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Subordinations and Norm Estimates for Functions Associated with Ma-Minda Subclasses

Aaisha Farzana Habibullah [1,†], Muthusamy Palani Jeyaraman [2,†] and Teodor Bulboacă [3,*,†]

[1] Department of Mathematics with Computer Applications, A. M. Jain College, Meenambakkam, Chennai 600061, Tamil Nadu, India
[2] Department of Mathematics, L. N. Government College, Ponneri, Chennai 601204, Tamil Nadu, India
[3] Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
* Correspondence: bulboaca@math.ubbcluj.ro; Tel.: +40-729087153
† These authors contributed equally to this work.

Abstract: For a function p analytic in the open unit disc and satisfying $p(0) = 1$, we prove certain subordination implications of the first order differential subordination $1 + zp'(z) \prec 1 + Mz$, which provides sufficient conditions for a function to belong to various subclasses of Ma-Minda starlike functions. The pre-Schwarzian norm estimate and inclusion criteria for certain subclasses of analytic function are also obtained. Additionally, using Gronwall's inequality we give a sufficient condition for a normalized function to belong to a class of functions with bounded arguments that extends the class of strongly α-Bazilevič functions of order γ studied by Gao in 1996.

Keywords: starlike; convex and close-to-convex functions; differential subordination; norm estimate; Schwarzian derivative; pre-Schwarzian derivative; Gronwall's inequality

MSC: 30C45; 30C50; 30C55

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane, with the usual normalization $f(0) = f'(0) - 1 = 0$. The class $\mathcal{S} \subset \mathcal{A}$ consists of all normalized and univalent functions in $\mathbb{D}$.

A function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*$ of *starlike functions*, or to the class $\mathcal{C}$ of *convex functions*, if f maps conformally the unit disc $\mathbb{D}$ onto the domains that are starlike with respect to the origin, or convex, and the analytical characterization of these classes are given by $\operatorname{Re}(zf'(z)/f(z)) > 0$ and $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ in $\mathbb{D}$, respectively.

If f and g are two analytic functions on $\mathbb{D}$, we say f *is subordinated to* g, written $f(z) \prec g(z)$, if $f = g \circ w$ for some analytic function $w : \mathbb{D} \to \mathbb{D}$, with $w(0) = 0$. If g is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ if, and only if, $f(\mathbb{D}) \subset g(\mathbb{D})$, and $f(0) = g(0)$ (cf. ([1] p. 90); see also [2]).

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$, and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z), \tag{1}$$

then p is called a *solution of the differential subordination*. The univalent function q is called a *dominant* of the solutions of the differential subordination, or more simply a dominant, if $p(z) \prec q(z)$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants q of (1) is said to be the *best dominant* (for details see [2]).

For a convex univalent function h defined on $\mathbb{D}$ with $h(0) = 1$, $\operatorname{Re} h(z) > 0$, $z \in \mathbb{D}$, in 1985 Padmanabhan and Parvatham [3], and Shanmugam [4] in 1989, introduced and studied certain subclasses defined in terms of convolution and subordination as follows, respectively:

$$\mathcal{S}^*_\alpha(h) := \left\{ f \in \mathcal{A} : \frac{z(k_\alpha * f)'(z)}{(k_\alpha * f)(z)} \prec h(z) \right\},$$

where $k_\alpha(z) = \dfrac{z}{(1-z)^\alpha}$, $\alpha \in \mathbb{R}$, and

$$\mathcal{S}^*_g(h) := \left\{ f \in \mathcal{A} : \frac{z(g * f)'(z)}{(g * f)(z)} \prec h(z) \right\},$$

where g is a fixed function. For $g(z) = \dfrac{z}{1-z}$ and $g(z) = \dfrac{z}{(1-z)^2}$, the corresponding subclasses $\mathcal{S}^*_g(h)$ are denoted by $\mathcal{S}^*(h)$ and $\mathcal{K}(h)$, respectively.

If h is a univalent function with positive real part, which maps $\mathbb{D}$ onto a domain symmetric with respect to the real axis and starlike with respect to $h(0) = 1$ and $h'(0) > 0$, in 1992 Ma and Minda [5] studied growth, distortion, and coefficient estimates for functions belonging to the above mentioned classes, $\mathcal{S}^*(h)$ and $\mathcal{K}(h)$. In recent years, many researchers have taken specific functions and studied numerous subordination implications that provide sufficient conditions for functions to belong to various subclasses of Ma-Minda starlike functions (see Table 1).

Table 1. Ma-Minda starlike subclasses $\mathcal{S}^*(h)$.

No.	Ma-Minda Starlike Subclasses $\mathcal{S}^*(h)$	Article
(i)	$\mathcal{S}^*(\sqrt{1+z}) =: \mathcal{S}^*_{\mathrm{L}}$	[6]
(ii)	$\mathcal{S}^*\left(z + \sqrt{1+z^2}\right) =: \mathcal{S}^*_c$	[7]
(iii)	$\mathcal{S}^*(\alpha + (1-\alpha)e^z) =: \mathcal{S}^*_{\mathrm{e}}$	[8]
(iv)	$\mathcal{S}^*\left(1 + \frac{4z}{3} + \frac{2z^2}{3}\right) =: \mathcal{S}^*_{\mathrm{car}}$	[9]
(v)	$\mathcal{S}^*(1 + z - \frac{z^3}{3}) =: \mathcal{S}^*_{\mathrm{Ne}}$	[10]
(vi)	$\mathcal{S}^*\left(1 + \frac{z}{k} \cdot \frac{z+k}{k-z}\right) =: \mathcal{S}^*_{\mathrm{R}}, k = 1 + \sqrt{2}$	[11]
(vii)	$\mathcal{S}^*(1 + \sin z) =: \mathcal{S}^*_{\mathrm{sin}}$	[12]
(viii)	$\mathcal{S}^*\left(\frac{2}{1+e^{-z}}\right) =: \mathcal{S}^*_{\mathrm{sig}}$	[13]
(ix)	$\mathcal{S}^*\left(1 + \sqrt{2}z + \frac{z^2}{2}\right) =: \mathcal{S}^*_{\mathrm{lim}}$	[14]
(x)	$\mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right) =: \mathcal{S}^*_{[A,B]}$	[15]

In 1996, Sokół and Stankiewicz [6] studied the radius of convexity of some subclasses of strongly starlike functions. Raina and Sokół in [7] have discussed certain geometric properties of the functions belonging to the class $\mathcal{S}^*_c$, while in [8] the authors have obtained sharp radii for functions belonging to the class $\mathcal{S}^*_e$. In [9–11], the authors used the method of differential subordination to obtain sufficient conditions for functions of the class $\mathcal{A}$ to belong to Ma-Minda subclasses of starlike functions.

On the other hand, subordination implications and subclasses of analytic functions have been recently studied in literature, such as [16] and [17].

Motivated by their works, in the Section 2, we have used the subordination as a tool to obtain bounds on M for an analytic function p, with $p(0) = 1$, such that $1 + zp'(z) \prec 1 + Mz$ implies the function p is subordinate to each of the following functions:

$$e^z, \quad \sqrt{1+z}, \quad z + \sqrt{1+z^2}, \quad 1 + \frac{4z}{3} + \frac{2z^2}{3}, \quad 1 + z - \frac{z^3}{3},$$
$$1 + \frac{z}{k} \cdot \frac{z+k}{k-z} \text{ with } k = 1 + \sqrt{2}, \quad 1 + \sin z, \quad \frac{2}{1+e^{-z}}, \quad 1 + \sqrt{2}z + \frac{z^2}{2}. \tag{2}$$

In the Section 3, in connection with [18,19], we determine the pre-Schwarzian norm estimates for function belonging to the classes

$$\mathcal{S}^*(\psi, \mu) := \left\{ f \in \mathcal{A} : \left(\frac{z}{f(z)} \right)^{1+\mu} f'(z) \prec \psi(z) \right\}, \tag{3}$$

and

$$K(\psi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \right\}, \tag{4}$$

where $\mu \in \mathbb{C}$, and ψ is an analytic function in $\mathbb{D}$, with $\psi(0) = 1$ and $\psi(z) \neq 0$ for all $z \in \mathbb{D}$; also, $\mathcal{S}^*(\psi, 0) =: \mathcal{S}^*(\psi)$. Moreover, norm estimates for various classes mentioned in the Table 1 are obtained.

In the Section 4, using the Gronwall's inequality [20] we obtained a sufficient condition involving the Schwarzian derivative, such that a given function belongs to a certain class associated with $\mathcal{S}^*(\psi, \mu)$.

2. Subordination Results for Some Special Dominants

To obtain bounds on M for an analytic function p, with $p(0) = 1$, such that $1 + zp'(z) \prec 1 + Mz$ implies the function p is subordinate to each of the following functions mentioned in (2) the following lemma will be used to prove our main results.

Lemma 1. *([2]) Let q be univalent in $\mathbb{D}$ and let ϕ and ν be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) := zq'(z)\phi(q(z))$ and $h(z) := \nu(q(z)) + Q(z)$. Suppose that:*

(i) either h is convex or Q is starlike univalent in $\mathbb{D}$,

and

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} > 0$ for $z \in \mathbb{D}$.

Let p be analytic in $\mathbb{D}$ with $p(0) = q(0)$ and $p(\mathbb{D}) \subset q(\mathbb{D})$. If p satisfies

$$\nu(p(z)) + zp'(z)\phi(p(z)) \prec \nu(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$, and q is the best dominant of the above subordination.

Theorem 1. *Let p be an analytic function in $\mathbb{D}$, with $p(0) = 1$, such that*

$$1 + zp'(z) \prec 1 + Mz.$$

Then, the following subordination results hold:

- *(a)* $p(z) \prec e^z$, *for* $0 < M \leq 1 - e^{-1}$;
- *(b)* $p(z) \prec \psi_{\mathrm{Ne}}(z) = 1 + z - \frac{z^3}{3}$, *for* $0 < M \leq \frac{2}{3}$;
- *(c)* $p(z) \prec \psi_{\mathrm{L}}(z) = \sqrt{1+z}$, *for* $0 < M \leq \sqrt{2} - 1$;
- *(d)* $p(z) \prec \psi_c(z) = z + \sqrt{1+z^2}$, *for* $0 < M \leq 2 - \sqrt{2}$;
- *(e)* $p(z) \prec \psi_{\mathrm{car}}(z) = 1 + \frac{4z}{3} + \frac{2z^2}{3}$, *for* $0 < M \leq \frac{2}{3}$;
- *(f)* $p(z) \prec \psi_{\mathrm{R}}(z) = 1 + \frac{z}{k} \cdot \frac{k+z}{k-z}$, *with* $k = 1 + \sqrt{2}$, *for* $0 < M \leq \frac{1}{3+2\sqrt{2}}$;

(g) $p(z) \prec \psi_{\sin}(z) = 1 + \sin z$, *for* $0 < M \leq \sin 1$;

(h) $p(z) \prec \psi_{\mathrm{sig}}(z) = \frac{2}{1+e^{-z}}$, *for* $0 < M \leq \frac{e-1}{e+1}$;

(i) $p(z) \prec \psi_{\lim}(z) = 1 + \sqrt{2}z + \frac{z^2}{2}$, *for* $0 < M \leq \sqrt{2} - \frac{1}{2}$.

The bounds of M are the best possible.

Proof. If we let the function $q_M : \mathbb{D} \to \mathbb{C}$ defined by

$$q_M(z) = 1 + Mz,\ M > 0,$$

then q_M is a solution for the first order differential equation $1 + zq'_M(z) = 1 + Mz$. If we take $\nu = \phi = 1$ in Lemma 1, then we have

$$Q(z) = zq'_M(z)\phi(z) = Mz =: h(z) - 1.$$

Since the function Q is starlike, and $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} > 0$, $z \in \mathbb{D}$, from Lemma 1 it follows that the subordination assumption $1 + zp'(z) \prec 1 + zq'_M(z)$ implies $p(z) \prec q_M(z)$.

Each of the conclusions in all the parts of this theorem are of the form $p(z) \prec \mathcal{G}(z)$, for appropriate choices of $\mathcal{G}$, whenever the subordination $q_M(z) \prec \mathcal{G}(z)$ holds.

(a) Letting $\mathcal{G}(z) := \psi(z) = e^z$, then $q_M(z) \prec \mathcal{G}(z)$ implies

$$\mathcal{G}(-1) = e^{-1} \leq 1 - M < 1 + M \leq e = \mathcal{G}(1).$$

A simple calculation yields that the above double inequality holds whenever $0 \leq M \leq 1 - e^{-1}$. Therefore, according to the Figure 1a made by MAPLE™ software, we see that $q_M(\mathbb{D}) \subset \mathcal{G}(\mathbb{D})$, and since $q_M(0) = \mathcal{G}(0) = 1$, it follows that $q_M(z) \prec \mathcal{G}(z)$, hence the subordination (a) is proved.

(b) If $\mathcal{G}(z) := \psi_{\mathrm{Ne}}(z) = 1 + z - \frac{z^3}{3}$, then $q_M(z) \prec \mathcal{G}(z)$ implies

$$\mathcal{G}(-1) = \frac{1}{3} \leq 1 - M < 1 + M \leq \frac{5}{3} = \mathcal{G}(1),$$

that holds whenever $0 < M \leq \frac{2}{3}$. From the Figure 1b made by MAPLE™ software, since we see that $q_M(\mathbb{D}) \subset \mathcal{G}(\mathbb{D})$, and combining with $q_M(0) = \mathcal{G}(0) = 1$, it follows that $q_M(z) \prec \mathcal{G}(z)$, thus the subordination (b) is holds.

For proving the items (c)–(i) we proceed similarly and obtain the required results.

The bounds are the best possible in each cases because, such as in the cases (a) and (b) presented above, the left and right hand side inequalities are attained for the mentioned values of M, respectively. □

For $f \in \mathcal{A}$, such that $f(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$, if we let $p(z) := \left(\dfrac{z}{f(z)}\right)^{1+\mu} f'(z)$ in Theorem 1, we obtain the following special cases:

Corollary 1. *If* $\mu \in \mathbb{C}$ *and the function* $f \in \mathcal{A}$ *satisfies the subordination*

$$1 + \left(\frac{z}{f(z)}\right)^{\mu} \left[(1+\mu)\frac{zf'(z)}{f(z)}\left(1 - \frac{zf'(z)}{f(z)}\right) + \frac{z^2 f''(z)}{f(z)}\right] \prec 1 + Mz,$$

then:

(a) $\left(\dfrac{z}{f(z)}\right)^{1+\mu} f'(z) \prec \psi_{\mathrm{Ne}}(z) = 1 + z - \frac{z^3}{3}$, *for* $0 < M \leq \frac{2}{3}$;

(b) $\left(\dfrac{z}{f(z)}\right)^{1+\mu} f'(z) \prec \psi_{\mathrm{L}}(z) = \sqrt{1+z}$, *for* $0 < M \leq \sqrt{2} - 1$;

(c) $\left(\dfrac{z}{f(z)}\right)^{1+\mu} f'(z) \prec \psi_c(z) = z + \sqrt{1+z^2}$, *for* $0 < M \leq 2 - \sqrt{2}$;

(d) $\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) \prec \psi_{\mathrm{car}}(z) = 1 + \frac{4z}{3} + \frac{2z^2}{3}$, *for* $0 < M \leq \frac{2}{3}$;

(e) $\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) \prec \psi_{\mathrm{R}}(z) = 1 + \frac{z}{k} \cdot \frac{k+z}{k-z}$, *with* $k = 1 + \sqrt{2}$, *for* $0 < M \leq \frac{1}{3+2\sqrt{2}}$;

(f) $\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) \prec \psi_{\sin}(z) = 1 + \sin z$, *for* $0 < M \leq \sin 1$;

(g) $\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) \prec \psi_{\mathrm{sig}}(z) = \frac{2}{1+e^{-z}}$, *for* $0 < M \leq \frac{e-1}{e+1}$;

(h) $\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) \prec \psi_{\lim}(z) = 1 + \sqrt{2}z + \frac{z^2}{2}$, *for* $0 < M \leq \sqrt{2} - \frac{1}{2}$.

The bounds of M are the best possible.

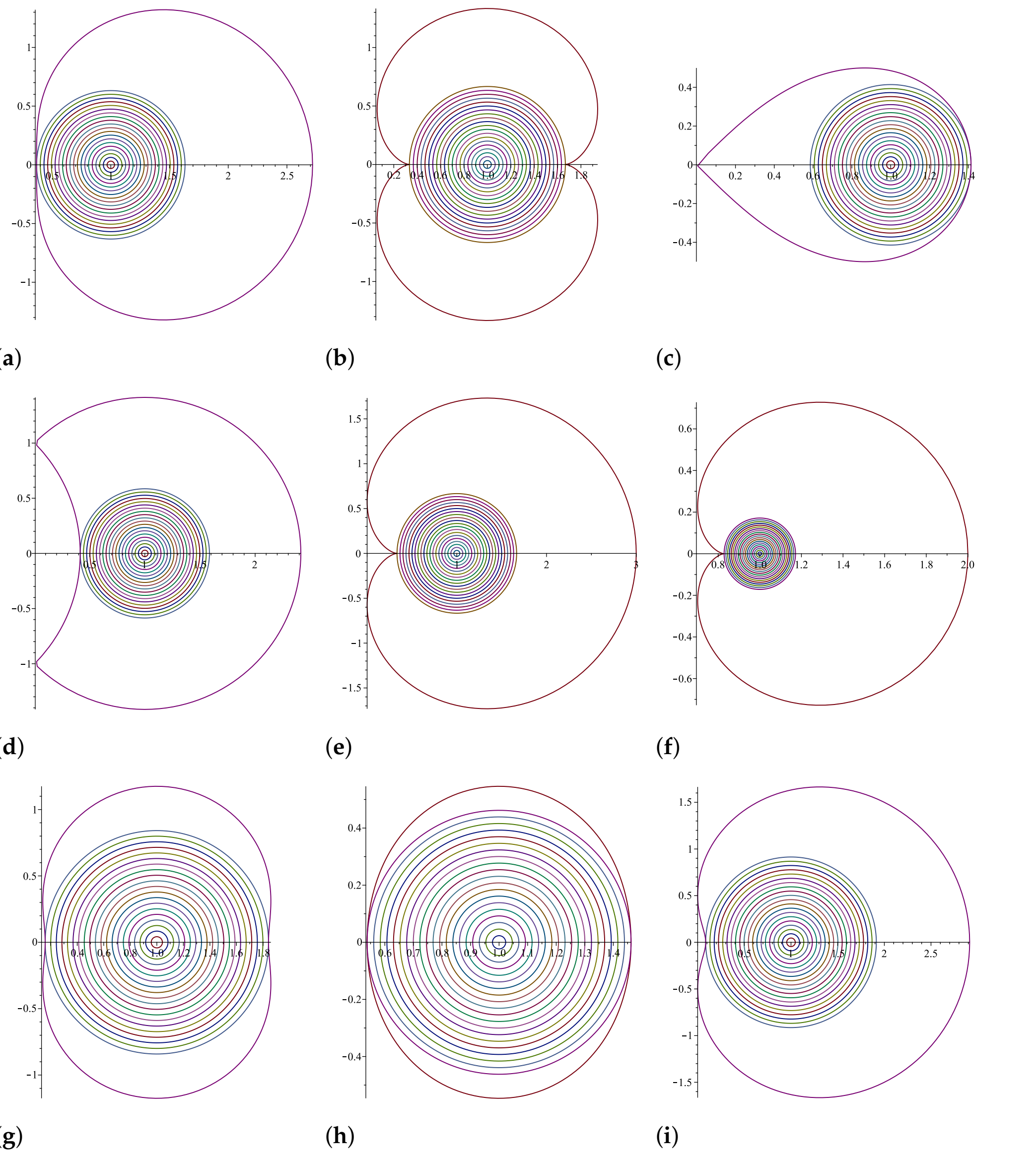

Figure 1. Figures for the Theorem 1. (**a**) Boundary curve of $\psi(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = 1 - \frac{1}{e}$. (**b**) Boundary curve of $\psi_{\mathrm{Ne}}(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = \frac{1}{e}$. (**c**) Boundary curve of $\psi_{\mathrm{L}}(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = \sqrt{2} - 1$. (**d**) Boundary curve of $\psi_c(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = 2 - \sqrt{2}$. (**e**) Boundary curve of $\psi_{\mathrm{car}}(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = \frac{2}{3}$. (**f**) Boundary curve of $\psi_{\mathrm{R}}(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = \frac{1}{3+2\sqrt{2}}$. (**g**) Boundary curve of $\psi_{\sin}(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = \sin 1$. (**h**) Boundary curve of $\psi_{\mathrm{sig}}(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = \frac{e-1}{e+1}$. (**i**) Boundary curve of $\psi_{\lim}(\mathbb{D})$ and $q_M(\mathbb{D})$ for $M = \sqrt{2} - \frac{1}{2}$.

Taking $\mu = 0$ in Corollary 1 we obtain the next sufficient condition for the function $f \in \mathcal{A}$ to belong to various Ma-Minda type subclasses:

Remark 1. *If the function $f \in \mathcal{A}$ satisfies the subordination*

$$1+\left[\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}\right)+\frac{z^2f''(z)}{f(z)}\right]\prec 1+Mz,$$

then:

(a) *If $0 < M \le \frac{2}{3}$, then $f \in \mathcal{S}^*_{\mathrm{Ne}}$;* (b) *If $0 < M \le \sqrt{2}-1$, then $f \in \mathcal{S}^*_{\mathrm{L}}$;*
(c) *If $0 < M \le 2-\sqrt{2}$, then $f \in \mathcal{S}^*_{c}$;* (d) *If $0 < M \le \frac{2}{3}$, then $f \in \mathcal{S}^*_{\mathrm{car}}$;*
(e) *If $0 < M \le \frac{1}{3+2\sqrt{2}}$, then $f \in \mathcal{S}^*_{\mathrm{R}}$;* (f) *If $0 < M \le \sin 1$, then $f \in \mathcal{S}^*_{\sin}$;*
(g) *If $0 < M \le \frac{e-1}{e+1}$, then $f \in \mathcal{S}^*_{\mathrm{sig}}$;* (h) *If $0 < M \le \sqrt{2}-\frac{1}{2}$, then $f \in \mathcal{S}^*_{\lim}$.*

The bounds of M are the best possible.

3. Norm Estimates for the Classes $\mathcal{S}^*(\psi,\mu)$ and $\mathcal{K}(\psi)$

In this section, we derive pre-Schwarzian norm estimate for the classes $\mathcal{S}^*(\psi,\mu)$ and $\mathcal{K}(\psi)$ defined by (3) and (4), respectively, followed by certain applications of the main results. Additionally, inclusion criteria for various subclasses of analytic function are obtained using Gronwall's inequality.

The *pre-Schwarzian derivative* of a locally univalent function f in the unit disc $\mathbb{D}$, i.e.,

$$T_f(z) := \frac{f''(z)}{f'(z)}$$

and the *Schwarzian derivative*

$$S_f(z) := T_f(z) - \frac{3}{2}\Big(T_f(z)\Big)^2,$$

and their respective norms

$$\|T_f\| := \sup_{|z|<1}\Big(1-|z|^2\Big)|T_f| \quad \text{and} \quad \|S_f\| = \sup_{|z|<1}\Big(1-|z|^2\Big)^2|S_f|$$

play a crucial role in the theory of Teichmüller spaces [21]. The Teichmüller space can be associated with the set of Schwarzian derivatives of univalent functions on $\mathbb{D}$ with quasiconformal extensions to the Riemann sphere, $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$. Moreover, for $0 \le k < 1$, if f extends to a k-quasiconformal mapping of $\widehat{\mathbb{C}}$, then $\|S_f\| \le 6k$, and, conversely, if $\|S_f\| \le 2k$, then f is a k-quasiconformal mapping of $\widehat{\mathbb{C}}$.

It is well known that $\|T_f\| \le 6$ for $f \in \mathcal{S}$ and that $\|T_f\| \le 4$ for $f \in \mathcal{C}$, and, conversely, for $f \in \mathcal{A}$, $\|T_f\| \le 1$ implies $f \in \mathcal{S}$ (the well-known Becker's theorem).

Yamashita [22] showed that if $f \in \mathcal{S}^*(\alpha)$ (i.e., f is starlike of order alpha), then $\|T_f\| \le 6-4\alpha$.

Let $\mathcal{M}$ denote the class of analytic functions $w : \mathbb{D} \to \mathbb{D}$, $w(0) = 0$, and for a given $z_0 \in \mathbb{D}$ let $\zeta_0 := w(z_0)$. The *Dieudonné Lemma* ([1] p. 198) which plays a pivotal role in the proof of our next results states that for a fixed pair of points z_0, $\zeta_0 \in \mathbb{D}$ with $|\zeta_0| \le |z_0| \ne 0$, the value $w'(z_0)$ belongs the closed disc centered at $\dfrac{\zeta_0}{z_0}$ with radius $\dfrac{|z_0|^2-|\zeta_0|^2}{|z_0|(1-|z_0|^2)}$. Moreover, if $w'(z_0)$ lies on the boundary of this disc, then w has the form:

$$w(z) = z\,\frac{\lambda\,\dfrac{z-z_0}{1-\bar{z}_0 z}+\dfrac{\zeta_0}{z_0}}{1+\lambda\,\dfrac{\bar{\zeta}_0}{\bar{z}_0}\,\dfrac{z-z_0}{1-\bar{z}_0 z}}, \tag{5}$$

where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Kim and Sugawa [19] obtained the pre-Schwarzian norm estimate for some close-to-convex functions of specified type. Additionally,

$$|w'(z_0)| \leq \left|\frac{\zeta_0}{z_0}\right| + \frac{|z_0|^2 - |\zeta_0|^2}{|z_0|(1 - |z_0|^2)} = K(|z_0|, |\zeta_0|), \tag{6}$$

and the sharp inequality for $w \in \mathcal{M}$ was obtained in [19].

The following two lemmas will be used to prove Theorem 2:

Lemma 2 ([19] Proposition 2.4., (2.3) and (2.4))**.** *For a continuous function F defined on the interval $[0,1)$, the maximal function of F, denoted by $\widehat{F}$, and defined by*

$$\widehat{F}(r) := \max_{0 \leq s \leq r} K(r,s)|F(s)|, \; 0 \leq r < 1,$$

where

$$K(r,s) := \frac{s}{r} + \frac{r^2 - s^2}{r(1-r^2)},$$

satisfies

$$\lim_{r \to 1-} \left(1 - r^2\right)\widehat{F}(r) = \sup_{0 \leq r < 1} \left(1 - r^2\right)\widehat{F}(r) = \sup_{0 \leq r < 1} \left(1 - r^2\right)|F(r)|.$$

Lemma 3 ([19] Corollary 2.5., (2.3) and (2.4))**.** *For two functions $F, G \in C([0,1))$, we have*

$$\sup_{0 \leq r < 1} \left(1 - r^2\right)\left(\widehat{F}(r) + \widehat{G}(r)\right) = \sup_{0 \leq r < 1} \left(1 - r^2\right)\widehat{F}(r) + \sup_{0 \leq r < 1} \left(1 - r^2\right)\widehat{G}(r).$$

Theorem 2. *For a function $f \in \mathcal{S}^*(\psi, \mu)$ we have*

$$\|T_f\| \leq \sup_{|z|<1} \left(1 - |z|^2\right)\left|(1+\mu)\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right)\right| + \sup_{|z|<1} \left(1 - |z|^2\right)\left|\frac{\psi'(z)}{\psi(z)}\right|. \tag{7}$$

Proof. For a function $f \in \mathcal{S}^*(\psi, \mu)$, from the definition of the subordination we have

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) = \psi(w(z)), \; z \in \mathbb{D},$$

where the function w is analytic in $\mathbb{D}$, such that $|w(z)| < 1, z \in \mathbb{D}$, and $w(0) = 0$. According to the definition of the class $\mathcal{S}^*(\psi, \mu)$, since $\psi(z) \neq 0$ for all $z \in \mathbb{D}$, it follows that f is locally univalent in $\mathbb{D}$.

By taking derivative on both sides of the last equation, a simple computation yields

$$\frac{f''(z)}{f'(z)} =: T_f(z) = (1+\mu)\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) + \frac{\psi'(w(z))}{\psi(w(z))} w'(z), \; z \in \mathbb{D},$$

which implies

$$\left|T_f(z)\right| \leq \left|(1+\mu)\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right)\right| + \left|\frac{\psi'(w(z))}{\psi(w(z))}\right| \left|w'(z)\right|, \; z \in \mathbb{D}.$$

Generally, for a function g analytic in $\mathbb{D}$ let define $Q_g(r) := \max\{|g(z)| : |z| = r\}$, and $\widehat{Q}(r, g)$ be the maximal function of $Q_g(r)$. Additionally, we denote

$$\Psi(z) := \frac{\psi'(z)}{\psi(z)}, \quad \text{and} \quad \mathcal{F}(z) := (1+\mu)\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right).$$

For a fixed point $z_0 \in \mathbb{D}$ with $|z_0| = r > 0$, let $\zeta_0 = w(z_0)$. Therefore, from the well-known *Schwarz lemma* we have $r_1 = |\zeta_0| = |w(z_0)| \leq |z_0| = r$, hence $w \in \mathcal{M}$. From the above inequality, using (6) it follows that

$$\begin{aligned}|T_f(z_0)| &\leq K(r,r)\left|(1+\mu)\left(\frac{f'(z_0)}{f(z_0)} - \frac{1}{z_0}\right)\right| + K(r,r_1)|\Psi(\zeta_0)| \\ &\leq K(r,r)Q_{\mathcal{F}}(r) + K(r,r_1)Q_{\Psi}(r_1) \\ &\leq \widehat{Q}(r,\mathcal{F}) + \widehat{Q}(r,\Psi).\end{aligned} \tag{8}$$

From here, using Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}\|T_f\| &\leq \sup_{0\leq r<1}\left(1-r^2\right)\left(\widehat{Q}(r,\mathcal{F}) + \widehat{Q}(r,\Psi)\right) \\ &= \sup_{0\leq r<1}\left(1-r^2\right)Q_{\mathcal{F}}(r) + \sup_{0\leq r<1}\left(1-r^2\right)Q_{\Psi}(r),\end{aligned}$$

that is (7) holds for all $f \in \mathcal{S}^*(\psi,\mu)$. $\square$

Remark 2. *The inequality (7) of Theorem 2 gives an estimation for the upper bound of the norm of the pre-Schwarzian derivative for the functions that belong to the class $\mathcal{S}^*(\psi,\mu)$. Since this estimation is not the best possible, to find the sharp result remains still an open question.*

Taking the function $\psi(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, the parameter $\mu = 0$ in Theorem 2 and using Lemma 4.2 of [19] we obtain the next result:

Corollary 2. *For $-1 \leq B < A \leq 1$, if $f \in \mathcal{S}^*_{[A,B]}$, then*

$$\|T_f\| \leq \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{f'(z)}{f(z)} - \frac{1}{z}\right| + \frac{2(A-B)}{1-AB+\sqrt{(1-A^2)(1-B^2)}}.$$

If we consider $B = 0$ in Corollary 2, then $\psi(z) = 1 + \lambda z$ with $0 < \lambda \leq 1$, and we obtain the following norm estimate for the class $\mathcal{S}^*(1+\lambda z)$:

Example 1. *For $0 \leq \lambda < 1$, if $f \in \mathcal{S}^*(1+\lambda z)$, then*

$$\|T_f\| \leq \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{f'(z)}{f(z)} - \frac{1}{z}\right| + \frac{2\lambda}{1+\sqrt{1-\lambda^2}}.$$

Letting in Theorem 2 the special case $\mu = 0$ and $\psi(z) = 1 + \dfrac{4z}{3} + \dfrac{2z^2}{3}$, we will obtain the norm estimate for $f \in \mathcal{S}^*_{\text{car}}$, as follows.

A simple computation shows that

$$S := \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{\psi'(z)}{\psi(z)}\right| = 4\sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{1+z}{3+4z+2z^2}\right|,$$

and

$$S = \max\{M_1; M_2\},$$

where

$$M_1 = 4\sup\left\{\left(1-r^2\right)\sqrt{\frac{r^2-2r+1}{4r^4-16r^3+28r^2-24r+9}} : r \in \left[0, \frac{\sqrt{2}}{2}\right]\right\},$$

and

$$M_2 = 4\sup\left\{\left(1-r^2\right)\sqrt{\frac{r^2+2r+1}{4r^4+16r^3+28r^2+24r+9}} : r \in \left[\frac{\sqrt{2}}{2}, 1\right]\right\}.$$

We obtain that

$$M_1 = 4\left(1-r^2\right)\sqrt{\frac{r^2-2r+1}{4r^4-16r^3+28r^2-24r+9}}\Bigg|_{r=0.1203851202\ldots} = 1.361155360\ldots,$$

and

$$M_2 = 4\left(1-r^2\right)\sqrt{\frac{r^2+2r+1}{4r^4+16r^3+28r^2+24r+9}}\Bigg|_{r=\frac{\sqrt{2}}{2}} = \frac{1}{2},$$

hence $S = 1.361155360\ldots$, and next result holds:

Corollary 3. *If $f \in \mathcal{S}^*_{\mathrm{car}}$, then*

$$\|T_f\| \le \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{f'(z)}{f(z)} - \frac{1}{z}\right| + 1.361155360\ldots.$$

Remark 3. *For $\mu = 0$ and various specializations of the function ψ, we know that the class $\mathcal{S}^*(\psi)$ represents various Ma-Minda type starlike subclasses showed in the Table 1. Additionally, as given in the above corollary, we can obtain estimates for the pre-Schwarzian norm for the functions belonging to $\mathcal{S}^*_{\mathrm{L}}$, $\mathcal{S}^*_c$, $\mathcal{S}^*_{\mathrm{e}}$, $\mathcal{S}^*_{\mathrm{Ne}}$, $\mathcal{S}^*_{\mathrm{R}}$, $\mathcal{S}^*_{\sin}$, $\mathcal{S}^*_{\mathrm{sig}}$, and $\mathcal{S}^*_{\lim}$.*

Theorem 3. *For a function $f \in K(\psi)$ we have*

$$\|T_f\| \le \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{\psi(z)-1}{z}\right|, \tag{9}$$

and the estimate is sharp if the inequality

$$\left|\frac{\psi(z)-1}{z}\right| \le \frac{\psi(\varepsilon|z|)-1}{\varepsilon|z|},$$

hold for all $z \in \mathbb{D}$, where $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$.

Proof. If $f \in K(\psi)$, then

$$1 + \frac{zf''(z)}{f'(z)} = \psi(w(z)),\ z \in \mathbb{D},$$

hence

$$\frac{f''(z)}{f'(z)} = \frac{\psi(w(z))-1}{z},\ z \in \mathbb{D},$$

where the function w is analytic in $\mathbb{D}$, such that $|w(z)| < 1$, $z \in \mathbb{D}$, and $w(0) = 0$. Therefore,

$$|T_f(z)| := \left|\frac{f''(z)}{f'(z)}\right| = \left|\frac{\psi(w(z))-1}{z}\right|,$$

and denote

$$\mathcal{H}(z) := \frac{\psi(w(z))-1}{z}.$$

As in the proof of Theorem 2, let us define $Q_g(r) := \max\{|g(z)| : |z| = r\}$, and $\widehat{Q}(r,g)$ be the maximal function of $Q_g(r)$.

For a fixed point $z_0 \in \mathbb{D}$ with $|z_0| = r > 0$, let $\zeta_0 = w(z_0)$. Thus, from the well-known *Schwarz lemma* we have $r_1 = |\zeta_0| = |w(z_0)| \leq |z_0| = r$, hence $w \in \mathcal{M}$. From the above inequality we have

$$|T_f(z_0)| \leq K(r,r)|\mathcal{H}(z_0)| \leq K(r,r)Q_{\mathcal{H}}(r) \leq \widehat{Q}(r,\mathcal{H}),$$

and using Lemma 2 we obtain

$$\|T_f\| \leq \sup_{0\leq r<1} \left(1-r^2\right)\widehat{Q}(r,\mathcal{H}) = \sup_{0\leq r<1} \left(1-r^2\right)Q_{\mathcal{H}}(r),$$

hence (9) holds for all $f \in K(\psi)$. Since the remaining part of the proof concerning the sharpness is similar to that proved for the Theorem 2 it will be omitted. $\square$

Example 2. *We will prove that the function*

$$f(z) = z + \frac{z^2}{4} + \frac{z^3}{6}$$

belongs to the class $\mathcal{S}^*\left(\frac{1+z}{1-z}, -2\right)$. *Thus, denoting*

$$P(z) := \frac{f'(z)f(z)}{z} = \left(1 + \frac{z}{2} + \frac{z^2}{4}\right)\left(1 + \frac{z}{4} + \frac{z^2}{6}\right),$$

like we could see in the Figure 2 we have

$$\operatorname{Re} P\left(e^{it}\right) > 0.2 > 0,\ t \in [0, 2\pi].$$

The function P *is analytic in* $\overline{\mathbb{D}}$, *hence* $\operatorname{Re} P$ *is a harmonic function in* $\overline{\mathbb{D}}$, *and thus it attains its extremal values of the boundary of* $\overline{\mathbb{D}}$. *Therefore, this inequality combined with* $P(0) = 1 > 0$ *leads to*

$$\operatorname{Re} P(z) > 0,\ z \in \mathbb{D},$$

hence $P(\mathbb{D}) \subset \{w \in \mathbb{C} : \operatorname{Re} w > 0\} = \varphi(\mathbb{D})$. *Since* $P(0) = \varphi(0)$ *and* φ *is a univalent function in* $\mathbb{D}$, *we conclude that* $P(z) \prec \varphi(z)$, *that is* $f \in \mathcal{S}^*\left(\frac{1+z}{1-z}, -2\right)$.

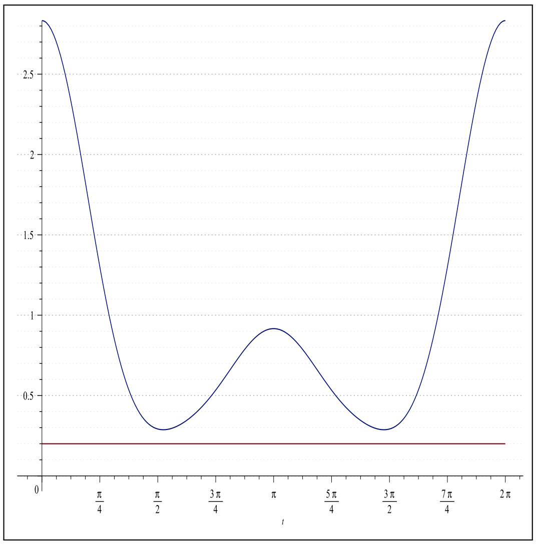

Figure 2. The graphics of $v = \operatorname{Re} P\left(e^{it}\right)$ for $t \in [0, 2\pi]$, and $v = 0.2$.

Since for $\psi(z) = \dfrac{1+z}{1-z}$ *we have*

$$\sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{\psi'(z)}{\psi(z)}\right| = 2, \tag{10}$$

and using the inequality (7) *it follows that*

$$\|T_f\| \le \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{3+4z}{12+3z+2z^2}\right| + 2,$$

Denoting

$$S := \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{4z+3}{2z^2+3z+12}\right|,$$

a simple computation shows that

$$S = \sup\left\{T(r) := \frac{(1-r^2)\sqrt{16r^2+24r+9}}{\sqrt{4r^4+12r^3+57r^2+72r+144}} : r \in [0,1)\right\}$$
$$= T(0.2924799068\ldots) = 0.2922326371\ldots,$$

hence $\|T_f\| \le 2.2922326371\ldots$.

Example 3. *We will prove that the function*

$$f(z) = \frac{2z}{1+e^{-z}}$$

belongs to the class $\mathcal{S}^*\left(\frac{1+z}{1-z}, 1+i\right)$. *Letting*

$$R(z) := \left(\frac{z}{f(z)}\right)^{2+i} f'(z) = \frac{1}{2}\left(\frac{1+e^{-z}}{2}\right)^{i}\left(1+e^{-z}+ze^{-z}\right),$$

from the Figure 3 *we get*

$$\operatorname{Re} R\left(e^{it}\right) > 0.3 > 0,\ t \in [0,2\pi].$$

Since $R(0) = 1 > 0$, *using similar reasons like in the proof of Example* 2, *the above inequality implies*

$$\operatorname{Re} R(z) > 0,\ z \in \mathbb{D},$$

that is $f \in \mathcal{S}^*\left(\frac{1+z}{1-z}, 1+i\right)$.

Using (10) *from the inequality* (7) *we obtain*

$$\|T_f\| \le \sqrt{5}\sup_{|z|<1}\frac{1-|z|^2}{|1+e^z|} + 2.$$

Letting

$$S := \sqrt{5}\sup_{|z|<1}\frac{1-|z|^2}{|1+e^z|},$$

and

$$S = \sup\left\{T(r;t) := \frac{(1-r^2)\sqrt{5}}{\sqrt{1+2e^{r\cos t}\cos(r\sin t)+e^{2r\cos t}}} : (r,t) \in [0,1)\times[0,2\pi]\right\}$$
$$= T(0.2132772440\ldots;\pi) = 1.1805507135\ldots,$$

hence

$$\|T_f\| \le 3.1805507135\ldots.$$

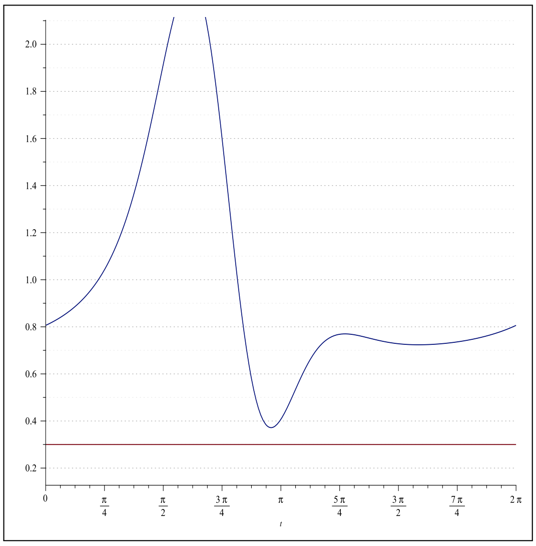

Figure 3. The graphics of $v = \operatorname{Re} R\left(e^{it}\right)$ for $t \in [0, 2\pi]$, and $v = 0.3$.

4. An Application of the Gronwall Inequality

Suppose that u_1 and u_2 are two linearly independent solutions with initial conditions

$$u_1(0) = u_2'(0) = 0 \quad \text{and} \quad u_2(0) = u_1'(0) = 1$$

of the second order differential equation

$$u''(z) + A(z)u = 0,\ z \in \mathbb{D}, \tag{11}$$

where $2A(z) := S(f; z)$, and $S(f; z)$ is the Schwarzian derivative of the function $f \in \mathcal{A}$, that is

$$S(f; z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Thus, the function f has the following representation (see also ([1] p. 259), [23,24])

$$f(z) = \frac{u_1(z)}{cu_1(z) + u_2(z)}, \quad \text{where} \quad c = -\frac{f''(0)}{2}, \tag{12}$$

hence

$$f'(z) = \frac{1}{(cu_1(z) + u_2(z))^2}. \tag{13}$$

An important tool used in this section is the following Gronwall's inequality.

Lemma 4 ([20] p. 19)**.** *Suppose that P and q are non-negative continuous real functions for $t \ge 0$. If $k > 0$ is a constant, then the inequality*

$$q(t) \le k + \int_0^t q(s)P(s)ds$$

implies

$$q(t) \le k \exp\left(\int_0^t P(s)ds\right),\ t > 0.$$

Using the integral representation of the fundamental solutions of (11) we have

$$\begin{aligned} u_1(z) &= z + \int_0^z (\eta - z) A(\zeta) u_1(\zeta) d\zeta, \\ u_2(z) &= 1 + \int_0^z (\eta - z) A(\zeta) u_2(\zeta) d\zeta, \end{aligned} \tag{14}$$

and using the Gronwall's inequality in [25] the author proved that if $f \in \mathcal{A}$, whenever $|A(z)| < \delta := \delta(\eta)$ for all $z \in \mathbb{D}$, where $\eta = \left|\frac{f''(0)}{2}\right|$ and $2\delta(\eta) = \sup_{z\in\mathbb{D}} |S(f,z)|$, then

$$\left|\frac{u_1(z)}{z} - 1\right| < \frac{1}{2}\delta\, e^{\frac{\delta}{2}}, \tag{15}$$

$$\begin{aligned} &|cu_1(z) + u_2(z)| < (1+c)e^{\frac{\delta}{2}}, \\ &|cu_1(z) + u_2(z) - 1| < \eta + \frac{1}{2}(1+\eta)\delta\, e^{\frac{\delta}{2}}, \end{aligned} \tag{16}$$

for all $z \in \mathbb{D}$ (for details see [24,25]).

Using the Gronwall's inequality Chiang [25] investigated conditions involving the Schwarzian derivatives for a function $f \in \mathcal{A}$ to belong to the class of strongly starlike functions, and to the class of convex functions. In [24], the authors have obtained inclusion criteria for various subclasses of analytic functions.

The subclass which is of interest in the next study is defined as follows:

$$\mathcal{R}_\mu^\tau(\gamma) := \left\{ f \in \mathcal{A} : \left| (1-\tau)\arg\left(\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) \right) + \tau \arg\left(1 + \frac{zf''(z)}{f'(z)}\right) \right| < \frac{\gamma\pi}{2},\ z \in \mathbb{D} \right\},$$

where $\mu \in \mathbb{R}$, and $\tau, \gamma \in [0,1)$.

Consequently, this class reduces to the class of starlike functions, and to the class of convex functions of order γ for specified values of τ and μ. For instance, if $\mu = -\alpha$ and $\tau = 0$ we obtain the class of strongly α-Bazilevič functions of order γ studied by Gao [26].

The next result represents a sufficient condition for a function $f \in \mathcal{A}$ to belong to the class $\mathcal{R}_\mu^\tau(\gamma)$.

Theorem 4. *Let $\mu \in \mathbb{R}$ and $\tau, \gamma \in (0,1]$. For $f \in \mathcal{A}$, let γ, μ, τ and $\eta := \left|\frac{f''(0)}{2}\right|$ satisfy the relation*

$$|1-\mu|(1-\tau)\sin^{-1}(\eta) + \tau \sin^{-1}\left(\frac{2\eta}{1-\eta}\right) < \frac{\gamma\pi}{2}. \tag{17}$$

If

$$2\delta := \sup_{|z|<1} |S(f;z)|,$$

and δ satisfies the inequality

$$\begin{aligned} (1-\tau)\left[|1+\mu|\sin^{-1}\left(\frac{1}{2}\delta e^{\frac{\delta}{2}}\right) + |1-\mu|\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)e^{\frac{\delta}{2}}\delta\right)\right] \\ + \tau\sin^{-1}\frac{2\left(\eta + (1+\eta)\delta e^{\frac{\delta}{2}}\right)}{1-\eta-\frac{1}{2}(1+\eta)\delta e^{\frac{\delta}{2}}} \le \frac{\gamma\pi}{2}, \end{aligned} \tag{18}$$

then $f \in \mathcal{R}_\mu^\tau(\gamma)$.

Proof. First, the condition (17) assures the existence of a real number $\delta \ge 0$ satisfying the inequality (18). This follows from the fact that, taking $\delta \to 0$ the assumption (17) shows that there exists a real number $\delta(\mu, \tau, \eta) > 0$ such that the inequality (18) holds for $0 < \delta < \delta(\mu, \tau, \eta)$.

Representing the function f as in (12) in terms of the linearly independent solutions u_1 and u_2 of the differential equation (11), and using (13) we obtain

$$\left|\arg\left(\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z)\right)\right| = \left|\arg\left(\left(\frac{z}{u_1(z)}\right)^{1+\mu} (cu_1(z) + u_2(z))^{\mu-1}\right)\right|$$
$$\leq |1+\mu| \left|\arg \frac{u_1(z)}{z}\right| + |1-\mu| \, |\arg(cu_1(z) + u_2(z))|, \; z \in \mathbb{D}.$$

Additionally, for $w \in \mathbb{C}$ we have that $|w-1| \leq r$ implies $|\arg w| \leq \sin^{-1} r$. Using this implication combined with the inequalities (15) and (16), the above inequality leads to

$$\left|\arg\left(\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z)\right)\right| < |1+\mu| \sin^{-1}\left(\frac{1}{2}\delta e^{\frac{\delta}{2}}\right)$$
$$+ |1-\mu| \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\frac{\delta}{2}}\right), \; z \in \mathbb{D}. \quad (19)$$

Since
$$\frac{zf''(z)}{f'(z)} = -2z\frac{cu_1'(z) + u_2'(z)}{cu_1(z) + u_2(z)}, \; z \in \mathbb{D},$$
following the method of proof given in ([24] Theorem 3.3, p. 69) we obtain

$$\left|\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right| \leq \frac{2\left(\eta + (1+\eta)\delta e^{\frac{\delta}{2}}\right)}{1 - \eta - \frac{1}{2}(1+\eta)\delta e^{\frac{\delta}{2}}}, \; z \in \mathbb{D},$$

hence
$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \sin^{-1}\left(\frac{2\left(\eta + (1+\eta)\delta e^{\frac{\delta}{2}}\right)}{1 - \eta - \frac{1}{2}(1+\eta)\delta e^{\frac{\delta}{2}}}\right), \; z \in \mathbb{D}. \quad (20)$$

Therefore, using the inequalities (19) and (20), from the assumption (18) we obtain

$$\left|\arg\left[(1-\tau)\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z)\right] + \arg\left[\tau\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\right|$$
$$\leq (1-\tau)\left|\arg\left(\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z)\right)\right| + \tau\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right|$$
$$< (1-\tau)|1+\mu| \sin^{-1}\left(\frac{1}{2}\delta e^{\frac{\delta}{2}}\right) + |1-\mu|(1-\tau) \sin^{-1}\left(\eta + (1+\eta)\frac{1}{2}\delta e^{\frac{\delta}{2}}\right)$$
$$+ \tau \sin^{-1}\left(\frac{2(\eta + (1+\eta)\delta e^{\frac{\delta}{2}})}{1 - \eta - \frac{1}{2}(1+\eta)\delta e^{\frac{\delta}{2}}}\right) \leq \frac{\gamma\pi}{2}, \; z \in \mathbb{D},$$

which implies $f \in \mathcal{R}_\mu^\tau(\gamma)$. $\square$

Taking $\tau = 0$ in Theorem 4, the sufficient condition for a function $f \in \mathcal{A}$ to belong to the class $\mathcal{S}^*\left(\left(\frac{1+z}{1-z}\right)^\gamma, \mu\right)$, with $0 < \gamma \leq 1$, is given by the following corollary.

Corollary 4. *Let $\mu \in \mathbb{R}$ and $\gamma \in (0,1]$. For $f \in \mathcal{A}$, let γ, μ and $\eta := \left|\frac{f''(0)}{2}\right|$ satisfy the relation*

$$|1-\mu| \sin^{-1}(\eta) < \frac{\gamma\pi}{2}.$$

If

$$2\delta := \sup_{|z|<1} |S(f;z)|,$$

and δ satisfies the inequality

$$|1+\mu| \sin^{-1}\left(\frac{1}{2}\delta e^{\frac{\delta}{2}}\right) + |1-\mu| \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)e^{\frac{\delta}{2}}\delta\right) \leq \frac{\gamma\pi}{2},$$

then $f \in \mathcal{S}^*\left(\left(\frac{1+z}{1-z}\right)^{\gamma}, \mu\right)$.

Remark 4. *(i) If we choose of $\mu = -\alpha$, with $\alpha > 0$, and $\tau > 0$ in Theorem 4 we obtain the result of ([24] Theorem 3.2, p. 68).*

(ii) For $\mu = 0$, if $\tau = 0$ or $\tau = 1$ we obtain the sufficient conditions for a function to be strongly starlike and strongly convex of order γ, with $\gamma \in (0,1]$, respectively.

5. Conclusions

Concluding, by using the general subordination theory and the graphical representations, we obtained bounds on M for an analytic function p, with $p(0) = 1$, such that $1 + zp'(z) \prec 1 + Mz$ implies the function p is subordinate to each of the functions given by (2).

Secondly, we determine the pre-Schwarzian norm estimates for function belonging to the classes given by (3) and (4), and we gave some simple applications of the main results given by Theorems 2 and 3.

The article ends with an application of the Gronwall's inequalities, that represents a sufficient condition for an analytic and normalized function to belong a subclass of functions with bounded arguments and connected with the class of strongly α-Bazilevič functions of order γ studied by Gao [26].

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References

1. Duren, P.L. Univalent Functions. In *Grundlehren der Mathematischen Wissenschaften*; Springer: New York, NY, USA, 1983; Volume 259.
2. Miller S.S.; Mocanu P.T. Differential subordinations. Theory and applications. In *Monographs and Textbooks in Pure and Applied Mathematics*; Marcel Dekker, Inc.: New York, NY, USA, 2000; Volume 225.
3. Padmanabhan, K.S.; Parvatham, R. Some applications of differential subordination. *Bull. Aust. Math. Soc.* **1985**, *32*, 321–330.
4. Shanmugam, T.N. Convolution and differential subordination. *Int. J. Math. Math. Sci.* **1989**, *12*, 333–340.
5. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions, In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; Li, Z., Ren, F., Yang, L., Zhang, S., Eds.; International Press: Cambridge, MA, USA, 1994; pp. 157–169.
6. Sokół, J.; Stankiewicz, J.; Radius of convexity of some subclasses of strongly starlike functions. *Zesz. Nauk. Politech. Rzesz. Mat.* **1996**, *19*, 101–105.
7. Raina, R.K.; Sokół, J. Some properties related to a certain class of starlike functions. *C. R. Math. Acad. Sci.* **2015**, *353*, 973–978.
8. Khatter, K.; Ravichandran, V.; Sivaprasad Kumar, S. Starlike functions associated with exponential function and the lemniscate of Bernoulli. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2019**, *113*, 233–253. [CrossRef]

9. Sharma, K.; Jain, N.K.; Ravichandran, V. Starlike functions associated with a cardioid. *Afr. Mat.* **2016**, *27*, 923–939.
10. Wani, L.A.; Swaminathan, A. Starlike and convex functions associated with a nephroid domain having cusps on the real axis. *arXiv* **2019**, arXiv:1912.05767.
11. Kumar, S.; Ravichandran, V. A subclass of starlike functions associated with a rational function. *Southeast Asian Bull. Math.* **2016**, *40*, 199–212.
12. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran, V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* **2019**, *45*, 213–232. [CrossRef]
13. Goel, P.; Sivaprasad Kumar, S. Certain class of starlike functions associated with modified sigmoid function. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 957–991. [CrossRef]
14. Yunus,Y.; Halim, S.A.; Akbarally A.B. Subclass of starlike functions associated with a limacon. *AIP Conf. Proc.* **2018**, *1974*, 030023.
15. Janowski, W. Some extremal problems for certain families of analytic functions I. *Ann. Polon. Math.* **1973**, *28*, 297–326. [CrossRef]
16. Aouf, M.K.; Seoudy, T. Certain class of Bi-Bazilevič functions with bounded boundary rotation involving Sălăgean operator. *Constr. Math. Anal.* **2020**, *3*, 139–149. [CrossRef]
17. Seoudy, T.; Aouf, M.K. Fekete-Szegő Problem for certain subclass of analytic functions with complex order defined by q-analogue of Ruscheweyh operator. *Constr. Math. Anal.* **2020**, *3*, 36–44. [CrossRef]
18. Dziok, J.; Raina, R.K.; Sokół, J. Applications of differential subordinations for norm estimates of an integral operator. *Proc. R. Soc. Edinb. Sect. A* **2018**, *148*, 281–291. [CrossRef]
19. Kim Y.C.; Sugawa, T. Norm estimates of the pre-Schwarzian derivatives for certain classes of univalent functions. *Proc. Edinb. Math. Soc.* **2006**, *49*, 131–143. [CrossRef]
20. Hille, E. *Lectures on Ordinary Differential Equations*; Addison-Wesley Publ. Co.: Reading, MA, USA, 1969.
21. Zhuravlev, I.V. A model of the universal Teichmüller space. *Siberian Math. J.* **1986**, *27*, 691–697. [CrossRef]
22. Yamashita, S. Norm estimates for function starlike or convex of order alpha. *Hokkaido Math. J.* **1999**, *28*, 217–230. [CrossRef]
23. Kim, J.A.; Sugawa T. Geometric properties of functions with small Schwarzian derivatives. (preprint) Available online: http://www.cajpn.org/pp04/0403.pdf (accessed on 24 January 2021).
24. Ali, R. M; Nargesi, M.M.; Ravichandran, V.; Swaminathan, A. Inclusion criteria for subclasses of functions and Gronwall's inequality. *Tamsui Oxf. J. Inf. Math. Sci.* **2013**, *29*, 61–75.
25. Chiang, Y.M. Properties of analytic functions with small Schwarzian derivative. *Complex Var. Theory Appl.* **1994**, *25*, 107–118. [CrossRef]
26. Gao, C. Fekete-Szegő problem for strongly Bazilevič functions. *Northeast Math. J.* **1996**, *12*, 469–474.