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Bounds for Incomplete Confluent Fox–Wright Generalized Hypergeometric Functions

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Abstract: We establish several new functional bounds and uniform bounds (with respect to the variable) for the lower incomplete generalized Fox–Wright functions by means of the representation formulae for the McKay I_ν Bessel probability distribution's cumulative distribution function. New cumulative distribution functions are generated and expressed in terms of lower incomplete Fox–Wright functions and/or generalized hypergeometric functions, whilst in the closing part of the article, related bounding inequalities are obtained for them.

Keywords: modified Bessel functions of the first kind; McKay's I_ν Bessel distribution; lower incomplete Fox–Wright functions; cumulative distribution function; functional bounding inequality

MSC: 26D15; 33C20; 33C47; 33E20; 60E05

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Citation: Pogány, T.K. Bounds for Incomplete Confluent Fox–Wright Generalized Hypergeometric Functions. *Mathematics* **2022**, *10*, 3106. https://doi.org/10.3390/math10173106

Academic Editor: Sitnik Sergey

Received: 13 August 2022
Accepted: 25 August 2022
Published: 29 August 2022

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1. Introduction and Motivation

The incomplete special functions having integral expressions with nonnegative integrands are obviously bounded above with their complete variants, when the incomplete variants integration domain is contained in the complete variant's integration domain, provided the considered integrals converge, as it happens, for instance, in the case of lower and upper incomplete gamma functions (p. 174, Equations (8).2.1-2, [1])

$$\gamma(p,x) = \int_0^x t^{p-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \qquad \Gamma(p,x) = \int_x^\infty t^{p-1}\,\mathrm{e}^{-t}\,\mathrm{d}t\,, \qquad x,\, \Re(p) > 0, \tag{1}$$

respectively, whose sum gives the Euler function of the second kind called also (complete) gamma function (p. 136, Equation (5).2.1, [1]):

$$\gamma(p,x) + \Gamma(p,x) = \Gamma(p) = \int_0^\infty t^{p-1}\,\mathrm{e}^{-t}\,\mathrm{d}t\,, \qquad \Re(p) > 0\,. \tag{2}$$

The straightforward consequence of these relations is

$$\max\big(\gamma(x,p), \Gamma(p,x)\big) \le \Gamma(p)\,.$$

However, the question of the existence of more precise upper and/or lower bounds for the incomplete versions of special functions is frequent in applications.

In this note, we derive upper bounds for a set of special functions coming from the hypergeometric family of functions, the class of lower incomplete confluent Fox–Wright generalized hypergeometric functions, which nowadays have numerous appearances in the mathematical literature, see, e.g., [2–5] and the relevant titles therein.

We start with the definition of the incomplete Fox–Wright function. The Fox–Wright generalized hypergeometric function consisting of p numerator parameter couples (a_1, A_1),

$\cdots, (a_p, A_p)$ and q denominator parameter pairs $(b_1, B_1), \cdots, (b_q, B_q)$, possesses the series form (pp. 286–287, [6])

$$ {}_p\Psi_q\Big[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p) \\ (b_1, B_1), \cdots, (b_q, B_q) \end{array} \Big| z \Big] = {}_p\Psi_q\Big[\begin{array}{c} (\mathbf{a}_p, \mathbf{A}_p) \\ (\mathbf{b}_q, \mathbf{B}_q) \end{array} \Big| z \Big] = \sum_{n\geq 0} \frac{\prod_{j=1}^{p} \Gamma(a_j + nA_j)}{\prod_{j=1}^{q} \Gamma(b_j + nB_j)} \frac{z^n}{n!}, \tag{3} $$

where $A_j, B_k \geq 0$, $j = 1, \ldots, p$, $k = 1, \ldots, q$. The series (3) converges for all $z \in \mathbb{C}$ when

$$ \Delta := 1 + \sum_{j=1}^{q} B_j - \sum_{k=1}^{p} A_k > 0. $$

When $\Delta = 0$, the series in (3) converges for $|z| < \nabla$ and $|z| = \nabla$ under the condition $\Re(\mu) > 1/2$, where

$$ \nabla := \left(\prod_{i=1}^{p} A_i^{-A_i} \right) \left(\prod_{j=1}^{q} B_j^{B_j} \right), \qquad \mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}. $$

Taking $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$ in (3), the Fox–Wright function reduces to the generalized hypergeometric function ${}_pF_q$, up to the multiplicative constant in the following way:

$$ {}_pF_q\Big[\begin{array}{c} \mathbf{a}_p \\ \mathbf{b}_q \end{array} \Big| z \Big] = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \, {}_p\Psi_q\Big[\begin{array}{c} (\mathbf{a}_p, \mathbf{1}) \\ (\mathbf{b}_q, \mathbf{1}) \end{array} \Big| z \Big]. \tag{4} $$

Now, we denote by ${}_p\Psi_q^{(\gamma)}[\cdot]$ the lower incomplete Fox–Wright function by replacing *one* gamma function out of p in the product in the numerator of (3) with a lower incomplete gamma function $\gamma(\mu + \cdot M, x)$, in which the new parameters μ, M, x take place. So, by this change, the defining power series (3) becomes (p. 196, Equation (6), [4]) (also see (p. 982, [5]))

$$ {}_p\Psi_q^{(\gamma)}\Big[\begin{array}{c} (\mu, M, x), (\mathbf{a}_{p-1}, \mathbf{A}_{p-1}) \\ (\mathbf{b}_q, \mathbf{B}_q) \end{array} \Big| z \Big] = \sum_{n\geq 0} \frac{\gamma(\mu + nM, x) \prod_{j=1}^{p-1} \Gamma(a_j + nA_j)}{\prod_{j=1}^{q} \Gamma(b_j + nB_j)} \frac{z^n}{n!}. \tag{5} $$

The parameters $M, A_j, B_k > 0$ should satisfy the constraint

$$ \Delta^{(\gamma)} = 1 + \sum_{j=1}^{q} B_j - M - \sum_{j=1}^{p-1} A_j \geq 0, $$

while the other convergence conditions remain the same as the ones for the complete Fox–Wright (3), which we have for $x = \infty$ in (5). We point out that the upper incomplete Fox–Wright generalized hypergeometric function ${}_p\Psi_q^{(\Gamma)}$ is presented with associated comments in Section 5 under A.

The probability distributions involving Bessel functions were pioneered by McKay [7] considering two classes of continuous distributions involving modified Bessel functions of the first and second kinds I_ν and K_ν, which we call today Bessel function distributions. However, we observe here McNolty's version [8] of McKay's I_ν Bessel distribution. The random variable (rv) X defined on a probability space $(\Omega, \mathscr{A}, \mathsf{P})$ behaves according to

McNolty's variant of McKay's distribution when the probability distribution function (pdf) is of the form (p. 496, Equation (13), [8])

$$f_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)}\, \mathrm{e}^{-bx} x^\nu I_\nu(ax), \qquad x \geq 0, \tag{6}$$

defined for all $\nu > -1/2$ and $b > a > 0$. The related cumulative distribution function (cdf) reads

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \int_0^x \mathrm{e}^{-bt} t^\nu I_\nu(at)\,\mathrm{d}t, \qquad x \geq 0, \tag{7}$$

where the power series form of the modified Bessel function of the first kind is (p. 13, [9])

$$I_\nu(x) = \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)\, n!}\left(\frac{x}{2}\right)^{2n+\nu}.$$

We write this correspondence as $X \sim \mathrm{McKayI}(a,b,\nu)$. We consider McNolty's pdf (6) and cdf (7) in our calculations.

New expressions for cdf of rv $X \sim \mathrm{McKayI}(a,b,\nu)$ were given recently in [2]. In turn, these results imply several by-products. For instance, we can deduce several functional and uniform bounds for the incomplete generalized Fox–Wright functions and other hypergeometric-type functions which are the building blocks of cdfs; we discuss these elsewhere. We derive the bounds by simple methods applying certain known and less known properties of cdfs.

2. The First Set of Results

Here, we report on a uniform and a functional bound for the incomplete confluent Fox–Wright function ${}_1\Psi_1^{(\gamma)}$ and for the generalized hypergeometric function ${}_1F_2[\cdot]$, consult (4).

Theorem 1. *For all $b > a > 0$ and $\nu > -1/2$, we have*

$${}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+1, 2, bx) \\ (\nu+1, 1) \end{array} \middle| \frac{a^2}{4b^2}\right] \leq \frac{4^\nu \left(\frac{1}{2}\right)_\nu b^{2\nu+1}}{(b^2-a^2)^{\nu+\frac{1}{2}}}.$$

Moreover , for $\nu \geq 0$ and $a \geq 1$, the following holds:

$${}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+1, 2, bx) \\ (\nu+1, 1) \end{array} \middle| \frac{a^2}{4b^2}\right] \leq \frac{b^{2\nu}x^{2\nu}(1-\mathrm{e}^{-bx})}{(2\nu+1)\,\Gamma(\nu+1)}\, {}_1F_2\left[\begin{array}{c} \nu+\frac{1}{2} \\ \nu+1,\, \nu+\frac{3}{2} \end{array} \middle| \frac{a^2x^2}{4}\right]. \tag{8}$$

Proof. According to the result of Theorem 1 in [2], for the rv $X \sim \mathrm{McKayI}(a,b,\nu)$, the related cdf reads

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{2^{2\nu}\, b^{2\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}\, {}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+1, 2, bx) \\ (\nu+1, 1) \end{array} \middle| \frac{a^2}{4b^2}\right], \qquad x \geq 0. \tag{9}$$

From $F_I(x;a,b;\nu) \leq 1$, the assertion of the theorem immediately follows. As to the functional upper bound (8), we apply the estimate (Equation 8.10.2, [1])

$$\gamma(a,t) \leq \frac{t^{a-1}}{a}\left(1-\mathrm{e}^{-t}\right), \qquad a \geq 1,\, t > 0\,. \tag{10}$$

This bound, taken in (9) for $a = 2\nu + 1 + 2n$ and $t = bx$, increases the sum and implies

$$\begin{aligned}
{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu+1,2,bx)\\(\nu+1,1)\end{array}\middle|\frac{a^2}{4b^2}\right] &\le \sum_{n\ge 0}\frac{(bx)^{2(\nu+n)}\left(1-\mathrm{e}^{-bx}\right)}{4^n(2\nu+1+2n)\Gamma(\nu+1+n)\,n!}\left(\frac{a^2}{b^2}\right)^n \\
&= \frac{(bx)^{2\nu}\left(1-\mathrm{e}^{-bx}\right)}{2\,\Gamma(\nu+1)}\sum_{n\ge 0}\frac{\Gamma(\nu+\frac{1}{2}+n)\,(ax)^{2n}}{4^n\,(\nu+1)_n\,\Gamma(\nu+\frac{3}{2}+n)\,n!} \\
&= \frac{(bx)^{2\nu}\left(1-\mathrm{e}^{-bx}\right)}{(2\nu+1)\Gamma(\nu+1)}\sum_{n\ge 0}\frac{(\nu+\frac{1}{2})_n\left(\frac{ax}{2}\right)^{2n}}{(\nu+1)_n(\nu+\frac{3}{2})_n\,n!}\,,
\end{aligned}$$

which is equivalent to the stated inequality (8). Finally, the constraint $1 \le a = 2\nu + 1 + 2n$ in (10), which holds for all $n \in \mathbb{N}_0$, shows that $\nu \ge 0$ is indeed the parameter range. The proof is complete. □

In the next part of this section, we establish a bilateral functional bound upon the lower incomplete confluent Fox–Wright function. In turn, the upper bound contains the same incomplete confluent Fox–Wright function whose argument is reciprocal.

Theorem 2. *Let $b > a > 0$ and $2\nu + 1 > 0$. Then, for all $x \ge 1$, the two-sided functional inequality holds:*

$$ {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] \le {}_1\Psi_1^{(\gamma)}[bx] \le {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] + \frac{2\Gamma(2\nu)\,b^{2\nu+1}}{\Gamma(\nu)\,(b^2-a^2)^{\nu+\frac{1}{2}}}\,, \tag{11}$$

where

$$ {}_1\Psi_1^{(\gamma)}[z] := {}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu+1,2,z)\\(\nu+1,1)\end{array}\middle|\frac{a^2}{4b^2}\right].$$

Proof. Let $X \sim F(x)$ be a continuous nonnegative random variable with $F(0) = 0$. Consider the rv $\max\left(X, X^{-1}\right)$. For the related cdf, the following holds:

$$\mathsf{P}\left\{\max\left(X,X^{-1}\right) < x\right\} = \mathsf{P}\left\{x^{-1} < X < x\right\} = F(x) - F(x^{-1})\,,$$

which implies that (p. 45, 2.1.8, [10])

$$G(x) = \begin{cases} F(x) - F(x^{-1}), & x \ge 1 \\ 0, & x < 1 \end{cases}$$

is also a cdf. Replacing the general rv with $X \sim \mathrm{McKayI}(a,b,\nu)$ and keeping our standard parameter space $b > a > 0$, $2\nu + 1 > 0$, we obtain that

$$0 \le G_I(x) = F_I(x;a,b;\nu) - F_I(x^{-1};a,b;\nu) \le 1\,, \qquad x \ge 1\,. \tag{12}$$

Hence,

$$ {}_1\Psi_1^{(\gamma)}[bx] - {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] \ge 0\,,$$

which implies the left-hand-side inequality in (11). Next, from $G_I(x) \le 1$, we conclude

$$\begin{aligned}
{}_1\Psi_1^{(\gamma)}[bx] &\le {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] + \frac{2^{2\nu}b^{2\nu+1}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,(b^2-a^2)^{\nu+\frac{1}{2}}} \\
&= {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] + \frac{2^{2\nu}b^{2\nu+1}\Gamma(\nu+\frac{1}{2})\Gamma(\nu+1)}{\sqrt{\pi}\,\Gamma(\nu+1)\,(b^2-a^2)^{\nu+\frac{1}{2}}}
\end{aligned}$$

$$= {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] + \frac{b^{2\nu+1}\Gamma(2\nu+1)}{\Gamma(\nu+1)\,(b^2-a^2)^{\nu+\frac{1}{2}}} \tag{13}$$

$$= {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] + \frac{2\,b^{2\nu+1}\Gamma(2\nu)}{\Gamma(\nu)(b^2-a^2)^{\nu+\frac{1}{2}}}\,, \tag{14}$$

where (13) is obtained by the Legendre duplication formula for the gamma function (Equation 5.5.5, [1])

$$\Gamma(2\nu) = \frac{2^{2\nu-1}}{\sqrt{\pi}}\,\Gamma(\nu)\,\Gamma\big(\nu+\tfrac{1}{2}\big), \qquad -2\nu \notin \mathbb{N}_0\,.$$

This explains at the same time that the quotient of gamma functions is well defined for the nonpositive values of $\nu \in (-\frac{1}{2}, 0]$ in (14). The rest is obvious. □

3. The Second Set of Results

The rv $X\sim\mathrm{McKayI}(a,b,\nu)$ possesses a counterpart result to the representation Formula (9) for the related cdf, also in terms of the lower incomplete confluent Fox–Wright generalized hypergeometric function ${}_1\Psi_1^{(\gamma)}$, the exponential function and the modified Bessel function of the first kind I_ν.

We establish bounding inequalities and monotonicity results applying the simple properties of the cdfs used in the previous section. Therefore, according to Theorem 1 of [2], we have

$$F_I(x;a,b;\nu) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{2^{2\nu-1}\,b^{2\nu+1}\Gamma\Big(\nu+\frac{1}{2}\Big)}\left\{{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu,2,bx)\\(\nu,1)\end{array}\,\middle|\,\frac{a^2}{4b^2}\right] - \frac{2^{\nu-1}b^{2\nu}\,x^\nu}{a^\nu}\,\mathrm{e}^{-bx}I_\nu(ax)\right\}. \tag{15}$$

for all $x \geq 0$.

Theorem 3. *For all $b > a > 0$, $\nu > -1/2$ and for all $x \geq 0$, it holds true that*

$$\frac{1}{2}\left(\frac{2b^2x}{a}\right)^\nu \mathrm{e}^{-bx}I_\nu(ax) \leq {}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu,2,bx)\\(\nu,1)\end{array}\,\middle|\,\frac{a^2}{4b^2}\right]$$
$$\leq \frac{1}{2}\left(\frac{2b^2x}{a}\right)^\nu \mathrm{e}^{-bx}I_\nu(ax) + \frac{2^{2\nu-1}b^{2\nu+1}\,(\frac{1}{2})_\nu}{(b^2-a^2)^{\nu+\frac{1}{2}}}\,.$$

Moreover, when $x \geq 1$, we have the bilateral functional inequality

$$H(x) \leq {}_1\Psi_1^{(\gamma)}[bx] - {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] \leq H(x) + \frac{\Gamma(2\nu)\,b^{2\nu+1}}{\Gamma(\nu)\,(b^2-a^2)^{\nu+\frac{1}{2}}}\,, \tag{16}$$

where

$$H(x) := \frac{1}{2}\left(\frac{2b^2}{a}\right)^\nu\left\{x^\nu\mathrm{e}^{-bx}\,I_\nu(ax) - x^{-\nu}\mathrm{e}^{-\frac{b}{x}}I_\nu\left(\frac{a}{x}\right)\right\},$$

and ${}_1\Psi_1^{(\gamma)}[z]$ denotes the same function as in Theorem 2.

Proof. Applying $0 \leq F_I(x;a,b;\nu) \leq 1$ to the representation Formula (15), the first statement of theorem immediately follows.

As to the bilateral inequality (16), we take into account the same property, now for the cdf $G_I(x)$, defined in (12). After some routine calculations, following the steps of the proof of Theorem 2, we arrive at the assertion (16). □

4. The Third Set of Results

Let us treat $F_I(x;a,b;\nu)$ by virtue of the property which holds for any continuous baseline cdf $F(x)$ and states that (p. 45, Equation (2).1.7, [10])

$$F_1(x) = \frac{1}{h}\int_x^{x+h} F(t)\,\mathrm{d}t\,, \qquad h > 0\,,$$

is also a cdf. Consequently, we can consider the newly generated cdf

$$F_{I,1}^{(1)}(x;h) = \frac{1}{h}\int_x^{x+h} F_I(t;a,b;\nu)\,\mathrm{d}t\,, \qquad h > 0\,. \tag{17}$$

The main result in this part of the article is the special function representation formula for the generated cdf $F_{I,1}(x;h)$.

Theorem 4. *Let the* rv $X\sim\mathrm{McKayI}(a,b,\nu)$*, and the* cdf $F_{I,1}^{(1)}(x;h)$ *defined by* (17) *be. Then, for all* $b > a > 0$, $\nu > -\frac{1}{2}$ *and* $x, h \geq 0$ *we have*

$$\begin{aligned} F_{I,1}^{(1)}(x;h) = \frac{C_\nu}{h}\Bigg\{ &(x+h)\,{}_2\Psi_2^{(\gamma)}\left[\begin{matrix} (2\nu+2, 2, b(x+h)),\,(\nu+\frac{1}{2},1) \\ (\nu+1,1),\,(\nu+\frac{3}{2},1)\end{matrix}\,\Bigg|\,\frac{a^2}{4b^2}\right] \\ &- \frac{2}{b}\,{}_1\Psi_1^{(\gamma)}\left[\begin{matrix} (2\nu+2, 2, b(x+h)) \\ (\nu+1,1)\end{matrix}\,\Bigg|\,\frac{a^2}{4b^2}\right] \\ &+ \frac{2[b(x+h)]^{2\nu+2}\mathrm{e}^{-b(x+h)}}{b\,(2\nu+1)\Gamma(\nu+1)}\,{}_1F_2\left[\begin{matrix} \nu+\frac{1}{2} \\ \nu+1,\,\nu+\frac{3}{2}\end{matrix}\,\Bigg|\,\frac{a^2}{4}(x+h)^2\right] \\ - x\,{}_2\Psi_2^{(\gamma)}&\left[\begin{matrix} (2\nu+2, 2, bx),\,(\nu+\frac{1}{2},1) \\ (\nu+1,1),\,(\nu+\frac{3}{2},1)\end{matrix}\,\Bigg|\,\frac{a^2}{4b^2}\right] \\ &+ \frac{2}{b}\,{}_1\Psi_1^{(\gamma)}\left[\begin{matrix} (2\nu+2, 2, bx) \\ (\nu+1,1)\end{matrix}\,\Bigg|\,\frac{a^2}{4b^2}\right] \\ &- \frac{2(bx)^{2\nu+2}\mathrm{e}^{-bx}}{b\,(2\nu+1)\Gamma(\nu+1)}\,{}_1F_2\left[\begin{matrix} \nu+\frac{1}{2} \\ \nu+1,\,\nu+\frac{3}{2}\end{matrix}\,\Bigg|\,\frac{a^2}{4}x^2\right]\Bigg\}, \end{aligned}$$

where

$$C_\nu = \frac{\sqrt{\pi}\,(b^2-a^2)^{\nu+\frac{1}{2}}}{4^\nu b^{2\nu+1}\,\Gamma(\nu+\frac{1}{2})}\,.$$

Proof. Because the baseline cdf $F_I(x;a,b;\nu)$ contains the incomplete confluent Fox–Wright term ${}_1\Psi_1^{(\gamma)}$, which is built by $\gamma(\cdot, bx)$, we should know the integral of this function, see (5). As (Equation 8.5.1, [1])

$$\gamma(\alpha, z) = \frac{z^\alpha}{\alpha}\,{}_1F_1\left[\begin{matrix}\alpha \\ \alpha+1\end{matrix}\,\Bigg|\,-z\right], \qquad -\alpha \notin \mathbb{N}_0\,,$$

we conclude

$$\int_0^x \gamma(\alpha,z)\,\mathrm{d}z = \frac{1}{\alpha}\sum_{k\geq 0}\frac{(-1)^k\,(\alpha)_k}{(\alpha+1)_k\,k!}\,\frac{x^{\alpha+k+1}}{\alpha+k+1} = \frac{x^{\alpha+1}}{\alpha(\alpha+1)}\sum_{k\geq 0}\frac{(\alpha)_k\,(-x)^k}{(\alpha+2)_k\,k!} = \frac{x^{\alpha+1}}{\alpha(\alpha+1)}\,{}_1F_1\left[\begin{matrix}\alpha \\ \alpha+2\end{matrix}\,\Bigg|\,-x\right],$$

that is

$$\int_0^x \gamma(\alpha,z)\,\mathrm{d}z = \frac{x-\alpha}{\alpha}\,\gamma(\alpha+1,x) + \frac{1}{\alpha}\,x^{\alpha+1}\,\mathrm{e}^{-x}\,. \tag{18}$$

The final formula follows by (p. 583, Equation (7).11.3.2, [11]).

By virtue of relations (9), (17) and (18), we infer:

$$\begin{aligned}
F_{I,1}^{(1)}(x;h) &= \frac{(b^2-a^2)^{\nu+\frac12}}{b^{2\nu+2}\,\Gamma(2\nu+1)\,h}\sum_{n\geq 0}\frac{a^{2n}\int\limits_{bx}^{b(x+h)}\gamma(2\nu+1+2n,t)\,\mathrm{d}t}{(2b)^{2n}\,(\nu+1)_n\,n!}\\
&= \frac{(b^2-a^2)^{\nu+\frac12}}{b^{2\nu+2}\,\Gamma(2\nu+1)\,h}\sum_{n\geq 0}\frac{\left(a^2/b^2\right)^n}{4^n\,(\nu+1)_n\,n!}\bigg\{\Big(\frac{b(x+h)}{2\nu+1+2n}-1\Big)\\
&\qquad\cdot\gamma\big(2\nu+2+2n,b(x+h)\big)+\frac{\big(b(x+h)\big)^{2\nu+2+2n}}{2\nu+1+2n}\,\mathrm{e}^{-b(x+h)}\\
&\qquad-\Big(\frac{bx}{2\nu+1+2n}-1\Big)\,\gamma\big(2\nu+2+2n,bx\big)-\frac{(bx)^{2\nu+2+2n}}{2\nu+1+2n}\,\mathrm{e}^{-bx}\bigg\}\\
&=: I_1(x+h)-I_2(x+h)+I_3(x+h)-I_1(x)+I_2(x)-I_3(x)\,.
\end{aligned}$$

This linear combination of six series we separate and sum up. Thus, not changing the order of the outcoming series the first (fourth) series can be expressed in terms of the lower incomplete Fox–Wright function ${}_2\Psi_2^{(\gamma)}$. Indeed, comparing with (5), this results in

$$\begin{aligned}
I_1(t) &= \frac{\left(1-\frac{a^2}{b^2}\right)^{\nu+\frac12}t}{\Gamma(2\nu+1)\,h}\sum_{n\geq 0}\frac{1}{(\nu+1)_n\,n!}\,\frac{\gamma\big(2\nu+2+2n,b\,t\big)}{2\nu+1+2n}\,\left(\frac{a^2}{4b^2}\right)^n\\
&= \frac{\left(1-\frac{a^2}{b^2}\right)^{\nu+\frac12}\Gamma(\nu+1)\,t}{2\,\Gamma(2\nu+1)\,h}\sum_{n\geq 0}\frac{\gamma\big(2\nu+2+2n,b\,t\big)\,\Gamma(\nu+\frac12+n)}{\Gamma(\nu+1+n)\,\Gamma(\nu+\frac32+n)\,n!}\left(\frac{a^2}{4b^2}\right)^n\\
&= \frac{\left(1-\frac{a^2}{b^2}\right)^{\nu+\frac12}\Gamma(\nu+1)\,t}{2\,\Gamma(2\nu+1)\,h}\,{}_2\Psi_2^{(\gamma)}\left[\begin{matrix}(2\nu+2,2,b\,t),\,(\nu+\frac12,1)\\(\nu+1,1),\,(\nu+\frac32,1)\end{matrix}\,\bigg|\,\frac{a^2}{4b^2}\right],
\end{aligned}$$

where $t\in\{x+h,x\}$. Now, with the lower incomplete confluent Fox–Wright function ${}_1\Psi_1^{(\gamma)}$ we get

$$\begin{aligned}
I_2(t) &= \frac{\left(b^2-a^2\right)^{\nu+\frac12}}{b^{2\nu+2}\Gamma(2\nu+1)\,h}\sum_{n\geq 0}\frac{\gamma\big(2\nu+2+2n,b\,t\big)}{(\nu+1)_n\,n!}\left(\frac{a^2}{4b^2}\right)^n\\
&= \frac{\left(1-\frac{a^2}{b^2}\right)^{\nu+\frac12}\Gamma(\nu+1)}{\Gamma(2\nu+1)\,bh}\sum_{n\geq 0}\frac{\gamma\big(2\nu+2+2n,b\,t\big)}{\Gamma(\nu+1+n)\,n!}\left(\frac{a^2}{4b^2}\right)^n\\
&= \frac{\left(1-\frac{a^2}{b^2}\right)^{\nu+\frac12}\Gamma(\nu+1)}{\Gamma(2\nu+1)\,bh}\,{}_1\Psi_1^{(\gamma)}\left[\begin{matrix}(2\nu+2,2,b\,t)\\(\nu+1,1)\end{matrix}\,\bigg|\,\frac{a^2}{4b^2}\right],
\end{aligned}$$

where $t\in\{x+h,x\}$ covers two integrals as well.

Finally, the third (sixth) series becomes

$$I_3(t)=\frac{(b^2-a^2)^{\nu+\frac12}\,t^{2\nu+2}\mathrm{e}^{-bt}}{\Gamma(2\nu+2)\,h}\,{}_1F_2\left[\begin{matrix}\nu+\frac12\\\nu+1,\,\nu+\frac32\end{matrix}\,\bigg|\,\frac{a^2}{4}t^2\right],\qquad t\in\{x+h,x\}\,.$$

Collecting these integrals, the expression for $F_{I,1}(x;h)$ is confirmed. $\square$

Remark 1. *The constant C_ν, whose form we owe to McNolty's* pdf (6) *and which appears for the first time in the proof of* Theorem 1 (9)*, is introduced in* Theorem 4*. This constant remains unchanged throughout, in all further results to the end of the exposition.*

The same questions occur for the cdf which is reported as (p. 45, 2.1.7, [10])

$$F_2(x) = \frac{1}{2h}\int_{x-h}^{x+h} F(t)\,\mathrm{d}t\,, \qquad h>0\,,$$

when we take the baseline cdf $F_I(x;a,b;\nu)$. This gives

$$F_{I,1}^{(2)}(x;h) = \frac{1}{2h}\int_{x-h}^{x+h} F_I(t;a,b;\nu)\,\mathrm{d}t\,. \tag{19}$$

As a consequence of Theorem 4, we deduce the following specified result.

Corollary 1. *Let the* rv $X\sim\mathrm{McKayI}(a,b,\nu)$*, and the* cdf $F_{I,1}^{(2)}(x;h)$ *defined by* (19) *be. Then, for all $b>a>0$, $\nu>-\frac{1}{2}$ and $x\geq 0$; $h>0$ the following holds true:*

$$\begin{aligned}
F_{I,1}^{(2)}(x;h) = \frac{C_\nu}{h}\Bigg\{ & \frac{x+h}{2}\, {}_2\Psi_2^{(\gamma)}\left[\begin{array}{c} (2\nu+2,2,b(x+h)),\,(\nu+\frac{1}{2},1) \\ (\nu+1,1),\,(\nu+\frac{3}{2},1)\end{array}\Bigg|\,\frac{a^2}{4b^2}\right] \\
& - \frac{1}{b}\,{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+2,2,b(x+h)) \\ (\nu+1,1)\end{array}\Bigg|\,\frac{a^2}{4b^2}\right] \\
& + \frac{[b(x+h)]^{2\nu+2}\mathrm{e}^{-b(x+h)}}{b\,(2\nu+1)\Gamma(\nu+1)}\,{}_1F_2\left[\begin{array}{c} \nu+\frac{1}{2} \\ \nu+1,\ \nu+\frac{3}{2}\end{array}\Bigg|\,\frac{a^2}{4}(x+h)^2\right] \\
& - \frac{x-h}{2}\, {}_2\Psi_2^{(\gamma)}\left[\begin{array}{c} (2\nu+2,2,b(x-h)),\,(\nu+\frac{1}{2},1) \\ (\nu+1,1),\,(\nu+\frac{3}{2},1)\end{array}\Bigg|\,\frac{a^2}{4b^2}\right] \\
& + \frac{1}{b}\,{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+2,2,b(x-h)) \\ (\nu+1,1)\end{array}\Bigg|\,\frac{a^2}{4b^2}\right] \\
& - \frac{(b(x-h))^{2\nu+2}\mathrm{e}^{-b(x-h)}}{b\,(2\nu+1)\Gamma(\nu+1)}\,{}_1F_2\left[\begin{array}{c} \nu+\frac{1}{2} \\ \nu+1,\ \nu+\frac{3}{2}\end{array}\Bigg|\,\frac{a^2}{4}(x-h)^2\right]\Bigg\}.
\end{aligned}$$

Remark 2. *The first kind of two-sided inequalities which we can obtain are the straightforward consequences of $0\leq F_{I,1}^{(j)}(x;h)\leq 1$; $j=1,2$ for the same parameter space $b>a>0; 2\nu+1>0$, $h>0$ as in* Theorem 4 *and* Corollary 4.1*, respectively.*

On the other hand, generating with the baseline cdfs $F_{I,1}^{(j)}(x;h)$*—mimicking* (12)*—another associated* cdfs $G_{I,1}^{(j)}(x;h); j=1,2$*, a new set of bilateral inequalities follow for* $\mathrm{supp}(G_{I,1}^{(j)}) = [1,\infty); j=1,2$ *for positive $h>0$. These results can also be understood as a kind of monotonicity with respect to the argument x since the* cdfs *are monotone nondecreasing* per definitionem.

Finally, we introduce a generalization of (17). Let $r\in\mathbb{N}$. We are looking for the cdf $F_{I,r}^{(1)}(x;h)$, which we build by r-tuple successive application of the integral operator $F_{I,r}^{(1)}$ to the baseline cdf F_I defined by (17). This gives

$$F_{I,r}^{(1)} = \underbrace{F_{I,1}^{(1)}\circ\cdots\circ F_{I,1}^{(1)}}_{r}\,(F_I)\,, \tag{20}$$

where under $u\circ v$, we mean the composition of functions u, v. Obviously, $F_{I,r}^{(1)}(x;h)$ is a cdf as well.

Theorem 5. *For all $b > a > 0$; $2\nu + 1 > 0$; $r \in \mathbb{N}$ and $x \geq 0$, $h > 0$, we have*

$$F_{I,r}^{(1)}(x;h) = \frac{C_\nu}{r!}\left(\frac{x}{h}\right)^r \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^{k+1}}{(b\,x)^k} \left\{ {}_1\Psi_1^{(\gamma)}\left[\begin{matrix} (2\nu+1+k, 2, bx) \\ (\nu+1,1) \end{matrix} \,\Big|\, \frac{a^2}{4b^2} \right] - \left(1+\frac{h}{x}\right)^{r-k} {}_1\Psi_1^{(\gamma)}\left[\begin{matrix} (2\nu+1+k, 2, b(x+h)) \\ (\nu+1,1) \end{matrix} \,\Big|\, \frac{a^2}{4b^2} \right] \right\}.$$

Proof. Introducing the shorthand $\mathbf{d}x^{r-1} := \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}_{x_{r-1}}$, we rewrite the operator $F_{I,r}^{(1)}$ in (20), as the r-tuple integral

$$F_{I,r}^{(1)}(x;h) = \frac{C_\nu}{h^r} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+1+n)\, n!} \left(\frac{a^2}{4b^2}\right)^n \int_{\prod_{j=1}^{r-1}[x_j, x_j+h]\times[x,x+h]} \gamma(2\nu+1+2n, bt)\, \mathrm{d}t\, \mathbf{d}x^{r-1}.$$

The use of the special form of the formula ((p. 23, Equation **1.2.1.1**), [12]) for $\min(\alpha,\beta) > 0$, $\lambda \geq 0$, implies

$$\int_0^x x^\lambda \gamma(\alpha, \beta x)\, \mathrm{d}x = \frac{x^{\lambda+1}}{\lambda+1}\, \gamma(\alpha,\beta x) - \frac{\gamma(\lambda+1+\alpha, \beta x)}{(\lambda+1)\, \beta^{\lambda+1}},$$

which provides the expression

$$I(\alpha,\beta;x) = \int_{\prod_{j=1}^{r-1}[0,x_j]\times[0,x]} \gamma(\alpha,\beta t)\, \mathrm{d}t\, \mathbf{d}x^{r-1} = \frac{x^r}{r!} \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^k}{(\beta x)^k}\, \gamma(\alpha+k, \beta x). \tag{21}$$

Indeed, the first few iterations read

$$\begin{aligned}
\int_0^x \gamma(\alpha,\beta t)\, \mathrm{d}t &= x\, \gamma(\alpha,\beta x) - \frac{1}{\beta}\, \gamma(\alpha+1, \beta x), \\
\int_0^x \int_0^{x_1} \gamma(\alpha,\beta t)\, \mathrm{d}t\, \mathrm{d}x_1 &= \frac{1}{2!}\Big[x^2\, \gamma(\alpha,\beta x) - \frac{2}{\beta}\, x\, \gamma(\alpha+1,\beta x) \\
&\qquad + \frac{1}{\beta^2}\, \gamma(\alpha+2,\beta x)\Big], \\
\int_0^x \int_0^{x_2} \int_0^{x_1} \gamma(\alpha,\beta t)\, \mathrm{d}t\, \mathrm{d}x_1\, \mathrm{d}x_2 &= \frac{1}{3!}\Big[x^3\, \gamma(\alpha,\beta x) - \frac{3}{\beta}\, x^2\, \gamma(\alpha+1,\beta x) \\
&\qquad + \frac{3}{\beta^2}\, x\, \gamma(\alpha+2,\beta x) - \frac{1}{\beta^3}\, \gamma(\alpha+3,\beta x)\Big], \quad \cdots
\end{aligned}$$

accordingly, we obtain (21) by induction. Consequently,

$$\begin{aligned}
F_{I,r}^{(1)}(x;h) &= \frac{C_\nu}{h^r} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+1+n)\, n!} \left(\frac{a^2}{4b^2}\right)^n \\
&\quad \cdot \big[I(2\nu+1+2n, b; x+h) - I(2\nu+1+2n, b; x)\big] \\
&= \frac{C_\nu\, x^r}{r!\, h^r} \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^{k+1}}{(b\,x)^k} \left\{ \sum_{n\geq 0} \frac{\gamma(2\nu+1+k+2n, bx)}{\Gamma(\nu+1+n)\, n!} \left(\frac{a^2}{4b^2}\right)^n \right. \\
&\quad \left. - \left(1+\frac{h}{x}\right)^{r-k} \sum_{n\geq 0} \frac{\gamma(2\nu+1+k+2n, b(x+h))}{\Gamma(\nu+1+n)\, n!} \left(\frac{a^2}{4b^2}\right)^n \right\} \\
&= \frac{C_\nu}{r!}\left(\frac{x}{h}\right)^r \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^{k+1}}{(b\,x)^k} \left\{ {}_1\Psi_1^{(\gamma)}\left[\begin{matrix} (2\nu+1+k, 2, bx) \\ (\nu+1,1) \end{matrix} \,\Big|\, \frac{a^2}{4b^2} \right] \right. \\
&\quad \left. - \left(1+\frac{h}{x}\right)^{r-k} {}_1\Psi_1^{(\gamma)}\left[\begin{matrix} (2\nu+1+k, 2, b(x+h)) \\ (\nu+1,1) \end{matrix} \,\Big|\, \frac{a^2}{4b^2} \right] \right\}.
\end{aligned}$$

In turn, this provides the statement of the theorem. □

Corollary 2. *Let the situation be the same as in* Theorem 5. *Denote*

$$\mathscr{A}_r\big({}_1\Psi_1^{(\gamma)};t\big) := \sum_{k=0}^{r} \frac{(-r)_k}{b^k k!} t^{r-k}\, {}_1\Psi_1^{(\gamma)}\Big[\begin{array}{c}(2\nu+1+k,2,bt)\\ (\nu+1,1)\end{array}\Big|\frac{a^2}{4b^2}\Big].$$

Then, for all $x>0$, we have

$$\mathscr{A}_r\big({}_1\Psi_1^{(\gamma)};x\big) \le \mathscr{A}_r\big({}_1\Psi_1^{(\gamma)};x+h\big) \le \mathscr{A}_r\big({}_1\Psi_1^{(\gamma)};x\big) + C_\nu\, r!\, h^r\,.$$

Proof. The statement follows, since $F^{(1)}_{I,r}(x;h)$ is a cdf having a unit interval codomain for the $\operatorname{supp}\big(F^{(1)}_{I,r}\big)=\mathbb{R}_+$ and any positive h. □

5. Concluding Remarks

A. Inserting the lower and upper incomplete gamma functions (1) from relation (2) into (5) we deduce the upper incomplete Fox–Wright function's power series definition:

$${}_p\Psi_q^{(\Gamma)}\Big[\begin{array}{c}(\mu,M,x),(\mathbf{a}_{p-1},\mathbf{A}_{p-1})\\ (\mathbf{b}_q,\mathbf{B}_q)\end{array}\Big|z\Big] = \sum_{n\ge0} \frac{\Gamma(\mu+nM,x)\prod_{j=1}^{p-1}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\,\frac{z^n}{n!}\,.$$

Consequently, it follows that

$$\begin{aligned}{}_p\Psi_q^{(\gamma)}\Big[\begin{array}{c}(\mu,M,x),(\mathbf{a}_{p-1},\mathbf{A}_{p-1})\\ (\mathbf{b}_q,\mathbf{B}_q)\end{array}\Big|z\Big] + {}_p\Psi_q^{(\Gamma)}\Big[\begin{array}{c}(\mu,M,x),(\mathbf{a}_{p-1},\mathbf{A}_{p-1})\\ (\mathbf{b}_q,\mathbf{B}_q)\end{array}\Big|z\Big]\\ = {}_p\Psi_q\Big[\begin{array}{c}(\mu,M),(\mathbf{a}_{p-1},\mathbf{A}_{p-1})\\ (\mathbf{b}_q,\mathbf{B}_q)\end{array}\Big|z\Big],\end{aligned}$$

consult (pp. 196–197 , Equations (6)–(7), [4]). Obviously, the parameter space remains unchanged. The reduction to the confluent function, which builds (9), results in

$${}_1\Psi_1^{(\gamma)}\Big[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\Big|\frac{a^2}{4b^2}\Big] + {}_1\Psi_1^{(\Gamma)}\Big[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\Big|\frac{a^2}{4b^2}\Big] = {}_1\Psi_1\Big[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\Big|\frac{a^2}{4b^2}\Big].$$

Now, let $X\sim\mathrm{McKayI}(a,b,\nu)$, where $b>a>0$, $2\nu+1>0$. The associated reliability (or survival) function and the hazard function, which also characterize the probability distributions, are

$$R_I(x;a,b;\nu) = 1 - F_I(x;a,b;\nu)\,, \qquad h_I(x;a,b;\nu) = \frac{f_I(x;a,b,\nu)}{R_I(x;a,b;\nu)}\,.$$

Hence, in conjunction with the pdf (6) and cdf (9), we perform for all $x\ge0$ the following formulae

$$\begin{aligned} R_I(x;a,b;\nu) &= \frac{\sqrt{\pi}\,(b^2-a^2)^{\nu+\frac12}}{2^{2\nu}b^{2\nu+1}(\nu+\frac12)}\,{}_1\Psi_1^{(\Gamma)}\Big[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\Big|\frac{a^2}{4b^2}\Big],\\ h_I(x;a,b;\nu) &= \Big(\frac{2b^2}{a}\Big)^{\nu} \frac{\mathrm{e}^{-bx}\, I_0(ax)}{{}_1\Psi_1^{(\Gamma)}\Big[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\Big|\frac{a^2}{4b^2}\Big]}\,.\end{aligned}$$

These expressions show that in fact, no novel quality can be achieved by applying the upper incomplete gamma and upper Fox–Wright functions instead of the lower ones. In turn, the obtained inequality bounds become reversed, bearing in mind the reliability function terminology. The problem of how to achieve these bounds, we leave to the interested reader.

B. The probabilistic research methodology is in fact unique with respect to the confluent Fox–Wright function ${}_1\Psi_1^{(\gamma)}$, since McNolty's pdf (6) and cdf (9) are expressible by this special case of (5). On the other hand, this strategy of considerations can lead to other useful bounds for special functions appearing in the formulae of the pdf and cdf for "classical" and/or newly introduced random variables.

C. New research directions can be formulated for other special functions which participate in representing either the rv $X \sim \mathrm{McKayI}(a, b, \nu)$, or its counterpart variable with the so-called $\mathrm{McKayK}(a, b, \nu)$ distribution, see [2,7,8]. However, these goals will be addressed and presented in future work.

Funding: No external funding support.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The author is immensely grateful to Dragana Jankov Maširević for valuable suggestions and comments which she provided during this research. Furthermore, the author owes sincerest thanks to all three unknown referees for constructive remarks, insightful questions and their crucial help with improving the article.

Conflicts of Interest: The author declares no conflict of interest.

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