E_1(G) = \{xz, zy \mid xy \in E(G) \setminus E_0(G) \text{ and } z \text{ is a crossing on } xy\}.$$

When discussing the graph $G^\times$, we will call the vertices in $V(G)$ *true vertices* and the vertices in $Z(G)$ *false vertices*. A k-vertex which is true will be called a *true k-vertex*. Note that $d_{G^\times}(v) = d_G(v)$ for each $v \in V(G)$, and $d_{G^\times}(v) = 4$ for each $v \in Z(G)$. Let $n_\times(v)$ be the number of false vertices adjacent to a vertex v.

Let $G^\times_T$ be a triangulation of $G^\times$, which can be obtained by performing the following steps.

1. For each $z \in Z(G)$ label the unique pair of crossing edges x_1x_2 and y_1y_2. For each $i \in \{1,2\}$ and $j \in \{1,2\}$, if z, x_i, y_j does not form a 3-face in $G^\times$, then add an edge x_iy_j so that $[zx_iy_j]$ is a 3-face in the new graph.
2. If needed, add additional edges so that the new graph is a triangulation.

Note that the newly added edges in step 1 and existing edges may form 2-cycles but that no 2-face is created. It is easy to see that $G^\times_T$ is a plane triangulation without loops and 2-faces, but it may contain a cycle of length 2. However, every 2-cycle is a separating cycle of $G^\times_T$. A face f of $G^\times_T$ will be called *false* if f is incident to at least one false vertex, and *normal* otherwise.

The following theorem is a generalization of Theorem 1 for 1-planar graphs. This theorem provides additional insight into the detailed structure of 1-planar graphs.

Theorem 2. *Every 1-planar graph G contains a k-vertex v with neighbors $v_1, v_2, \ldots, v_k$ with $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$ such that one of the following conditions holds:*

(C1) $k \leq 2$;
(C2) $k = 3$ with $d_G(v_1) \leq 23$;
(C3) $k = 4$ with $d_G(v_1) \leq 14$ and $d_G(v_2) \leq 23$;
(C4) $k = 5$ with $d_G(v_1) \leq 10$, $d_G(v_2) \leq 14$, and $d_G(v_3) \leq 23$;

(C5) $k = 6$ *with* $d_G(v_1) \le 9$, $d_G(v_2) \le 10$, $d_G(v_3) \le 14$, *and* $d_G(v_4) \le 23$;
(C6) $k = 7$ *with* $d_G(v_1) \le 7$, $d_G(v_2) \le 9$, $d_G(v_3) \le 10$, $d_G(v_4) \le 14$, *and* $d_G(v_5) \le 23$.

Proof. Assume that Theorem 2 is false. Let G be a connected counterexample. Obviously, if G is a 1-planar counterexample graph and u, v are a pair of non-adjacent vertices of G and $G' = G + uv$ is still a 1-planar graph, then G' is still a counterexample of Theorem 2. This implies that we may assume G is 2-connected. Let $G_T^\times$ be a plane triangulation of G obtained by the previous operations. It is possible that there are separating 2-cycles in $G_T^\times$. If $G_T^\times$ has no 2-cycles, let $H = G_T^\times$. Otherwise let C be a separating 2-cycle of $G_T^\times$ such that the number of interior vertices of C, $|V_{\text{int}}(C)|$, is minimum. Let $H = G[V(C) \cup V_{\text{int}}(C)]$. Clearly, H does not contain any 2-cycles other than C. If H contains a 2-cycle, let f_0 denote the outer face of H, and write $F^0(H) = F(H) \setminus \{f_0\}$ and $V^0(H) = V(H) \setminus V(C)$. Otherwise let $F^0(H) = F(H)$ and $V^0(H) = V(H)$. Vertices in $V^0(H)$ will be called *interior vertices* and faces in $F^0(H)$ will be called *interior faces*. For $v \in V^0(H)$, it holds obviously that $d_H(v) = d_{G_T^\times}(v)$. We have the following observation. □

Observation 1.
(1) *No two false vertices are adjacent in* H.
(2) *Let* $v \in V(H)$ *be a true k-vertex. Then* $n_\times(v) = 0$ *if* $k = 3$; $n_\times(v) \le 1$ *if* $k = 4$, *and* $n_\times(v) \le \lfloor \frac{k}{2} \rfloor$ *if* $k \ge 5$.

Two true vertices which are adjacent in H will be referred to as *direct neighbors*. Suppose that $x \in V^0(H)$ is a false vertex which is incident to four 3-faces $[xuy]$, $[xyv]$, $[xvz]$, and $[xzu]$ in clockwise order. We say that u is an *indirect neighbor* of v and both y and z are *type* 1 *direct neighbors* of v. Moreover, if w is a true neighbor of v incident to two normal 3-faces, then w is called a *type* 2 *direct neighbor* of v. The related configuration for a vertex v is depicted in Figure 1, where f_1 and f_2 are normal 3-faces of H. In all figures, we use • to denote a vertex that has no edges incident to it other than those shown in the graph, ∘ to denote a vertex that may have edges connected to other vertices that are not in the graph, and ⊗ to denote false vertices.

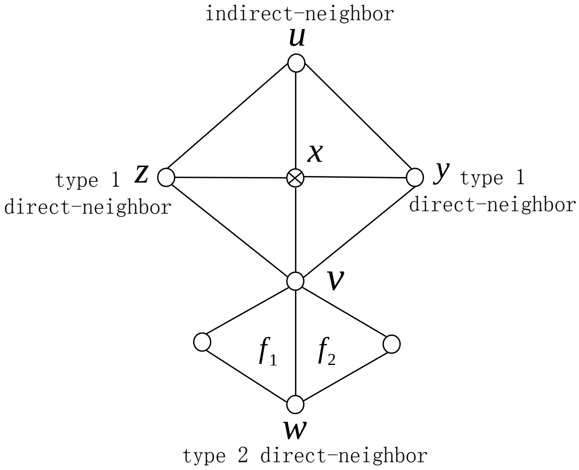

Figure 1. Direct neighbors and indirect neighbor for a vertex v.

It is not hard to see that $n_\times(v)$ is equal to the number of indirect neighbors of v in H. A vertex v of H is called *small* if $d_H(v) \le 7$, and *big* otherwise. The following identity is deduced from applying Euler's formula on H.

$$\sum_{v \in V(H)} (d_H(v) - 4) + \sum_{f \in F(H)} (d_H(f) - 4) = -8 \quad (1)$$

We now define a weight function on H. Let $w(x) = d_H(x) - 4$ for each $x \in V(H) \cup F(H)$. Next, we shall redistribute the weight among vertices and faces of H and keep the sum of all weights unchanged. We will show that the resultant weight function w' satisfies the following two conditions.

(I) $w'(x) \geq 0$ for all $x \in V^0(H) \cup F^0(H)$;

(II) if H contains a 2-cycle, then $w'(f_0) + \sum_{x \in V(C)} w'(x) \geq -7$.

This leads to the following inequalities.

$$-7 \leq \sum_{x \in V(H) \cup F(H)} w'(x) = \sum_{x \in V(H) \cup F(H)} w(x) = -8 \tag{2}$$

This contradiction implies there is no such counterexample exists and so the theorem holds.

The discharging rules are defined as follows. Note that the rules (R0) and (R1.2.1) only apply if H contains a 2-cycle C. For a vertex $v \in V^0(H)$, we use $v_1, v_2, \ldots, v_{k_v}$ to denote the internal (direct or indirect) neighbors of v with $d_H(v_1) \leq d_H(v_2) \leq \cdots \leq d_H(v_{k_v})$, where $k_v \leq d_H(v)$.

(R0) Every vertex $v \in V(C)$ sends $\frac{1}{2}$ to each internal (direct or indirect) neighbor and to each incident 3-face.

(R1) Let $f = [u_1 u_2 u_3]$ be a 3-face.

(R1.1) If f is false, then f gets $\frac{1}{2}$ from each of its incident internal true vertices.

(R1.2) If f is normal, then we carry out the following two sub-rules:

(R1.2.1) If $|V(f) \cup V(C)| = 1$ and let $u_1 \in V(C)$, then at least one of u_2 and u_3, say u_2, is big, then f gets $\frac{1}{2}$ from u_2. Otherwise, f gets $\frac{1}{4}$ from each of u_2 and u_3.

(R1.2.2) If $|V(f) \cup V(C)| = 0$, then we may assume that $d_H(u_1) \leq d_H(u_2) \leq d_H(u_3)$.

- If $d_H(u_1) \geq 8$, or $d_H(u_3) \leq 7$, then f gets $\frac{1}{3}$ from each of u_1, u_2, u_3.
- If $d_H(u_1) \leq 7$ and $d_H(u_2) \geq 8$, then f gets $\frac{1}{2}$ from each of u_2, u_3.
- If $d_H(u_2) \leq 7$ and $d_H(u_3) \geq 8$, then f gets $\frac{1}{2}$ from u_3 and $\frac{1}{4}$ from each of u_1, u_2.

(R2) If $d_H(v) = 3$, then v gets $\frac{1}{3}$ from each of $v_1, v_2, \ldots, v_{k_v}$.

(R3) $d_H(v) = 4$.

(R3.1) If $d_H(v_1) \geq 15$, then v gets $\frac{1}{4}$ from each of $v_1, v_2, \ldots, v_{k_v}$.

(R3.2) Suppose $d_H(v_1) \leq 14$ and $d_H(v_2) \geq 24$. If $d_H(v_1) \leq 7$, $n_\times(v) = 1$, and v_1 is a type 2 direct neighbor of v, then v gets $\frac{1}{2}$ from each of $v_2, v_3, \ldots, v_{k_v}$. Otherwise v gets $\frac{5}{12}$ from each of $v_2, v_3, \ldots, v_{k_v}$, see Figure 2a.

(R4) $d_H(v) = 5$.

(R4.1) If $d_H(v_1) \geq 11$, then v gets $\frac{1}{5}$ from each of $v_1, v_2, \ldots, v_{k_v}$.

(R4.2) If $d_H(v_1) \leq 10$ and $d_H(v_2) \geq 15$, v gets $\frac{5}{16}$ from each of $v_2, v_3, \ldots, v_{k_v}$ when $n_\times(v) = 2$. Otherwise v gets $\frac{1}{8}$ from each of $v_2, v_3, \ldots, v_{k_v}$.

(R4.3) $d_H(v_1) \leq 10$, $d_H(v_2) \leq 14$, and $d_H(v_3) \geq 24$.

(R4.3.1) If $n_\times(v) \leq 1$, then v gets $\frac{5}{18}$ from each of $v_3, \ldots, v_{k_v}$.

(R4.3.2) If $n_\times(v) = 2$, then v gets $\frac{4}{9}$ from each of $v_3, \ldots, v_{k_v}$ when $d_H(v_1), d_H(v_2) \leq 7$ and $[v v_1 v_2]$ is a normal 3-face. Otherwise v gets $\frac{5}{12}$ from each of $v_3, \ldots, v_{k_v}$, see Figure 2b.

(R5) $d_H(v) = 6$.

(R5.1) If $d_H(v_1) \geq 10$, then v gets $\frac{1}{6}$ from each of $v_1, v_2, \ldots, v_{k_v}$.

(R5.2) If $d_H(v_1) \leq 9$ and $d_H(v_2) \geq 11$, then v gets $\frac{1}{5}$ from each of $v_2, v_3, \ldots, v_{k_v}$.

(R5.3) If $d_H(v_1) \leq 9$, $d_H(v_2) \leq 10$, and $d_H(v_3) \geq 15$, then v gets $\frac{1}{4}$ from each of $v_3, v_4, \ldots, v_{k_v}$ when $n_\times(v) = 3$. Otherwise v gets $\frac{7}{48}$ from each of $v_3, v_4, \ldots, v_{k_v}$.

(R5.4) If $d_H(v_1) \leq 9$, $d_H(v_2) \leq 10$, $d_H(v_3) \leq 14$ and $d_G(v_4) \geq 24$, then v gets $\frac{1}{3}$ from each of $v_4, \ldots, v_{k_v}$ when $n_\times(v) = 3$. Otherwise v gets $\frac{2}{9}$ from each of $v_4, \ldots, v_{k_v}$.

(R6) $d_H(v) = 7$. Then v gets

$\frac{1}{24}$ from each of $v_2, v_3, \ldots, v_{k_v}$ if $d_H(v_2) \geq 10$;

$\frac{1}{15}$ from each of $v_3, v_4, \ldots, v_{k_v}$ if $d_H(v_2) \leq 9$ and $d_H(v_3) \geq 11$;

$\frac{1}{12}$ from each of $v_4, v_5, \ldots, v_{k_v}$ if $d_H(v_2) \leq 9$, $d_H(v_3) \geq 10$ and $d_H(v_4) \geq 15$;

$\frac{1}{9}$ from each of $v_5, v_6, \ldots, v_{k_v}$ if $d_H(v_2) \leq 9$, $d_H(v_3) \geq 10$, $d_H(v_4) \leq 14$ and $d_H(v_5) \geq 24$.

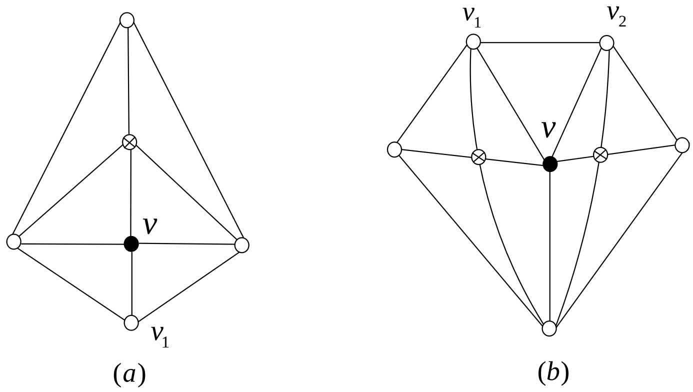

Figure 2. Configurations (**a**) in (R3.2) and (**b**) in (R4.3.2).

Let w' denote the resultant weight function after (R0)–(R6) are performed on H. First, we prove that $w'(x) \geq 0$ for all $x \in V^0(H) \cup F^0(H)$.

First we consider $x = f \in F^0(H)$. Let $f = [x_1x_2x_3]$ be a 3-face. Note that if H has a 2-cycle C, the vertices in $V(C)$ are true vertices. If f is false, then f is incident to exactly two true vertices. By (R0) and (R1.1), $w'(f) \geq -1 + 2 \times \frac{1}{2} = 0$. We may assume that f is normal. If $|V(f) \cap V(C)| = 2$, then $w'(f) \geq -1 + 2 \times \frac{1}{2} = 0$ by (R0). If $|V(f) \cap V(C)| = 1$, say $x_1 \in V(C)$. If at least one of x_2, x_3 is big, then $w'(f) \geq -1 + 2 \times \frac{1}{2} = 0$ by (R0) and (R1.2.1). Otherwise, $w'(f) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by (R0) and (R1.2.1). The only case to be considered is $|V(f) \cap V(C)| = 0$. Without loss of generality, we may assume that $d_H(x_1) \leq d_H(x_2) \leq d_H(x_3)$. If $d_H(x_1) \geq 8$, or $d_H(x_3) \leq 7$, then $w'(f) \geq -1 + 3 \times \frac{1}{3} = 0$ by (R1.2.2). If $d_H(x_2) \geq 8$, then $w'(f) \geq -1 + 2 \times \frac{1}{2} = 0$ by (R1.2.2). If $d_H(x_3) \geq 8$, then $w'(f) \geq -1 + \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by (R1.2.2).

Secondly, we shall consider the case that $x = v$ is an interior vertex of H. Let $v_1, v_2, \ldots, v_{k_v}$ denote the interior (direct or indirect) neighbors of v with $d_H(v_1) \leq d_H(v_2) \leq \cdots \leq d_H(v_{k_v})$, where $d_H(v) - 2 \leq k_v \leq d_H(v)$. Since (C1) is not satisfied, $d_H(v) \geq 3$. The proof is divided into the following cases based on $d_H(v)$.

Case 1 $d_H(v) = 3$.

Then $w(v) = -1$. By Observation 1(2), $n_\times(v) = 0$. Since (C2) is not satisfied, $v_1, v_2, \ldots, v_{k_v}$ are true and big vertices. By (R0) and (R2), $w'(v) \geq -1 + 3 \times \frac{1}{3} = 0$.

Case 2 $d_H(v) = 4$.

Clearly, $w(v) = 0$. If v is false, then $w'(v) = w(v) = 0$. If v is true, then $n_\times(v) \leq 1$ by Observation 1(2). Hence v is incident to at most two false 3-faces. If $d_H(v_1) \geq 15$, then $w'(v) \geq 0 - 2 \times \frac{1}{2} + 4 \times \frac{1}{4} = 0$ by (R0), (R1), and (R3.1). We may assume that $d_H(v_1) \leq 14$. Since (C3) is not satisfied, $d_H(v_2) \geq 24$. If $n_\times(v) = 1$, $d_H(v_1) \leq 7$, and v_1 is a type 2 direct neighbor of v, then $w'(v) \geq 0 - 2 \times \frac{1}{2} - 2 \times \frac{1}{4} + 3 \times \frac{1}{2} = 0$ by (R0), (R1), and (R3.2). Otherwise, $w'(v) \geq 0 - 2 \times \frac{1}{2} - \frac{1}{4} + 3 \times \frac{5}{12} = 0$ by (R0), (R1), and (R3.2).

Case 3 $d_H(v) = 5$.

Clearly, $w(v) = 1$. By Observation 1(2), $n_\times(v) \leq 2$ and it follows that v is incident to at most four false 3-faces. If $d_H(v_1) \geq 11$, then $w'(v) \geq 1 - 4 \times \frac{1}{2} + 5 \times \frac{1}{5} = 0$ by (R0), (R1), and (R4.1). Otherwise, we may assume that $d_H(v_1) \leq 10$.

Suppose that $d_H(v_2) \geq 15$. If $n_\times(v) = 2$, then $w'(v) \geq 1 - 4 \times \frac{1}{2} - \frac{1}{4} + 4 \times \frac{5}{16} = 0$ by (R0), (R1), and (R4.2). Otherwise, $w'(v) \geq 1 - 2 \times \frac{1}{2} - 2 \times \frac{1}{4} + 4 \times \frac{1}{8} = 0$ by (R0), (R1), and (R4.2). Suppose that $d_H(v_2) \leq 14$. Since (C4) is not satisfied, we must have $d_H(v_3) \geq 24$. If $n_\times(v) \leq 1$, then $w'(v) \geq 1 - 2 \times \frac{1}{2} - \frac{1}{3} - 2 \times \frac{1}{4} + 3 \times \frac{5}{18} = 0$ by (R0), (R1), and (R4.3.1). Now consider the case where $n_\times(v) = 2$. If $d_H(v_1), d_H(v_2) \leq 7$ and $[vv_1v_2]$ is a normal 3-face, then $w'(v) \geq 1 - 4 \times \frac{1}{2} - \frac{1}{3} + 3 \times \frac{4}{9} = 0$ by (R0), (R1), and (R4.3.2). Otherwise, $w'(v) \geq w(v) - 4 \times \frac{1}{2} - \frac{1}{4} + 3 \times \frac{5}{12} = 0$ by (R0), (R1), and (R4.3.2).

Case 4 $d_H(v) = 6$.

Clearly, $w(v) = 2$. By Observation 1(2), $n_\times(v) \le 3$. It follows that v is incident to at most six false 3-faces. If $d_H(v_1) \ge 10$, then $w'(v) \ge 2 - 6 \times \frac{1}{2} + 6 \times \frac{1}{6} = 0$ by (R0), (R1), and (R5.1). So we may assume that $d_H(v_1) \le 9$. If $d_H(v_2) \ge 11$, then $w'(v) \ge 2 - 6 \times \frac{1}{2} + 5 \times \frac{1}{5} = 0$ by (R0), (R1), and (R5.2). Otherwise, $d_H(v_2) \le 10$.

Suppose that $d_H(v_3) \ge 15$. If $n_\times(v) = 3$, then $w'(v) \ge 2 - 6 \times \frac{1}{2} + 4 \times \frac{1}{4} = 0$ by (R0), (R1), and (R5.3). Otherwise, we assume that $n_\times(v) \le 2$. It follows that v is incident to at most four false 3-faces. Consequently, $w'(v) \ge 2 - 4 \times \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + 4 \times \frac{7}{48} = 0$ by (R0), (R1), and (R5.3).

Suppose that $d_H(v_3) \le 14$. Since (C5) is not satisfied, we have $d_H(v_4) \ge 24$. If $n_\times(v) = 3$, then $w'(v) \ge 2 - 6 \times \frac{1}{2} + 3 \times \frac{1}{3} = 0$ by (R0), (R1), and (R5.4). Now we consider the case that $n_\times(v) \le 2$. It follows that v is incident to at most four false 3-faces. Consequently, $w'(v) \ge 2 - 4 \times \frac{1}{2} - 2 \times \frac{1}{3} + 3 \times \frac{2}{9} = 0$ by (R0), (R1), and (R5.4).

Case 5 $d_H(v) = 7$.

Clearly, $w(v) = 3$. By Observation 1(2), $n_\times(v) \le 3$. It follows that v is incident to at most six false 3-faces. If $d_H(v_1) \ge 8$, then $w'(v) \ge 3 - 6 \times \frac{1}{2} = 0$ by (R0), (R1), and (R6). Suppose that $d_H(v_1) \le 7$. If $d_H(v_2) \ge 10$, then $w'(v) \ge 3 - 6 \times \frac{1}{2} - \frac{1}{4} + 6 \times \frac{1}{24} = 0$ by (R0), (R1), and (R6). Now we may assume that $d_H(v_2) \le 9$. If $d_H(v_3) \ge 11$, then $w'(v) \ge 3 - 6 \times \frac{1}{2} - \frac{1}{3} + 5 \times \frac{1}{15} = 0$ by (R0), (R1), and (R6). Suppose that $d_H(v_3) \le 10$. If $d_H(v_4) \ge 15$, then $w'(v) \ge 3 - 6 \times \frac{1}{2} - \frac{1}{3} + 4 \times \frac{1}{12} = 0$ by (R0), (R1), and (R6). Suppose that $d_H(v_4) \le 14$. Since (C6) is not satisfied, $d_H(v_5) \ge 24$. Consequently, $w'(v) \ge 3 - 6 \times \frac{1}{2} - \frac{1}{3} + 3 \times \frac{1}{9} = 0$ by (R0), (R1), and (R6).

Case 6 $8 \le d_H(v) \le 9$.

Notice that v sends nothing to any small vertex. Thus, $w'(v) \ge w(v) - \frac{1}{2}d_H(v) = d_H(v) - 4 - \frac{1}{2}d_H(v) = \frac{1}{2}d_H(v) - 4 \ge 0$.

From now on, we assume that $d_H(v) \ge 10$. Let $y_0, x_0^1, x_0^2, \ldots, x_0^{m_0}, y_1, x_1^1, x_1^2, \ldots, x_1^{m_1}, y_2, \ldots, y_{k-1}, x_{k-1}^1, x_{k-1}^2, \ldots, x_{k-1}^{m_{k-1}}$ denote the (direct or indirect) neighbors of v in H in clockwise order, where for every $i = 0, 1, \ldots, k-1$ and $j = 1, 2, \ldots, m_i$, x_i^j is a small vertex, and for each $i = 0, 1, \ldots, k-1$ either $y_i \in V(C)$ or y_i is an interior big vertex. Define

$X_i = \{x_i^j | 0 \le j \le m_i\}$,
$X = \bigcup_{i=1}^{k-1} X_i$,
$Y = \{y_i | 0 \le i \le k-1\}$.

By (R1)–(R6), each v transfers at most $\frac{1}{2}$ to every (direct or indirect) neighbor and to every incident 3-face. If $|Y| \ge 8$, then $w'(v) \ge w(v) - \frac{1}{2}d_H(v) - \frac{1}{2}(d_H(v) - |Y|) = \frac{1}{2}(|Y| - 8) \ge 0$. Hence, we will assume that $0 \le |Y| \le 7$ in the remaining cases.

For $x, y \in V(H)$, let $\tau(x \to y)$ denote the weight transferring from x to y according to (R0) and (R2)–(R6). Let $\sigma(v)$ denote the total weight that an internal vertex v sends to all its (direct or indirect) neighboring vertices according to rules (R2)–(R6). Moreover, let $\eta_0(v)$ denote the number of vertices in X which receive no charge from v.

Case 7 $d_H(v) = 10$.

It is easy to see from (R1)–(R6) that v only transfers $\frac{1}{6}$ to each adjacent 6-vertex or 7-vertex. Hence $w'(v) \ge 10 - 4 - \frac{1}{2} \times 10 - \frac{1}{6} \times (10 - |Y| - \eta_0(v))$ by (R1), (R5), and (R6). It is sufficient to show $|Y| + \eta_0(v) \ge 4$. Noting that for $x \in X$ if $|N(x) \cap X| \ge 2$, then by (R5) and (R6), v transfers no charge to x. It follows that $\eta_0(v) \ge 10 - 3|Y|$. Therefore $|Y| + \eta_0(v) \ge 4$, a contradiction.

Case 8 $11 \le d_H(v) \le 14$.

By inspecting our discharging rules, we find that v needs only to transfer at most $\frac{1}{5}$ to each adjacent 5-vertex, 6-vertex, or 7-vertex. Thus,

$$\begin{aligned} w'(v) &\geq d_H(v) - 4 - \frac{1}{2}d_H(v) - \frac{1}{5}(d_H(v) - |Y|) \\ &= \frac{1}{10}(3d_H(v) + 2|Y| - 40). \end{aligned}$$

If $|Y| \geq 4$, we can immediately conclude that $w'(v) \geq 0$. Since (C2)–(C4) are not satisfied, for each $i = 0, 1, \ldots, k-1$, every vertex $x \in X_i \setminus \{x_i^1, x_i^{m_i}\}$ is a 6-vertex or a 7-vertex, therefore $\tau(v \to x) \leq \frac{1}{15}$ by (R5) and (R6). Thus,

$$\begin{aligned} w'(v) &\geq d_H(v) - 4 - \frac{1}{2}d_H(v) - \frac{1}{5}(d_H(v) - |Y| - (d_H(v) - 3|Y|)) - \frac{1}{15}(d_H(v) - 3|Y|) \\ &= \frac{1}{30}(13d_H(v) - 6|Y| - 120) \\ &\geq \frac{1}{30}(13d_H(v) - 138) > 0. \end{aligned}$$

Case 9 $15 \leq d_H(v) \leq 23$.

Since (C2)–(C4) are not satisfied, for each $i = 0, 1, \ldots, k-1$, every vertex $x \in X_i \setminus \{x_i^1, x_i^{m_i}\}$ is a 6-vertex or a 7-vertex, therefore $\tau(v \to x) \leq \frac{1}{4}$ by (R5) and (R6). Moreover, $\tau(v \to x_i^1) \leq \frac{5}{16}$ and $\tau(v \to x_i^{m_i}) \leq \frac{5}{16}$ by (R3)–(R6). Thus,

$$\begin{aligned} w'(v) &\geq d_H(v) - 4 - \frac{1}{2}d_H(v) - \frac{5}{16}(2|Y|) - \frac{1}{4}(d_H(v) - 3|Y|) \\ &= \frac{1}{4}d_H(v) - 4 + \frac{1}{8}|Y|. \end{aligned}$$

If $d_H(v) \geq 16$, or $d_H(v) = 15$ and $|Y| \geq 2$, then it holds obviously that $w'(v) \geq 0$. Suppose that $d_H(v) = 15$ and $0 \leq |Y| \leq 1$.

Claim 1. *If $\tau(v \to x_0^j) = \frac{1}{4}$ for some index j with $3 \leq j \leq m_0 - 2$, then $\tau(v \to x_0^{j-1}) \leq \frac{7}{48}$ and $\tau(v \to x_0^{j+1}) \leq \frac{7}{48}$.*

Proof. Since $\tau(v \to x_0^j) = \frac{1}{4}$, it follows from (R5) and (R6) that $d_H(x_0^j) = 6$ and $n_\times(x_0^j) = 3$. By symmetry, it will be sufficient to show that $\tau(v \to x_0^{j-1}) \leq \frac{7}{48}$. Suppose $\tau(v \to x_0^{j-1}) > \frac{7}{48}$. It follows from (R5) and (R6) that $d_H(x_0^{j-1}) = 6$ and $n_\times(x_0^{j-1}) = 3$.

If x_0^j is a direct neighbor of v in H. Since $n_\times(x_0^j) = 3$ and false vertices can not be adjacent, it must be the case that $x_0^j x_0^{j-2}$ and $x_0^j x_0^{j+2}$ are two crossing edges of G. So x_0^j is adjacent to four small vertices $x_0^{j-2}, x_0^{j-1}, x_0^{j+1}, x_0^{j+2}$, which implies that x_0^j satisfies (C5), a contradiction. Otherwise, x_0^j is an indirect neighbor of v in H and $n_\times(x_0^{j-1}) = 3$, it follows that x_0^{j-1} must be a direct neighbor of v. Similar to the previous argument, x_0^{j-1} is adjacent to at least three small vertices and v, contradicting (C5). This completes the proof of Claim 1. □

If $|Y| = 0$, then Claim 1 implies that $w'(v) \geq 15 - 4 - \frac{1}{2} \times 15 - \frac{1}{4} \times 7 - \frac{7}{48} \times 8 = \frac{7}{12}$.

If $|Y| = 1$, then by Claim 1 and (R4), $w'(v) \geq 15 - 4 - \frac{1}{2} \times 15 - \frac{5}{16} \times 2 - \frac{1}{4} \times 6 - \frac{7}{48} \times 6 = \frac{1}{2}$.

Case 10 $d_H(v) \geq 24$.

We divide the proof into the following two subcases.

Subcase 10.1 $|Y| = 0$.

Then X consists of all (direct or indirect) neighbors of v in H. Let $x \in X$. Then $3 \le d_H(x) \le 7$. Since x is adjacent to at least two other vertices in X and neither (C2) nor (C3) holds, we must have $d_H(x) \ne 3, 4$. Hence $5 \le d_H(x) \le 7$.

Claim 2. *Let x_0, x_1, x_2, x_3, x_4 be five consecutive vertices in X in clockwise order with $5 \le d_H(x_i) \le 7$ for $i = 1, 3$. If $\tau(v \to x_2) \in \{\frac{4}{9}, \frac{5}{12}\}$, then $\tau(v \to x_i) \le \frac{2}{9}$ for $i = 1, 3$.*

Proof. Note that x_2 is (directly or indirectly) adjacent to v, x_1, x_3 in H. Since $\tau(v \to x_2) = \frac{4}{9}$ or $\tau(v \to x_2) = \frac{5}{12}$, then it is easy to verify that $d_H(x_2) = 5$, $n_\times(x_2) = 2$, by (R4)-(R6). Let u_1, u_2 be the neighbors of x_2 other than v, x_1, x_3 such that $vx_1u_1u_2x_3v$ is a 5-cycle in G. Since x_1, x_3 are small and (C4) is not satisfied, it follows that $d_H(u_1), d_H(u_2) \ge 8$ and thus $u_1, u_2 \notin X$. This implies that $u_2 \ne x_4$ and $u_1 \ne x_0$.

If $\tau(v \to x_2) = \frac{4}{9}$, then (R4.3.2) shows that $[vx_1x_3]$ is a normal 3-face of H, but this is impossible because of the existence of x_2. Thus, $\tau(v \to x_2) = \frac{5}{12}$. By (R4.3.2), v is an indirect neighbor of x_2, and exactly one of u_1, u_2 is an indirect neighbor of x_2, say u_2. Thus, $[x_1x_2u_1]$ is a normal 3-face of H.

It is easy to see that $d_H(x_3) \ge 6$. If $d_H(x_3) = 7$, then $\tau(v \to x_3) \le \frac{1}{9}$ by (R6). Suppose that $d_H(x_3) = 6$. If $n_\times(x_3) \le 2$, then $\tau(v \to x_3) \le \frac{2}{9}$ by (R5) and we are done. Otherwise, $n_\times(x_3) = 3$. This implies that there is a crossing edge between u_2 and v in G, therefore u_2 is an indirect neighbor of v. This is impossible because $u_2 \notin X$. Thus, we always have that $\tau(v \to x_3) \le \frac{2}{9}$.

Now we consider the vertex x_1. If $d_H(x_1) = 7$, then $\tau(v \to x_1) \le \frac{1}{9}$ by (R6). If $d_H(x_1) = 6$, then $n_\times(x_1) \le 2$ since x_1 is incident to a formal 3-face $[x_1x_2u_1]$. By (R5), $\tau(v \to x_1) \le \frac{2}{9}$. If $d_H(x_1) = 5$, then x_1 is adjacent to three small vertices x_0, x_2, x_3, thus x_1 satisfies (C4), a contradiction. Thus, we have $\tau(v \to x_1) \le \frac{2}{9}$. This completes the proof. □

Claim 3. *Let x_1, x_2 be consecutive vertices in X. Then $\tau(v \to x_1) + \tau(v \to x_2) \le \frac{2}{3}$.*

Proof. Since (C2) and (C3) are not satisfied, $5 \le d_H(x_i) \le 7$ for $i = 1, 2$. If $\tau(v \to x_1) \in \{\frac{4}{9}, \frac{5}{12}\}$, then $\tau(v \to x_2) \le \frac{2}{9}$ by Claim 2. Thus, $\tau(v \to x_1) + \tau(v \to x_2) \le \frac{4}{9} + \frac{2}{9} = \frac{2}{3}$. If $\tau(v \to x_2) \in \{\frac{4}{9}, \frac{5}{12}\}$, a similar argument holds. This implies that v gives at most $\frac{1}{3}$ to each of x_1 and x_2 by (R4)–(R6), so that $\tau(v \to x_1) + \tau(v \to x_2) \le \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. □

Claim 4. $\sigma(v) \le \frac{1}{3}|X|$.

Proof. Let $p = |X| = d_H(v)$. If p is even, then X can be divided into $\frac{p}{2}$ 2-sets $\{x_0^1, x_0^2\}$, $\{x_0^3, x_0^4\}, \ldots, \{x_0^{p-1}, x_0^p\}$. By Claim 3, $\sigma(v) \le \frac{2}{3} \times \frac{p}{2} = \frac{1}{3}p = \frac{1}{3}|X|$. If p is odd, then we may choose x_0^p satisfying $\tau(v \to x_0^p) \le \frac{1}{3}$ by Claim 3. Now we can divide X into $\frac{p-1}{2}$ 2-sets $\{x_0^1, x_0^2\}, \{x_0^3, x_0^4\}, \ldots, \{x_0^{p-2}, x_0^{p-1}\}$ and one single set $\{x_0^p\}$. By Claim 3, $\sigma(v) \le \frac{2}{3} \times \frac{p-1}{2} + \frac{1}{3} = \frac{1}{3}p = \frac{1}{3}|X|$. This proves Claim 4. □

Using (R2) and Claim 4, we have the following:

$$\begin{aligned} w'(v) &\ge w(v) - \frac{1}{2}d_H(v) - \sigma(v) \\ &\ge d_H(v) - 4 - \frac{1}{2}d_H(v) - \frac{1}{3}|X| \\ &= \frac{1}{2}d_H(v) - 4 - \frac{1}{3}d_H(v) \\ &= \frac{1}{6}(d_H(v) - 24) \ge 0. \end{aligned}$$

Subcase 10.2 $|Y| \ge 1$.

For $i = 0, 1, \dots, k-1$, we use $\sigma(y_i, y_{i+1})$ to denote the total weight that v sends to all the vertices in X_i, where $k = |Y|$ and indices are taken as modulo k.

Claim 5. $\sigma(y_i, y_{i+1}) \le \frac{1}{3}(|X_i| + 1)$ *for* $i = 0, 1, \dots, k-1$.

Proof. By symmetry, it suffices to consider the case $i = 0$, i.e., we are going to show that $\sigma(y_0, y_1) \le \frac{1}{3}(|X_0| + 1)$. Note that if $|Y| = 1$, then y_0 is identical to y_1.

If $|X_0| = 0$, then $\sigma(y_0, y_1) = 0 < \frac{1}{3}$. If $|X_0| = 1$, then $\sigma(y_0, y_1) \le \frac{1}{2} < \frac{2}{3} = \frac{1}{3}(1+1)$ by (R2)-(R6). If $|X_0| = 2$, then $\sigma(y_0, y_1) \le 2 \times \frac{1}{2} = \frac{1}{3}(2+1)$ by (R2)-(R6).

If $|X_0| = 3$, let $X_0 = \{x_0^1, x_0^2, x_0^3\}$. Since $d_H(v) \ge 24$, $y_1 \ne y_0$, so $|Y| \ge 2$. If every vertex in X_0 gets at most $\frac{4}{9}$ from v, then $\sigma(y_0, y_1) \le 3 \times \frac{4}{9} = \frac{1}{3}(|X_0| + 1)$. Otherwise, (R2)-(R6) imply that at least one of x_0^1 and x_0^3 gets $\frac{1}{2}$ from v, say $\tau(v \to x_0^1) = \frac{1}{2}$. Now it follows easily that y_0 is an indirect neighbor of v, and x_0^2 is a type 2 direct neighbor of x_0^1. So $x_0^1 x_0^2$ is a common edge of two normal 3-faces in H. Note that $d_H(x_0^2) \ge 5$ since (C2) and (C3) are not satisfied. If $6 \le d_H(x_0^2) \le 7$, then $\tau(v \to x_0^2) \le \frac{1}{3}$ by (R5) and (R6). Otherwise, we may assume that $d_H(x_0^2) = 5$. Since x_0^2 is incident to at least two normal 3-faces, it follows that $n_\times(x_0^2) \le 1$, and hence $\tau(v \to x_0^2) \le \frac{5}{18}$ by (R4). Thus, $\tau(v \to x_0^2) \le \frac{1}{3}$ holds. Consequently, $\sigma(y_0, y_1) \le 2 \times \frac{1}{2} + \frac{1}{3} = \frac{4}{3} = \frac{1}{3}(|X_0| + 1)$.

Now we may assume that $|X_0| \ge 4$. We will prove the following claim first. □

Claim 6. $\tau(v \to x_0^1) + \tau(v \to x_0^2) \le \frac{5}{6}$.

Proof. If each of x_0^1 and x_0^2 gets from v at most $\frac{5}{12}$, the result holds. Otherwise, by (R3) and (R4), we need only to discuss the following cases.

• $d_H(x_0^1) = 4$ such that $\tau(v \to x_0^1) = \frac{1}{2}$. According to the previous analysis, we can similarly derive that $\tau(v \to x_0^2) \le \frac{1}{3}$. It yields that $\tau(v \to x_0^1) + \tau(v \to x_0^2) \le \frac{5}{6}$.

• $d_H(x_0^1) = 5$ such that $\tau(v \to x_0^1) = \frac{4}{9}$. Note that $d_H(x_0^2) \ge 5$ by (C2) and (C3). If $\tau(v \to x_0^2) \le \frac{5}{6} - \frac{4}{9} = \frac{7}{18}$, we are done. Otherwise, (R4)-(R6) implies that $d_H(x_0^2) = 5$, $n_\times(x_0^2) = 2$, and $\tau(v \to x_0^2) \in \{\frac{5}{12}, \frac{4}{9}\}$. Let w_1 and w_2 be the other (direct or indirect) neighbors of x_0^1 other than v, y_0, x_0^2 in H so that $v y_0 w_1 w_2 x_0^2 v$ forms a 5-cycle in G. Since $\tau(v \to x_0^1) = \frac{4}{9}$, we can conclude that $n_\times(x_0^1) = 2$; v, w_1 are indirect neighbors of x_0^1, $d_H(w_2) \le 7$, and $[x_0^1 x_0^2 w_2]$ is a normal 3-face of H. If $w_2 \ne x_0^3$, then x_0^2 will be adjacent to three small vertices x_0^1, x_0^3, w_2. If $w_2 = x_0^3$, then x_0^2 is adjacent to three small vertices x_0^1, x_0^3, x_0^4, this contradicts with (C4).

• $d_H(x_0^2) = 5$ and $\tau(v \to x_0^2) = \frac{4}{9}$. Then $n_\times(x_0^2) = 2$, y_0, x_0^4 are indirect neighbors of x_0^2 in H, and $[x_0^1 x_0^2 x_0^3]$ is a normal 3-face of H. This implies that x_0^2 is adjacent to three small vertices, a contradiction with (C4). This completes the proof. □

Now we continue the proof of Claim 5. Recall that $m_0 = |X_0|$.

First, suppose that m_0 is even. We partition X_0 into $\frac{m_0}{2}$ 2-sets $\{x_0^1, x_0^2\}$, $\{x_0^3, x_0^4\}, \dots, \{x_0^{m_0-3}, x_0^{m_0-2}\}, \{x_0^{m_0-1}, x_0^{m_0}\}$. It is easy to check that Claims 2 and 3 in Subase 10.1 still hold for 2-sets $\{x_0^3, x_0^4\}, \{x_0^5, x_0^6\}, \dots, \{x_0^{m_0-3}, x_0^{m_0-2}\}$. By Claim 6, $\sigma(y_0, y_1) \le \frac{5}{6} + \frac{2}{3} \times \frac{m_0-4}{2} + \frac{5}{6} = \frac{1}{3}(m_0 + 1) = \frac{1}{3}(|X_0| + 1)$.

Next, suppose that m_0 is odd. We partition X_0 into $\frac{m_0-3}{2}$ 2-sets $\{x_0^1, x_0^2\}, \{x_0^3, x_0^4\}, \dots, \{x_0^{m_0-6}, x_0^{m_0-5}\}, \{x_0^{m_0-4}, x_0^{m_0-3}\}$ and one 3-set $\{x_0^{m_0-2}, x_0^{m_0-1}, x_0^{m_0}\}$.

If $\tau(v \to x_0^{m_0-2}) \le \frac{1}{3}$, then by Claim 6 and Claim 3, $\sigma(y_0, y_1) \le \frac{5}{6} + \frac{2}{3} \times \frac{m_0-5}{2} + + \frac{1}{3} + \frac{5}{6} = \frac{1}{3}(m_0 + 1) = \frac{1}{3}(|X_0| + 1)$.

If $\tau(v \to x_0^{m_0-2}) > \frac{1}{3}$, then $x_0^{m_0-2}$ is a 5-vertex with $n_\times(x_0^{m_0-2}) = 2$ and $\tau(v \to x_0^{m_0-2}) \in \{\frac{4}{9}, \frac{5}{12}\}$. Similarly to the proof of Claim 2, we obtain that $\tau(v \to x_0^{m_0-1}) \le \frac{2}{9}$, and hence $\tau(v \to x_0^{m_0-2}) + \tau(v \to x_0^{m_0-1}) \le \frac{4}{9} + \frac{2}{9} = \frac{2}{3}$. By Claim 6 and Claim 2, $\sigma(y_0, y_1) \le \frac{5}{6} + \frac{2}{3} \times \frac{m_0-5}{2} + \frac{2}{3} + \frac{1}{2} = \frac{1}{3}(m_0 + 1) = \frac{1}{3}(|X_0| + 1)$. This completes the proof of Claim 5. □

Using Claim 5 and noting that $d_H(v) = |Y| + \sum_{i=0}^{k-1} |X_i|$, we can establish the following:

$$
\begin{aligned}
w'(v) &\geq w(v) - \frac{1}{2}d_H(v) - \sum_{i=0}^{k-1} \sigma(y_i, y_{i+1}) \\
&\geq \frac{1}{2}d_H(v) - 4 - \sum_{i=0}^{k-1} \frac{1}{3}(|X_i| + 1) \\
&= \frac{1}{2}d_H(v) - 4 - \frac{1}{3}|Y| - \frac{1}{3}\sum_{i=0}^{k-1} |X_i| \\
&= \frac{1}{2}d_H(v) - 4 - \frac{1}{3}|Y| - \frac{1}{3}(d_H(v) - |Y|) \\
&= \frac{1}{6}(d_H(v) - 24) \geq 0.
\end{aligned}
$$

Up to now, we have proved the statement (I). That is, $w'(x) \geq 0$ for all $x \in V^0(H) \cup F^0(H)$. To show statement (II), we first notice that $w'(f_0) = w(f_0) = d_H(f_0) - 4 = 2 - 4 = -2$. Let $v \in V(C)$. Since H is a plane triangulation, $d_H(v) \geq 3$. Note that v has exactly $d_H(v) - 2$ internal (direct or indirect) neighbors and has $d_H(v) - 1$ internal faces incident with it. By (R0), $w'(v) \geq d_H(v) - 4 - \frac{1}{2}(d_H(v) - 2) - \frac{1}{2}(d_H(v) - 1) = -\frac{5}{2}$. Consequently, $w'(f_0) + \sum_{x \in V(C)} w'(x) \geq -2 - \frac{2}{5} \times 2 = -7$. This completes the proof of Theorem 2. □

Given a graph G, for a vertex $v \in V(G)$, let $n_2(v)$ denote the number of 2-vertices that are adjacent to v.

Theorem 3. *Every nonempty* 1*-planar graph G contains a k-vertex v with neighbors $v_1, v_2, \ldots, v_k$ with $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$ such that one of the following conditions holds:*

(B1) $k = 1$;

(B2) $k = 2$ with $n_2(v_1) \geq d_G(v_1) - 23$ or $n_2(v_2) \geq d_G(v_2) - 23$;

(B3) $k = 3$ with $d_G(v_1) \leq 23$;

(B4) $k = 4$ with $d_G(v_1) \leq 14$ and $d_G(v_2) \leq 23$;

(B5) $k = 5$ with $d_G(v_1) \leq 10$, $d_G(v_2) \leq 14$, and $d_G(v_3) \leq 23$;

(B6) $k = 6$ with $d_G(v_1) \leq 9$, $d_G(v_2) \leq 10$, $d_G(v_3) \leq 14$, and $d_G(v_4) \leq 23$;

(B7) $k = 7$ with $d_G(v_1) \leq 7$, $d_G(v_2) \leq 9$, $d_G(v_3) \leq 10$, $d_G(v_4) \leq 14$, and $d_G(v_5) \leq 23$.

Proof. Suppose that Theorem 3 is not true. Let G be a counterexample, which is a 1-planar graph and contains no vertex v satisfying (B1)–(B7). Then $\delta(G) \geq 2$ since (B1) is not satisfied. Theorem 2 implies that at least one of (C1)–(C6) is satisfied. Note that the only difference between these two theorems is the case where $\delta(G) = 2$. Let H be the graph obtained by removing all 2-vertices from G. Clearly, H remains to be a 1-planar graph.

If $\delta(H) = 0$, then (B2) is obviously true for G. Now we consider the case $\delta(H) = 1$. Let $x \in V(H)$ with $d_H(x) = 1$. Since $2 = \delta(G) \leq d_G(x) = d_H(x) + n_2(x) = 1 + n_2(x)$, it follows that $n_2(x) \geq 1$. Let $x_1, x_2, \ldots, x_m, y$ be the neighbors of x in G, where $d_G(x_i) = 2$ for $i = 1, 2, \ldots, m$ (≥ 1), and $d_G(y) \geq 2$. So, x_1 is a 2-vertex of G with x as its neighbor satisfying $n_2(x) = d_G(x) - d_H(x) = d_G(x) - 1 \geq d_G(x) - 23$, a contradiction with (B2).

Suppose that $\delta(H) = 2$. Let $u \in V(H)$ with $d_H(u) = 2$. We claim that $d_G(u) \geq 3$. Suppose, on contrary, if $d_G(u) = 2$, then u will be removed from G by the construction of H. Thus, u is adjacent to at least one 2-vertex, say u_1, in G. Since $n_2(u) = d_G(u) - d_H(u) = d_G(u) - 2 \geq d_G(u) - 23$, a contradiction with (B2).

Finally if $\delta(H) \geq 3$, then H contains a k-vertex y, $3 \leq k \leq 7$ by Theorem 2 and it has neighbors $y_1, y_2, \ldots, y_k$ where $d_H(y_1) \leq d_H(y_2) \leq \cdots \leq d_H(y_k)$ such that this vertex v satisfies at least one of (C2)-(C6). Note that $d_H(y_{k-2}) \leq 23$ by the definition of (C2)–(C6). Since (B2) is not satisfied, no vertex $x \in V(H)$ with $d_H(x) \leq 23$ is adjacent to a 2-vertex in G. It follows that $d_G(y) = d_H(y) = k$ and $d_G(y_i) = d_H(y_i)$ for $i = 1, 2, \ldots, k-2$. This shows that y is a vertex of G satisfying at least one of (B3)–(B7), a contradiction. □

3. Acyclic Edge Coloring

A *multiset* S is a generalized set in which an element can appear many times. Let $\mathrm{mul}_S(x)$ denote the number of times that an element $x \in S$ appears in S. The union, denoted by $S \uplus S'$, of two multisets S and S' is a multiset that consists of all elements of S and S'. Moreover, we define $\mathrm{mul}_{S \uplus S'}(x) = \mathrm{mul}_S(x) + \mathrm{mul}_{S'}(x)$ for any $x \in S \uplus S'$.

Suppose that G is a 1-planar graph with an acyclic edge-k-coloring ϕ using the color set $C = \{1, 2, \ldots, k\}$. For a vertex $v \in V(G)$, let $C_\phi(v)$ denote the subset of colors in C that are assigned to the edges incident to v. For an edge $e = uv$, let $C_\phi(v; e) = C_\phi(v) \setminus \{\phi(e)\}$. Given a partial acyclic edge coloring π of G, an (i, j)-*maximal bichromatic path* on π is a maximal path consisting of edges colored with i and j alternately. An $(i, j; u, v)$-*critical bichromatic path* of π is an (i, j)-maximal bichromatic path which starts at u with i and ends at v with i.

Lemma 1 (Basavaraju and Chandran, [8])**.** *Given any two colors i and j with respect to a partial acyclic edge-coloring of a graph G, then there exists at most one (i, j)-maximal bichromatic path containing a particular vertex v.*

Lemma 2 (Esperet and Parreau, [5])**.** *Every graph G with $\Delta \geq 2$ has $\chi'_a(G) \leq 4\Delta - 4$.*

Theorem 4. *If G is a 1-planar graph with maximum degree Δ, then $\chi'_a(G) \leq \Delta + 36$.*

Proof. The proof is proceeded by induction on the edge number $|E(G)|$ of G. If $|E(G)| \leq \Delta + 36$, then it is trivial that G is acyclically edge-$(\Delta + 36)$-colorable. We may assume that G is a connected 1-planar graph with $|E(G)| \geq \Delta + 37$. If $\Delta \leq 13$, then Lemma 2 confirms that $\chi'_a(G) \leq 4\Delta - 4 \leq \Delta + 36$. So we may assume that $\Delta \geq 14$.

If $\delta(G) = 1$, let $x \in V(G)$ with $d_G(x) = 1$ and let y be the unique neighbor of x in G. Consider the graph $H = G - x$. Then H is a 1-planar graph with $|E(H)| = |E(G)| - 1$ and $\Delta(H) \leq \Delta$. By the induction hypothesis, H admits an acyclic edge-$(\Delta + 36)$-coloring ϕ using the color set $C = \{1, 2, \ldots, \Delta + 36\}$. Based on ϕ, we can color xy with a color in $C \setminus C_\phi(y)$ to extend ϕ to G.

Assume that $\delta(G) \geq 2$. By Theorem 3, there exists a k-vertex $v \in V(G)$, $2 \leq k \leq 7$ with neighbors $v_1, v_2, \ldots, v_k$ where $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_k)$ such that at least one of the conditions (B2)–(B7) holds.

First, suppose that (B2) holds. That is, there exists a 2-vertex v with neighbors v_1, v_2 such that $n_2(v_i) \geq d_G(v_i) - 23$ for some $i = 1, 2$, say $i = 1$. Let $y_1, y_2, \ldots, y_m, y_{m+1}, \ldots, y_t$ be the neighbors of v_1, where $t = d_G(v_1) - 1$, such that $d_G(y_i) \geq 3$ for $i = 1, 2, \ldots, m$, and $d_G(y_i) = 2$ for $i = m+1, m+2, \ldots, t$. Since $n_2(v_1) \geq d_G(v_1) - 23$, we derive that $m = d_G(v_1) - n_2(v) \leq 23$. Let $H = G - v$. Then H is a 1-planar graph with $|E(H)| = |E(G)| - 2$ and $\Delta(H) \leq \Delta$. By the induction hypothesis, H admits an acyclic edge-$(\Delta + 36)$-coloring ϕ using the color set $C = \{1, 2, \ldots, \Delta + 36\}$. Set $S_1 = \{\phi(v_1y_1), \phi(v_1y_2), \ldots, \phi(v_1y_m)\}$ and $S_2 = \{\phi(v_1y_{m+1}), \phi(v_1y_{m+2}), \ldots, \phi(v_1y_t)\}$. To extend ϕ to G, we first color the edge vv_2 with a color $a \in C \setminus (C_\phi(v_2) \cup S_1)$. Since $|C_\phi(v_2) \cup S_1| \leq |C_\phi(v_2)| + |S_1| \leq d_G(v_2) - 1 + m \leq \Delta - 1 + m \leq \Delta + 22$. Since $|C| = \Delta + 36$, there is the color which can be assigned to vv_2. If $a \notin S_2$, then we color vv_1 with any color in $C \setminus (C_\phi(v_1) \cup \{a\})$. Otherwise, we may assume that $a = \phi(v_1y_{m+1})$. Let z be the neighbor of y_{m+1} other than v_1. We color vv_1 with a color in $C \setminus (C_\phi(v_1) \cup \{\phi(zy_{m+1})\})$. It is easy to check that any newly created cycle by adding vv_1 and vv_2 has at least three colors and, in turn, the extended coloring is an acyclic edge-$(\Delta + 36)$-coloring of G.

Next assume that one of (B3)-(B7) holds. Let $H = G - vv_1$. Then H is a 1-planar graph with $|E(H)| = |E(G)| - 1$ and $\Delta(H) \leq \Delta$. By the induction hypothesis, H admits an acyclic edge-$(\Delta + 36)$-coloring ϕ using the color set $C = \{1, 2, \ldots, \Delta + 36\}$. Let $C^* =$

$C \setminus (C_\phi(v) \cup C_\phi(v_1))$, and $r = |C_\phi(v) \cap C_\phi(v_1)|$. Of all the possible acyclic edge-$(\Delta+36)$-colorings of H choose the coloring ϕ which minimizes r. The following inequality holds.

$$\begin{aligned} |C^*| &= |C| - |C_\phi(v)| - |C_\phi(v_1)| + |C_\phi(v) \cap C_\phi(v_1)| \\ &\geq (\Delta+36) - (d_G(v)-1) - (d_G(v_1)-1) + r \\ &\geq \Delta + 36 - (7-1) - (23-1) + r = \Delta + r + 8. \end{aligned}$$

If $r = 0$, then we can color vv_1 with any color in C^* such that no bichromatic cycle is produced in G. Otherwise, assume that $r \geq 1$. Set $d_i = d_G(v_i)$ for $i = 1, 2, \ldots, k$. Let $u_1, u_2, \ldots, u_{d_1-1}$ denote the neighbors of v_1 other than v, and assume that $\phi(v_1u_i) = i$ for $i \in \{1, 2, \ldots, d_1 - 1\}$. Define a multiset S_v as follows:

$$S_v = C_\phi(v; vv_2) \uplus C_\phi(v; vv_3) \uplus \cdots \uplus C_\phi(v; vv_k)$$

First we show that $\sum_{i=1}^{k-2} d_i \leq 77 - k - d_1$. Let $\theta_k = \sum_{i=1}^{k-2} d_i + d_1 + k$. It remains to show that $\theta_k \leq 77$ for any $k \in \{3,4,5,6,7\}$. If $k = 3$, then $d_1 \leq 23$ and hence $\theta_3 = d_1 + d_1 + k \leq 23 + 23 + 3 = 49$. If $k = 4$, then $d_1 \leq 14$, $d_2 \leq 23$, and hence $\theta_4 = 2d_1 + d_2 + k \leq 2 \times 14 + 23 + 4 = 55$. If $k = 5$, then $d_1 \leq 10$, $d_2 \leq 14$, $d_3 \leq 23$, and hence $\theta_5 = 2d_1 + d_2 + d_3 + k \leq 2 \times 10 + 14 + 23 + 5 = 62$. If $k = 6$, then $d_1 \leq 9$, $d_2 \leq 10$, $d_3 \leq 14$, $d_4 \leq 23$ and hence $\theta_6 = 2d_1 + d_2 + d_3 + d_4 + k \leq 2 \times 9 + 10 + 14 + 23 + 6 = 71$. If $k = 7$, then $d_1 \leq 7$, $d_2 \leq 9$, $d_3 \leq 10$, $d_4 \leq 14$, $d_5 \leq 23$ and hence $\theta_7 = 2d_1 + d_2 + d_3 + d_4 + +d_5 + k \leq 2 \times 7 + 9 + 10 + 14 + 23 + 7 = 77$.

If $r = 1$, then we may assume that $\phi(vv_2) = \phi(v_1u_1) = 1$. It follows that we can color vv_1 with a color $a \in C^*$ missing the neighbors of v_2 such that no bichromatic cycle is created in G. This is possible because $d_G(v) = 7$ and $|C^*| \geq \Delta + 9$.

Now suppose that $r \geq 2$. Let $\phi(vv_{i+1}) = \phi(v_1u_i) = i$ for $i = 1, 2, \ldots, r$. Clearly, $r \leq k - 1$. If there exists a color $a \in C^*$ such that assigning a to vv_1 does not result in a bichromatic cycle in G, we are done. Otherwise, for any $a \in C^*$, there exists a $(i, a; v_1, v_{i+1})$-critical bichromatic path for every $i \in \{1, 2, \ldots, r\}$. Note that $|C^*| \geq (\Delta+36) - (d_G(v) - 1) - (d_G(v_1) - 1) + 2 = \Delta + 40 - k - d_1$. The fact of $\sum_{i=1}^{k-2} d_i \leq 77 - k - d_1$ and $d_{k-1}, d_k \leq \Delta$ implies that

$$\begin{aligned} \sum_{i=2}^{k}(d_i - 1) &= \sum_{i=1}^{k-2} d_i - (k-1) + d_{k-1} + d_k - d_1 \\ &\leq 77 - k - d_1 - k + 1 + 2\Delta - d_1 \\ &= 2\Delta + 78 - 2d_1 - 2k \\ &< 2\Delta + 80 - 2d_1 - 2k \\ &= 2(\Delta + 40 - k - d_1) \\ &\leq 2|C^*|. \end{aligned}$$

The above inequality implies that there exists a color $\beta \in C^*$ such that it appears at most once in S_v, say $\beta \in C_\phi(v_2; vv_2)$. Since there exists a $(1, \beta; v, v_1)$-critical bichromatic path through v, there is no $(1, \beta; v, v_3)$-critical bichromatic path through v_2 by Lemma 1. We can recolor vv_3 with β to yield a new acyclic edge-$(\Delta+36)$-coloring ϕ' of H. This implies that $|C_{\phi'}(v) \cap C_{\phi'}(v_1)| < |C_\phi(v) \cap C_\phi(v_1)|$, a contradiction with the choice of ϕ. This completes the proof of the theorem. □

4. Concluding Remarks

In this paper, we established the structural theorem on 1-planar graphs using the discharging method in Section 2, then show that every 1-planar graph G has $\chi'_a(G) \leq \Delta + 36$ in Section 3 by the structural theorem 3. We strengthen the following two results:

(1) G is a 1-planar graph, then $\chi'_a(G) \leq \max\{2\Delta - 2, \Delta + 83\}$ in [17];
(2) G is a triangle-free 1-planar graph, then $\chi'_a(G) \leq \Delta + 16$ in [19].

Conjecture 1 seems difficult. As yet, it has been verified only for several restricted classes of planar graphs. A little work on 1-planar graphs has been done, there is a certain gap between our research results and the conjectured results ($\Delta + 2$); however, this result is the best research so far. We believe that this bound is not tight, and this bound can be reduced to $\Delta + 22$ or less by a good method.

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