


Article

# Algebraic Construction of the Sigma Function for General Weierstrass Curves

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**Abstract:** The Weierstrass curve  $X$  is a smooth algebraic curve determined by the Weierstrass canonical form,  $y^r + A_1(x)y^{r-1} + A_2(x)y^{r-2} + \cdots + A_{r-1}(x)y + A_r(x) = 0$ , where  $r$  is a positive integer, and each  $A_j$  is a polynomial in  $x$  with a certain degree. It is known that every compact Riemann surface has a Weierstrass curve  $X$ , which is birational to the surface. The form provides the projection  $\omega_r : X \rightarrow \mathbb{P}^1$  as a covering space. Let  $R_X := \mathbf{H}^0(X, \mathcal{O}_X(*\infty))$  and  $R_{\mathbb{P}^1} := \mathbf{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(*\infty))$ . Recently, we obtained the explicit description of the complementary module  $R_X^c$  of  $R_{\mathbb{P}^1}$ -module  $R_X$ , which leads to explicit expressions of the holomorphic form except  $\infty$ ,  $\mathbf{H}^0(\mathbb{P}^1, \mathcal{A}_{\mathbb{P}^1}(*\infty))$  and the trace operator  $p_X$  such that  $p_X(P, Q) = \delta_{P, Q}$  for  $\omega_r(P) = \omega_r(Q)$  for  $P, Q \in X \setminus \{\infty\}$ . In terms of these, we express the fundamental two-form of the second kind  $\Omega$  and its connection to the sigma function for  $X$ .

**Keywords:** Weierstrass canonical form; fundamental two-form of the second kind; sigma function; plane and space curves with the higher genera

**MSC:** 14H42; 14H05; 14H55; 14H50



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## 1. Introduction

In Weierstrass' elliptic function theory, the algebraic properties associated with an elliptic curve of Weierstrass' standard equation  $y^2 = 4x^3 - g_2x - g_3$  are connected with the transcendental properties defined on its Jacobi variety via the  $\sigma$  function since  $\left(\wp(u) = -\frac{d^2}{du^2} \log \sigma(u), \frac{d\wp(u)}{du}\right)$  is identical to a point  $(x, y)$  in the curve [1,2]. These algebraic and transcendental properties are equivalently obtained by the identity and play a central role in the elliptic function theory. Via the equivalence, the elliptic function theory affects several fields in mathematics, science, and technology. In other words, in Weierstrass' elliptic function theory, the equivalence between the algebraic objects of the curve and the transcendental objects on its Jacobi variety is crucial.

Weierstrass himself extended the picture to general algebraic curves [3], but it failed due to difficulties. Some of the purposes of mathematics in the XX-th century were to overcome the difficulties that were achieved. We have studied the generalization of this picture to algebraic curves with higher genera in the series of studies [4–6] following Mumford's studies for the hyperelliptic curves based on modern algebraic geometry [7,8], i.e., the unification of the theory of algebraic curves in the XIX-th century with the modern one.

The elliptic theta function was generalized by Riemann for an Abelian variety. In contrast, its equivalent function  $A_1$  was defined for any hyperelliptic curve by Weierstrass, which was refined by Klein using only the data of the hyperelliptic Riemann surface and Jacobi variety as a generalization of the elliptic sigma function [9]. Baker re-constructed

Klein's sigma functions by using the data of hyperelliptic curves algebraically [10]. Buchstaber, Enolskii, and Leykin extend the sigma functions to certain plane curves, so-called  $(n, s)$  curves, based on Baker's construction, which we call the EEL construction due to work by Eilbeck, Enolskii, and Leykin [11] [ref. [12] and its references]. For the  $(n, s)$  curves with cyclic symmetry, the direct relations between the affine rings and the sigma functions were obtained as the Jacobi inversion formulae [13,14]. Additionally, we generalized the sigma functions and the formulae to a particular class of the space curves using the EEL-construction [4,6,15].

We have studied further generalization of the picture in terms of the Weierstrass canonical form [16,17]. The Weierstrass curve  $X$  is a normalized curve of the curve given by the Weierstrass canonical form,  $y^r + A_1(x)y^{r-1} + A_2(x)y^{r-2} + \dots + A_{r-1}(x)y + A_r(x) = 0$ , where  $r$  is a positive integer, and each  $A_j$  is a polynomial in  $x$  of a certain degree (c.f. Proposition 2) so that the Weierstrass non-gap sequence at  $\infty \in X$  is given by the numerical semigroup  $H_X$  whose generator contains  $r$  as its minimal element. It is known that every compact Riemann surface has a Weierstrass curve  $X$ , which is birational to the surface. We also simply call the Weierstrass curve *W-curve*.

It provides the projection  $\omega_r : X \rightarrow \mathbb{P}$  as a covering space. Let  $R_X := \mathbf{H}^0(X, \mathcal{O}_X(*\infty))$  and  $R_{\mathbb{P}} := \mathbf{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(*\infty))$ . In [17], we have the explicit description of the complementary module  $R_X^c$  of  $R_{\mathbb{P}}$ -module  $R_X$ , which leads the explicit expressions of the holomorphic one form except  $\infty$ ,  $\mathbf{H}^0(\mathbb{P}, \mathcal{A}_{\mathbb{P}}(*\infty))$ .

Recently, D. Korotkin and V. Shramchenko [18] and Nakayashiki [19] defined the sigma function of every compact Riemann surface as a generalization of Klein's sigma function transcendently. Every compact Riemann surface can be characterized by the Weierstrass non-gap sequence, which is described by a numerical semigroup  $H$  called *Weierstrass semigroup*. Nakayashiki defined the sigma function for every compact Riemann surface with Weierstrass semigroup  $H$  [19] based on Sato's theory on the universal Grassmannian manifolds (UGM) [20,21].

In this paper, we use our recent results on the complementary module of the  $W$ -curve [17] to define the trace operator  $p_X$  such that  $p_X(P, Q) = \delta_{P, Q}$  for  $\omega_r(P) = \omega_r(Q)$  for  $P, Q \in X \setminus \{\infty\}$ . In terms of them, we express the fundamental two-form of the second kind  $\Omega$  algebraically in Theorem 3, and finally obtain a connection to Nakayashiki's sigma function by modifying his definition in Theorem 4, which means an algebraic construction of the sigma function for every  $W$ -curve as Baker performed for Klein's sigma function following Weierstrass' elliptic function theory.

The contents are as follows: Section 2 reviews the Weierstrass curves ( $W$ -curves) based on [17] (1) the numerical semigroup in Section 2.1, (2) Weierstrass canonical form in Section 2.2, (3) their relations to the monomial curves in Section 2.3, (4) the properties of the  $R_{\mathbb{P}}$ -module  $R_X$  in Section 2.4, (5) the covering structures in  $W$ -curves in Section 2.5, and (6) especially the complementary module  $R_X^c$  of  $R_X$  in Section 2.6; the explicit description of  $R_X^c$  is the first main result in [17]. Section 3 provides the first and the second theorems in this paper on the  $W$ -normalized Abelian differentials on  $X$ . In Section 3.1, we review the second main result in [17] on the  $W$ -normalized Abelian differentials  $\mathbf{H}^0(X, \mathcal{A}_X(*\infty))$ , which contain the Abelian differentials of the first kind. Further, we extend the trace operator  $p \in R_X \otimes_{R_{\mathbb{P}}} R_X$  introduced in [17] to  $R_X \otimes_{\mathbb{C}} R_X$  in Section 3.2. The trace operator  $p$  enables us to define a proper one-form  $\Sigma$  and its differential  $d\Sigma$  in Section 3.3. After investigating  $d\Sigma$ , we find the  $W$ -normalized Abelian differentials of the second kind in Theorems 1 and 2. Further, in Theorem 3, we mention our results on the fundamental differential of the second kind  $\Omega$  in our  $W$ -curves and the Abelian differentials of the third kind. We obtain the generalized Legendre relation in Proposition 18. Using them, we show the connection of the sigma function for  $X$  with  $R_X^c$  in the  $W$ -curves  $X$  in Section 4. As studied in [5], we introduce the shifted Abelian integrals in Section 4.1. Section 4.2 shows the properties of the Riemann theta functions of  $W$ -curves and its Riemann–Kempf theorem as in Proposition 22. In Section 4.3, we define the sigma function for a  $W$ -curve  $X$

by modifying the definition of Nakayashiki (Definition 9) in [19] and show its properties in Theorem 4 as our main results in this paper.

## 2. Weierstrass Canonical Form and Weierstrass Curves (W-Curves)

### 2.1. Numerical and Weierstrass Semigroup

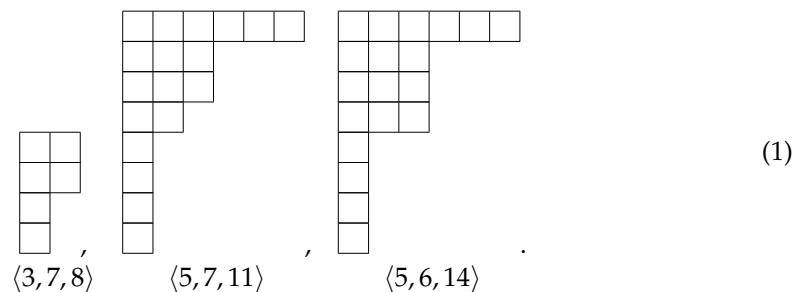
This subsection is on numerical and Weierstrass semigroups based on [4,22]. An additive sub-monoid of the monoid of the non-negative integers  $\mathbb{N}_0$  is called *numerical semigroup* if its complement in  $\mathbb{N}_0$  is a finite set. In this subsection, we review the numerical semigroups associated with algebraic curves.

In general, a numerical semigroup  $H$  has a unique (finite) minimal set of generators,  $M = M(H)$ , ( $H = \langle M \rangle$ ) and the finite cardinality  $g$  of  $H^c = \mathbb{N}_0 \setminus H$ ;  $g$  is the genus of  $H$  or  $H^c$  and  $H^c$  is called gap-sequence. We let  $r_{\min}(H)$  be the smallest positive integer of  $M(H)$ . We call the semigroup  $H$  an  $r_{\min}(H)$ -semigroup, so that  $\langle 3, 7, 8 \rangle$  is a 3-semigroup and  $\langle 6, 13, 14, 15, 16 \rangle$  is a 6-semigroup. Let  $N(i)$  and  $N^c(i)$  be the  $i$ -th ordered element of  $H = \{N(i) \mid i \in \mathbb{N}_0\}$  and  $H^c = \{N^c(i) \mid i = 0, 1, \dots, g - 1\}$  satisfying  $N(i) < N(i + 1)$ , and  $N^c(i) < N^c(i + 1)$ , respectively.

Further, the conductor  $c_H$  of  $H$  is defined by the minimal natural number satisfying  $c_H + \mathbb{N}_0 \subset H$ . The number  $c_H - 1$  is known as the Frobenius number, which is the largest element of  $H^c$ , i.e.,  $c_H = N^c(g - 1) + 1$ .

By letting the row lengths be  $\Lambda_i = N^c(g - i) - g + i$ , ( $i = 1, 2, \dots, g$ ), we have the Young diagram of the semigroup,  $\Lambda := (\Lambda_1, \dots, \Lambda_g)$ , ( $\Lambda_i \leq \Lambda_{i+1}$ ). The Young diagram  $\Lambda$  is a partition of  $\sum_i \Lambda_i$ . We say that such a Young diagram is associated with the numerical

semigroup. If for a given Young diagram  $\Lambda$ , we cannot find any numerical semigroups  $H$  such that  $\Lambda_i = N^c(g - i) - g + i$ , we say that  $\Lambda$  is not associated with the numerical semigroup. It is obvious that, in general, the Young diagrams are not associated with the numerical semigroups. We show examples of the Young diagrams associated with numerical semigroups:



The Young diagram and the associated numerical semigroup are called symmetric if the Young diagram is invariant under reflection across the main diagonal. It is known that the numerical semigroup is symmetric if and only if  $2g - 1$  occurs in the gap sequence. It means that if  $c_H = 2g$ ,  $H$  is symmetric.

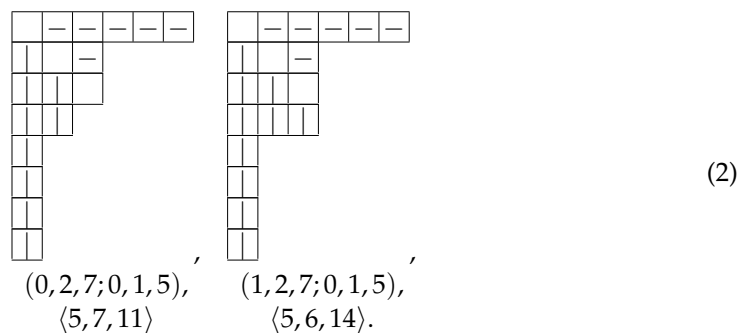
We obviously have the following proposition:

**Proposition 1.** *The following holds:*

1.  $N(n) - n \leq g$  for every  $n \in \mathbb{N}_0$ ,
2.  $N(n) - n = g$  for  $N(n) \geq c_H = N(g)$  or  $n \geq g$ ,
3.  $N(n) - n < g$  for  $0 \leq N(n) < c_H$  or  $n < g$ ,
4.  $\#\{n \mid N^c(n) \geq g\} = \#\{n \mid N(n) < g\}$
5. for  $N(i) < N^c(j)$ ,  $N^c(j) - N(i) \in H^c$ , and
6. when  $H$  is symmetric,  $c_H = N(g) = 2g$  and  $c_H - N(i) - 1 = N^c(g - i - 1)$  for  $0 \leq i \leq g - 1$ .

**Proof.** 1–3 and 5 are obvious. Noting  $\#H^c = g, 4$  means that what is missing must be filled later for  $H^c$ , and 6 is left to [22].  $\square$

The length  $r_\Lambda$  of the diagonal of the Young diagram  $\Lambda$  is called the rank of  $\Lambda$ . The number of boxes below and to the right of the  $i$ -th box of the diagonal from lower right to upper left are assumed as  $a_{r_\Lambda-i+1}$  and  $b_{r_\Lambda-i+1}$ , respectively. The Young diagram is represented by  $(a_{r_\Lambda}, \dots, a_2, a_1; b_{r_\Lambda}, \dots, b_2, b_1)$ , which is known as Frobenius representation or characteristics of  $\Lambda$ . Then  $\ell_i := a_i + b_i + 1$  is called the hook length of the characteristics. For  $\Lambda = (6, 3, 3, 2, 1, 1, 1, 1)$  associated with  $H = \langle 5, 7, 11 \rangle$ , it is  $(0, 2, 7; 0, 1, 5)$  and the rank of  $\Lambda$  is three. For  $\Lambda = (6, 3, 3, 3, 1, 1, 1, 1)$  associated with  $H = \langle 5, 6, 14 \rangle$ , it is  $(1, 2, 7; 0, 1, 5)$  and rank of  $\Lambda$  is three. We show their examples of the Young diagram:



**Definition 1.** For a given Young diagram  $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_g)$ , we define

$$\Lambda^{(k)} := (\Lambda_1, \Lambda_2, \dots, \Lambda_k), \quad \Lambda^{[k]} := (\Lambda_{k+1}, \Lambda_{k+2}, \dots, \Lambda_g).$$

We show the properties of the Young diagram associated with the numerical semigroup  $H$  in the following lemma, which is geometrically obvious:

- Lemma 1.** 1. If we put the number on the boundary of the Young diagram  $\Lambda$  from the lower to upper right as in (3), each number in the right side box of the  $i$ -th row corresponds to the gap number in  $N^c(g - i) \in H^c$  or  $\Lambda_i + g - i$ , i.e.,  $N^c(g - i) \in H^c = \Lambda_i + g - i$ , for  $0 < i \leq g$ , and each number in the top numbered box of the  $i$ -th column corresponds to  $N(i)$  for  $0 < i < \Lambda_1$ .
2. The hook length  $\ell_i$  is given by  $\ell_i = N^c(g - i) - N(i - 1)$  ( $i = 1, \dots, r_\Lambda$ ), which belongs to  $H^c$ , and thus let  $L_i \in \{0, 1, \dots, g - 1\}$  such that  $N^c(g - L_i) = \ell_i$ . Then,  $\Lambda_{L_i} + g - L_i = \ell_i$ , and  $L_1 = 1$ .
3. For  $\Lambda := (\Lambda_1, \dots, \Lambda_g)$  associated with the numerical semigroup  $H := \langle M \rangle$ , the truncated Young diagram  $\Lambda^{[k]} := (\Lambda_{k+1}, \dots, \Lambda_g)$  is also associated with a numerical semigroup  $H'$ , i.e.,  $H'$  is generated by

$$M \cup \{N^c(g - k), N^c(g - k + 1), \dots, N^c(g - 1)\},$$

though the generators are not minimal in general. Further, its compliment  $H'^c$  is a subset of  $H^c$ , i.e.,  $H'^c \subset H^c$ .

4. For  $\Lambda := (a_{r_\Lambda}, \dots, a_2, a_1; b_{r_\Lambda}, \dots, b_2, b_1)$  associated with the numerical semigroup  $H$ ,  $\Lambda[\ell] := (a_{r_\Lambda}, \dots, a_{\ell+1}, a_\ell; b_{r_\Lambda}, \dots, b_{\ell+1}, b_\ell)$ , and then  $\Lambda[\ell]$  is, in general, not associated with the numerical semigroup unless  $\ell = 1$ .

Further by letting  $|\Lambda| := \sum_{i=1}^{r_\Lambda} (a_i + b_i + 1)$ , we have  $|\Lambda| := \sum_{i=1}^{g_\Lambda} \Lambda_i$ .

$\Lambda^{[1]}$  of the numerical semigroup  $\langle 5, 7, 11 \rangle$  is associated with the numerical semigroup  $\langle 5, 7, 11, 13 \rangle$ . We show their examples of the Young diagram:

$$\begin{array}{ccc}
 \begin{array}{|c|c|} \hline & 5 \\ \hline 3 & 4 \\ \hline 2 & \\ \hline 1 & \\ \hline \end{array} & , & \begin{array}{|c|c|c|} \hline & & 9 \\ \hline & 7 & 8 \\ \hline 5 & 6 & \\ \hline 4 & & \\ \hline 3 & & \\ \hline 2 & & \\ \hline 1 & & \\ \hline \end{array} & , & \begin{array}{|c|c|c|} \hline & & 9 \\ \hline & & 8 \\ \hline 5 & 6 & 7 \\ \hline 4 & & \\ \hline 3 & & \\ \hline 2 & & \\ \hline 1 & & \\ \hline \end{array} \\
 \langle 3, 7, 8 \rangle & & \langle 5, 7, 11, 13 \rangle & & \langle 5, 6, 13, 14 \rangle
 \end{array} \tag{3}$$

In this paper, we mainly consider the  $r$ -numerical semigroup,  $H$ . We introduce the tools as follows:

- Definition 2.** 1. Let  $\mathbb{Z}_r := \{0, 1, 2, \dots, r - 1\}$  and  $\mathbb{Z}_r^\times := \mathbb{Z}_r \setminus \{0\}$ .  
 2. Let  $\tilde{e}_i := \min\{h \in H \mid i \equiv h \pmod r\}$ ,  $i \in \mathbb{Z}_r$ .  
 3. Let  $\tilde{\mathfrak{E}}_H := \{\tilde{e}_i \mid i \in \mathbb{Z}_r\}$  be the standard basis of  $H$ . Further, we define the ordered set  $\mathfrak{E}_H := \{e_i \in \tilde{\mathfrak{E}}_H \mid e_i < e_{i+1}\}$ , and  $\mathfrak{E}_H^\times := \mathfrak{E}_H \setminus \{0\}$ , e.g.,  $e_0 = \tilde{e}_0 = 0$ .  
 4. Let  $\overline{H}^c := H^c \cup (-\mathbb{N})$ , where  $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ .

We have the following elementary but essential results (Lemmas 2.8 and 2.9) in [17]:

**Lemma 2.** For  $a \in \mathbb{N}_0$ , we define

$$[a]_r := \{a + kr \mid k \in \mathbb{N}_0\}, \quad \overline{[a]_r}^c := \{a - kr \mid k \in \mathbb{N}\}, \quad [a]_r^c := \overline{[a]_r}^c \cap \mathbb{N}.$$

1. We have the following decomposition;

- (a)  $H = \bigoplus_{i \in \mathbb{Z}_r} [e_i]_r$ ,
- (b)  $\overline{H}^c = \bigoplus_{i \in \mathbb{Z}_r} \overline{[e_i]_r}^c$ ,  $H \cup \overline{H}^c = \mathbb{Z}$ ,
- (c)  $H^c = \bigoplus_{i \in \mathbb{Z}_r} [e_i]_r^c = \bigoplus_{i \in \mathbb{Z}_r^\times} [e_i]_r^c$ ,  $H \cup H^c = \mathbb{N}_0$ ,

2. for every  $x_i \in [e_i]_r$  ( $i \in \mathbb{Z}_r$ ),

$$\{x_i \text{ modulo } r \mid i \in \mathbb{Z}_r\} = \mathbb{Z}/r\mathbb{Z},$$

and especially for  $x \in \overline{[e_i]_r}^c$ ,  $x = i$  modulo  $r$ .

The following is obvious:

**Lemma 3.** For the generators  $r$  and  $s$  in the numerical semigroup  $H$ , there are positive integers  $i_s$  and  $i_r$  such that  $i_s s - i_r r = 1$ .

2.2. Weierstrass Canonical Form

We recall the ‘‘Weierstrass canonical form’’ (‘‘Weierstrass normal form’’) based on [5,6,16,17], which is a generalization of Weierstrass’ standard form for elliptic curves and whose origin came from Abel’s insight; Weierstrass investigated its primitive property [3,23]. Baker (Ch. V, §§60–79) in Baker97 gives its complete review, proof, and examples, but we refer to Kato [24,25], who also produces this representation from a modern viewpoint.

**Proposition 2** ([24,25]). For a pointed curve  $(X, \infty)$  with Weierstrass semigroup  $H_X := H(X, \infty)$  for which  $r_{\min}(H_X) = r$ , and  $\mathbf{e}_i \in \mathfrak{E}_{H_X}$ ,  $(i \in \mathbb{Z}_r^\times)$  in Definition 2, and we let  $s := \min_{i \in \mathbb{Z}_r^\times} \{\mathbf{e}_i \in \mathfrak{E}_{H_X} \mid (\mathbf{e}_i, r) = 1\}$  and  $s = \mathbf{e}_{\ell_s}$ .  $(X, \infty)$  is defined by an irreducible equation,

$$f_X(x, y) = 0, \tag{4}$$

for a polynomial  $f_X \in \mathbb{C}[x, y]$  of type,

$$f_X(x, y) := y^r + A_1(x)y^{r-1} + A_2(x)y^{r-2} + \dots + A_{r-1}(x)y + A_r(x), \tag{5}$$

where the  $A_i(x)$ 's are polynomials in  $x$ ,  $A_0 = 1$ ,  $A_i = \sum_{j=0}^{\lfloor is/r \rfloor} \lambda_{i,j}x^j$ , and  $\lambda_{i,j} \in \mathbb{C}$ ,  $\lambda_{r,s} = -1$ .

In this paper, we call the curve in Proposition 2 a *Weierstrass curve* or a *W-curve*. The Weierstrass canonical form characterizes the W-curve, which has only one infinity point  $\infty$ . The infinity point  $\infty$  is a Weierstrass point if  $H_X^c = H^c(X, \infty) = \{N(i)\}$  differs from  $\{1, 2, \dots, g\}$  [26]. Since every compact Riemann surface of the genus,  $g (> 1)$ , has a Weierstrass point whose Weierstrass gap sequence with genus  $g$  [27], it characterizes the behavior of the meromorphic functions at the point, and thus, there is a Weierstrass curve, which is bi-rationally equivalent to the compact Riemann surface.

Further, Proposition 2 is also applicable to a pointed compact Riemann surface  $(Y, P)$  of genus  $g$  whose point  $P$  is a non-Weierstrass point rather than the Weierstrass point; its Weierstrass gap sequence at  $P$  is  $H^c(Y, P) = \{1, 2, \dots, g\}$ . Even for the case, we find the Weierstrass canonical form  $f_X$  and the W-curve  $X$  with  $H_X^c = \{1, 2, \dots, g\}$ , which is bi-rational to  $Y$ .

**Remark 1.** Let  $R_{X^\circ}^\circ := \mathbb{C}[x, y]/(f_X(x, y))$  for (4) and its normalized ring be  $R_X^\circ$  if  $X^\circ := \text{Spec } R_{X^\circ}^\circ$  is singular.  $R_X^\circ$  is the coordinate ring of the affine part of  $X \setminus \{\infty\}$ , and we identify  $R_X^\circ$  with  $R_X = \mathbf{H}^0(X, \mathcal{O}_X(*\infty))$ . Then, the quotient field  $\mathbb{C}(X) := \mathcal{Q}(R_X)$  of  $R_X$  is considered as an algebraic function field on  $X$  over  $\mathbb{C}$ .

By introducing  $R_{\mathbb{P}} := \mathbf{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(*\infty)) = \mathbb{C}[x]$  and its quotient field  $\mathbb{C}(x) := \mathcal{Q}(R_{\mathbb{P}})$ ,  $\mathcal{Q}(R_X)$  is considered a finite extension of  $\mathcal{Q}(R_{\mathbb{P}})$ . We regard  $R_X$  as a finite extended ring of  $R_{\mathbb{P}}$  of rank  $r$ , e.g.,  $R_{X^\circ}^\circ = R_{\mathbb{P}}[y]/(f_X(x, y))$  as mentioned in Section 2.5 [28].

For the local ring  $R_{X,P}$  of  $R_X$  at  $P \in X$ , we have the ring homomorphism,  $\varphi_P : R_X \rightarrow R_{X,P}$ . We note that  $R_{X,\infty}$  plays crucial roles in the Weierstrass canonical form. We let the minimal generator  $M_X = \{r_1, r_2, \dots, r_{m_X}\}$  of the numerical semigroup  $H_X = H(X, \infty)$ . The Weierstrass curve admits a local cyclic  $\mathfrak{E}_r = \mathbb{Z}/r\mathbb{Z}$ -action at  $\infty$  c.f., Section 2.3. The genus of  $X$  is denoted by  $g_X$ , briefly  $g$  and the conductor of  $H_X$  is denoted by  $c_X := c_{H_X}$ ; the Frobenius number  $c_X - 1$  is the maximal gap in  $H(X, \infty)$ . We let,  $\overline{H}_X^c := \mathbb{Z} \setminus H_X$ .

**Projection from  $X$  to  $\mathbb{P}$**

There is the natural projection,

$$\omega_r : X \rightarrow \mathbb{P}, \quad (\omega_r(x, y_{r_2}, \dots, y_{r_{m_X}}) = x = y_r) \tag{6}$$

such that  $\omega_r(\infty) = \infty \in \mathbb{P}$ .

Let  $\{y_\bullet\} := \{y_s = y_{r_2}, y_{r_3}, \dots, y_{r_{m_X}}\}$  and  $\mathbb{C}[x, y_\bullet] := \mathbb{C}[x, y_s = y_{r_2}, y_{r_3}, \dots, y_{r_{m_X}}]$ .

**2.3. The Monomial Curves and W-Curves**

This subsection shows the monomial curves and their relation to W-curves based on [5,6,16].

For a given W-curve  $X$  with the Weierstrass semigroup  $H = H_X$ , and its generator  $M_X = \{r = r_1, r_2, \dots, r_{m_X}\}$ , the behavior of singularities of the elements in  $R_X$  at  $\infty$  is described by a monomial curve  $X_H^Z$ . For the numerical semigroup  $H = \langle M_X \rangle$ , the numerical semigroup ring  $R_H$  is defined as  $R_H := \mathbb{C}[z^{r_1}, z^{r_2}, \dots, z^{r_{m_X}}]$ .

Following a result of Herzog’s [29], we recall the well-known proposition for a polynomial ring  $\mathbb{C}[Z] := \mathbb{C}[Z_{r_1}, Z_{r_2}, \dots, Z_{r_{m_X}}]$ .

**Proposition 3.** For the  $\mathbb{C}$ -algebra homomorphism  $\tilde{\varphi}_H^Z : \mathbb{C}[Z] \rightarrow R_H$ , the kernel of  $\tilde{\varphi}_H^Z$  is generated by  $f_i^H = 0$  ( $i = 1, 2, \dots, k_X$ ) of a certain binomial  $f_i^H \in \mathbb{C}[Z]$  and a positive integer  $k_X$  ( $m_X - 1 \leq k_X < \infty$ ), i.e.,  $\ker \tilde{\varphi}_H^Z = (f_1^H, f_2^H, \dots, f_{k_X}^H)$ , and

$$R_H \simeq \mathbb{C}[Z] / \ker \tilde{\varphi}_H^Z =: R_H^Z.$$

We call  $R_H^Z = \mathbb{C}[Z] / \ker \tilde{\varphi}_H^Z$  a monomial ring. Sending  $Z_r$  to  $1/x$  and  $Z_{r_i}$  to  $1/y_{r_i}$ , the monomial ring  $R_H^Z$  determines the structure of the gap sequence of  $X$  at  $\infty$  [29,30]. Bresinsky showed that  $k_X$  can be any finitely large number if  $m_X > 3$  [31].

Let  $X_H := \text{Spec } R_H^Z$ , which we call a monomial curve. We also define the ring isomorphism on  $R_H^Z$  induced from  $\tilde{\varphi}_H^Z$ , which is denoted by  $\varphi_H^Z$ ,

$$\varphi_H^Z : \mathbb{C}[Z] / \ker \tilde{\varphi}_H^Z = R_H^Z \rightarrow R_H. \tag{7}$$

Further, we let  $\{Z_\bullet\} := \{Z_{r_2}, Z_{r_3}, \dots, Z_{r_{m_X}}\}$ , and  $\mathbb{C}[Z_r, Z_\bullet] := \mathbb{C}[Z_r, Z_{r_2}, Z_{r_3}, \dots, Z_{r_{m_X}}]$ . A monomial curve is an irreducible affine curve with  $\mathbb{G}_m$ -action, where  $\mathbb{G}_m$  is the multiplicative group of the complex numbers;  $Z_a \mapsto g^a Z_a$  for  $g \in \mathbb{G}_m$ , and it induces the action on the monomial ring  $R_H^Z$ .

The following cyclic action of order  $r$  plays a crucial role in this paper.

**Lemma 4.** The cyclic group  $\mathfrak{C}_r$  of order  $r$  acts on the monomial ring  $R_H^Z$ ; the action of the generator  $\widehat{\zeta}_r \in \mathfrak{C}_r$  on  $Z_a$  is defined by sending  $Z_a$  to  $\zeta_r^a Z_a$ , where  $\zeta_r$  is a primitive  $r$ -th root of unity. By letting  $\mathfrak{v}_i^* := (r, r_i)$ ,  $\mathfrak{v}_i := r / \mathfrak{v}_i^*$ , and  $\bar{\mathfrak{v}}_i := r_i / \mathfrak{v}_i^*$ , the orbit of  $Z_{r_i}$  forms  $\mathfrak{C}_{\mathfrak{v}_i}$ ; especially for the case that  $(r, r_i) = 1$ , it recovers  $\mathfrak{C}_r$ .

Thus, in  $R_H^Z$ ,

$$f_H^{(j)}(Z_r, Z_{r_j}) = 0, \quad f_H^{(j)} := Z_{r_j}^{\mathfrak{v}_j} - Z_r^{\bar{\mathfrak{v}}_j}, \quad (j = 2, \dots, m_X). \tag{8}$$

For a ring  $R$ , let its quotient field be denoted by  $\mathcal{Q}(R)$ . Further, we obviously have the identity in  $\mathcal{Q}(R_H^Z \otimes_{\mathbb{C}[Z_r]} R_H^Z)$ ,

$$p_H^{(j)}(Z_{r_j}, Z'_{r_j}) := \frac{1}{\mathfrak{v}_j Z_{r_j}^{\mathfrak{v}_j - 1}} \frac{Z_{r_j}^{\mathfrak{v}_j} - Z_{r_j}'^{\mathfrak{v}_j}}{Z_{r_j} - Z_{r_j}'} = \begin{cases} 1 & \text{for } Z_{r_j} = Z_{r_j}' \\ 0 & \text{for } Z_{r_j} \neq Z_{r_j}' \end{cases} \tag{9}$$

as a  $\mathbb{C}[Z_r]$ -module.

Corresponding to the standard basis of  $H_X$  in Definition 2, we find the monic monomial  $\mathfrak{z}_{\mathfrak{e}_i} \in \mathbb{C}[Z]$  such that  $\varphi_H^Z(\mathfrak{z}_{\mathfrak{e}_i}) = z^{\mathfrak{e}_i}$ , and the standard basis  $\{\mathfrak{z}_{\mathfrak{e}_i} \mid i \in \mathbb{Z}_r\}$ ;  $\mathfrak{z}_{\mathfrak{e}_0} = 1$ .

**Lemma 5.** The  $\mathbb{C}[Z_r]$ -module  $\mathbb{C}[Z]$  is given by

$$\mathbb{C}[Z] = \mathbb{C}[Z_r] \oplus \mathbb{C}[Z_r]\mathfrak{z}_{\mathfrak{e}_1} \oplus \dots \oplus \mathbb{C}[Z_r]\mathfrak{z}_{\mathfrak{e}_{r-1}},$$

and thus  $\mathfrak{z}_H := \{\mathfrak{z}_{\mathfrak{e}_0}, \mathfrak{z}_{\mathfrak{e}_1}, \dots, \mathfrak{z}_{\mathfrak{e}_{r-1}}\}$  is the basis of the  $\mathbb{C}[Z_r]$ -module  $\mathbb{C}[Z]$ . Then there is a monomial  $b_{ijk} \in \mathbb{C}[Z_r]$  such that

$$\mathfrak{z}_{\mathfrak{e}_i} \mathfrak{z}_{\mathfrak{e}_j} = \sum_{k \in \mathbb{Z}_r} b_{ijk} \mathfrak{z}_{\mathfrak{e}_k}.$$

**Lemma 6.** By defining an element in  $\mathcal{Q}(R_H^Z \otimes_{\mathbb{C}[Z_r]} R_H^Z)$  by  $p_H(Z_r, Z_\bullet, Z'_\bullet) := \prod_{j=2}^{m_X} p_H^{(j)}(Z_{r_j}, Z'_{r_j})$ ,

we have the identity

$$p_H(Z_r, Z_\bullet, Z'_\bullet) = \begin{cases} 1 & \text{for } Z_\bullet = Z'_\bullet, \\ 0 & \text{for } Z_\bullet \neq Z'_\bullet, \end{cases} \tag{10}$$

as  $\mathbb{C}[Z_r]$ -module properties and

$$\varphi_Z^H(\tilde{p}_H(Z_r, Z_\bullet, Z'_\bullet)) = \frac{1}{r} \sum_{i=0}^r \frac{z'^i \epsilon_i}{z^{\epsilon_i}}. \tag{11}$$

To construct our curve  $X$  from  $R_H$  or  $\text{Spec } R_H$ , we could follow Pinkham’s strategy [30] with an irreducible curve singularity with  $\mathbb{G}_m$  action, though we will not mention it in this paper. Pinkham’s investigations provide the following proposition [30] (Proposition 3.7) in [17]:

**Proposition 4.** For a given  $W$ -curve  $X$  and its associated monomial ring,  $R_H^Z = \mathbb{C}[Z]/(f_1^H, f_2^H, \dots, f_{k_X}^H)$ , there are a surjective ring-homomorphism [22] (p. 80)

$$\varphi_H^X : R_X \rightarrow R_H^Z$$

such that  $R_X/\mathfrak{m}_\mathbb{A}$  is isomorphic to  $R_H^Z$ , where  $\mathfrak{m}_\mathbb{A}$  is the maximal ideal  $(\lambda_{ij})$  in coefficient ring  $\mathbb{C}[\lambda_{ij}]$ , and  $\varphi_H^X(y_{r_i}) = Z_{r_i}$ , and a polynomial  $\{f_i^X\}_{i=1, \dots, k_X} \in \mathbb{C}[x, y_\bullet]$  satisfying

1.  $\varphi_H^X(f_i^X) (= \varphi_H^X(f_i^X \text{ modulo } \mathfrak{m}_\mathbb{A})) = f_i^H$  for  $(i = 1, \dots, k_X)$ ,
2. the affine part of  $R_X$  is given by  $R_X = \mathbb{C}[x, y_\bullet]/(f_1^X, f_2^X, \dots, f_{k_X}^X)$ , and
3. the rank of the matrix  $\left( \frac{\partial f_i^X}{\partial y_{r_j}} \right)_{i=1,2, \dots, k_X, j=1,2, \dots, m_X}$  is  $m_X - 1$  for every point  $P$  in  $X$ .

**Definition 3.** 1. Recalling Lemma 3, we define the arithmetic local parameter at  $\infty$  by  $t = \frac{x^{i_r}}{y^{i_s}}$  [32].

2. The degree at  $\mathcal{Q}(R_{X,\infty})$  as the order of the singularity with respect to  $t$  is naturally defined by

$$\text{wt} = \text{deg}_\infty : \mathcal{Q}(R_X) \rightarrow \mathbb{Z}, \tag{12}$$

which is called the Sato-Weierstrass weight [23].

3. In the ring of the formal power series  $\mathbb{C}[[t_1, \dots, t_\ell]]$ , we define the symbol  $d_{>n}(t_1, \dots, t_\ell)$  so that it belongs to the ideal,

$$d_{>n}(t_1, \dots, t_\ell) \in \left\{ \sum a_{i_1, \dots, i_\ell} t_1^{i_1} \cdots t_\ell^{i_\ell} \mid a_{i_1, \dots, i_\ell} = 0 \text{ for } i_1 + \dots + i_\ell \leq n \right\}.$$

The weight of  $y_{r_i}$  is given by

$$\text{wt}(y_{r_i}) = -r_i, \quad (i = 1, 2, \dots, m_X), \quad y_{r_i} = \frac{1}{t^{r_i}} (1 + d_{>0}(t)).$$

**Lemma 7.** We have the decomposition of  $R_X$  as a  $\mathbb{C}$ -vector space,

$$R_X = \bigoplus_{i=0} \mathbb{C}\phi_i, \tag{13}$$

where  $\phi_i$  is a monomial in  $R_X$  satisfying the inequalities  $-\text{wt } \phi_i < -\text{wt } \phi_j$  for  $i < j$ , i.e.,  $\phi_0 = 1$ ,  $\phi_1 = x, \dots$



Further, by assigning a certain weight on each coefficient  $\lambda_{i,j}$  in (4) so that (4) is a homogeneous equation of weight  $rs$ , we also define another weight,

$$\text{wt}_\lambda : R_X \rightarrow \mathbb{Z}. \tag{14}$$

**Definition 4.** We define  $S_X := \{\phi_i \mid i = 0, 1, 2, \dots\}$  by the basis of  $R_X$  as in (13).

Then  $N(i) = -\text{wt}(\phi_i)$ , for  $\{N(i) \mid i \in \mathbb{N}_0\} = H_X$ .

**Lemma 8.** Let  $t$  be the arithmetic local parameter at  $\infty$  of  $R_X$ .

1. By the isomorphism  $\varphi_{\text{inv}} : z \mapsto \frac{1}{t}$ ,  $\varphi_{\text{inv}}(R_H) (\cong R_H^Z)$  is a subring of  $\mathbb{C} \left[ \frac{1}{t} \right]$ ; for  $g(z) \in R_H$ ,  $g\left(\frac{1}{t}\right) \in \mathbb{C} \left[ \frac{1}{t} \right]$ .
2. There is a surjection of ring  $\varphi_\infty : R_X \rightarrow R_H (\cong R_H^Z)$ ; for  $f \in R_X$ , there is  $g(z) \in R_H$  such that

$$(f)_\infty = g\left(\frac{1}{t}\right) (1 + d_{>0}(t)) \in R_{X,\infty},$$

where  $(f)_\infty$  means the germ at  $\infty$  or  $(f)_\infty \in R_{X,\infty}$  via  $\varphi_H^X$  in Proposition 4. It induces the surjection  $R_{X,\infty} \rightarrow R_H (\cong R_H^Z)$ .

**Proof.** By letting  $g = \varphi_H^Z \circ \varphi_H^X(f)$ , the existence of  $g$  is obvious.  $\square$

#### 2.4. $R_{\mathbb{P}}$ -Module $R_X$

$R_X$  is an  $R_{\mathbb{P}}$ -module, and its affine part is given by the quotient ring of  $R_{\mathbb{P}}[y_\bullet]$ .

We recall Definition 2 and Lemma 5, and apply them to  $W$ -curves and then we obtain the following (Proposition 3.11) in [17].

**Proposition 5.** For  $\epsilon_i \in \mathfrak{E}_{H_X}$ , we let  $\eta_{\epsilon_i}$  be the monic monomial in  $R_X$  whose weight is  $-\epsilon_i$ , ( $\eta_{\epsilon_0} = 1$ ) satisfying

$$R_X = R_{\mathbb{P}} \oplus \bigoplus_{i=1}^{r-1} R_{\mathbb{P}} \eta_{\epsilon_i} = \bigoplus_{i=0}^{r-1} R_{\mathbb{P}} \eta_{\epsilon_i} = \langle \eta_{\epsilon_0}, \eta_{\epsilon_1}, \dots, \eta_{\epsilon_{r-1}} \rangle_{R_{\mathbb{P}}}$$

with the relations,

$$\eta_{\epsilon_i} \eta_{\epsilon_j} = \sum_{k=0}^{r-1} \alpha_{ijk} \eta_{\epsilon_k}, \tag{15}$$

where  $\alpha_{ijk} \in R_{\mathbb{P}}$ ,  $\alpha_{ijk} = \alpha_{ijk}$ , especially  $\alpha_{0jk} = \alpha_{j0k} = \delta_{jk}$ .

#### 2.5. The Covering Structures in $W$ -Curves

We will follow [28,33] to investigate the covering structure in  $W$ -curves.

##### 2.5.1. Galois Covering

As mentioned in Remark 1, let us consider the Riemann sphere  $\mathbb{P}$  and  $R_{\mathbb{P}} = \mathbf{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(*\infty))$ . We identify  $R_{\mathbb{P}}$  with its affine part  $R_{\mathbb{P}}^\circ = \mathbb{C}[x]$ , and its quotient field is denoted by  $\mathbb{C}(x) = \mathcal{Q}(R_{\mathbb{P}})$ . The quotient field  $\mathcal{Q}(R_X) = \mathbb{C}(X)$  of  $R_X$  is an extension of the field  $\mathbb{C}(x)$ .

Following the above description, we consider the  $W$ -curve  $X$ . The covering  $\omega_r : X \rightarrow \mathbb{P} ((x, y_\bullet) \mapsto x)$  is obviously a holomorphic  $r$ -sheeted covering. When we obtain the Galois group on  $X$ , i.e.,  $\text{Gal}(\mathcal{Q}(R_X)/\mathcal{Q}(R_{\mathbb{P}})) = \text{Aut}(X/\mathbb{P}) = \text{Aut}(\omega_r)$ , this is denoted by  $G_X$ . The  $\omega_r$  is a finite branched covering. A ramification point of  $\omega_r$  is defined as a point that is not biholomorphic. The image  $\omega_r$  of the ramification point is called the branch point of  $\omega_r$ . The number of finite ramification points is denoted by  $\ell_{\mathfrak{B}}$ .

We basically focus on the holomorphic  $r$ -sheeted covering  $\omega_x = \omega_r : X \rightarrow \mathbb{P}$ .  $G_x$  denotes the finite group action on  $\omega_r^{-1}(x)$  for  $x \in \mathbb{P}$ , referred to as group action at  $x$  in this paper.

**Definition 5.** Let  $\mathfrak{B}_X := \mathfrak{B}_{X,r} = \{B_i\}_{i=0,\dots,\ell_{\mathfrak{B}}}$  and  $\mathfrak{B}_{\mathbb{P}} := \omega_r(\mathfrak{B}_X) = \{b_i\}_{i=0,\dots,\ell_{\mathfrak{B}}}$ , where  $\ell_{\mathfrak{B}} := \#\mathfrak{B}_X - 1$ ,  $B_0 = \infty \in X$  and  $b_i := \omega_r(B_i)$ .

2.5.2. Riemann–Hurwitz Theorem

Let us consider the behaviors of the covering  $\omega_r : X \rightarrow \mathbb{P}$ , including the ramification points. The Riemann–Hurwitz theorem [28],

$$2g - 2 = -2r + \sum_{i=1}^{\ell_{\mathfrak{B}}} (e_{B_i} - 1) + (r - 1) = \sum_{i=1}^{\ell_{\mathfrak{B}}} (e_{B_i} - 1) - (r + 1), \tag{16}$$

where  $e_{B_i}$  is the ramification index at  $B_i$ , shows the following:

**Corollary 1.** The divisor of  $dx$  is given by,

$$\text{div}(dx) = \sum_{i=1}^{\ell_{\mathfrak{B}}} (e_{B_i} - 1)B_i - (r + 1)\infty.$$

2.5.3. Embedding of  $X$  into  $\mathbb{P}^{2(m_X-1)}$

By identifying  $R_X = R_X^\circ = \mathbb{C}[x, y_\bullet] / (f_1^X, \dots, f_{k_X}^X)$ , it is obvious that  $R_{X^\circ}^\circ = \mathbb{C}[x, y_s] / f_X(x, y_s)$  is a subring of  $R_X$  because  $R_X$  is a normalized ring of  $R_{X^\circ}^\circ$ . There is a projection  $\omega_{r,r_2} : X \rightarrow X^\circ$ . Thus, we can find the subring  $R_{X^{(i)}} = \mathbb{C}[x, y_{r_i}] / (f_X^{(i)}(x, y_{r_i}))$  of  $R_X$  as its normalized ring, as we prove this fact in (Proposition 3.14) in [17] precisely. The image of  $f_X^{(i)}(x, y_{r_i})$  by  $\varphi_H^X$  is (8). There are injective ring-homomorphisms  $R_{\mathbb{P}} \xrightarrow{\iota_r^{(i)}} R_{X^{(i)}} \xrightarrow{\iota_{r,r_i}} R_X$  and they induce the projections  $\omega_{r,r_i} : X \rightarrow X^{(i)} ((x, y_\bullet) \mapsto (x, y_{r_i}))$  and  $\omega_r^{(i)} : X^{(i)} \rightarrow \mathbb{P} ((x, y_{r_i}) \mapsto x)$  where  $X^{(i)} := \text{Spec } R_{X^{(i)}}$ . They satisfy the commutative diagrams,

$$\begin{array}{ccc} R_X & \xleftarrow{\iota_{r,r}} & R_{X^{(j)}} \\ \uparrow \iota_{r_i,r} & \swarrow \iota_r & \uparrow \iota_r^{(j)} \\ R_{X^{(i)}} & \xleftarrow{\iota_r^{(i)}} & R_{\mathbb{P}} \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\omega_{r,r_j}} & X^{(j)} \\ \omega_{r,r_i} \downarrow & \searrow \omega_r & \downarrow \omega_r^{(j)} \\ X^{(i)} & \xrightarrow{\omega_r^{(i)}} & \mathbb{P}. \end{array} \tag{17}$$

Further, we also define the tensor product of these rings  $R_{X^{(2)}} \otimes_{R_{\mathbb{P}}} R_{X^{(3)}} \otimes_{R_{\mathbb{P}}} \dots \otimes_{R_{\mathbb{P}}} R_{X^{(m_X)}}$ , and its geometrical picture  $X_{\mathbb{P}}^{[m_X-1]} := X^{(2)} \times_{\mathbb{P}} X^{(3)} \times_{\mathbb{P}} \dots \times_{\mathbb{P}} X^{(m_X)}$ . By identifying  $\mathbb{C}[x, y_\bullet] / (f_X^{(2)}, \dots, f_X^{(m_X)}) = R_X^{\otimes [m_X-1]}$  with a ring  $R_{X^{(2)}} \otimes_{R_{\mathbb{P}}} R_{X^{(3)}} \otimes_{R_{\mathbb{P}}} \dots \otimes_{R_{\mathbb{P}}} R_{X^{(m_X)}}$ , we have the natural projection  $\varphi_{R_X^{\otimes [m_X-1]}} : R_X^{\otimes [m_X-1]} \rightarrow R_X$ , i.e.,  $R_X = R_X^{\otimes [m_X-1]} / (f_1^X, \dots, f_{k_X}^X)$ , and the injection  $\iota_{R_X^{\otimes [m_X-1]}} : R_{\mathbb{P}} \hookrightarrow R_X^{\otimes [m_X-1]}$ . It induces the injection  $\iota_{X_{\mathbb{P}}^{[m_X-1]}} : X \rightarrow X_{\mathbb{P}}^{[m_X-1]}$  and the projection  $\prod_i \omega_{X^{(i)}} : X_{\mathbb{P}}^{[m_X-1]} \rightarrow \mathbb{P}$ .

Moreover, we also define the direct product of these rings  $R_X^{[m_X-1]} := R_{X^{(2)}} \times R_{X^{(3)}} \times \dots \times R_{X^{(m_X)}}$ , and its geometrical picture  $X^{[m_X-1]} := \prod X^{(i)} \subset \mathbb{P}^{2(m_X-1)}$ . Then we have an embedding,

$$\varphi_{R_X^{[m_X-1]}} : R_X^{[m_X-1]} \rightarrow R_X, \quad \iota_{X^{[m_X-1]}} : X \hookrightarrow X^{[m_X-1]} (\subset \mathbb{P}^{2(m_X-1)}).$$

2.6. Complementary Module  $R_X^c$  of  $R_X$

By introducing  $p_X^{(i)} := \frac{1}{f_{X,y}^{(i)}} \frac{f_X^{(i)}(x, y_{r_i}) - f_X^{(i)}(x, y'_{r_i})}{y_{r_i} - y'_{r_i}}$ , where  $f_{X,y}^{(i)}(x, y) = \frac{\partial f_X^{(i)}(x, y)}{\partial y}$ , as

an  $R_X$ -analog of (9), let us consider  $p_{R_X} := \prod_{j=2}^{m_X} p_X^{(j)}$  as an element of  $\mathcal{Q}(R_X \otimes_{R_{\mathbb{P}}} R_X)$ . The following is obvious:

**Proposition 6.** For  $(P, Q) \in X \times_{\mathbb{P}} X$ ,

$$p_{R_X}(P, Q) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \neq Q, \end{cases}$$

However, some parts in its numerator and denominator are canceled because they belong to  $R_{\mathbb{P}}$ . Thus, we introduce an element  $h(x, y_{\bullet}, y'_{\bullet}) \in R_X \otimes_{R_{\mathbb{P}}} R_X$  such that  $h(x, y_{\bullet}, y'_{\bullet}) / h(x, y_{\bullet}, y_{\bullet})$  reproduces  $p_{R_X}$ .

**Lemma 9.** Lemma 4.14 in [17]. For a point  $(P = (x, y_{\bullet}), P' = (x, y'_{\bullet})) \in X \times_{\mathbb{P}} X$ , there is a polynomial  $\tilde{h}_{R_X}(x, y_{\bullet}, y'_{\bullet}) \in R_X \otimes_{R_{\mathbb{P}}} R_X$  such that

1. by regarding the element  $a$  in  $R_X$  as  $a \otimes 1$  in  $R_X \otimes_{R_{\mathbb{P}}} R_X$ ,  $\tilde{h}_{R_X}(x, y_{\bullet}, y_{\bullet})$  and  $\tilde{h}_{R_X}(x, y_{\bullet}, y'_{\bullet})$  are coprime as elements in  $R_X \otimes_{R_{\mathbb{P}}} R_X$ ,
2. for a group action  $\hat{\zeta} \in G_X$ ,  $\frac{\tilde{h}_{R_X}(x, \hat{\zeta}y_{\bullet}, \hat{\zeta}y'_{\bullet})}{\tilde{h}_{R_X}(x, \hat{\zeta}y_{\bullet}, \hat{\zeta}y_{\bullet})} = \frac{\tilde{h}_{R_X}(x, y_{\bullet}, y'_{\bullet})}{\tilde{h}_{R_X}(x, y_{\bullet}, y_{\bullet})}$ .
3. it satisfies  $\frac{\tilde{h}_{R_X}(x, y_{\bullet}, y'_{\bullet})}{\tilde{h}_{R_X}(x, y_{\bullet}, y_{\bullet})} = p_{R_X}(x, y_{\bullet}, y'_{\bullet})$ , and
4.  $\varphi_H^X(\tilde{h}_{R_X}(x, y_{\bullet}, y'_{\bullet})) =: \tilde{h}_H(Z_r, Z_{\bullet}, Z'_{\bullet})$  ( $\text{wt}(\tilde{h}_{R_X}(x, y_{\bullet}, y'_{\bullet})) = d_h$ ), then

$$\frac{\tilde{h}_H(Z_r, Z_{\bullet}, Z'_{\bullet})}{\tilde{h}_H(Z_r, Z_{\bullet}, Z_{\bullet})} = \tilde{p}_H^{(j)}(Z_r, Z_{\bullet}, Z'_{\bullet}),$$

and

$$\varphi_H^Z(\tilde{h}_H(Z_r, Z_{\bullet}, Z'_{\bullet})) = z^{d_h} \sum_{i \in \mathbb{Z}_r} z^{i} z^{-e_i}, \quad \varphi_H^Z(\tilde{h}_H(Z_r, Z_{\bullet}, Z_{\bullet})) = rz^{d_h}.$$

**Definition 6.** Let  $h_X(x, y_{\bullet}) := \tilde{h}_{R_X}(x, y_{\bullet}, y_{\bullet})$ .

We have the expression of  $\tilde{h}_{R_X}(x, y_{\bullet}, y'_{\bullet})$  following (Proposition 4.16, Lemma 4.20) in [17].

**Proposition 7.**  $\tilde{h}_{R_X} \in R_X \otimes_{R_{\mathbb{P}}} R_X$  is expressed by

$$\begin{aligned} \tilde{h}_{R_X}(x, y_{\bullet}, y'_{\bullet}) &= \hat{Y}_0 \cdot 1 + \hat{Y}_1 \eta'_{e_1} + \dots + \hat{Y}_{r-1} \eta'_{e_{r-1}} \\ &= 1 \cdot \hat{Y}'_0 + \eta_{e_1} \hat{Y}'_1 + \dots + \eta_{e_{r-1}} \hat{Y}'_{r-1} \\ &= \hat{\eta}_{e_0} \eta'_{e_0} + \hat{\eta}_{e_1} \eta'_{e_1} + \dots + \hat{\eta}_{e_{r-1}} \eta'_{e_{r-1}} \\ &\quad + \text{lower weight terms with respect to } -\text{wt} \end{aligned}$$

as an  $R_{\mathbb{P}}$ -module. Here  $\eta'_{e_0} = 1$ , and each  $\hat{Y}_i$  has the following properties

1.  $\hat{Y}_i = \sum_{j=0}^{r-1} b_{i,j} \eta_{e_j}$ , with certain  $b_{i,j} \in \mathbb{C}[x]$ ,

2.  $\widehat{Y}_i = \widehat{\eta}_{\epsilon_i} +$  lower weight terms with respect to  $-\text{wt}$ , where  $\widehat{\eta}_{\epsilon_i} = \widetilde{\delta}_i(x)\eta_{\epsilon_{\ell_i}^*}$  with an element  $\ell \in \mathbb{Z}_r$ , and a monic polynomial  $\widetilde{\delta}_i(x) \in \mathbb{C}[x]$  whose weight is  $-\delta_i r$ , (especially,  $\widehat{\eta}_{\epsilon_0} = \widetilde{\delta}_0(x)\eta_{\epsilon_{\ell,0}^*} = \widetilde{\delta}_0(x)\eta_{\epsilon_\ell}$  such that

$$\widehat{\eta}_{\epsilon_0} = \widehat{\eta}_{\epsilon_i}\eta_{\epsilon_i} + \text{lower weight terms with respect to } -\text{wt}$$

for  $i \in \mathbb{Z}_r$ ,  $\text{wt}(\widehat{\eta}_{\epsilon_i}) = -(d_h - \epsilon_i)$  where  $b_{i,j}$  is a certain element in  $R_{\mathbb{P}}$  for  $(i, j)$ , and

3.  $\langle \widehat{Y}_0, \widehat{Y}_1, \dots, \widehat{Y}_{r-1} \rangle_{R_{\mathbb{P}}} = \langle \widehat{Y}_1, \dots, \widehat{Y}_{r-1} \rangle_{R_X}$ .

We introduce more convenient quantities  $\widehat{\eta}_{\epsilon_i}$  ( $i = 0, 1, \dots, r - 1$ ):

**Definition 7.** For  $i \in \mathbb{Z}_r$ , we define a truncated polynomial  $\widehat{\eta}_{\epsilon_i}$  of  $\widehat{Y}_i$  such that the weight  $-\text{wt}$  of  $\widehat{Y}_i - \widehat{\eta}_{\epsilon_i}$  is less than  $-\text{wt}(\widehat{\eta}_{\epsilon_i})$ , i.e.,  $\widehat{\eta}_{\epsilon_i} = \widehat{\eta}_{\epsilon_i} +$  certain terms, and the number of the terms is minimal satisfying the relations as  $R_{\mathbb{P}}$ -modules,

$$\langle \widehat{Y}_1, \dots, \widehat{Y}_{r-1} \rangle_{R_X} = \langle \widehat{\eta}_{\epsilon_1}, \dots, \widehat{\eta}_{\epsilon_{r-1}} \rangle_{R_X} = \langle \widehat{\eta}_{\epsilon_0}, \dots, \widehat{\eta}_{\epsilon_{r-1}} \rangle_{R_{\mathbb{P}}}, \quad \tau_h(\widehat{\eta}_{\epsilon_i}) = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since some of  $f_{X,y}^{(j)}(P) = 0$  at  $P = B_i \in \mathfrak{B}_X \setminus \{\infty\}$ ,  $h_X(x, y_{\bullet}) \in R_X$  vanishes only at the ramification point  $B_i \in X$  and the construction of  $h_X$ , we have the following lemma from Dedekind’s different theorem Proposition 4.27 in [17].

**Lemma 10.**

$$\text{div}(h_X(x, y_{\bullet})) = \sum_{B_i \in \mathfrak{B}_X \setminus \{\infty\}} d_{B_i} B_i - d_h \infty,$$

where  $d_{B_i} := \text{deg}_{\mathfrak{B}_{B_i,0}}(h_x) \geq (e_{B_i} - 1)$ , and  $d_h = \sum_{B_i \in \mathfrak{B}_X \setminus \{\infty\}} d_{B_i} = -\text{wt}(h_X)$ .

**Definition 8.** The effective divisor,  $\sum_{B_i \in \mathfrak{B}_X \setminus \{\infty\}} (d_{B_i} - e_{B_i} + 1) B_i$ , is denoted by  $\mathfrak{K}_X$ , i.e.,  $\mathfrak{K}_X > 0$

and let  $\mathfrak{k}_X := \sum_{B_i \in \mathfrak{B}_X \setminus \{\infty\}} (d_{B_i} - e_{B_i} + 1) = \text{deg}(\mathfrak{K}_X) \geq 0$ .

**Lemma 11.** The divisor of  $\frac{dx}{h_X}$  is expressed by  $(2g - 2 + \mathfrak{k}_X)\infty - \mathfrak{K}_X$ , and  $2g - 2 + \mathfrak{k}_X = d_h - r - 1$  or  $\mathfrak{k}_X = d_h - 2g - r + 1$ .  $\varphi_{\infty}\left(\frac{dx}{h_X}\right) = t^{d_h-r-1} dt$ .

From Corollary 1, we note that these  $\mathfrak{K}_X$  and  $\mathfrak{k}_X$  play crucial roles in the investigation of the differentials on  $X$ .

**Proposition 8.**  $\mathfrak{k}_X$  is equal to zero if  $d_h$  is symmetric whereas  $\mathfrak{k}_X$  is not zero otherwise.

We recall  $\epsilon_i$  in Definition 2 in the standard basis in Lemma 5 and Proposition 5, and  $\widehat{\eta}_{\epsilon_i}$  in Definition 7.

From Proposition 4.32 in [17], we have the properties of  $\frac{x^k \widehat{\eta}_{\epsilon_i}(x, y_{\bullet}) dx}{h_X(x, y_{\bullet})}$ :

**Proposition 9.**

$$\text{div}\left(\frac{x^k \widehat{\eta}_{\epsilon_i}(x, y_{\bullet}) dx}{h_X(x, y_{\bullet})}\right) = k \text{div}_0(x) + \text{div}_0(\widehat{\eta}_{\epsilon_i}) - \mathfrak{K}_X + (\epsilon_i - (k + 1)r - 1)\infty,$$

where  $\text{div}_0(\widehat{\eta}_{e_i}) - \mathfrak{K}_X \geq 0$ . We have

$$\left\{ \text{wt} \left( \frac{x^k \widehat{\eta}_{e_i}(x, y_\bullet) dx}{h_X(x, y_\bullet)} \right) + 1 \mid i \in \mathbb{Z}_r, k \in \mathbb{N}_0 \right\} = \overline{H}_X^c.$$

### 3. W-Normalized Abelian Differentials on X

#### 3.1. W-Normalized Abelian Differentials $\mathbf{H}^0(X, \mathcal{A}_X(*\infty))$

Following K. Weierstrass [3], H. F. Baker [10], V.M. Buchstaber, D.V. Leykin and V.Z. Enolskii [12], J.C. Eilbeck, V.Z. Enolskii and D.V. Leykin [11] and our previous results [4,6,16], we construct the Abelian differentials of the first kind and the second kind  $\mathbf{H}^0(X, \mathcal{A}_X(*\infty))$  on X for more general W-curves based on Proposition 9 [17].

We consider the Abelian differentials of the first kind on a W-curve. Due to the Riemann–Roch theorem, there is the  $i$ -th holomorphic one-form whose behavior at  $\infty$  is given by

$$\left( t^{N^c(g-i)-1} (1 + d_{>0}(t)) \right) dt, \tag{18}$$

where  $N^c(i) \in H_X^c$  ( $i = 1, 2, \dots, g$ ) satisfying  $N^c(i) < N^c(i + 1)$ , and  $t$  is the arithmetic local parameter at  $\infty$ . We call this normalization the *W-normalization*. Similarly we find the differentials or the basis of  $\mathbf{H}^0(X, \mathcal{A}_X(*\infty))$  associated with  $\overline{H}_X^c$ .

The W-normalized holomorphic one-forms are directly obtained from Proposition 9:

**Lemma 12.** For  $x^k \widehat{\eta}_{e_i}$  in Proposition 9, we have the relation,

$$\left\langle \frac{x^k \widehat{\eta}_{e_i}}{h_X(x, y_\bullet)} dx \mid i \in \mathbb{Z}_r^{\times}, k \in \mathbb{N}_0 \right\rangle_{\mathbb{C}} = \mathbf{H}^0(X, \mathcal{A}_X(*\infty)).$$

By re-ordering  $x^k \widehat{\eta}_{e_i}$  with respect to the weight at  $\infty$ , we define the ordered set  $\{\widehat{\phi}_i\}$ :

**Definition 9. 1.** Let us define the ordered subset  $\widehat{S}_X$  of  $R_X$  by

$$\widehat{S}_X = \{\widehat{\phi}_i \mid i \in \mathbb{N}_0\}$$

such that  $\widehat{\phi}_i$  is ordered by the Sato–Weierstrass weight, i.e.,  $-\text{wt}\widehat{\phi}_i < -\text{wt}\widehat{\phi}_j$  for  $i < j$ , and  $\widehat{S}_X$  is equal to  $\{x^k \widehat{\eta}_{e_i} \mid i \in \mathbb{Z}_r, k \in \mathbb{N}_0\}$  as a set.

2. Let  $\widehat{R}_X$  be an  $R_X$ -module generated by  $\widehat{S}_X$ , i.e.,  $\widehat{R}_X := \langle \widehat{S}_X \rangle_{R_X} \subset R_X$ .
3. Recalling  $\mathfrak{K}_X$  and  $\mathfrak{k}_X$  in Definition 8, we let  $\widehat{N}(n) := -\text{wt}(\widehat{\phi}_n) - \mathfrak{k}_X$ ,  $\widehat{H}_X := \{-\text{wt}(\widehat{\phi}_n) \mid n \in \mathbb{N}_0\}$ , and we define the dual conductor  $\widehat{c}_X$  as the minimal integer satisfying  $\widehat{c}_X + \mathbb{N}_0 \subset \widehat{H}_X - \mathfrak{k}_X$ .
4. We define  $\widehat{S}_X^{(g)} := \{\widehat{\phi}_0, \widehat{\phi}_1, \dots, \widehat{\phi}_{g-1}\}$ , and the W-normalized holomorphic one form, or W-normalized Abelian differentials of the first kind  $v_i^I$  as the canonical basis of X,

$$\left\langle v_i^I := \frac{\widehat{\phi}_{i-1} dx}{h_X} \mid \widehat{\phi}_{i-1} \in \widehat{S}_X^{(g)} \right\rangle_{\mathbb{C}} = \mathbf{H}^0(X, \mathcal{A}_X). \tag{19}$$

We note that at  $\infty$ ,  $v_i^I$  behaves like  $v_i^I = (t^{N^c(g-i-1)-1} (1 + d_{>0}(t))) dt$  for the arithmetic local parameter  $t$  at  $\infty$ , and further  $\frac{\widehat{\phi}_{i-1} dx}{h_X} = t^{N^c(g-i-1)-1} (1 + d_{>0}(t)) dt$  where  $N^c(i)$  indicates the element in  $\overline{H}_X^c$  such that  $N^c(-i) = -i$  for  $i \in \mathbb{N}$ ; they are W-normalized Abelian differentials.

We summarize them:

**Lemma 13. 1.**  $\mathbf{H}^0(X, \mathcal{A}_X(*\infty)) = \bigoplus_{i=0}^{\widehat{c}_X} \mathbb{C} \frac{\widehat{\phi}_i dx}{h_X} = R_X^c dx = \frac{\widehat{R}_X dx}{h_X}$ .

$$2. \quad \overline{H}_X^c = \left\{ \text{wt} \left( \frac{\widehat{\phi}_i dx}{h_X} \right) + 1 \mid i \in \mathbb{N}_0 \right\} = \{ \text{wt}(\widehat{\phi}_i) + d_h - r \mid i \in \mathbb{N}_0 \} \\ = \{ \text{wt}(\widehat{\phi}_i) + 2g - 1 - \mathfrak{k}_X \mid i \in \mathbb{N}_0 \}.$$

By the Abel–Jacobi theorem [26],  $\mathfrak{K}_X$  in Definition 8 can be divided into two pieces, which are related to the spin structure in  $X$ .

**Definition 10.** Let  $\mathfrak{K}_s$  and  $\mathfrak{K}_X^c$  be the effective divisors that satisfy

$$\mathfrak{K}_X - \mathfrak{k}_X \infty \sim 2\mathfrak{K}_s - 2\mathfrak{k}_s \infty, \quad \mathfrak{K}_X + \mathfrak{K}_X^c - (\mathfrak{k}_X + \mathfrak{k}_X^c) \infty \sim 0$$

as the linear equivalence, where  $\mathfrak{k}_s$  and  $\mathfrak{k}_X^c$  are the degrees of  $\mathfrak{K}_s$  and  $\mathfrak{K}_X^c$ , respectively.

Since the  $W$ -normalized holomorphic one form is given by the basis (19), Definition 10 shows the canonical divisor:

**Proposition 10.** The canonical divisor is given by

$$K_X \sim (2g - 2 + \mathfrak{k}_X) \infty - \mathfrak{K}_X \sim (2g - 2 + 2\mathfrak{k}_s) \infty - 2\mathfrak{K}_s \\ \sim (2g - 2 - \mathfrak{k}_X^c) \infty + \mathfrak{K}_X^c.$$

From (Lemmas 5.8 and 5.9) in [17], we show the properties of these parameters:

- Lemma 14.** 1.  $\{-\text{wt}(\widehat{\phi}_i)\} = \{d_h - \mathfrak{c}_i + kr \mid i \in \mathbb{Z}_r, k \in \mathbb{N}_0\}$ .  
 2.  $\text{div}(\widehat{\phi}_i) \geq (\mathfrak{K}_X - (2g - 2 + \mathfrak{k}_X) \infty)$  for every  $\widehat{\phi}_i \in \widehat{S}_R^{(g)}$ , ( $i = 0, 1, 2, \dots, g - 1$ ).  
 3.  $\text{div}(\widehat{\phi}_i) \geq (\mathfrak{K}_X - (g + \mathfrak{k}_X + i) \infty)$  for every  $\widehat{\phi}_i \in \widehat{S}_R$ , ( $i \geq g$ ).

- Lemma 15.** 1.  $-\text{wt} \widehat{\phi}_0 = (0 \text{ if } H_X \text{ is symmetric}) = \widehat{c}_X + \mathfrak{k}_X - c_X = d_h - r - c_X + 1$ ,  
 2.  $-\text{wt} \widehat{\phi}_{g-1} = \widehat{c}_X + \mathfrak{k}_X - 2 = d_h - r - 1 = (2g - 2) + \mathfrak{k}_X$ , i.e.,  $\widehat{N}(g - 1) = 2g - 2$ ,  
 3.  $\widehat{c}_X = 2g = d_h - \mathfrak{k}_X - r + 1$ ,  $-\text{wt} \widehat{\phi}_g = 2g + \mathfrak{k}_X = \widehat{c}_X + \mathfrak{k}_X = d_h - r + 1$ ,  
 4.  $-\text{wt} \widehat{\phi}_{g-1} + \text{wt} \widehat{\phi}_0 = \mathfrak{e}_{r-1} - r - 1 = c_X - 2$ , and  $c_X = \mathfrak{e}_{r-1} - r + 1$ .

3.2. Extension  $p \in R_X \otimes_{R_{\mathbb{P}}} R_X$  to  $R_X \otimes_{\mathbb{C}} R_X$

As we have  $\widetilde{h}_{R_X}$  in Lemma 9, we extend it in  $R_X \otimes_{R_{\mathbb{P}}} R_X$  to an element in  $R_X \otimes_{\mathbb{C}} R_X$ , though the extension is not unique; there are two different  $h(x, y_{\bullet}, x', y'_{\bullet})$  and  $h'(x, y_{\bullet}, x', y'_{\bullet})$  in  $R_X \otimes_{\mathbb{C}} R_X$  such that  $h(x, y_{\bullet}, x, y'_{\bullet}) = h'(x, y_{\bullet}, x, y'_{\bullet})$  in  $R_X \otimes_{R_{\mathbb{P}}} R_X$ . Since they are quasi-isomorphic, we select one of them, and thus, it is well-defined in the meaning of Proposition 15-4 and 16-4.

**Definition 11.** Using  $\widehat{Y}_i$  in Proposition 7, for a point  $(P = (x, y_{\bullet}), P' = (x', y'_{\bullet})) \in X \times X$ , we define a polynomial  $\widetilde{h}_X(x, y_{\bullet}, y'_{\bullet}) \in R_X \otimes_{\mathbb{C}} R_X$  by

$$\widetilde{h}_X(x, y_{\bullet}, y'_{\bullet}) := \widehat{Y}_0 \cdot 1 + \widehat{Y}_1 \mathfrak{y}'_{\mathfrak{e}_1} + \dots + \widehat{Y}_{r-1} \mathfrak{y}'_{\mathfrak{e}_{r-1}}.$$

Then the following lemma is evident from Lemma 9.

**Lemma 16.** For a point  $(P = (x, y_{\bullet}), P' = (x', y'_{\bullet})) \in X \times X$ ,  $\widetilde{h}_X(x, y_{\bullet}, y'_{\bullet})$  satisfies

1. for the case  $x' = x$  and a group action  $\widehat{\zeta} \in G_x$ ,  $\frac{\widetilde{h}_X(x, \widehat{\zeta} y_{\bullet}, \widehat{\zeta} y'_{\bullet})}{\widetilde{h}_X(x, \widehat{\zeta} y_{\bullet}, \widehat{\zeta} y_{\bullet})} = \frac{\widetilde{h}_X(x, y_{\bullet}, y'_{\bullet})}{\widetilde{h}_X(x, y_{\bullet}, y_{\bullet})}$ ,
2.  $\lim_{(x,y) \rightarrow \infty} \frac{\widetilde{h}_X(x, y_{\bullet}, y'_{\bullet})}{\widetilde{h}_X(x, y_{\bullet}, y_{\bullet})} = \frac{1}{r}$  for  $P' \in X \setminus \{\infty\}$ ,

3. when  $x = x'$  or  $\tilde{h}_X(x, y_\bullet, x', y'_\bullet)$  belongs to  $R_X \otimes_{\mathbb{R}_\mathbb{P}} R_X$ , it satisfies

$$\tilde{h}_X(x, y_\bullet, y'_\bullet) = \tilde{h}_{R_X}(x, y_\bullet, y'_\bullet),$$

4.  $h_X(x, y_\bullet) = \tilde{h}_X(x, y_\bullet, y_\bullet)$ .

**Definition 12.** Using  $\tilde{h}_X(x, y_\bullet, y'_\bullet)$  for a point  $(P = (x, y_\bullet), P' = (x', y'_\bullet)) \in X \times X$ , we define

$$p_\omega(P, P') := \frac{\tilde{h}_X(x, y_\bullet, y'_\bullet)}{\tilde{h}_X(x, y_\bullet)}.$$

It is obvious that  $p_\omega(P, Q)$  belongs to  $\mathfrak{h}_{X,P}R_{X,P}$  as a function of  $P$  at  $P \in X$  and thus Proposition 9, whose origin is Dedekind’s different theorem (Proposition 4.27) in [17], shows the proposition:

**Proposition 11.**  $e_{B_i} - 1 = -\text{deg}_{P=B_i}(p_\omega(P, Q))$  for  $Q \in X \setminus \{\infty\}$ .

Further, the direct computations provide the following proposition:

**Proposition 12.** For  $Q \in X \setminus \{\infty\}$ ,

$$p_\omega(P, Q) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \neq Q \text{ and } \omega_x(P) = \omega_x(Q), \end{cases}$$

$$\lim_{P \rightarrow \infty} p_\omega(P, Q) = \frac{1}{r}, \quad \lim_{Q \rightarrow \infty} p_\omega(P, Q) = \infty.$$

**Remark 2.** We should remark that  $p(P, Q) \in R_X \otimes_{\mathbb{C}} R_X$ , which holds the relations in Proposition 12 is not unique. The problem comes from the fact that there are infinitely many different  $p'(P, Q)$  from  $p(P, Q)$  such that  $p(P, Q) = p'(P, Q)$  for  $\omega_r(P) = \omega_r(Q)$ .

**Proposition 13.** For a point  $Q \in X \setminus \mathfrak{B}_X$ ,

$$\text{div}_P(p_\omega(P, Q)) = \sum_{\zeta \in G_{X, \omega_x(Q)}, \zeta \neq e} \zeta Q + E_Q - \sum_{i=1}^{\ell_{\mathfrak{B}}} (e_{B_i} - 1) B_i,$$

where  $E_Q$  is an effective divisor such that  $\text{deg } E_Q = \left( \sum_{i=1}^{\ell_{\mathfrak{B}}} (e_{B_i} - 1) - r + 1 \right) = 2g$ .

Further, this relation is extended to the condition  $Q \in \mathfrak{B}_X \setminus \{\infty\}$  by considering the multiplicity of the action  $G_{\omega_r(Q)}$ .

**Proof.** At the ramification point  $B_i$  of  $\omega_x : X \rightarrow \mathbb{P}$ , Proposition 11 shows the third term. We note that  $p_\omega(\infty, Q) = 1/r$ . From the Riemann–Hurwitz theorem (16), there exist the first and the second terms.  $\square$

For an element  $f \otimes f'$  in  $R_X \otimes_{\mathbb{C}} R_X$ , we define the weight  $\text{wt}(f \otimes f')$  by  $\text{wt}(f) + \text{wt}(f')$ .

**Lemma 17.**  $\tilde{h}_X(x, y_\bullet, x', y'_\bullet)$  is a homogeneous element in  $R_X \otimes_{\mathbb{C}} R_X$  whose weight is  $d_h$  or  $-\text{wt } \tilde{h}_X(x, y_\bullet, x', y'_\bullet) = -\text{wt } h_X(x, y_\bullet) = -d_h$  and thus, the weight of  $p_\omega$  is zero in  $\mathcal{Q}(R_X \otimes_{\mathbb{C}} R_X)$ .

For later convenience, we introduce  $\varphi_\infty(R_X \otimes_{\mathbb{C}} R_X)$ .

**Definition 13.** We define a polynomial in  $R_H^Z \otimes_{\mathbb{C}} R_H^Z$  such that

$$\tilde{h}_H(Z, Z') := \varphi_H^X(\tilde{h}_X(x, y_{\bullet}, x', y'_{\bullet})) =: \sum_{i=0}^{r-1} Z_r^{\delta_i} \mathfrak{z}_{\epsilon_i}^* \mathfrak{z}'_{\epsilon_i},$$

and an element,

$$p_H(Z, Z') := \frac{\tilde{h}_H(Z, Z')}{h_H(Z_r, Z_{\bullet})} \in \mathcal{Q}(R_H^Z \otimes_{\mathbb{C}} R_H^Z).$$

This  $\tilde{h}_H(Z, Z')$  is uniquely defined in the meaning of Proposition 15 4. and 16 4.

**Proposition 14.** When  $Z_r = Z'_r$ ,  $\tilde{h}_H(Z, Z')$  and  $p_H(Z, Z')$  agree with  $\tilde{h}_H(Z_r, Z_{\bullet}, Z'_r)$  in Lemma 9 and  $p_H(Z_r, Z_{\bullet}, Z'_r)$  in Lemma 6, respectively,  $\varphi_H^Z(\tilde{h}_H(Z, Z')) = z^{d_h} \sum_{i \in \mathbb{Z}_r} \frac{z'^{\epsilon_i}}{z^{\epsilon_i}}$ .

### 3.3. W-Normalized Differentials of the Second and the Third Kinds

#### 3.3.1. The One-Form $\Sigma$ on $X$

We construct an algebraic representation of the fundamental W-normalized differential of the second kind in (Corollary 2.6) in [34], namely, a two-form  $\Omega(P_1, P_2)$  on  $X \times X$ , which is symmetric and has quadratic singularity as in Theorem 3.

Following K. Weierstrass [3], H. F. Baker [10], V.M. Buchstaber, D.V. Leykin and V.Z. Enolskii [ref. [12] and therein], J.C. Eilbeck, V.Z. Enolskii, and D.V. Leykin [11], we have  $\Omega$  using a meromorphic one-form  $\Sigma(P, Q)$  on  $X \times X$  for the hyperelliptic curves and plane Weierstrass curves (W-curves). In this subsection, we extend it to more general W-curves based on Definition 12 and Proposition 12 to introduce  $\Sigma(P, Q)$  on  $X \times X$ .

**Proposition 15.** For a point  $(P, Q) \in X \times X$ ,

$$\Sigma(P, Q) := \frac{dx_P}{(x_P - x_Q)} p_{\omega}(P, Q) = \frac{dx_P}{(x_P - x_Q)} \frac{\tilde{h}_X(x_P, y_{\bullet P}, y_{\bullet Q})}{h_X(x_P, y_{\bullet P})} \tag{20}$$

has the following properties:

1. For a group action  $\tilde{\zeta} \in G_{\omega_r(P)}$ ,  $\Sigma(\tilde{\zeta}P, \tilde{\zeta}Q) = \Sigma(P, Q)$  if  $\omega_r(P) = \omega_r(Q)$ .
2.  $\Sigma(P, Q)$  is holomorphic over  $X$  except  $Q$  and  $\infty$  as a function of  $P$  such that
  - (a) at  $Q$ , in terms of the local parameters  $t_Q(P) = 0$  and  $t_P(P) = 0$ , it behaves like

$$\Sigma(P, Q) = \frac{dt_P}{t_P} (1 + d_{>0}(t_P, t_Q)),$$

- (b) at  $\infty$ , the local parameter  $t_P$  ( $t_P(\infty) = 0$ ), it behaves like

$$\Sigma(P, Q) = -\frac{dt_P}{t_P} (1 + d_{>0}(t_P)).$$

3.  $\Sigma(P, Q)$  as a function of  $Q$  is singular at  $P$  and  $\infty$  such that
  - (a) at  $P$ , in terms of the local parameters  $t_Q(P) = 0$  and  $t_P(P) = 0$ , it behaves like

$$\Sigma(P, Q) = -\frac{dt_P}{t_Q} (1 + d_{>0}(t_P, t_Q)),$$

- (b) at  $\infty$ , the arithmetic local parameter  $t_Q$ , ( $t_Q(\infty) = 0$ ), it behaves like

$$\Sigma(P, Q) = \frac{dx_P}{\eta_{\epsilon_{r-1}P} t_Q^{\epsilon_{r-1}-r}} (1 + d_{>0}(t_Q)) = \frac{dx_P}{\eta_{\epsilon_{r-1}P} t_Q^{\epsilon_X-1}} (1 + d_{>0}(t_Q)).$$



4. Let  $\tilde{h}'_X(x_P, y_{\bullet P}, y_{\bullet Q})$  be an element in  $R_X \otimes_{\mathbb{C}} R_X$  satisfying the conditions in Lemma 16, i.e.,  $\tilde{h}'_X(x_P, y_{\bullet P}, y_{\bullet Q}) = \tilde{h}_X(x_P, y_{\bullet P}, y_{\bullet Q})$  at  $\omega_x(P) = \omega_x(Q)$ , and let

$$\Sigma'(P, Q) := \frac{dx_P}{(x_P - x_Q)} \frac{\tilde{h}'_X(x_P, y_{\bullet P}, y_{\bullet Q})}{h'_X(x_P, y_{\bullet P})}.$$

Then  $\Sigma'(P, Q) - \Sigma(P, Q)$  belongs to the set,

$$\mathbf{H}^0(X, \mathcal{A}_X) \otimes_{\mathbb{C}} \mathbf{H}^0(X, \mathcal{O}_X(*\infty)) = \mathbf{H}^0(X, \mathcal{A}_X) \otimes_{\mathbb{C}} R_X.$$

**Proof.** Lemma 16 1 shows 1. Noting the properties in Proposition 13 and Corollary 1, the numerator of  $\Sigma$  is zero with the first order at the points where  $P \neq Q$  and  $P \in \omega_x^{-1}(\omega_x(Q))$  and thus,  $\Sigma$  behaves in a finite one-form there. At  $P = Q$ , the numerator is equal to one, and thus, we have  $\Sigma = dx_P/(x_P - x_Q)$ , which means  $\Sigma = dt_P/t_P(1 + d_{>0}(t_P))$  in 2.(a) and 3.(a).

Recalling  $\varphi_{\infty} : R_X \rightarrow R_H$  in Lemma 8, let  $t_P$  and  $t_Q$  be the local parameters at  $\infty$  corresponding to  $P$  and  $Q$ , respectively. Proposition 12 shows  $\lim_{P \rightarrow \infty} p_{\omega}(P, Q) = \frac{1}{r}$  whereas  $\lim_{P \rightarrow \infty} \frac{dx_P}{(x_P - x_Q)} = -r(1 + d_{>0}(t_P))dt_P$ , and thus, we have 2.(b). On the other hand, the fol-

lowing Lemma 18-3 shows that  $\Sigma(P, Q)$  behaves  $\sum_{k=1}^{\infty} \sum_{i=0}^{r-1} \frac{t_P^{\epsilon_i - rk}}{t_P^{\epsilon_i - rk}} \frac{dt_P}{t_P} (1 + d_{>0}(t_P))$ . The maximum of  $\{\epsilon_i - r\}_{i \in \mathbb{Z}_r}$  is  $\epsilon_{r-1} - r$ , which is equal to  $c_X - 1$  due to Lemma 15-4, and thus  $\varphi_{\infty}\left(\frac{dx}{\eta_{\epsilon_{r-1}}}\right) = \frac{t_P^{\epsilon_{r-1}}}{rt_P^{r+1}} dt_P = \frac{t_P^{c_X - 2}}{r} dt_P$ . We obtain 3.b.

Let us consider 4. Since both  $\Sigma'(P, Q)$  and  $\Sigma(P, Q)$  satisfy these properties 1–3, their difference is holomorphic over  $X$  with respect to  $P$ , and over  $X \setminus \{\infty\}$  with respect to  $Q$ . It shows 4.  $\square$

### 3.3.2. The One-Form $\Sigma$ at $\infty$

Noting Lemma 8, we consider a derivation in  $\mathcal{Q}(R_H^Z \otimes R_H^Z)$  for the monomial curve  $X_H \times X_H$  instead of  $R_{X,\infty}$  using surjection  $\varphi_{\infty}$ .

In order to investigate the behavior of  $\Sigma$  at  $\infty$ , we consider the differential in monomial curve  $X_H \times X_H$  and element in  $\mathcal{Q}(R_H^Z \otimes_{\mathbb{C}} R_H^Z)$  (rather than  $\mathcal{Q}(R_H^Z \otimes_{\mathbb{C}[Z_r]} R_H^Z)$ ),

$$\frac{dZ_r}{(Z_r - Z'_r)} p_H(Z_r, Z_{\bullet}, Z'_r). \tag{21}$$

Noting  $p_H(Z_r, Z_{\bullet}, Z'_r) = \frac{\tilde{h}_H(Z_r, Z_{\bullet}, Z'_r)}{\tilde{h}_H(Z_r, Z_{\bullet}, Z_r)}$  and Lemmas 8, 6, and 17, we define

$$\Sigma_H(t_P, t_Q) := -\frac{t_Q^r}{t_Q^r - t_P^r} \sum_{i=0}^{r-1} \frac{t_P^{\epsilon_i}}{t_Q^{\epsilon_i}} \frac{dt_P}{t_P} \tag{22}$$

as an element in  $\mathcal{Q}(R_H \otimes_{\mathbb{C}} R_H)$  using isomorphism  $\varphi_H^Z$  and the parameters  $t_P = 1/z_P$  and  $t_Q = 1/z_Q$ ;  $\Sigma_H$  is regarded as a derivation in  $\mathcal{Q}(R_{X,\infty} \otimes_{\mathbb{C}} R_{X,\infty})$ .

Then, the direct computations lead to the following results:

**Lemma 18.** 1.  $\varphi_{\infty}(\Sigma(P, Q)) = \Sigma_H(t_P, t_Q)$  with the local parameters  $t_P$  and  $t_Q$  at  $\infty$ , and it shows the behavior at  $\infty$ , i.e.,

$$\Sigma(P, Q) = \Sigma_H(t_P, t_Q)(1 + d_{>0}(t_P, t_Q))$$

for the local parameters at  $\infty$ ,  $t_P(\infty) = t_Q(\infty) = 0$ .

2. for the case  $|t_P| < |t_Q|$ ,  $\Sigma_H(t_P, t_Q) = - \sum_{k=0}^{\infty} \sum_{i=0}^{r-1} \frac{t_P^{e_i+rk}}{t_Q^{e_i+rk}} \frac{dt_P}{t_P}$ , and
3. for the case  $|t_Q| < |t_P|$ ,  $\Sigma_H(t_P, t_Q) = \sum_{k=1}^{\infty} \sum_{i=0}^{r-1} \frac{t_P^{e_i-rk}}{t_Q^{e_i-rk}} \frac{dt_P}{t_P}$ , and  $|e_i - kr| \neq 0, k > 0$ .

**Proof.** 1 is obvious. The relations  $\frac{t_Q^r}{t_Q^r - t_P^r} = \frac{1}{1 - (t_P/t_Q)^r} = \sum_{k=0}^{\infty} \left(\frac{t_P}{t_Q}\right)^{rk}$ , and  $\frac{t_Q^r}{t_Q^r - t_P^r} = - \frac{t_Q^r}{t_P^r} \frac{1}{1 - (t_Q/t_P)^r} = - \frac{t_Q^r}{t_P^r} \sum_{k=0}^{\infty} \left(\frac{t_Q}{t_P}\right)^{rk}$  show 2 and 3.  $\square$

### 3.3.3. The differential $d\Sigma$ on $X$

In order to define  $\Omega$ , we consider the derivative  $d\Sigma$  in this subsection.

#### Proposition 16.

$$d_Q \Sigma(P, Q) := dx_Q \frac{\partial}{\partial x_Q} \Sigma(P, Q) \tag{23}$$

has the following properties:

1.  $d_Q \Sigma(P, Q)$  as a function of  $P$  is holomorphic over  $X$  except  $Q$  such that
  - (a) at  $Q$ , in terms of the local parameter  $t_P(Q) = 0$ , it behaves like

$$d_Q \Sigma(P, Q) = \frac{dt_P dt_Q}{t_P^2} (1 + d_{>0}(t_P)),$$

- (b) at  $\infty$ , in terms of the arithmetic parameters  $t_P$  and  $t_Q$ , it behaves like

$$d_Q \Sigma(P, Q) = r \frac{t_P^{r-1}}{t_Q^{r+1}} dt_P dt_Q (1 + d_{>0}(t_P, t_Q)),$$

and  $d_Q \Sigma(P, Q)$  is holomorphic at  $\infty$ .

2.  $d_Q \Sigma(P, Q)$  as a function of  $Q$  is holomorphic over  $X$  except  $P$  and  $\infty$  such that
  - (a) at  $P$ , in terms of the local parameter  $t_Q(P) = 0$ , it behaves like

$$d_Q \Sigma(P, Q) = \frac{dt_P dt_Q}{t_Q^2} (1 + d_{>0}(t_Q)),$$

- (b) at  $\infty$ , using the arithmetic parameter  $t_Q(\infty) = 0$ , it behaves like

$$d_Q \Sigma(P, Q) = \frac{1}{t_Q^{c_X}} dx_P dt_Q (1 + d_{>0}(t_Q)).$$

3. We have the following relations,

- (a)  $d_Q \Sigma(P, Q) - d_P \Sigma(Q, P) \in \mathbf{H}^0(X, \mathcal{A}_X(*\infty))^{2\otimes \mathbb{C}}$ ,

- (b)  $\oint_{Q=\infty} \Sigma(P, Q) = 0$ ,

- (c)  $\oint_{\infty} \left( \int^Q d_{Q'} \Sigma(P, Q) \right) = \oint_{P=\infty} \Sigma(P, Q) = 2\pi\sqrt{-1}$ , and

4. For  $\Sigma'(P, Q)$  defined in Proposition 15 4, we have

$$d_Q \Sigma'(P, Q) - d_Q \Sigma(P, Q) \in \mathbf{H}^0(X, \mathcal{A}_X) \otimes_{\mathbb{C}} d\mathbf{H}^0(X, \mathcal{O}_X(*\infty)) = \mathbf{H}^0(X, \mathcal{A}_X) \otimes_{\mathbb{C}} dR_X.$$

**Proof.** 1.(a), 2.(a), and finite part of 3.(a) are directly obtained from Proposition 15. The properties of  $d\Sigma$  on  $X_H \times X_H$  determine the behavior at  $\infty$  in the following subsection, Section 3.3.4.

We use the facts:  $dx_Q \frac{\partial}{\partial x_Q} = dt_Q \frac{\partial}{\partial t_Q}$ , and  $dt_Q \frac{\partial}{\partial t_Q} t_Q^\ell = \ell t_Q^{\ell-1} dt_Q$ . Lemmas 19 and 18 show 1.(b), 2.(b) and the behavior at  $\infty$  of 3.(a). Lemma 18 3 shows 3.(b).

Lemma 23 shows 3.(c), and Proposition 15 4 shows 4.  $\square$

It means that there exist  $F_{d\Sigma}(P, Q) \in R_X \otimes_{\mathbb{C}} R_X$  such that

$$d_Q \Sigma(P, Q) = \frac{F_{d\Sigma}(P, Q) dx_P \otimes dx_Q}{(x_P - x_Q)^2 h_X(x_P, y_{\bullet P}) h_X(x_Q, y_{\bullet Q})}. \tag{24}$$

### 3.3.4. The Differential $d\Sigma_H$ on $X_H$

Noting Lemma 8, we also consider  $d\Sigma$  as a two-form in  $\mathcal{Q}(R_H^Z \otimes_{\mathbb{C}} R_H^Z)$  and the monomial curve  $X_H \times X_H$  instead of  $R_{X, \infty}$ .

We consider the differential of (21),

$$d\Sigma_H = dZ'_r \frac{\partial}{\partial Z'_r} \frac{dZ_r}{(Z_r - Z'_r)} \frac{\tilde{h}_H(Z_r, Z_{\bullet}, Z'_r)}{\tilde{h}_H(Z_r, Z_{\bullet}, Z_{\bullet})}$$

which is equal to

$$\frac{dZ_r}{(Z_r - Z'_r)^2} \frac{\tilde{h}_H(Z_r, Z_{\bullet}, Z'_r)}{\tilde{h}_H(Z_r, Z_{\bullet}, Z_{\bullet})} + \frac{dZ_r}{(Z_r - Z'_r) \tilde{h}_H(Z_r, Z_{\bullet}, Z_{\bullet})} \frac{\partial}{\partial Z'_r} \tilde{h}_H(Z_r, Z_{\bullet}, Z'_r).$$

We recall the correspondence between  $z$  of  $R_H$  and  $t$  of  $R_{X, \infty}$  in Lemma 8. Using (22), we have

$$d_Q \Sigma_H(t_P, t_Q) = -dt_Q \frac{\partial}{\partial t_Q} \frac{t_Q^r}{t_Q^r - t_P^r} \sum_{i=0}^{r-1} \frac{t_P^{\epsilon_i}}{t_Q^{\epsilon_i}} \frac{dt_P}{t_P}.$$

**Lemma 19. 1.** We have the equality:

$$d_Q \Sigma_H(t_P, t_Q) = \frac{dt_P dt_Q}{(t_Q^r - t_P^r)^2 t_P t_Q} \left[ \sum_{i=0}^{r-1} \left( \epsilon_i \frac{t_P^{\epsilon_i}}{t_Q^{\epsilon_i - 2r}} - (\epsilon_i - r) \frac{t_P^{\epsilon_i + r}}{t_Q^{\epsilon_i - r}} \right) \right].$$

2. For the case  $|t_Q| < |t_P|$ , we have the expansion,

$$d_Q \Sigma_H(t_P, t_Q) = dt_P dt_Q \left[ \sum_{i=0}^{r-1} \sum_{k=1}^{\infty} \left( -(\epsilon_i - kr) \frac{t_P^{\epsilon_i - rk - 1}}{t_Q^{\epsilon_i - rk + 1}} \right) \right]. \tag{25}$$

3. For the case  $|t_P| < |t_Q|$ , we have the expansion,

$$d_Q \Sigma_H(t_P, t_Q) = dt_P dt_Q \left[ \sum_{i=0}^{r-1} \sum_{k=1}^{\infty} \left( (\epsilon_i + kr) \frac{t_P^{\epsilon_i + rk - 1}}{t_Q^{\epsilon_i + rk + 1}} \right) \right]. \tag{26}$$

**Proof. 1:** Using the relation,  $\frac{\partial}{\partial t_Q} \frac{t_Q^r}{t_Q^r - t_P^r} = \left( r - \frac{rt_Q^r}{t_Q^r - t_P^r} \right) \frac{t_Q^{r-1}}{t_Q^r - t_P^r} = -\frac{rt_Q^r t_P^r}{(t_Q^r - t_P^r)^2 t_Q^r}$ , we have

$$d_Q \Sigma_H(t_P, t_Q) = \left[ \left( \frac{rt_P^r}{t_Q^r - t_P^r} \right) \sum_{i=0}^{r-1} \frac{t_P^{\epsilon_i}}{t_Q^{\epsilon_i - r}} + \sum_{i=0}^{r-1} \epsilon_i \frac{t_P^{\epsilon_i}}{t_Q^{\epsilon_i - r}} \right] \frac{1}{t_Q^r - t_P^r} \frac{dt_P dt_Q}{t_P t_Q}.$$

The relation  $rt_P^r + \epsilon_i(t_Q^r - t_P^r) = \epsilon_i t_Q^r - (\epsilon_i - r)t_P^r$  enables us to obtain 1.

2: Since  $\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{(1-x)} = \sum_{k=1}^{\infty} kx^{k-1}$ , for a polynomial  $H(t_P, t_Q) \in \mathbb{C}[t_P, t_Q]$ , we have

$$\frac{H(t_P, t_Q)}{(t_Q^r - t_P^r)^2} = \frac{H(t_P, t_Q)}{t_P^{2r}} \sum_{k=1}^{\infty} k \frac{t_Q^{r(k-1)}}{t_P^{r(k-1)}} = \frac{H(t_P, t_Q)}{t_P^{2r}} \left( \sum_{k=1}^N k \frac{t_Q^{r(k-1)}}{t_P^{r(k-1)}} + \sum_{k=N+1}^{\infty} k \frac{t_Q^{r(k-1)}}{t_P^{r(k-1)}} \right).$$

and

$$d_Q \Sigma_H(t_P, t_Q) = dt_P dt_Q \left[ \sum_{i=0}^{r-1} \sum_{k=1}^{\infty} \left( k \epsilon_i \frac{t_P^{\epsilon_i - r(k+1) - 1}}{t_Q^{\epsilon_i - r(k+1) + 1}} - k(\epsilon_i - r) \frac{t_P^{\epsilon_i - rk - 1}}{t_Q^{\epsilon_i - rk + 1}} \right) \right].$$

The relation  $(k-1)\epsilon_i - k(\epsilon_i - r) = -(\epsilon_i - kr)$  shows 2. Similarly, we prove 3.  $\square$

**Lemma 20.**  $d_Q \Sigma_H(t_P, t_Q) - d_P \Sigma_H(t_Q, t_P)$  is equal to

$$dt_P dt_Q \sum_{i=0}^{r-1} \frac{\epsilon_i (t_P^{2\epsilon_i - r} t_Q^r - t_Q^{2\epsilon_i - r} t_P^r) - (\epsilon_i - r)(t_P^{2\epsilon_i} - t_Q^{2\epsilon_i})}{(t_Q^r - t_P^r) t_P^{\epsilon_i - r + 1} t_Q^{\epsilon_i - r + 1}}.$$

When  $t_P = t_Q + \epsilon$ , it vanishes for the limit  $\epsilon \rightarrow 0$ .

**Proof.** The direct computations show them.  $\square$

For the case  $|t_Q| < |t_P|$ , noting  $\varphi_{\infty}(h_X(P)) = r t_P^{-d_h}$ , and  $\varphi_{\infty}(dx_P) = r t_P^{-r-1} dt_P$ , (25) in Lemma 19 leads to the expression,

$$d_Q \Sigma_H(t_P, t_Q) = \frac{dt_P}{t_P^{r+1}} \frac{dt_Q}{t_Q^{r+1}} \left[ \sum_{i=0}^{r-1} \sum_{k=1}^{\infty} \left( -(\epsilon_i - kr) t_P^{d_h} t_Q^{\epsilon_i - d_h - r(k-1)} \frac{t_P^{\epsilon_i - d_h - r(k-1)}}{t_Q^{\epsilon_i + d_h - r(k+1)}} \right) \right]. \tag{27}$$

**Lemma 21.** Letting  $\widehat{H}^{(g)} := \{d_h - \epsilon_i + kr \mid i \in \mathbb{Z}_r, k \in \mathbb{N}_0, \epsilon_i - (k+1)r > 0\}$ , and  $\widehat{H}_X^{(g)*} := \{d_h + \epsilon_i - r(k+2) \mid d_h - \epsilon_i + rk \in \widehat{H}^{(g)}\}$ , we have

$$\widehat{H}_X^{(g)} \subset \widehat{H}_X, \quad \widehat{H}_X^{(g)*} \subset \widehat{H}_X, \quad \#\widehat{H}_X^{(g)} = g, \quad \#\widehat{H}_X^{(g)*} = g.$$

**Proof.**  $\widehat{H}^{(g)}$  corresponds to the weight of  $\widehat{\phi}_i$  consisting of  $\mathbf{H}^0(X, \mathcal{A}_X)$  and thus  $\widehat{H}_X^{(g)} \subset \widehat{H}_X$  and its cardinality is  $g$ . On the other hand, the condition in  $\widehat{H}^{(g)*}$  means that  $d_h + \epsilon_i - r(k+2) \geq (d_h - r + 1) + (\epsilon_i - r) > \widehat{c}_X + \mathfrak{k}_X$  because of  $\widehat{c}_X + \mathfrak{k}_X - 2 = d_h - r - 1$  from Lemma 15. Thus,  $\widehat{H}^{(g)*} \subset \widehat{H}_X$ , and  $\#\widehat{H}_X^{(g)*} = g$ .  $\square$

Since the order of the singularity of  $\Sigma_H(t_P, t_Q)$  at  $t_Q = \infty$  is  $c_X - 1 = \epsilon_{r-1} - r$ , we introduce the two-form,

$$\Omega_H^{(0)}(t_P, t_Q) := dt_P dt_Q \left[ \sum_{i=0}^{r-1} \sum_{\substack{k=1 \\ d_h - \epsilon_i + rk \in \widehat{H}^{(g)}}} \left( (\epsilon_i - kr) \frac{t_P^{\epsilon_i - rk - 1}}{t_Q^{\epsilon_i - rk + 1}} \right) \right].$$

**Lemma 22.**  $\Omega_H^{(1)}(t_P, t_Q) := d \Sigma_H(t_P, t_Q) - \Omega_H^{(0)}(t_P, t_Q)$  has no singularity at  $t_Q = \infty$ .

**Lemma 23.** Then each term in  $\Omega_H^{(0)}(t_P, t_Q)$  has the property,

$$\oint_{t_Q=0} \left[ \int_0^{t_Q} -(\epsilon_i - kr) \frac{t_P^{\epsilon_i - kr - 1}}{t_Q^{\epsilon_i - kr + 1}} dt_P \right] dt_Q = -2\pi\sqrt{-1}.$$

**Lemma 24.** For  $i \in \mathbb{Z}_r$  and  $k \in \mathbb{N}$  such that  $\widehat{H}^{(g)} \ni d_h - \epsilon_i + rk$ , there are  $\widehat{\phi}_j$  ( $j < g$ ) and  $\widehat{\phi}_{j'}$  ( $j' \geq g$ ) such that

$$\varphi_\infty \left( \frac{\widehat{\phi}_j dx}{h_X} \right) = -t^{d_h - (\epsilon_i + (k-1)r) - r - 1} dt (1 + d_{>0}(t)),$$

$$\varphi_\infty \left( \frac{\widehat{\phi}_{j'} dx}{h_X} \right) = -\frac{dt}{t^{\epsilon_i - kr + 1}} (1 + d_{>0}(t)) = -\frac{t^{d_h} dt}{t^{d_h + \epsilon_i - (k+1)r + (r+1)}} (1 + d_{>0}(t)).$$

**Proof.** Lemma 21 shows the existence.  $\square$

### 3.3.5. W-Normalized Differentials of the Second Kind

We introduce the W-normalized differentials of the second kind using this  $d_Q \Sigma(P, Q)$ .

**Definition 14.** We consider a sufficiently small closed contour  $C_\epsilon$  at  $\infty$ . Let  $D_\epsilon$  be the inner side of  $C_\epsilon = \partial D_\epsilon$ , including  $\infty$ , and  $\epsilon'$  be a point in  $D_\epsilon$  such that  $\epsilon' \neq \infty$ . For differentials  $v$  and  $v'$  in  $\mathbf{H}^0(X, \mathcal{A}(*\infty))$ , we define a pairing:

$$\langle v, v' \rangle := \lim_{\epsilon' \rightarrow \infty} \frac{1}{2\pi\sqrt{-1}} \oint_{C_\epsilon} \left( \int_{\epsilon'}^P v(Q) \right) v'(P).$$

The following is obtained from the primitive investigation of complex analysis on a compact Riemann surface [26].

**Lemma 25.**  $\langle v, v' \rangle = -\langle v', v \rangle$ , and  $\langle v, v' \rangle$  does not depend on  $\epsilon$ .

**Definition 15.** We define the pre-normalized differentials  $\widehat{v}_i^\Pi \in \mathbf{H}^0(X, \mathcal{A}_X(*\infty))$  of the second kind ( $i = 1, 2, \dots, g$ ), which satisfies the relations (if they exist)

$$\langle v_i^I, \widehat{v}_j^\Pi \rangle = \delta_{ij}, \quad \langle \widehat{v}_i^\Pi, \widehat{v}_j^\Pi \rangle = 0. \tag{28}$$

It is obvious that from Lemma 13 2 and Lemma 24,  $\widehat{v}_j^\Pi$  with  $j'$  in Lemma 24 expressed

as  $\widehat{v}_j^\Pi = \frac{\sum_{i=0}^{j'} a_{j,i} \widehat{\phi}_i dx}{h_X(x, y_\bullet)}$ .

Noting Proposition 16 3.(b), we have the following relations:

**Corollary 2.**  $\widehat{v}_g^\Pi = \frac{\widehat{\phi}_g}{h_X(x, y_\bullet)} dx$ ,  $\text{wt } \widehat{v}_g^\Pi = -2$ , and  $\text{wt } \widehat{v}_1^\Pi = -c_X$ .

**Proof.**  $-\text{wt}(\widehat{\phi}_g) + \text{wt}(\widehat{\phi}_{g-1}) = 2$  and thus

$$\frac{1}{2\pi\sqrt{-1}} \oint \left( \int^{(x,y_\bullet)} \frac{\widehat{\phi}_{g-1}}{h_X(x', y'_\bullet)} dx' \right) \frac{\widehat{\phi}_g}{h_X(x, y_\bullet)} dx = 2\pi\sqrt{-1}.$$

On the other hand,  $-\text{wt}(\widehat{\phi}_0) = d_h - r + 1 - c_X$  in Lemma 15 means

$$\text{wt } v_1^I = \text{wt} \left( \frac{\widehat{\phi}_0 dx}{h_X(x, y_\bullet)} \right) = c_X - 2.$$

Hence,  $\text{wt } \widehat{v}_1^{\text{II}} = -c_X$  and we obtain

$$\widehat{v}_1^{\text{II}} = \frac{\widehat{\phi}_0^* dx}{h_X}, \quad \widehat{\phi}_0^* = \frac{1}{t^{d_h+c_X-r+1}} (1 + d_{>0}(t)), \quad dx = \frac{dt}{t^{r+1}} (1 + d_{>0}(t)).$$

□

**Theorem 1.** *There exist the pre-normalized differentials  $v_j^{\text{II}}$  ( $j = 1, 2, \dots, g$ ) of the second kind such that they have a simple pole at  $\infty$  and satisfy the relation,*

$$d_Q \Sigma(P, Q) - d_P \Sigma(Q, P) = \sum_{i=1}^g \left( v_i^I(Q) \otimes v_i^{\text{II}}(P) - v_i^I(P) \otimes v_i^{\text{II}}(Q) \right), \tag{29}$$

where the set of differentials  $\{v_1^{\text{II}}, v_2^{\text{II}}, v_3^{\text{II}}, \dots, v_g^{\text{II}}\}$  is determined modulo the linear space spanned by  $\langle v_j^I \rangle_{j=1, \dots, g}$  and  $dR_X$ .

We call these  $v_i^{\text{II}}$ 's W-normalized differentials of the second kind (In Ônishi's articles and our previous articles [ref. [4,13,14,32,35] and references therein], the definition of W-normalized differentials  $v^{\text{II}}$  of the second kind differs from this definition by its sign. The difference is not significant but has an effect on the Legendre relation in Proposition 18 and the sign of the quadratic form in the definition of the sigma function in Definition 21).

**Proof.** Noting Proposition 17, Lemmas 22 and 23 show the facts. □

**Theorem 2.**

$$(\mathbf{H}^0(X, \mathcal{A}_X(*\infty)) / d\mathbf{H}^0(X, \mathcal{O}_X(*\infty))) = \bigoplus_{i=1}^g (\mathbb{C}v_i^I \oplus \mathbb{C}v_i^{\text{II}})$$

and thus,  $\dim_{\mathbb{C}}(\mathbf{H}^0(X, \mathcal{A}_X(*\infty)) / d\mathbf{H}^0(X, \mathcal{O}_X(*\infty))) = 2g$ .

**Proof.** It is obvious that  $d\mathbf{H}^0(X, \mathcal{O}_X(*\infty)) \subset \mathbf{H}^0(X, \mathcal{A}_X(*\infty))$ ; ( $j : dR_X \hookrightarrow \mathbf{H}^0(X, \mathcal{A}_X(*\infty))$ ). We recall Lemma 13 2 and 7.  $\mathbf{H}^0(X, \mathcal{A}_X(*\infty)) \setminus \mathbf{H}^0(X, \mathcal{A}_X)$  is the set of differentials of the second kind. In other words, there is no differential with the first-order singularity at  $\infty$ .

The embedding  $j$  is realized by  $j : d\phi_i \mapsto \frac{h_X d\phi_i}{h_X}$ . Let  $H_{dR_X} := \{\text{wt}(j(d\phi_i)) \mid i \in \mathbb{N}_0\}$ . Then  $H_{dR_X} \subset -\mathbb{N}$  and  $\#((-\mathbb{N}) \setminus (H_{dR_X} + 1)) = g$ , whereas  $\#H_X^c = g$ . Thus  $\#(\overline{H}_X^c \setminus H_{dR_X}) = 2g$ . Due to the Riemann–Roch theorem, for every  $\ell$  in  $H_X^c$ , there is an element  $k$  in  $(-\mathbb{N}) \setminus (H_{dR_X} + 1)$  such that  $\ell + k = 0$ . It shows the relations. □

**Lemma 26.**  $d_{P_2} \Sigma(P_1, P_2) + \sum_{i=1}^g v_i^I(P_1) \otimes v_i^{\text{II}}(P_2)$  is holomorphic over  $X \setminus \{P_2\}$  as a function of  $P_1$  and is holomorphic over  $X \setminus \{P_1\}$  as a function of  $P_2$ .

**Proof.** From Proposition 16,  $d_{P_2} \Sigma(P_1, P_2)$  is holomorphic over  $X \setminus \{P_2\}$  as a function of  $P_1$  whereas  $d_{P_2} \Sigma(P_1, P_2)$  is holomorphic over  $X \setminus (\{P_2\} \cup \{\infty\})$  as a function of  $P_2$ . The order of the singularity at  $Q = \infty$  is  $2g$  and thus can be canceled by  $v_i^{\text{II}}(P_2)$ . Since the numerator of  $d_{P_2} \Sigma(P_1, P_2)$  in (24) consists of the elements in  $R_X \otimes_{\mathbb{C}} R_X dx_1 \otimes dx_2$ , from Proposition 16, there is no term whose weight is  $-1$  in  $d_{P_2} \Sigma(P_1, P_2)$  as a function in  $Q$ . Noting the homogeneous property of  $\tilde{h}_X$  from Lemma 14, we have the result. □

**Theorem 3.** 1. The one-form,

$$v_{P_1, P_2}^{\text{III}}(P) := \Sigma(P, P_1) - \Sigma(P, P_2),$$

is a differential of the third kind whose only (first-order) poles are  $P = P_1$  and  $P = P_2$ , with residues  $+1$  and  $-1$ , respectively.

2. The symmetric two-form,

$$\Omega(P_1, P_2) := d_{P_2} \Sigma(P_1, P_2) + \sum_{i=1}^g v_i^{\text{I}}(P_1) \otimes v_i^{\text{II}}(P_2), \tag{30}$$

is the fundamental differential of the second kind, which has the properties:

- (a)  $\Omega(P, Q) = \Omega(Q, P)$ ,
- (b) for any  $\zeta \in G_{\omega_r(P)}$ ,  $\Omega(\zeta P, \zeta Q) = \Omega(P, Q)$  if  $\omega_r(P) = \omega_r(Q)$ ,
- (c)  $\Omega(P, Q)$  is holomorphic except  $Q$  as a function of  $P$  and behaves like

$$\Omega(P, Q) = \frac{dt_P dt_Q}{(t_P - t_Q)^2} (1 + d_{>0}(t_P, t_Q)),$$

(d)

$$\int_{P_2}^{P_1} v_{Q_1, Q_2}^{\text{III}} = \int_{P_2}^{P_1} \int_{Q_2}^{Q_1} \Omega(P, Q).$$

**Proof.** 1 is directly obtained by Proposition 15, and 2 is obvious from (29) and Proposition 16.  $\square$

We note the  $W$ -normalized differentials of the first kind and the second kind in Definitions 9 and 15.

**Definition 16.** 1.  $v_{P_1, P_2}^{\text{III}}$  is called the  $W$ -normalized differential of the third kind

2.  $\Omega(P, Q)$  is the  $W$ -normalized fundamental differential of the second kind and when it is expressed by

$$\Omega(P_1, P_2) = \frac{F_{\Omega}(P, Q) dx_P \otimes dx_Q}{(x_P - x_Q)^2 h_X(x_P, y_{\bullet P}) h_X(x_Q, y_{\bullet Q})}, \tag{31}$$

where  $F_{\Omega}$  is called Klein fundamental form in  $R_X \otimes_{\mathbb{C}} R_X$ .

**Lemma 27.** We have

$$\lim_{P_1 \rightarrow \infty} \frac{F_{\Omega}(P_1, P_2)}{\widehat{\phi}_{g-1}(P_1)(x_1 - x_2)^2} = \widehat{\phi}_g(P_2). \tag{32}$$

**Proof.** The expression (30) and Corollary 2 give the result.  $\square$

For the connection of these algebraic tools with the sigma functions, we define  $\Pi$

$$\begin{aligned} \Pi_{Q_1, Q_2}^{P_1, P_2} &:= \int_{P_2}^{P_1} \int_{Q_2}^{Q_1} \Omega(P, Q) \\ &= \int_{P_2}^{P_1} (\Sigma(P, Q_1) - \Sigma(P, Q_2)) + \sum_{i=1}^g \int_{P_2}^{P_1} v_i^{\text{I}}(P) \int_{Q_2}^{Q_1} v_i^{\text{II}}(P). \end{aligned} \tag{33}$$

It has the properties.

**Lemma 28.**  $\Pi_{Q_1, Q_2}^{P_1, P_2} = -\Pi_{Q_1, Q_2}^{P_2, P_1}$ ,  $\Pi_{Q_1, Q_2}^{P_1, P_2} = \Pi_{P_1, P_2}^{Q_1, Q_2}$ .

We have the generalized Legendre relation given as follows. (The sign of this relation depends on the sign of the  $W$ -normalized Abelian differentials of the second kind because

there are variant definitions). We introduce the homological basis  $\{\alpha_i, \beta_i\}_{i=1, \dots, g}$  of  $H_0(X, \mathbb{Z})$  satisfying

$$\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0, \quad \langle \alpha_i, \beta_j \rangle = \delta_{ij} \quad (i, j = 1, 2, \dots, g). \tag{34}$$

**Definition 17.** We define the complete Abelian integrals of the first kind and the second kind by

$$\omega'_{ij} := \int_{\alpha_j} v_i^I, \quad \omega''_{ij} := \int_{\beta_j} v_i^I, \quad \eta'_{ij} := \int_{\alpha_j} v_i^{II}, \quad \eta''_{ij} := \int_{\beta_j} v_i^{II},$$

and the Jacobian (Jacobi variety) by  $J_X := \mathbb{C}^g / \Gamma_X$  with  $\kappa_j : \mathbb{C}^g \rightarrow J_X$  and  $\Gamma_X := \langle \omega', \omega'' \rangle_{\mathbb{Z}}$ .

**Proposition 17.** If by using  $\Sigma'(P, Q)$  defined in Proposition 15 4, we define  $\{v_i^{III}\}_{i=1, \dots, g}$ , these  $v_i^{II}$  and  $v_i^{III}$  give the same period matrices in Definition 17.

**Proof.** From Proposition 16 4 and Theorem 1, it is obvious.  $\square$

Let  $v_{Q_1, Q_2}^{III\circ}$  be the normalized Abelian differential of the third kind, i.e.,  $\oint_{\alpha_i} v_{Q_1, Q_2}^{III\circ} = 0$  [26]. The following lemma corresponds to Corollary 2.6 (ii) in [34].

**Lemma 29.**

$$\Pi_{Q_1, Q_2}^{P_1, P_2} = \int_{P_2}^{P_1} v_{Q_1, Q_2}^{III\circ} - \sum_{i=1}^g \sum_{j=1}^g \gamma_{i,j} \int_{P_2}^{P_1} v_i^I \int_{Q_2}^{Q_1} v_j^I,$$

where  $\gamma = \eta' \omega'^{-1}$  and  ${}^t\gamma = \gamma$ .

**Proof.** See Proposition 5.1 in [4].  $\square$

**Proposition 18.**

$$\omega'' {}^t\eta' - \omega' {}^t\eta'' = \frac{\pi}{2} \sqrt{-1} 1_g,$$

where  $1_g$  is the unit  $g \times g$  matrix.

The following matrix satisfies the generalized Legendre relation:

$$M := \begin{bmatrix} 2\omega' & 2\omega'' \\ 2\eta' & 2\eta'' \end{bmatrix}, \quad M \begin{bmatrix} & 1_g \\ -1_g & \end{bmatrix} {}^tM = \frac{2\pi}{\sqrt{-1}} \begin{bmatrix} & 1_g \\ -1_g & \end{bmatrix}. \tag{35}$$

**Proof.** It is the same as Proposition 5.1 in [4].  $\square$

From Definitions 14 and 15, we have the following corollary, which is the dual of the homological relations (34):

**Corollary 3.**  $\langle v_i^I, v_j^I \rangle = \langle v_i^{II}, v_j^{II} \rangle = 0, \langle v_i^I, v_j^{II} \rangle = \delta_{ij}$  for  $i, j = 1, 2, \dots, g$ .

The Galois action on the basis of the Homology  $H_1(X, \mathbb{Z})$  shows the actions of these period matrices  $(\omega', \omega'')$  geometrically:

**Lemma 30.** If  $X$  is the Galois covering on  $\mathbb{P}$ , for the Galois action  $\widehat{\zeta} \in G_X$ , i.e.,  $\widehat{\zeta} : X \rightarrow X$ , its associated element  $\rho_{\widehat{\zeta}}$  of  $\text{Sp}(2g, \mathbb{Z})$  acts on  $(\omega', \omega'')$  and  $(\eta', \eta'')$  by

$$\widehat{\zeta}(\omega', \omega'') = (\omega', \omega'') {}^t\rho_{\widehat{\zeta}}, \quad \widehat{\zeta}(\eta', \eta'') = (\eta', \eta'') {}^t\rho_{\widehat{\zeta}},$$

and the generalized Legendre relation (35) is invariant for the action.



**Proof.** Due to the definition of  $\text{Sp}(2g, \mathbb{Z})$ , we have  ${}^t\rho_{\tilde{\zeta}} \begin{pmatrix} & -1_g \\ 1_g & \end{pmatrix} {}^t\rho_{\tilde{\zeta}} = \begin{pmatrix} & -1_g \\ 1_g & \end{pmatrix}$  and thus, (35) is invariant.  $\square$

**4. Sigma Function for W-curves**

4.1. W-Normalized Shifted Abelian Integrals

Since the non-symmetric W-curves have the non-trivial  $R_X$ -module,  $\hat{R}_X$  in Definition 9, (and properties in Proposition 10), the Abel-Jacobi map [26] for  $\hat{R}_X$  naturally appears. Let  $\tilde{X}$  be the Abelian universal covering of  $X$ , which is constructed by the path space of  $\tilde{X}$ ;  $\kappa_X : \tilde{X} \rightarrow X$  with  $\iota_X : X \rightarrow \tilde{X}$ . Thus, recalling Definitions 8 and 10, we introduce the shifted Abelian integral  $\tilde{w}_s$  and the Abel-Jacobi map  $w_s$  as an extension of the W-normalized Abelian integral  $\tilde{w} : S^k \tilde{X} \rightarrow \mathbb{C}^g$ ,  $\left(\tilde{w}(\gamma_1, \dots, \gamma_k) = \sum_i \int_{\gamma_i} \nu^I\right)$ ,  $\tilde{w}^\circ := \frac{1}{2} \omega'^{-1} \tilde{w}$ , and the Abel-Jacobi map  $w : S^k X \rightarrow J_X$ ,  $\left(w(P_1, \dots, P_k) = \sum_i \int_{\infty}^{P_i} \nu^I\right)$ ,  $w^\circ := \frac{1}{2} \omega'^{-1} w$  as mentioned in [5]:

**Definition 18.** We define  $\tilde{w}_s$  and  $w_s$  by

$$\tilde{w}_s : S^k \tilde{X} \rightarrow \mathbb{C}^g, \quad \tilde{w}_s(\gamma_1, \dots, \gamma_k) := \left(\sum_{i=1}^k \tilde{w}(\gamma_i)\right) + \tilde{w}(\iota_X \mathfrak{R}_s) = \sum_{i=1}^k \int_{\gamma_i} \nu^I + \int_{\iota_X \mathfrak{R}_s} \nu^I,$$

$$w_s := \kappa_J \circ \tilde{w}_s \circ \iota_X : S^k X \rightarrow J_X,$$

and  $\tilde{w}_s^\circ$  and  $w_s^\circ$  by

$$\tilde{w}_s^\circ := \frac{1}{2} \omega'^{-1} \tilde{w}_s : S^k \tilde{X} \rightarrow \mathbb{C}^g, \quad w_s^\circ := \frac{1}{2} \omega'^{-1} w_s : S^k X \rightarrow J_X^\circ := \mathbb{C}^g / \Gamma_X^\circ, \quad \Gamma_X^\circ := \omega'^{-1} \Gamma_X.$$

For symmetric numerical semigroup case  $H_X$ ,  $\tilde{w}_s = \tilde{w}$ ,  $\tilde{w}_s^\circ = \tilde{w}^\circ$ ,  $w_s = w$ , and  $w_s^\circ = w^\circ$ .

For given a divisor  $D$ , let  $|D|$  be the set of effective divisors linearly equivalent to  $D$  and identified with  $\mathbf{H}^0(X, \mathcal{O}_X(D))$ .

**Definition 19.** We define the Wirtinger varieties by

$$W_X^k := w(S^k X), \quad W_{X,s}^k := w_s(S^k X) \subset J_X, \quad W_X^{ok} := w^\circ(S^k X), \quad W_{X,s}^{ok} := w_s^\circ(S^k X) \subset J_X^\circ,$$

and their strata,

$$W_X^{k,1} := w(S_1^k X), \quad W_{X,s}^{k,1} := w_s(S_1^k X), \quad W_X^{k,1} := w^\circ(S_1^k X), \quad W_{X,s}^{k,1} := w_s^\circ(S_1^k X),$$

where  $S_m^n X := \{D \in S^n X \mid \dim |D| \geq m\}$ .

We would encounter several results that are obtained via the embedding  $\iota_X : S^k X \rightarrow S^k \tilde{X}$ . For such cases, we sometimes omit  $\iota_X$  for maps  $\tilde{w} \circ \iota_X$  and  $\tilde{w}_s \circ \iota_X$ , e.g., for  $\tilde{w} \circ \iota_X : S^k X \rightarrow \mathbb{C}^g$ , we simply write  $\tilde{w}(P_1, P_2, \dots, P_k)$  rather than  $\tilde{w}(\iota_X(P_1, P_2, \dots, P_k))$ .

4.2. Riemann Theta Function of W-Curves

The Riemann theta function, analytic in both variables  $z \in \mathbb{C}^g$  and  $\tau := \omega'^{-1} \omega''$ , is defined by

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp\left(2\pi\sqrt{-1}\left({}^t n z + \frac{1}{2} {}^t n \tau n\right)\right). \tag{36}$$

(For a given  $W$ -curve  $X$ , we simply write it as  $\theta(z) = \theta(z, \tau)$  by assuming that the Homology basis is implicitly fixed.) By letting  $\Gamma_X^\circ := \langle 1_g, \tau \rangle_{\mathbb{Z}}$  and  $J_X^\circ := \mathbb{C}^g / \Gamma_X^\circ$ , the zero-divisor of  $\theta$  modulo  $\Gamma_X^\circ$  is denoted by  $\Theta_X^\circ := \kappa_J \operatorname{div}(\theta) \subset J_X^\circ$ .

The  $\theta$  function with characteristic  $\delta', \delta'' \in \mathbb{R}^g$  is defined as: (there is another definition, e.g., in [7,8], in which  $\delta''$  and  $\delta'$  are exchanged in our definition)

$$\theta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left[ \pi \sqrt{-1} \left\{ {}^t(n + \delta'') \tau (n + \delta'') + 2 {}^t(n + \delta'') (z + \delta') \right\} \right]. \tag{37}$$

We also basically write it as  $\theta[\delta](z) = \theta[\delta](z, \tau)$ .

The shifted Abelian integral  $\tilde{w}_s^\circ$  and the Abel-Jacobi map  $w_s^\circ$  in Definition 18 lead to the shifted Riemann constant [5]:

**Proposition 19.** 1. If  $\epsilon_X$  in Definition 8 is not zero or  $X$  is not symmetric, the Riemann constant  $\zeta_X$  is not a half period of  $\Gamma_X^\circ$ .

2. The shifted Riemann constant  $\zeta_{X,s} := \zeta_X - \tilde{w}_s^\circ(\iota_X \mathfrak{R}_s)$  for every  $W$ -curve  $X$  is the half period of  $\Gamma_X^\circ$ .

3. By using the shifted Abel-Jacobi map, we have

$$\Theta_X^\circ = w_s^\circ(S^{g-1}X) + \zeta_{X,s} \quad \text{modulo } \Gamma_X^\circ,$$

i.e., for  $P_i \in X$ ,  $\theta(\tilde{w}_s^\circ \circ \iota_X(P_1, \dots, P_{g-1}) + \zeta_{X,s}) = 0$  for every  $W$ -curve  $X$ .

4. The following holds

$$\Theta_X^\circ = w_s^\circ(S^{g-1}X) + \zeta_{X,s} = w^\circ(S^{g-1}X) + \zeta_X \quad \text{modulo } \Gamma_X^\circ,$$

5. There is a  $\theta$ -characteristic  $\delta_X$  of a half period, which represents the shifted Riemann constant  $\zeta_{X,s}$ , i.e.,  $\theta[\delta_X] \left( \frac{1}{2} \omega'^{-1} \tilde{w}_s^\circ \circ \iota_X(P_1, \dots, P_{g-1}) \right) = 0$ , i.e.,  $\delta_X = \begin{bmatrix} \delta''_X \\ \delta'_X \end{bmatrix}$ ,  $\zeta_{X,s} \equiv \delta'_X + \tau_X \delta''_X$  modulo  $\Gamma_X^\circ$ .

The following comes from the investigation of the truncated Young diagram and the Schur polynomials in [14]; though we did not consider the Young diagram associated to the plane curve in the paper [14], the investigation is easily generalized to general Young diagrams associated with any numerical semigroups (c.f. Lemma 1). Thus, we state the facts without proofs.

**Proposition 20.** For the Young diagram  $\Lambda_X$  associated with the numerical semigroup  $H_X$  of genus  $g$ , an integer  $k$  ( $0 \leq k < g$ ), and the characteristics of the partition of  $\Lambda_X^{[k]} = (\Lambda_{k+1}, \Lambda_{k+2}, \dots, \Lambda_g) = (a_{n_k}, \dots, a_2, a_1; b_{n_k}, \dots, b_2, b_1)$ , ( $n_k := r_{\Lambda_X^{[k]}}$ , the rank of  $r_{\Lambda_X^{[k]}}$ ), the following holds:

1.  $N^c(g - k - i) - N(i - 1)$  ( $i = 1, 2, \dots, n_k$ ) is an element of the gap sequence  $H_X^c$ , thus let  $N^c(L_i^{[k]}) := N^c(g - k - i) - N(i - 1)$ , and then we have

$$\Lambda_{g-L_i^{[k]}} + g - L_i^{[k]} = a_i + b_i + 1$$

for every  $i = 1, \dots, n_k$ , and

2.  $L_1^{[k]} = k + 1$ .

**Definition 20.** Let  $\operatorname{Index}(g, \ell) := \{1, \dots, g\}^\ell$ . We define the sequences  $\mathfrak{h}_{X,k}$  (simply  $\mathfrak{h}_k$ ) and  $\mathfrak{h}_{X,k}^{(i)}$  (simply  $\mathfrak{h}_k^{(i)}$ ) as elements in  $\operatorname{Index}(g, n_k)$  given by

$$\mathfrak{h}_k = \mathfrak{h}_{X,k} := \{L_1^{[k]}, L_2^{[k]}, \dots, L_{n_k}^{[k]}\}, \quad \mathfrak{h}_k^{(i)} = \mathfrak{h}_{X,k}^{(i)} := (\mathfrak{h}_{X,k} \setminus \{k + 1\}) \cup \{i\}.$$

Let us consider  $\mathbb{C}[u_1, \dots, u_g]$  and the symmetric polynomials, e.g., the power symmetric polynomials,  $T_k^{(\ell)} := \frac{1}{k} \sum_{i=1}^{\ell} u_i^k$  for  $u = {}^t(u_1, \dots, u_g)$ , and  $T_k = T_k^{(g)}$ .

For a Young diagram  $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_n)$ , the Schur function  $s_{\Lambda}$  is defined by the ratio of determinants of  $n \times n$  matrices [14],

$$s_{\Lambda}(u) = \frac{|u_i^{\Lambda_j+n-j}|}{|u_i^{j-1}|}.$$

When  $\Lambda$  is associated with the semigroup  $H$  as in Section 2.1, it can be also regarded as a function of  $T = {}^t(T_{\Lambda_1+n-1}, \dots, T_{\Lambda_n})$  [14], and thus, we express it by

$$S_{\Lambda}(T) = s_{\Lambda}(u).$$

We recall the truncated Young diagrams  $\Lambda_X^{(k)}$  and  $\Lambda_X^{[k]}$  for the Young diagram  $\Lambda_X$  associated with the  $W$ -curve  $X$  in Definition 1. We define  $u^{[k]} = (u_1^{[k]}, \dots, u_g^{[k]})$ , where  $u_i^{[k]} := T_{\Lambda_i+g-i}^{(k)}$ , and let

$$s_{\Lambda_X^{(k)}}(u^{[k]}) := S_{\Lambda_X^{(k)}}(T^{(k)})|_{T_{\Lambda_i+g-i}^{(k)}=u_i^{[k]}}.$$

We also write the decomposition,  $u^{[g]} = u^{[k]} + u^{[g;k]} \in \mathbb{C}^g$ .

**Proposition 21.** For  $\mathfrak{d}_{X,k}$  in Definition 20,

$$s_{\Lambda_X^{(k)}}(u^{[k]}) = \varepsilon_{\Lambda, \mathfrak{d}_{X,k}} \left( \prod_{i \in \mathfrak{d}_{X,k}} \frac{\partial}{\partial u_i^{[g]}} \right) s_{\Lambda_X^{(g)}}(u^{[g]})|_{u^{[g]}=u^{[k]}}. \tag{38}$$

Following Nakayashiki’s results in [19], we state the Riemann–Kempf theorem [27,35] of the  $W$ -curves.

**Proposition 22.** For  $u \in \mathbb{C}^g$  and a multi-index  $J \subset \text{Index}(g, \ell)$ , we define  $\partial_I := \prod_{j \in J} \frac{\partial}{\partial u_j}$ . Let  $\Lambda_X$

be the Young diagram of the  $W$ -curve  $X$  and for given  $k \in \{0, 1, \dots, g\}$ , let  $\Lambda_X = \Lambda_X^{[k]} \cup \Lambda_X^{(k)}$ . For every multi-index  $I = \{\alpha_1, \dots, \alpha_m\} \in \text{Index}(g, m)$ ,  $m < n_k = r_{\Lambda_X^{[k]}}$ ,  $N_k := |\Lambda_X^{[k]}|$ , and  $u \in \Theta_X^k$ ,

1.  $\partial_I \theta((2\omega')^{-1}u + \zeta_X, \tau) = 0$ , whereas  $\partial_{\mathfrak{d}_{X,k}} \theta((2\omega')^{-1}u + \zeta_X, \tau) \neq 0$ ,
2. for  $\ell < N_k$   $\partial_{u_g}^{\ell} \theta((2\omega')^{-1}u + \zeta_X, \tau) = 0$ , whereas  $\partial_{u_g}^{N_k} \theta((2\omega')^{-1}u + \zeta_X, \tau) \neq 0$ .

**Proof.** See Corollary 3 in [19].  $\square$

### 4.3. Sigma Function and $W$ -Curves

We now define the sigma function following Nakayashiki (Definition 9) in [19]. We remark that due to the shifted Riemann constant, our definition differs from Nakayashiki’s so that our sigma function has the natural properties, including the parity and Galois action and the fact that the point of expansion by Schur polynomials is also shifted, as mentioned in Theorem 4. In other words, we employ some parts of the definition of the sigma function by Korotkin and Shramchenko [18], who defined several sigma functions with spin structures based on Klein’s transcendental approaches.

**Definition 21.** We define  $\sigma$  as an entire function of (a column-vector)  $u = {}^t(u_1, u_2, \dots, u_g) \in \mathbb{C}^g$ ,

$$\begin{aligned} \sigma(u) = \sigma(u; M) &= \frac{\varepsilon_{\Lambda, \mathfrak{h}_{X,0}} \exp\left(\frac{1}{2} {}^t u \eta' \omega'^{-1} u\right) \vartheta\left[\begin{smallmatrix} \delta'' \\ \delta' \end{smallmatrix}\right]\left(\frac{1}{2} \omega'^{-1} u; \omega'^{-1} \omega''\right)}{\partial_{\mathfrak{h}_{X,0}} \vartheta\left[\begin{smallmatrix} \delta'' \\ \delta' \end{smallmatrix}\right]\left(\frac{1}{2} \omega'^{-1} u; \omega'^{-1} \omega''\right)\Big|_{u=\tilde{w}(t_X, \mathfrak{R}_s)}} \\ &= \frac{\varepsilon_{\Lambda, \mathfrak{h}_{X,0}} \exp\left(\frac{1}{2} {}^t u \eta' \omega'^{-1} u\right)}{\partial_{\mathfrak{h}_{X,0}} \vartheta\left[\begin{smallmatrix} \delta'' \\ \delta' \end{smallmatrix}\right]\left(\frac{1}{2} \omega'^{-1} u; \omega'^{-1} \omega''\right)\Big|_{u=\tilde{w}(t_X, \mathfrak{R}_s)}} \\ &\quad \times \sum_{n \in \mathbb{Z}^g} \exp\left[\pi \sqrt{-1} \left\{ {}^t(n + \delta'') \omega'^{-1} \omega''(n + \delta'') + {}^t(n + \delta'')(\omega'^{-1} u + 2\delta') \right\}\right], \end{aligned} \tag{39}$$

where  $\varepsilon_{\Lambda, \mathfrak{h}_{X,k}}$  is defined in (38) and  $\mathfrak{h}_{X,0}$  is defined in Definition 20.

Then, we have the following theorem, i.e., Theorem 4.

It is worthwhile noting that the following (41) obviously leads to the Jacobi inversion formulae on the Jacobian  $J_X$  and its strata as mentioned in [4,6,13,14]; though we omit the inversion formulae for the reason of space, we can easily obtain them as its corollary following [13,14]. Since  $\Pi_{P_i, P'_j}^{P, Q}$  in (41) can be expressed in terms of  $R_X$  in (33), we can represent the elements of  $R_X$  by using the differentials of the sigma functions. More explicitly, since the Jacobi inversion formulae on  $J_X$  provide that the multi-variable differentials of the sigma are equal to the meromorphic functions of  $\tilde{R}_X$  as predicted in [16], they imply that if the formulae are integrable, the sigma function is, in principle, obtained by integrating the meromorphic functions on  $S^g X$ ; since the integrability is obvious, the sigma function for every  $W$ -curve can be, in principle, algebraically obtained as the elliptic sigma function in Weierstrass' elliptic function theory. They also show the equivalence between the algebraic and transcendental properties of the meromorphic functions on  $X$ . The sigma function is defined for every compact Riemann surface by Nakayashiki following Klein's construction of his sigma functions [9]. Klein defined his sigma functions using only the data of hyperelliptic Riemann surfaces, following Riemann's approach. On the other hand, Weierstrass criticized Riemann's approach and insisted on the algebraic ways, associated with Weierstrass curves (see Weierstrass's words in a letter to Schwarz (Werke II, 235) cited by Poincare [36]: "Plus je réfléchis aux principes de la théorie des fonctions—et c'est ce que je fais sans cesse—plus je suis solidement convaincu qu'ils sont bâtis sur le fondement des vérités algébriques et que, par conséquent, ce n'est pas le véritable chemin, si inversement ou fait appel au transcendant pour établir les théorèmes simples et fondamentaux de l'Algèbre ; et cela reste vrai, quelque pénétrantes que puissent paraître au premier abord les considérations par lesquelles Riemann a découvert tant d'importantes propriétés des fonctions algébriques". ("The more I think about the principles of function theory—and I do continuously—the more I am convinced that this must be built on the foundations of algebraic truths [my emphasis], and that it is consequently not correct to resort on the contrary to the 'transcendent', to express myself briefly, as the basis of simple and fundamental algebraic propositions. This view seems so attractive at first sight, in that through it Riemann was able to discover so many important properties of algebraic functions" [37])). Unifying Klein's and Weierstrass' views, Baker reformulated Klein's sigma functions after defining explicit algebra curves and connected the sigma functions and the meromorphic functions of the curves as in Weierstrass' elliptic function theory [10]. Thus, we emphasize that the following theorem implies completing Weierstrass' program by succeeding Baker's approaches.

**Theorem 4.**  $\sigma(u)$  has the following properties:

1. it is modular invariant,

2. it obeys the translational formula; for  $u, v \in \mathbb{C}^g$ , and  $\ell (= 2\omega' \ell' + 2\omega'' \ell'') \in \Gamma_X$ , if we define  $L(u, v) := 2 {}^t u (\eta' v' + \eta'' v'')$ ,  $\chi(\ell) := \exp[\pi \sqrt{-1} (2({}^t \ell' \delta'' - {}^t \ell'' \delta') + {}^t \ell' \ell'')$ ], the following holds

$$\sigma(u + \ell) = \sigma(u) \exp(L(u + \frac{1}{2}\ell, \ell)) \chi(\ell), \tag{40}$$

3. its divisor is  $\kappa_J^{-1} \Theta_{X, \mathfrak{s}} \subset \mathbb{C}^g$ , where  $\Theta_{X, \mathfrak{s}} := w_{\mathfrak{s}}(S^{g-1} X) \subset J_X$ ,
4. it satisfies the Jacobi–Riemann fundamental relation, for  $(P, Q, P_i, P'_i) \in X^2 \times (S^g(X) \setminus S_1^g(X)) \times (S^g(X) \setminus S_1^g(X))$ ,

$$u := \tilde{w}_{\mathfrak{s}}(P_1, \dots, P_g), \quad v := \tilde{w}_{\mathfrak{s}}(P'_1, \dots, P'_g),$$

$$\begin{aligned} \exp\left(\sum_{i,j=1}^g \Pi_{P_i, P'_j}^{P, Q}\right) &= \frac{\sigma(\tilde{w}(P) - u) \sigma(\tilde{w}(Q) - v)}{\sigma(\tilde{w}(Q) - u) \sigma(\tilde{w}(P) - v)} \\ &= \frac{\sigma(\tilde{w}(P) - \tilde{w}_{\mathfrak{s}}(P_1, \dots, P_g)) \sigma(\tilde{w}(Q) - \tilde{w}_{\mathfrak{s}}(P'_1, \dots, P'_g))}{\sigma((\tilde{w}(Q) - \tilde{w}_{\mathfrak{s}}(P_1, \dots, P_g)) \sigma(\tilde{w}(P) - \tilde{w}_{\mathfrak{s}}(P'_1, \dots, P'_g))} \end{aligned} \tag{41}$$

which generates the Jacobi inversion formulae for  $S^k X$ ,

5. the leading term in the Taylor expansion of the  $\sigma$  function associated with  $X$ , with normalized constant factor  $c$ , is expressed by the Schur function of  $\Lambda_X$

$$\sigma(u + \tilde{w}({}_X \mathfrak{R}_{\mathfrak{s}})) = S_{\Lambda_X}(T)|_{T_{\Lambda_i + g - i = u_i}} + \sum_{|\mathfrak{w}_g(\alpha)| > |\Lambda_X|} a_{\alpha} u^{\alpha},$$

where  $a_{\alpha} \in \mathbb{C}[\lambda_{ij}]$ ,  $\alpha = (\alpha_1, \dots, \alpha_g)$ ,  $u^{\alpha} = u_1^{\alpha_1} \dots u_g^{\alpha_g}$ , and  $\mathfrak{w}_g(\alpha) = \sum \alpha_i \text{wt}(u_i)$ . Here,  $S_{\Lambda}(T)$  is the lowest-order term in the  $w$ -degree of the  $u_i$ ;  $\sigma(u)$  is homogeneous of degree  $|\Lambda_X|$  with respect to  $\text{wt}_{\Lambda}$ ,

6.  $\sigma(-u) = \pm \sigma(u)$ , and
7. If  $\tilde{\zeta} \in G_X$  satisfies  $\tilde{\zeta}^{\ell} = \text{id}$ , and  $\tilde{\zeta}[\sigma(u + \tilde{\ell}) / \sigma(u)] = \sigma(u + \tilde{\ell}) / \sigma(u)$  for  $\tilde{\ell} \in \Gamma_X$  and  $u \in \mathbb{C}^g$ , the action provides the one-dimensional representation such that

$$\tilde{\zeta} \sigma(u) = \rho_{\tilde{\zeta}} \sigma(u),$$

where  $\rho_{\tilde{\zeta}}^{\ell} = 1$ .

**Proof.** 1 and 5 are obtained by Theorem 13 in [19] by noting the difference of the definition of our sigma function in Definition 21 from Nakayashiki’s (Definition 9) in [19]. 3 is due to Proposition 19. 2 is standard and can be obtained by direct computations [13]. 4 is the same as Proposition 4.4 in [13]. 6 and 7 are the same as Lemma 3.6, and Lemma 4.1 in [35].  $\square$

#### 4.4. Conclusions and Discussions

We have considered the Weierstrass curves (W-curves), which are algebraic expressions of compact Riemann surfaces; the set of W-curves represents the set of compact Riemann surfaces.

By using the algebraic tools we constructed in this paper, we have a connection between the sigma function for W-curve  $X$  and the meromorphic functions on  $X$  as in Theorem 4. Since the Jacobi inversion formulae via Theorem 4.4 are given by the differential identity, integrating it, in principle, it provides that the sigma function is constructed by an integral formula of the meromorphic functions on the W-curve  $X$ . In other words, we give an algebraic construction of the sigma function, or so-called the EEL-construction [11] in this paper.

It is noted that this construction is based on our recent result on the trace structure of the affine ring  $R_X$  [17].

Further, we also discuss the mathematical meaning of our result as follows. We also note that for an ordinary point  $P$  in every  $W$ -curve  $Y$  with  $H_Y$  at  $\infty \in Y$  of genus  $g$ , the Weierstrass gap sequence at  $P$  is given by the numerical semigroup  $H^c = \{1, 2, \dots, g\}$ , and there is a  $W$ -curves  $X$ , which is birationally equivalent to  $X$  such that  $\infty \in X$  corresponds to  $P \in Y$  and  $H_X^c = H^c$ . Then there appear two sigma functions  $\sigma_Y$  for the  $W$ -curve  $Y$  with non-trivial Weierstrass semigroup  $H_Y$  and  $\sigma_X$  for the  $W$ -curve  $X$  with  $H_X = H$ . By some arguments on the both Jacobians  $J_X$  and  $J_Y$ , we find that  $\sigma_X$  and  $\sigma_Y$ , due to the translational formulae and so on, are the same functions, and the both sifted Abelian integrals agree. Then the above theorem (5) means that we have the expansions of the sigma function at the Abelian image of the ordinary point  $P$  in  $Y$ . It means that the problem of finding the expansion of the sigma function for a point  $u$  in Jacobian  $J_Y$  is reduced to the problem that we should find the birational curves associated with the preimage of the Abelian integral. Since in Weierstrass' elliptic function theory, we often encounter the reductions of the transcendental problems to the algebraic problems, we also remark that this reduction has the same origin, i.e., the equivalence between algebraic objects and transcendental objects in the Abelian function theory.

With Theorem 4, we recognize that this theorem is the goal Weierstrass had in mind, and at the same time, with it, we also recognize that we finally reached the starting point for the development of the Weierstrass program to construct an Abelian function theory for every  $W$ -curve  $X$  as in his elliptic function theory.

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## References

1. Weierstrass, K. *Theorie der Elliptischen Functionen*; Mathematische Werke V, 1875-6; Georg Ohms: Hildesheim, Germany, 1915.
2. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis*; Cambridge University Press: Cambridge, UK, 1927.
3. Weierstrass, K. *Theorie der Abelschen Transcendenten*; Mathematische Werke IV, 1875-6; Georg Ohms: Hildesheim, Germany, 1902.
4. Komeda, J.; Matsutani, S.; Previato, E. The sigma function for Weierstrass semigroups  $\langle 3, 7, 8 \rangle$  and  $\langle 6, 13, 14, 15, 16 \rangle$ . *Int. J. Math.* **2013**, *24*, 1350085. [[CrossRef](#)]
5. Komeda, J.; Matsutani, S.; Previato, E. The Riemann constant for a non-symmetric Weierstrass semigroup. *Arch. Math.* **2016**, *107*, 499–509. [[CrossRef](#)]
6. Komeda, J.; Matsutani, S.; Previato, E. The sigma function for trigonal cyclic curves. *Lett. Math. Phys.* **2019**, *109*, 423–447. [[CrossRef](#)]
7. Mumford, D. *Tata Lectures on Theta Volume I*; Birkhäuser: Basel, Switzerland, 1981.
8. Mumford, D. *Tata Lectures on Theta Volume II*; Birkhäuser: Basel, Switzerland, 1984.
9. Klein, F. Ueber hyperelliptische sigmafunctionen. *Math. Ann.* **1886**, *27*, 431–464; Ueber hyperelliptische sigmafunctionen (Zweite Abhandlung). *Math. Ann.* **1888**, *32*, 351–380. [[CrossRef](#)]

10. Baker, H.F. *Abelian Functions: Abel's Theorem and the Allied Theory of Theta Functions*; Reprint of the 1897 original; Cambridge University Press: Cambridge, UK, 1995.
11. Eilbeck, J.C.; Enolskii, V.Z.; Leykin, D.V. On the Kleinian construction of abelian functions of canonical algebraic curves. In *Proceedings of the SIDE III—Symmetries and Integrability of Difference Equations (Sabaudia, 1998)*, Sabaudia, Italy, 16–22 May 1998; CRM: Providence, RI, USA, 2000; pp. 121–138.
12. Buchstaber, V.M.; Enolski, V.Z.; Leykin, D.V.  $\sigma$  functions: Old and new results. In *Integrable Systems and Algebraic Geometry Volume 2*; London Math Society Lecture Note Series; Donagi, R., Shaska, T., Eds.; Cambridge University Press: Cambridge, UK, 2020; Volume 459, pp. 175–214.
13. Matsutani, S.; Previato, E. Jacobi inversion on strata of the Jacobian of the  $C_{rs}$  curve  $y^r = f(x)$ . *J. Math. Soc. Jpn.* **2008**, *60*, 1009–1044. [[CrossRef](#)]
14. Matsutani, S.; Previato, E. Jacobi inversion on strata of the Jacobian of the  $C_{rs}$  curve  $y^r = f(x)$  II. *J. Math. Soc. Jpn.* **2014**, *66*, 647–692. [[CrossRef](#)]
15. Matsutani, S.; Komeda, J. Sigma functions for a space curve of type (3,4,5). *J. Geom. Symmetry Phys.* **2013**, *30*, 75–91.
16. Komeda, J.; Matsutani, S. Jacobi inversion formulae for a curve in Weierstrass normal form. In *Integrable Systems and Algebraic Geometry Volume 2*; London Math Society Lecture Note Series; Donagi, R., Shaska, T., Eds.; Cambridge University Press: Cambridge, UK, 2020; Volume 459, pp. 383–404.
17. Komeda, J.; Matsutani, S.; Previato, E. Complementary modules of Weierstrass canonical forms. *arXiv* **2022**, arXiv:2207.01905.
18. Korotkin, D.; Shramchenko, V. On higher genus Weierstrass sigma-function. *Phys. D* **2012**, *241*, 2086–2094. [[CrossRef](#)]
19. Nakayashiki, A. Tau function approach to theta functions. *Int. Math. Res. Not. IMRN* **2016**, *2016*, 5202–5248. [[CrossRef](#)]
20. Sato, M.; Noumi, M. *Soliton Equation and Universal Grassmann Manifold*; Sophia University Kokyuroku in Mathematics: Tokyo, Japan, 1984; Volume 18, pp. 1–131. (In Japanese)
21. Segal, G.; Wilson, G. Loop groups and equations of KdV type. *IHES* **1985**, *61*, 5–65. [[CrossRef](#)]
22. Assi, A.; D'Anna, M.; García-Sánchez, P. *Numerical Semigroups and Applications (RSME Springer Series 3)*; Springer: New York, NY, USA, 2020.
23. Weierstrass, K. *Über Normalformen algebraischer Gebilde*; Mathematische Werke III, 297–307; Georg Ohms: Hildesheim, Germany, 1897.
24. Coppens, M.; Kato, T. Weierstrass points with first two non-gaps equal to  $n$  and  $n + 2$ . *Kyushu J. Math.* **2014**, *68*, 139–147. [[CrossRef](#)]
25. Kato, T. Weierstrass normal form of a Riemann surface and its applications. *Sûgaku* **1980**, *32*, 73–75. (In Japanese)
26. Farkas, H.M.; Kra, I. *Riemann Surfaces (GTM 71)*; Springer: Berlin/Heidelberg, Germany, 1991.
27. Arbarello, E.; Cornalba, M.; Griffiths, P.A.; Harris, J. *Geometry of Algebraic Curves Volume I*; Springer: New York, NY, USA, 1985.
28. Kunz, E. *Introduction to Plane Algebraic Curves*; Belshoff, R.G., Translator; Birkhäuser: Boston, MA, USA, 2004.
29. Herzog, J. Generators and relations of Abelian semigroup and semigroup ring. *Manuscripta Math.* **1970**, *3*, 175–193. [[CrossRef](#)]
30. Pinkham, H.C. Deformation of algebraic varieties with  $G_m$  action. *Astérisque* **1974**, *20*, 1–131.
31. Bresinsky, H. On prime ideals with generic zero  $x_i = t^{n_i}$ . *Proc. Am. Math. Soc.* **1975**, *47*, 329–332.
32. Ônishi, Y. Arithmetical power series expansion of the sigma function for a plane curve. *Proc. Edinb. Math. Soc.* **2018**, *61*, 995–1022. [[CrossRef](#)]
33. Stichtenoth, H. *Algebraic Function Fields and Codes (GTM 254)*, English ed.; Springer: Berlin/Heidelberg, Germany, 2008.
34. Fay, J.D. *Theta Functions on Riemann Surfaces*; Lectures Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1973; Volume 352.
35. Ônishi, Y. Determinant expressions for hyperelliptic functions. *Proc. Edinb. Math. Soc.* **2005**, *48*, 705–742. [[CrossRef](#)]
36. Poincaré, H. *Savants et Écrivains*; Flammarion: Paris, France, 1910.
37. Jahnke, H.N. *A History of Analysis (History of Mathematics)*; American Mathematical Society: Providence, RI, USA, 2003.