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Two Inverse Eigenproblems for Certain Symmetric and Nonsymmetric Pentadiagonal Matrices

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Abstract: In this paper, we give sufficient conditions for the construction of certain symmetric and nonsymmetric pentadiagonal matrices from particular spectral information. The construction of the symmetric pentadiagonal matrix considers the extreme eigenvalues of its leading principal submatrices and a prescribed entry, and the construction of the nonsymmetric pentadiagonal matrix also considers an eigenvector and two prescribed entries.

Keywords: inverse eigenvalue problem; symmetric pentadiagonal matrices; nonsymmetric pentadiagonal matrices; leading principal submatrices

MSC: 15A42; 65F15; 65F18

Citation: Arela-Pérez, S.; Lozano, C.; Nina, H.; Pickmann-Soto, H. Two Inverse Eigenproblems for Certain Symmetric and Nonsymmetric Pentadiagonal Matrices. *Mathematics* **2022**, *10*, 3054. https://doi.org/10.3390/math10173054

Academic Editors: Lorentz Jäntschi and Virginia Niculescu

Received: 8 July 2022
Accepted: 21 August 2022
Published: 24 August 2022

1. Introduction

The structured inverse eigenvalue problem (SIEP) consists of determining sufficient and necessary conditions for a data set to be the spectral information of a structured matrix. Some structured matrices considered in the SIEP are Jacobi inverse eigenvalue problems, Toeplitz inverse eigenvalue problems, nonnegative inverse eigenvalue problems, stochastic inverse eigenvalue problems, and inverse singular value problems [1]. If the matrix is required to be nonnegative and symmetric, it is called a symmetric nonnegative inverse eigenvalue problem. Two particular SIEPs for symmetric matrices were introduced in [2], one is for constructing an up-arrow symmetric matrix from the smallest and largest eigenvalue of its principal principal submatrices, and the other for constructing a symmetric matrix arrowhead up from the largest eigenvalue of its principal principal submatrices and an eigenvector associated with its largest eigenvalue. These kinds of problems are also called extreme inverse eigenvalue problems. In the last years, extreme inverse eigenvalue problems for certain symmetric matrices such as tridiagonal, Jacobi, bordered diagonal, and acyclic matrices, among others, have been considered (see e.g., [3–6]). In [7–10], the authors advance the extreme inverse eigenvalue problem by studying nonsymmetric cases. The symmetric pentadiagonal matrix, i.e., the symmetric matrix with $2p+1$ bands with $p=2$, appears in the inverse problem for a vibrating beam [11]. In this case, the pentadiagonal matrix A involves the stiffness data of the beam, in which the first subdiagonals must be negative and the second subdiagonals positive. Given

$$\sigma(A) = (\lambda_i)_1^n,\ \sigma(A_1) = (\mu_i)_1^{n-1},\ \sigma(A_{1,2}) = (\nu_i)_1^{n-2}, \tag{1}$$

where $\sigma(A)$ denotes the spectrum of A, and $A_1, A_{1,2}$ the matrices when their first row and column, and their first two rows and columns, respectively, are removed, such that the eigenvalues are strictly interleaved, Gladwell constructs a pentadiagonal matrix A such that (1) holds [12].

The inverse extreme eigenvalue problem for a symmetric pentadiagonal matrix arises from the inverse problem for a discrete beam which occurs in the structural design of beams, buildings, and bridges, among others. In constructing the mass-spring systems, the problem of inferring the bending stiffness and density of a beam from its eigenfrequencies when one or both ends are clamped is studied as the inverse problem of extreme eigenvalues for symmetric pentadiagonal matrices [13]. This discrete beam problem considers some variables such as masses, stiffness, and lengths of a discrete beam. A relevant and widely studied problem is the Euler–Bernoulli beam problem, which presents some variations such as modes of vibration, an arbitrary number of concentrated open cracks [14,15], an online system of masses and springs with a minimum mass for total stiffness, and a sand cantilever beam system in bending vibration [16].

In this paper, we consider the following kinds of pentadiagonal matrices:

$$A = \begin{pmatrix} a_1 & b_1 & b_2 & & & \\ b_1 & a_2 & 0 & b_3 & & \\ b_2 & 0 & a_3 & \ddots & \ddots & \\ & b_3 & \ddots & \ddots & 0 & b_{n-1} \\ & & \ddots & 0 & a_{n-1} & b_n \\ & & & b_{n-1} & b_n & a_n \end{pmatrix}, \tag{2}$$

with $b_j > 0$, $j = 1, 2 \ldots, n-1$, and

$$B = \begin{pmatrix} a_1 & b_1 & b_2 & & & \\ c_1 & a_2 & 0 & b_3 & & \\ c_2 & 0 & a_3 & \ddots & \ddots & \\ & c_3 & \ddots & \ddots & 0 & b_{n-1} \\ & & \ddots & 0 & a_{n-1} & b_n \\ & & & c_{n-1} & c_n & a_n \end{pmatrix}, \tag{3}$$

where a_j, b_j, and c_j are real numbers with $b_j c_j > 0$, $j = 1, 2 \ldots, n$ (see [17]).

Remark 1.

1. *The matrix A defined in (2) is similar to a symmetric tridiagonal matrix [17]. However, the eigenvalues of its leading principal submatrices are not preserved under such similarity.*
2. *Notice that the matrix B of the form (3) is diagonally similar to the symmetric pentadiagonal matrix*

$$DBD^{-1} = \begin{pmatrix} a_1 & \sqrt{b_1c_1} & \sqrt{b_2c_2} & & & \\ \sqrt{b_1c_1} & a_2 & 0 & \ddots & & \\ \sqrt{b_2c_2} & 0 & a_3 & \ddots & \ddots & \\ & \ddots & 0 & \ddots & 0 & \sqrt{b_{n-1}c_{n-1}} \\ & & \ddots & 0 & a_{n-1} & \sqrt{b_nc_n} \\ & & & \sqrt{b_{n-1}c_{n-1}} & \sqrt{b_nc_n} & a_n \end{pmatrix} \tag{4}$$

with $D = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$*, and*

$$\alpha_{j=} \begin{cases} \left(\prod_{k=\lceil \frac{j}{2} \rceil}^{\lfloor \frac{n}{2} \rfloor} \frac{c_{2k}}{b_{2k}} \right)^{\frac{1}{2}}, & j \, old \\ \left(\prod_{k=\lceil \frac{j+1}{2} \rceil}^{\lceil \frac{n}{2} \rceil} \frac{c_{2k-1}}{b_{2k-1}} \right)^{\frac{1}{2}}, & j \, even \end{cases} \tag{5}$$

$j = 1, 2, \ldots, n-1$*, where* $\lceil x \rceil$ *and* $\lfloor x \rfloor$ *denote the least integer greater or equal to* x *and the greater integer least than or equal to* x*, respectively,* $\alpha_n = 1$*, and*

$$\prod_{k=1}^{\lceil \frac{n}{2} \rceil} \frac{b_{2k-1}}{c_{2k-1}} = \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{b_{2k}}{c_{2k}}, \tag{6}$$

which implies that, under this transformation, the eigenvalues of the leading principal submatrices B_j*,* $j = 1, 2, \ldots, n$ *of the matrix* B *remain invariant.*

In the sequel, we deal with nonsymmetric pentadiagonal matrices such that (6) holds.

Throughout the text, given an $n \times n$ symmetric matrix M_n, for $j = 1, 2, \ldots, n$, M_j denotes the $j \times j$ leading principal submatrix of M_n, $\sigma(M_j) = \{\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_j^{(j)}\}$ the spectrum of M_j, $P_j(\lambda)$ the characteristic polynomial of M_j, $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ the smallest and largest eigenvalue of M_j, respectively (also called extreme eigenvalues), and I_j the identity matrix of order j.

In this paper, we consider the following two extremal inverse eigenvalue problems to be similar but more general than the one considered by Gladwell in [12]:

Problem 1. *Given the set of real numbers*

$$\left\{ \lambda_1^{(n)}, \ldots, \lambda_1^{(j)}, \ldots, \lambda_1^{(2)}, \lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_j^{(j)}, \ldots, \lambda_n^{(n)} \right\},$$

and a positive real number d*, construct a symmetric pentadiagonal matrix* A *of the form (2) such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are, respectively, the smallest and largest eigenvalue of the leading principal submatrix* A_j*,* $j = 1, 2, \ldots, n$*, and* $b_n = d$*.*

Problem 2. *Given the set of real numbers*

$$\left\{ \lambda_1^{(n)}, \ldots, \lambda_1^{(j)}, \ldots, \lambda_1^{(2)}, \lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_j^{(j)}, \ldots, \lambda_n^{(n)} \right\},$$

a nonzero vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ *and two positive real numbers* d_1 *and* d_2*, construct a nonsymmetric pentadiagonal matrix* B *of the form (3) such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are, respectively, the smallest and largest eigenvalue of the leading principal submatrix* B_j*,* $j = 1, 2, \ldots, n$*,* $\left(\lambda_n^{(n)}, \mathbf{x} \right)$ *is an eigenpair of* B*,* $b_n = d_1$*, and* $c_n = d_2$*.*

The paper is organized as follows: in Section 2, we give sufficient conditions for the existence and construction of a symmetric pentadiagonal matrix A of the form (2) from the extreme eigenvalues of its leading principal submatrices. In Section 3, we determine a relationship between the entries of the eigenvector of a nonsymmetric pentadiagonal matrix B of the form (3) associated with its largest eigenvalue. Then, we give sufficient conditions for the construction of a matrix B of the form (3) from the extreme eigenvalues of its leading principal submatrices and an eigenpair. Throughout the paper, some illustrative examples are presented.

2. Symmetric Pentadiagonal Matrices from Extremal Eigenvalues

In this section, we show that the interleaving of the extreme eigenvalues of the leading principal submatrices of a symmetric pentadiagonal matrix is sufficient to guarantee a solution to Problem 1. In particular, we give a sufficient condition for the construction of a symmetric pentadiagonal matrix from the extreme eigenvalues of its leading principal submatrices and a prescribed entry. Moreover, a solution set is given.

Lemma 1. *Let A be an $n \times n$ symmetric pentadiagonal matrix of the form (2) and let A_j be the $j \times j$ principal submatrix of A with characteristic polynomial $P_j(\lambda)$, $j = 1, 2, \ldots, n$. Then the sequence $\{P_j(\lambda)\}_{j=1}^{n}$ satisfies the recurrence relation:*

$$\begin{aligned} P_1(\lambda) &= \lambda - a_1, && (7)\\ P_j(\lambda) &= (\lambda - a_j)P_{j-1}(\lambda) - b_{j-1}^2 Q_{j-1}(\lambda), \quad j = 2, 3, \ldots, n-1, && (8)\\ P_n(\lambda) &= (\lambda - a_n)P_{n-1}(\lambda) - b_n^2 P_{n-2}(\lambda) - b_{n-1}^2 Q_{n-1}(\lambda) - 2\prod_{i=1}^{n} b_i && (9) \end{aligned}$$

where $Q_1(\lambda) = 1$ and $Q_{j-1}(\lambda)$, $j = 3, \ldots, n$ is the characteristic polynomial of the principal submatrix of A obtained by deleting the $(j-2)$-th row and column of the leading principal submatrix A_{j-1}.

Proof. It is immediate by expanding the determinant. □

Hereafter, we will adopt the following notations

$$\left.\begin{aligned} R_j &= P_{j-2}\left(\lambda_1^{(j)}\right)Q_{j-1}\left(\lambda_j^{(j)}\right) - P_{j-2}\left(\lambda_j^{(j)}\right)Q_{j-1}\left(\lambda_1^{(j)}\right)\\ S_n &= Q_{n-1}\left(\lambda_n^{(n)}\right) - Q_{n-1}\left(\lambda_1^{(n)}\right)\\ T_n &= P_{n-1}\left(\lambda_n^{(n)}\right) - P_{n-1}\left(\lambda_1^{(n)}\right)\\ U_j &= P_{j-1}\left(\lambda_j^{(j)}\right)Q_{j-1}\left(\lambda_1^{(j)}\right) - P_{j-1}\left(\lambda_1^{(j)}\right)Q_{j-1}\left(\lambda_j^{(j)}\right)\\ V_j &= P_{j-2}\left(\lambda_1^{(j)}\right)P_{j-1}\left(\lambda_j^{(j)}\right) - P_{j-2}\left(\lambda_j^{(j)}\right)P_{j-1}\left(\lambda_1^{(j)}\right)\\ W_j &= \left(\lambda_j^{(j)} - \lambda_1^{(j)}\right)P_{j-1}\left(\lambda_1^{(j)}\right)P_{j-1}\left(\lambda_j^{(j)}\right)\\ Z_j &= \lambda_j^{(j)}P_{j-1}\left(\lambda_j^{(j)}\right)Q_{j-1}\left(\lambda_1^{(j)}\right) - \lambda_1^{(j)}P_{j-1}\left(\lambda_1^{(j)}\right)Q_{j-1}\left(\lambda_j^{(j)}\right)\\ \gamma &= \prod_{k=1}^{\lceil \frac{n}{2} \rceil} c_{2k-1} \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} b_{2k}, \text{ for all } j = 2, \ldots, n-1. \end{aligned}\right\} \quad (10)$$

The following lemma will be very useful in our results.

Lemma 2 ([4])**.** *Let $P(\lambda)$ be a monic polynomial of degree n with all real zeros. If λ_1 and λ_n are, respectively, the smallest and largest zeros of $P(\lambda)$, then*

(1) If $\mu < \lambda_1$, we have that $(-1)^n P(\mu) > 0$,
(2) If $\mu > \lambda_n$, we have that $P(\mu) > 0$.

Theorem 1. *Let the set of real numbers $\left\{\lambda_1^{(j)}, \lambda_j^{(j)}\right\}_{j=1}^{n}$, and a positive real number d satisfy*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(j)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_j^{(j)} < \cdots < \lambda_n^{(n)}, \quad (11)$$

where U_n, V_n, and W_n are as in (10). Then, there exists a symmetric pentadiagonal matrix A of the form (2), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the smallest and largest eigenvalues of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$.

Proof. It is immediate that $a_1 = \lambda_1^{(1)}$. To show the existence of a symmetric pentadiagonal matrix A with the required properties is equivalent to showing that:

On the one hand, the system of equations

$$\begin{cases} P_j\left(\lambda_1^{(j)}\right) = \left(\lambda_1^{(j)} - a_j\right)P_{j-1}\left(\lambda_1^{(j)}\right) - b_{j-1}^2 Q_{j-1}\left(\lambda_1^{(j)}\right) = 0, \\ P_j\left(\lambda_j^{(j)}\right) = \left(\lambda_j^{(j)} - a_j\right)P_{j-1}\left(\lambda_j^{(j)}\right) - b_{j-1}^2 Q_{j-1}\left(\lambda_j^{(j)}\right) = 0. \end{cases} \tag{12}$$

has real solutions a_j and b_{j-1} with $b_{j-1}^2 > 0$, $j = 1, 2, \ldots, n-1$. In effect, from (11) and Lemma 2, the determinant of the system (12)

$$-U_j = P_{j-1}\left(\lambda_1^{(j)}\right)Q_{j-1}\left(\lambda_j^{(j)}\right) - P_{j-1}\left(\lambda_j^{(j)}\right)Q_{j-1}\left(\lambda_1^{(j)}\right)$$

is nonzero. Solving (12), for $j = 1, 2, \ldots, n-1$, we have

$$a_j = \frac{Z_j}{U_j}, \quad b_{j-1}^2 = \frac{W_j}{-U_j}. \tag{13}$$

Moreover, from Lemma 2 we have $b_{j-1}^2 > 0$, then

$$b_{j-1} = \sqrt{\frac{W_j}{-U_j}}, \; j = 2, \ldots, n-1. \tag{14}$$

On the other hand, the system of equations

$$\begin{cases} P_n\left(\lambda_1^{(n)}\right) = \left(\lambda_1^{(n)} - a_n\right)P_{n-1}\left(\lambda_1^{(n)}\right) - b_n^2 P_{n-2}(\lambda_1^{(n)}) - b_{n-1}^2 Q_{n-1}\left(\lambda_1^{(n)}\right) - 2\prod_{i=1}^{n} b_i = 0, \\ P_n\left(\lambda_n^{(n)}\right) = \left(\lambda_n^{(n)} - a_n\right)P_{n-1}\left(\lambda_n^{(n)}\right) - b_n^2 P_{n-2}(\lambda_n^{(n)}) - b_{n-1}^2 Q_{n-1}\left(\lambda_n^{(n)}\right) - 2\prod_{i=1}^{n} b_i = 0, \end{cases} \tag{15}$$

has real solutions a_n, b_{n-1}, and b_n. Indeed, by solving (15) we obtain

$$U_n b_{n-1}^2 + 2cT_n b_{n-1} b_n + V_n b_n^2 + W_n = 0, \tag{16}$$

where $c = \prod_{i=1}^{n-2} b_i$. This implies that $(X, Y) = (b_{n-1}, b_n)$ must belong to the conic

$$\mathcal{C} = \{(X, Y) \in \mathbb{R}^2 : U_n X^2 + 2cT_n XY + V_n Y^2 + W_n = 0\},$$

which always exists, whether degenerate or not, i.e., $\mathcal{C} \neq \emptyset$. Actually, Equation (16) can be written as

$$\mathbf{L}M\mathbf{L}^T = 0 \tag{17}$$

with

$$\mathbf{L} = \begin{pmatrix} X & Y & 1 \end{pmatrix}, \quad M = \begin{pmatrix} N & 0 \\ 0^T & W_n \end{pmatrix}, \quad \text{and} \quad N = \begin{pmatrix} U_n & cT_n \\ cT_n & V_n \end{pmatrix}.$$

Therefore, the conic $\mathcal{C}$ is degenerate if $\det M = 0$, and does not exist, i.e., $\mathcal{C} = \emptyset$, if $\det N > 0$ and $(U_n + V_n)\det M > 0$. From Lemma 2 and condition (11), we obtain

$$-(U_n + V_n)W_n > 0.$$

Then, if $\det N > 0$, we have

$$(U_n + V_n)\det M = (U_n + V_n)W_n \det N < 0,$$

i.e., the conic $\mathcal{C}$ always exists. Thus, b_n and b_{n-1} that satisfy (15) exist. Moreover, as $P_{n-1}\left(\lambda_i^{(n)}\right) \neq 0$, a_n exists. □

Next, we give a particular solution to Problem 1.

Theorem 2. *Given the set of* $2n-1$ *real numbers* $\left\{\lambda_1^{(j)}, \lambda_j^{(j)}\right\}_{j=1}^{n}$ *that satisfy (11), and a positive real number* d*, if*

$$U_n V_n < 0, \tag{18}$$

and

$$U_n(V_n d^2 + W_n) < 0, \tag{19}$$

where U_n, V_n*, and* W_n *are as in (10), then there exists a symmetric pentadiagonal matrix* A *of the form (2), such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are, respectively, the smallest and largest eigenvalues of* A_j*,* $j = 1, 2, \ldots, n$*, and* $b_n = d$*.*

Proof. By Theorem 1 there exists a symmetric pentadiagonal matrix A of the form (2), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the smallest and largest eigenvalues of A_j, $j = 1, 2, \ldots, n$. Furthermore, we get

$$a_j = \frac{Z_j}{U_j}, \tag{20}$$

and

$$b_{j-1} = \sqrt{\frac{-T_j}{U_j}}, \quad j = 2, \ldots, n-1. \tag{21}$$

On the other hand, if $b_n = d$, $X = b_{n-1}$ is a real solution of the equation

$$U_n X^2 + 2cT_n dX + V_n d^2 + W_n = 0, \tag{22}$$

since (18) holds. Solving (22), we obtain

$$b_{n-1} = [-cT_n d \pm \sqrt{d^2(c^2 T_n^2 - U_n V_n) - U_n W_n}]U_n^{-1}, \tag{23}$$

and from (19) we choose $b_{n-1} > 0$.
Finally, solving (15) for a_n, we obtain

$$a_n = \frac{\lambda_i^{(n)} P_{n-1}\left(\lambda_i^{(n)}\right) - d^2 P_{n-2}\left(\lambda_i^{(n)}\right) - b_{n-1}^2 Q_{n-1}\left(\lambda_i^{(n)}\right) - 2\beta}{P_{n-1}\left(\lambda_i^{(n)}\right)}, \tag{24}$$

for $i = 1$ or $i = n$, where $\beta = \prod_{i=1}^{n} b_i$. □

Remark 2.

1. *Note that in Theorem 2, when constructing a symmetric pentadiagonal matrix with the required properties, all entries in the matrix are unique except* a_n*.*
2. *Theorem 2 guarantees that the conic* C *always exists, whether it is degenerate or not. Setting a value for* b_n*, say* d*, is equivalent to considering in the plane the line* $X = d$*, which may or may not intersect the conic* $\mathcal{C}$*. The condition (18) on* d *in Theorem 2 guarantees that this line intersects the conic at least one point.*

Corollary 1. *Under the same hypothesis and notations of the Theorem 1.*

1. *If $c^2T_n^2 - U_nV_n \geq 0$, then for all $d > 0$ there exists a symmetric pentadiagonal matrix A of the form (2) such that the smallest and largest eigenvalues of its leading principal submatrices A_j, $\lambda_1^{(j)}$, and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, satisfy (11) and $b_n = d$.*
2. *If $U_n(V_nd^2 + W_n) < 0$, then for all*

$$d \in \left(0, \sqrt{\frac{U_nW_n}{c^2T_n^2 - U_nV_n}}\right]$$

there exists a symmetric pentadiagonal matrix A of the form (2) such that the smallest and largest eigenvalues of its leading principal submatrices A_j, $\lambda_1^{(j)}$, and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, satisfy (11) and $b_n = d$.

Proof. The first part is immediate. The second part follows from (18). □

Remark 3. *The conic $\mathcal{C}$ can be degenerate or not, but the type of conic is determined by its invariants. Consequently, Corollary 1 establishes the different types of conics that can be presented, and for which values of d, the line $X = d$ intersects it. Indeed, we have the following cases:*

1. *In the first case for $U_n(V_nd^2 + W_n) = 0$, $\det M = \det N = 0$, we have that the conic $\mathcal{C}$ is degenerate and as $W_n(U_n + V_n) < 0$, the conic consist of two nonvertical parallel lines. Then, in this case, any line $X = d$ intersects the conic. For $U_n(V_nd^2 + W_n) > 0$, $\det M \neq 0$ and $\det N < 0$. The conic $\mathcal{C}$ is a hyperbola with directrix parallel to axis X. Again, in this case, any line $X = d$ intersects the hyperbola.*
2. *In the second case, $\det N > 0$, then $(U_n + V_n)\det M < 0$. The conic $\mathcal{C}$ is an ellipse. As the center of the ellipse is in the axis Y, any line $X = d > 0$, with d in an appropriate interval intersects the ellipse.*

Example 1. *In Table 1, we consider uniformly distributed random numbers generated using the Matlab rand function*

Table 1. Random extreme spectral data.

j	1	2	3	4	5	6	7
$\lambda_1^{(j)}$	0.1367	0.1359	−0.0287	−0.2883	−0.3880	−1.2466	−1.2533
$\lambda_j^{(j)}$	0.1367	0.6363	0.9552	0.9553	1.9018	2.2674	2.2842

which satisfy conditions (11) and (18) and $d = 0.1753$. Then from the procedure of Theorem 1, we obtain the following symmetric pentadiagonal matrix of the form (2)

$$A = \begin{pmatrix} 0.1367 & 0.0192 & 0.3674 & & & & \\ 0.0192 & 0.6356 & 0 & 0.2339 & & & \\ 0.3674 & 0 & 0.7901 & 0 & 1.0563 & & \\ & 0.2339 & 0 & 0.3684 & 0 & 1.7356 & \\ & & 1.0563 & 0 & 0.8239 & 0 & 0.4728 \\ & & & 1.7356 & 0 & 0.6527 & 0.1753 \\ & & & & 0.4728 & 0.1753 & 0.9782 \end{pmatrix}$$

whose spectra of its the leading principal submatrices A_j, $j = 1, \ldots, 7$, are:

$$
\begin{aligned}
\sigma(A_1) &= \{\mathbf{0.1367}\}, \\
\sigma(A_2) &= \{\mathbf{0.1359}, \mathbf{0.6363}\}, \\
\sigma(A_3) &= \{-\mathbf{0.0287}, 0.6358, \mathbf{0.9552}\}, \\
\sigma(A_4) &= \{-\mathbf{0.2883}, 0.2329, 0.7714, \mathbf{0.9553}\}, \\
\sigma(A_5) &= \{-\mathbf{0.3880},\ 0.2259,\ 0.2432,\ 0.7718, \mathbf{1.9018}\} \\
\sigma(A_6) &= \{-\mathbf{1.2466},\ -0.3880,\ 0.2361,\ 0.6366,\ 1.9018,\ \mathbf{2.2674}\}, \\
\sigma(A_7) &= \{-\mathbf{1.2533},\ -0.4449,\ 0.18895,\ 0.6345,\ 0.9704,\ 2.0056,\ \mathbf{2.2842}\}
\end{aligned}
$$

Example 2. *In Table 2, we show the errors in the construction of symmetric pentadiagonal matrices of the form (2), from uniformly distributed random numbers using the Matlab rand function, which satisfy conditions (11) and (18). $\hat{A}$ denotes the constructed matrix, $\hat{\lambda}$ the vector with extreme values of $\hat{A}$, λ the vector of data obtained randomly, and $e_\lambda = \log(\|\lambda - \hat{\lambda}\|/\|\lambda\|)$.*

Table 2. Relative errors in the construction of symmetric pentadiagonal matrix.

n	5	10	15	20	30	40	50
e_λ	−14.9713	−14.9746	−14.8271	−14.8057	−14.7293	−14.6482	−14.6037

3. Nonsymmetric Pentadiagonal Matrices from Extremal Eigenvalues and an Eigenpair

In this section, we show that each component of an eigenvector associated with the largest eigenvalue of matrix B is a linear combination of the first and second components. We then give sufficient conditions for the construction of a nonsymmetric pentadiagonal matrix from (3), the extreme eigenvalues of its leading principal submatrices, an eigenpair, and two prescribed entries.

Remark 4. *Note that if $(\lambda_n^{(n)}, \mathbf{x})$ is an eigenpair of a nonsymmetric pentadiagonal matrix B of the form (3), we have*

$$B\mathbf{x} = \lambda_n^{(n)}\mathbf{x},$$

equivalently

$$
\begin{aligned}
a_1x_1 + b_1x_2 + b_2x_3 &= \lambda_n^{(n)}x_1, && (25)\\
c_1x_1 + a_2x_2 + b_3x_4 &= \lambda_n^{(n)}x_2, && (26)\\
c_{j-1}x_{j-2} + a_jx_j + b_{j+1}x_{j+2} &= \lambda_n^{(n)}x_j,\ j = 3,\dots,n-2, && (27)\\
c_{n-2}y_{n-3} + a_{n-1}x_{n-1} + b_nx_n &= \lambda_n^{(n)}x_{n-1}, && (28)\\
c_{n-1}x_{n-2} + c_nx_{n-1} + a_nx_n &= \lambda_n^{(n)}x_n.
\end{aligned}
$$

Next, we give a characterization of an eigenvector of the nonsymmetric pentadiagonal matrix B.

Lemma 3. *If $\mathbf{x}^T = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $(\lambda, \mathbf{x})$ is an eigenpair of the nonsymmetric pentadiagonal matrix B of the form (3), then $|x_1| + |x_2| > 0$ and*

$$
x_j = \begin{cases}
\dfrac{r_{j-2}(\lambda)x_1 - s_{j-2}(\lambda)b_1x_2}{\prod_{\ell=1}^{\lfloor \frac{j}{2} \rfloor} b_{2\ell}}, & \text{for } j \text{ odd} \\[2ex]
\dfrac{r_{j-2}(\lambda)x_2 - s_{j-2}(\lambda)c_1x_1}{\prod_{\ell=1}^{\lfloor \frac{j-1}{2} \rfloor} b_{2\ell+1}}, & \text{for } j \text{ even}
\end{cases} \qquad (29)
$$

where $r_{-1} = r_0(\lambda) = 1$, $r_1(\lambda) = \lambda - a_1$, $r_2(\lambda) = \lambda - a_2$,

$$r_j(\lambda) = (\lambda - a_j)r_{j-2}(\lambda) - b_{j-1}^2 r_{j-4}(\lambda), \quad j = 3, \ldots, n-2,$$

and $s_1(\lambda) = s_2(\lambda) = 1$, $s_3(\lambda) = \lambda - a_3$, $s_4(\lambda) = \lambda - a_4$,

$$s_j(\lambda) = (\lambda - a_j)s_{j-2}(\lambda) - b_{j-1}^2 s_{j-4}(\lambda), \quad j = 5, \ldots, n-2.$$

Proof. From (25) and (26), we have

$$x_3 = \frac{r_1(\lambda)x_1 - s_1(\lambda)b_1x_2}{b_2}, \quad x_4 = \frac{r_2(\lambda)x_2 - s_2(\lambda)c_1x_1}{b_3}.$$

Then, from (27) for $j = 3$, we have

$$x_5 = \frac{r_3(\lambda)x_1 - s_3(\lambda)b_1x_2}{b_2b_4}.$$

Now, suppose that (29) is true for $j = 5, \ldots, k$. Then, if k is odd,

$$\begin{aligned} x_{k+2} &= \frac{(\lambda - a_k)}{b_{k+1}}\left[\frac{r_{k-2}(\lambda)x_1 - s_{k-2}(\lambda)b_1x_2}{\beta_k}\right] - \frac{c_{k-1}}{b_{k+1}}\left[\frac{r_{k-4}(\lambda)x_1 - s_{k-4}(\lambda)b_1x_2}{\beta_{k-2}}\right] \\ &= \frac{(\lambda - a_k)r_{k-2}(\lambda) - b_{k-1}c_{k-1}r_{k-4}(\lambda)}{\beta_{k+1}}x_1 - \frac{(\lambda - a_k)s_{k-2}(\lambda) - b_{k-1}c_{k-1}s_{k-4}(\lambda)}{\beta_{k+1}}b_1x_1 \\ &= \frac{r_k(\lambda)x_1 - s_k(\lambda)b_1x_1}{\beta_{k+1}}, \end{aligned}$$

where $\beta_{k+1} = \prod_{\ell=1}^{\lfloor \frac{k}{2} \rfloor} b_{2\ell}$. It is similar if k even, since $\mathbf{x}$ is an eigenvector of B, $\mathbf{x} \neq 0$. If $x_1 = x_2 = 0$, then from (29), all other entries of $\mathbf{x}$ must be zero. Therefore $|x_1| + |x_2| > 0$. □

Lemma 4. *Let B be an $n \times n$ nonsymmetric pentadiagonal matrix of the form (3) and let B_j be the $j \times j$ leading principal submatrix of B with characteristic polynomial $P_j(\lambda) = \det(\lambda I_j - B_j)$, $j = 1, 2, \ldots, n$. Then the sequence $\{P_j(\lambda)\}_{j=1}^n$ satisfies the recurrence relation:*

$$\begin{aligned} P_1(\lambda) &= \lambda - a_1, && (30)\\ P_j(\lambda) &= (\lambda - a_j)P_{j-1}(\lambda) - b_{j-1}c_{j-1}Q_{j-1}(\lambda), \ j = 2, 3, \ldots, n-1, && (31) \end{aligned}$$

$$\begin{aligned} P_n(\lambda) =& (\lambda - a_n)P_{n-1}(\lambda) - b_nc_nP_{n-2}(\lambda) - b_{n-1}c_{n-1}Q_{n-1}(\lambda) \\ & - \prod_{k=1}^{\lceil \frac{n}{2} \rceil} b_{2k-1} \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} c_{2k} - \prod_{k=1}^{\lceil \frac{n}{2} \rceil} c_{2k-1} \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} b_{2k} \end{aligned} \tag{32}$$

where $Q_1(\lambda) = 1$ and $Q_{j-1}(\lambda)$, $j = 3, \ldots, n$ is the characteristic polynomial of the principal submatrix of A obtained by deleting the $(j-2)$-th row and column of the leading principal submatrix A_{j-1}.

Theorem 3. *Let $\left\{\lambda_1^{(j)}, \lambda_j^{(j)}\right\}_{j=1}^n$ be a set of $2n-1$ real numbers, d_1 and d_2 be two positive numbers, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be a positive vector that satisfies (11) with*

$$d_1 \neq (\lambda_n^{(n)} - \lambda_1^{(1)})\frac{x_1}{x_3}. \tag{33}$$

If

$$(-1)^{n-1}[W_n + b_nd_2V_n + 2\gamma U_n] > 0, \tag{34}$$

then there exists a unique nonsymmetric pentadiagonal matrix B of the form (3), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the smallest and largest eigenvalues of the leading principal submatrix B_j of B, $j = 1, 2, \ldots, n$, $(\lambda_n^{(n)}, \mathbf{x})$ is an eigenpair of B, and $b_1 = d_1$ and $c_n = d_2$.

Proof. It is immediate that $a_1 = \lambda_1^{(1)}$. To show that the existence of a nonsymmetric pentadiagonal matrix B with the required properties is equivalent to proving that the systems of equations

$$\begin{cases} P_j\left(\lambda_i^{(j)}\right) = 0, \quad \text{for } j = 1, 2, \ldots, n,\ i = 1, j, \\ B\mathbf{x} = \lambda_n^{(n)}\mathbf{x}, \end{cases} \tag{35}$$

where $P_j(\lambda) = \det(\lambda I_j - B_j)$, $j = 1, 2, \ldots, n$ satisfies Lemma 4 and has real solutions a_j, b_j, and c_j with $b_j c_j > 0$.

The first expression of the system (35) can be written as

$$\begin{cases} P_j\left(\lambda_1^{(j)}\right) = \left(\lambda_1^{(j)} - a_j\right) P_{j-1}\left(\lambda_1^{(j)}\right) - b_{j-1}c_{j-1}Q_{j-1}\left(\lambda_1^{(j)}\right) = 0, \\ P_j\left(\lambda_j^{(j)}\right) = \left(\lambda_j^{(j)} - a_j\right) P_{j-1}\left(\lambda_j^{(j)}\right) - b_{j-1}c_{j-1}Q_{j-1}\left(\lambda_j^{(j)}\right) = 0, \end{cases} \tag{36}$$

for $j = 2, 3, \ldots, n-1$. For n being even

$$\begin{cases} P_n\left(\lambda_1^{(n)}\right) = \left(\lambda_1^{(n)} - a_n\right) P_{n-1}\left(\lambda_1^{(n)}\right) - b_n c_n P_{n-2}\left(\lambda_1^{(n)}\right) \\ \qquad\qquad -b_{n-1}c_{n-1}Q_{n-1}\left(\lambda_1^{(n)}\right) - 2\gamma = 0, \\ P_n\left(\lambda_n^{(n)}\right) = \left(\lambda_n^{(n)} - a_n\right) P_{n-1}\left(\lambda_n^{(n)}\right) - b_n c_n P_{n-2}\left(\lambda_n^{(n)}\right) \\ \qquad\qquad -b_{n-1}c_{n-1}Q_{n-1}\left(\lambda_n^{(n)}\right) - 2\gamma = 0, \end{cases} \tag{37}$$

since (6) holds. For n being odd, a similar system to (37) is obtained.

Moreover, from the second expression of system (35), and by (25)–(27) and (33), we obtain

$$b_2 = \left(\lambda_n^{(n)} - a_1\right)\frac{x_1}{x_3} - d_1\frac{x_2}{x_3} \neq 0, \tag{38}$$

$$b_3 = \left(\lambda_n^{(n)} - a_2\right)\frac{x_2}{x_4} - c_1\frac{x_1}{x_4}, \tag{39}$$

$$b_{j+1} = \left(\lambda_n^{(n)} - a_j\right)\frac{x_j}{x_{j+2}} - c_{j-1}\frac{x_{j-2}}{x_{j+2}}, \quad j = 2, 3, \ldots, n-2, \tag{40}$$

$$b_n = \left(\lambda_n^{(n)} - a_n\right)\frac{x_{n-1}}{x_n} - c_{n-2}\frac{x_{n-3}}{x_n}. \tag{41}$$

Then, from (36), we obtain

$$c_{j-1} = -\frac{W_j}{b_{j-1}R_j} \tag{42}$$

and

$$a_j = \frac{Z_j}{R_j}, \tag{43}$$

for $j = 2, \ldots, n-1$.

By Lemma 2 and condition (11), we have

$$b_{j-1}c_{j-1} = -\frac{(-1)^{j-1}W_j}{(-1)^{j-1}R_j} > 0,$$

for $j = 2, \ldots, n-1$.

Now, from (37) and Lemma 4, it follows that

$$c_{n-1} = \frac{1}{b_{n-1}} \frac{W_n + b_n d_2 V_n + 2\gamma U_n}{-R_n}, \tag{44}$$

and

$$a_n = \frac{Z_n + b_n d_2 T_n + 2\gamma S_n}{R_n}. \tag{45}$$

Finally, from (34) we conclude $b_{n-1}c_{n-1} > 0$. □

Example 3. *Given random real numbers in the Table 3*

Table 3. Extreme spectral data.

j	**1**	**2**	**3**	**4**	**5**	**6**
$\lambda_1^{(j)}$	1	−0.4142	−1	−1.1357	−1.236	−1.5615
$\lambda_j^{(j)}$	1	2.4142	3	3.1357	3.236	3.5615

and $d_1 = 3$, $d_2 = 1$, *and the vector*

$$\mathbf{x} = (2,\ 1.2807,\ 1.2807,\ 1.2807,\ 1.2807,\ 0.0177)^T$$

satisfying the conditions of Theorem 3. According to Formulas (38)–(45), there exist a nonsymmetric pentadiagonal matrix B of the form (3) given by

$$B = \begin{pmatrix} 1 & 3 & 3 & & & \\ 1 & 1 & 0 & 2 & & \\ 1 & 0 & 1 & 0 & 2 & \\ & 1 & 0 & 1 & 0 & 2 \\ & & 1 & 0 & 1 & 2 \\ & & & 1 & 1 & 1 \end{pmatrix}$$

whose spectra of its the leading principal submatrices B_j, $j = 1, \ldots, 6$, are:

$$\begin{aligned} \sigma(B_1) &= \{\mathbf{1}\}, \\ \sigma(B_2) &= \{-\mathbf{0.4142}, \mathbf{2.4142}\}, \\ \sigma(B_3) &= \{-\mathbf{1}, 1, \mathbf{3}\}, \\ \sigma(B_4) &= \{-\mathbf{1.1357}, 0.3378, 1.6621, \mathbf{3.1357}\}, \\ \sigma(B_5) &= \{-\mathbf{1.236},\ 0,\ 1,\ 2,\ \mathbf{3.236}\}, \\ \sigma(B_6) &= \{-\mathbf{1.5615},\ -0.5615,\ 0,\ 2,\ 2.5615,\ \mathbf{3.5615}\}, \end{aligned}$$

and $B\mathbf{x} = (\mathbf{3.5615})\mathbf{x}$.

Example 4. *In Table 4, we show the errors in the construction of nonsymmetric pentadiagonal matrices of the form (3), from uniformly distributed random numbers using the Matlab rand function, which satisfy conditions (11), (33), and (34). Here, we adopt the notations from Example 2. In addition,* $e_x = \log(\|\hat{B}x - \hat{\lambda}_n^{(n)} x\| / \|\lambda_n^{(n)} x\|)$.

Table 4. Relative errors in the construction of a nonsymmetric pentadiagonal matrix.

n	**5**	**10**	**15**	**20**	**30**	**40**	**50**
e_λ	−14.6143	−14.4017	−14.2956	−14.0792	−13.8641	−13.6728	−12.9366
e_x	−14.4871	−14.2250	−13.8901	−13.0411	−12.7655	−12.4002	−12.1874

4. Conclusions

In this paper, we give sufficient conditions for the existence of some symmetric and nonsymmetric pentadiagonal matrices considering the extreme eigenvalues of their leading principal submatrices in the first case, and, additionally, an eigenvector for the second case is considered. Our results, being constructive, provide an algorithm to determine the solution matrix.

Author Contributions: All the authors have contributed equally to the work. All authors have read and agreed to the published version of the manuscript.

Funding: S. Arela-Pérez was supported by Universidad de Tarapacá, Arica, Chile, Proyecto Mayor de Investigación Científica y Tecnológica UTA Mayor 4761-22. Charlie Lozano was supported by IIIMAT of the Universidad Mayor de San Andrés, La Paz, Bolivia, within the project Grafos, Matrices y Sistemas Dinámicos 2022. H. Nina was partially supported by Programa Regional MATHAMSUD, MATH2020003 and Universidad de Antofagasta UA INI-17-02. H. Nina also thanks the hospitalidad of IIMAT of Universidad Mayor de San Andrés, La Paz, Bolivia, where some of this work was done. H. Pickmann-Soto was supported by Universidad de Tarapacá, Arica, Chile, Proyecto Mayor de Investigación Científica y Tecnológica UTA Mayor 4762-22.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referees for their constructive suggestions that improved the final version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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