On the Dynamic Geometry of Kasner Quadrilaterals with Complex Parameter

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Abstract: We explore the dynamics of the sequence of Kasner quadrilaterals \((A_n B_n C_n D_n)_{n \geq 0}\) defined via a complex parameter \(\alpha\). We extend the results concerning Kasner triangles with a fixed complex parameter obtained in earlier works and determine the values of \(\alpha\) for which the generated dynamics are convergent, divergent, periodic, or dense.

Keywords: dynamical systems; Kasner quadrilaterals; convergence; orbits; nested quadrilaterals

MSC: 51P99; 60A99

1. Introduction

For a real number \(\alpha\) and an initial quadrilateral \(A_0B_0C_0D_0\), one can construct the quadrilateral \(A_1B_1C_1D_1\) such that \(A_1, B_1, C_1,\) and \(D_1\) divide the segments \([A_0B_0], [B_0C_0], [C_0D_0],\) and \([D_0A_0]\), respectively, in the ratio \(1 - \alpha : \alpha\). Continuing this process, one obtains the terms \(A_n B_n C_n D_n, n \geq 0\) whose terms are referred to as Kasner (or nested) quadrilaterals (after E. Kasner (1878–1955) who initiated these studies). A natural problem is to find the numbers \(\alpha\) for which the sequence \((A_n B_n C_n D_n)_{n \geq 0}\) is convergent.

The related dynamic geometries inspired by simple iterations (especially for triangles) are reviewed in the article [1]: generated by the incircle and the circumcircle of a triangle, the pedal triangle [2], the orthic triangle, and the incentral triangle. Similar recursive systems describing dynamic geometries are considered by S. Abbot [3], G. Z. Chang and P. J. Davis [4], R. J. Clarke [5], J. Ding, L. R. Hitt, and X-M. Zhang [6], L. R. Hitt and X-M. Zhang [7], and D. Ismailescu and J. Jacobs [8], or in the works by Dionisi et al. [9] and Roeschel [10]. In the paper [1], we proved that the sequence of Kasner triangles is convergent if and only if \(\alpha \in (0, 1)\), also providing the order of convergence.

Here, we prove similar results for the Kasner quadrilaterals, given by the complex coordinates of their vertices \(A_n(a_n), B_n(b_n), C_n(c_n), D_n(d_n), n \geq 0\) (see the notation in [11]). The iterations are defined recursively for \(n \geq 0\) as:

\[
\begin{align*}
    a_{n+1} &= a_n + (1 - \alpha)b_n \\
    b_{n+1} &= \alpha b_n + (1 - \alpha)c_n \\
    c_{n+1} &= \alpha c_n + (1 - \alpha)d_n \\
    d_{n+1} &= \alpha d_n + (1 - \alpha)a_n.
\end{align*}
\]

In this paper, we investigate the dynamic geometry generated by the sequence \((A_n B_n C_n D_n)_{n \geq 0}\), when \(\alpha\) is a complex number. Notice that when \(\alpha\) is complex, the quadrilaterals \(A_n B_n C_n D_n\) are not always nested. The work extends results for triangles in [12], preparing the ground for the study of the general case of Kasner polygons.
2. Preliminaries

The system (1) can be written in matrix form as

\[ X_{n+1} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & \alpha & 1-\alpha & 0 \\ 0 & 0 & \alpha & 1-\alpha \\ 1-\alpha & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix} = T X_n, \]

where \( X_n = (a_n, b_n, c_n, d_n)^T, n \geq 0 \). In this notation, one can write

\[ X_n = T^n X_0. \]

The matrix \( T \) has the characteristic polynomial

\[ p_T(u) = (u-\alpha)^4 - (1-\alpha)^4 \]

\[ = u^4 - 4u^3\alpha + 6u^2\alpha^2 - 6u\alpha^3 + \alpha^4 - (1-\alpha)^4 \]

\[ = (u-1)(u-2\alpha+1)(u^2-2u\alpha+2\alpha^2-2\alpha+1), \]

whose roots can be written as

\[ u_1 = \alpha + (1-\alpha)i = (1-i)\left(\alpha - \frac{1-i}{2}\right) \]

\[ u_2 = \alpha - (1-\alpha) = 2\left(\alpha - \frac{1}{2}\right) \]

\[ u_3 = \alpha - (1-\alpha)i = (1+i)\left(\alpha - \frac{1+i}{2}\right). \]

A direct computation shows that

\[ T = F^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 \\ 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & u_3 \end{pmatrix} F, \]

where the matrices \( F \) and \( F^{-1} \) are given by

\[ F = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}, \]

\[ F^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -i & 1 & -1 \\ 1 & -1 & 1 & i \end{pmatrix}. \]

By using (7), for every positive integer \( n \), we have the following relations

\[ T^n = F^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1^n & 0 & 0 \\ 0 & 0 & u_2^n & 0 \\ 0 & 0 & 0 & u_3^n \end{pmatrix} F, \]

\[ = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -i & 1 & -1 \\ 1 & -1 & 1 & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_1^n & 0 & 0 \\ 0 & 0 & u_2^n & 0 \\ 0 & 0 & 0 & u_3^n \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -i & 1 & -1 \\ 1 & -1 & 1 & i \end{pmatrix}. \]
By Formula (3), one obtains

\[a_n = g_0 + Mu_1^n + Nu_2^n + Pu_3^n,\]
\[b_n = g_0 + (Mi)u_1^n + (-N)u_2^n + (-Pi)u_3^n,\]
\[c_n = g_0 + (-M)u_1^n + Nu_2^n + (-P)u_3^n,\]
\[d_n = g_0 + (-Mi)u_1^n + (-N)u_2^n + (Pi)u_3^n,\]

where \(g_0 = \frac{a_0 + b_0i + c_0 + d_0i}{4}\), where multiplying (9) by \((a_0, b_0, c_0, d_0)^T\) we obtain

\[M = \frac{a_0 - b_0i - c_0 + d_0i}{4}, \quad N = \frac{a_0 - b_0 + c_0 - d_0i}{4}, \quad p = \frac{a_0 + b_0i - c_0 - d_0i}{4}.\]

From these formulae (but also from (1)), notice that \(a_n + b_n + c_n + d_n = 4g_0, n \geq 0\); hence, all polygons \(A_nB_nC_nD_n\) have the same centroid \(G_0\). Clearly, when \(M, N, P \neq 0\), the terms \(u_1^n, u_2^n,\) and \(u_3^n\) appear explicitly in (10).

3. Dynamical Properties in the Case of Real Parameter

In this section, we study the convergence of the sequence of the Kasner quadrilaterals when \(\alpha\) is a real number. By Formula (10), the sequences \((a_n)_{n \geq 0}, (b_n)_{n \geq 0}, (c_n)_{n \geq 0}\), and \((d_n)_{n \geq 0}\) are convergent if and only if \(|u_1| < 1, |u_2| < 1, |u_3| < 1\), that is,

\[|u_1| = \left|\left(1 - i\left(\alpha - \frac{1 - i}{2}\right)\right)\right| = \sqrt{2}\left|\alpha - \frac{1 - i}{2}\right| < 1,\]
\[|u_2| = 2\left|\alpha - \frac{1}{2}\right| < 1,\]
\[|u_3| = \left|\left(1 + i\left(\alpha - \frac{1 + i}{2}\right)\right)\right| = \sqrt{2}\left|\alpha - \frac{1 + i}{2}\right| < 1.\]

First, one can easily check that the condition \(|u_2| < 1\) is equivalent to \(\alpha \in (0, 1)\). Then, because \(\alpha\) is real, we clearly have \(|u_1| = |u_3|\); hence, the conditions \(|u_1| < 1\) and \(|u_3| < 1\) become equivalent to \(|u_1u_3| < 1\), that is,

\[|u_1u_3| = 2\left|\alpha - \frac{1 - i}{2}\right|\left|\alpha - \frac{1 + i}{2}\right| = 2\left(\alpha^2 - \alpha + \frac{1}{2}\right) < 1,\]

which is equivalent to \(\alpha(1 - \alpha) < 0\), that is, \(\alpha \in (0, 1)\).

4. Dynamical Properties in the Case of Complex Parameter

We now discuss the dynamics obtained when \(\alpha\) is a complex number.

It is convenient to define the following points

\[z_1 = \frac{1}{2} - \frac{1}{2}i, \quad z_2 = \frac{1}{2}, \quad z_3 = \frac{1}{2} + \frac{1}{2}i,\]

representing the centres of the open disks

\[D_1\left(z_1, \frac{\sqrt{2}}{2}\right), \quad D_2\left(z_2, \frac{1}{2}\right), \quad D_3\left(z_3, \frac{\sqrt{2}}{2}\right),\]

and of the circles depicted in Figure 1

\[C_1\left(z_1, \frac{\sqrt{2}}{2}\right), \quad C_2\left(z_2, \frac{1}{2}\right), \quad C_3\left(z_3, \frac{\sqrt{2}}{2}\right).\]
Considering the real numbers \( r_1, r_2, r_3, \theta_1, \theta_2, \theta_3 \), by (4), (5), and (6), we obtain
\[
\begin{align*}
U_1 &= r_1 e^{2\pi i \theta_1} = \sqrt{2}(\alpha - z_1)e^{-\pi i 4}, \\
U_2 &= r_2 e^{2\pi i \theta_2} = 2(\alpha - z_2), \\
U_3 &= r_3 e^{2\pi i \theta_3} = \sqrt{2}(\alpha - z_3)e^{\pi i 4}.
\end{align*}
\] (16)

By (16), we deduce that for a given \( j = 1, 2, 3 \), if \( \alpha \in D_j \), then we have \( r_j < 1 \). Moreover, if \( \alpha \in C_j \), then it follows that \( r_j = 1 \). The distinct behaviours below emerge:
1. If \( \alpha \in D_1 \cap D_2 \cap D_3 \), then \( 0 < r_1, r_2, r_3 < 1 \).
   One can easily check the set inclusion \( D_1 \cap D_3 \subseteq D_2 \).
2. If \( \alpha \) is in the interior of the complement of \( D_1 \cap D_3 \), then \( \max\{r_1, r_3\} > 1 \).
3. If \( \alpha \in C_1 \cap C_2 \cap C_3 \), then \( \alpha \in \{0, 1\} \).

The boundary of the shaded region in Figure 1 consists of two arcs
\[
U_1 = C_1 \cap D_3, \quad U_3 = C_3 \cap D_1,
\]
which can be parametrised as
\[
\alpha(t) = \begin{cases} 
  z_1 + \sqrt{2} \left( \cos t + i \sin t \right), & t \in \left[ \frac{\pi}{4}, \frac{3\pi}{4} \right] \\
  z_3 + \sqrt{2} \left( \cos t + i \sin t \right), & t \in \left[ \frac{5\pi}{4}, \frac{7\pi}{4} \right].
\end{cases}
\] (17)

To describe the orbits of the sequences \((a_n)_{n \geq 0}, (b_n)_{n \geq 0}, (c_n)_{n \geq 0}, \) and \((d_n)_{n \geq 0}\), one first needs to understand the behaviour of the sequence \((z^n)_{n \geq 0}\), where \( z \in \mathbb{C} \) (see, for example, Lemma 2.1 in [13], or Lemma 5.2 in [14]), which is shown in Figure 2.

**Lemma 1.** Let \( z = re^{2\pi i \theta} \), where \( r \geq 0, \theta \in \mathbb{R} \). The orbit of \((z^n)_{n \geq 0}\) is:
(a) A spiral convergent to 0 for \( r < 1 \);
(b) A divergent spiral for \( r > 1 \);
(c) A regular \( k \)-gon if \( z \) is a primitive \( k \)-th root of unity, \( k \geq 3 \);
(d) A dense subset of the unit circle if \( r = 1 \) and \( \theta \in \mathbb{R} \setminus \mathbb{Q} \).

When \( \theta = j/k \in \mathbb{Q} \) is irreducible, then the terms of the spirals obtained in (a) and (b) align along \( k \) rays.
1. Convergent if $0 < r_1, r_2, r_3 < 1$; 
2. Divergent if $\max\{r_1, r_3\} > 1$; 
3. Periodic if $r_1 = r_3 = 1$ (that is, when $\alpha = 0$ or $\alpha = 1$); 
4. There are two distinct patterns when $0 < \min\{r_1, r_3\} < \max\{r_1, r_3\} = 1$. 

Denoting $\theta = \theta_1$ if $r_1 = 1$ or $\theta = \theta_3$ if $r_3 = 1$, then the orbit:

(a) Has $k$ convergent subsequences if $\theta = \frac{k}{n}$ is an irreducible fraction;
(b) Is dense within a circle when $\theta$ is irrational.

The details of the geometric patterns obtained in each case are presented below.

In all figures, we consider the initial polygon of complex coordinates

$$A_0(-4 + 12i), \quad B_0(0), \quad C_0(8), \quad D_0(12 + 1i),$$

for which Formula (11) gives the values

$$G = 4 + 5i, \quad M = -5 + 6i, \quad N = -2 + i, \quad P = -1.$$  

The position of $\alpha$ relative to relevant boundaries is indicated in the left diagram with a star, while the iterations of the polygon are displayed on the right, where the star indicates the position of the centroid. All the simulations have been implemented in Matlab® 2021b.

4.1. Convergent Orbits

If $0 < r_1, r_2, r_3 < 1$, then by (16), the sequences $u^n_1$, $u^n_2$, and $u^n_3$ are convergent if and only if $\alpha \in D_1 \cap D_3$. Hence, by (10), we obtain that $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$ converge to $g_0$. We can formulate the following result.
Theorem 1. The following assertions hold:

1. The sequence \((A_n B_n C_n D_n)_{n \geq 0}\) is convergent if and only if \(\alpha \in D_1 \cap D_2\).
2. When the sequence \((A_n B_n C_n D_n)_{n \geq 0}\) is convergent, its limit is the degenerated quadrilateral at \(G_0\), the centroid of the initial quadrilateral \(A_0 B_0 C_0 D_0\).

Proof. The intersection \(D_1 \cap D_3\) is shaded in Figure 1.

(1) Clearly, \(\alpha \in D_1 \cap D_3\) is equivalent to \(r_1 < 1\) and \(r_2 < 1\) (in this case, one also has \(\alpha \in D_2\)). The relation (10) shows that the sequences \((a_n)_{n \geq 0}\), \((b_n)_{n \geq 0}\), \((c_n)_{n \geq 0}\), and \((d_n)_{n \geq 0}\) are convergent if and only if \((u_1^n)_{n \geq 0}\), \((u_2^n)_{n \geq 0}\), and \((u_3^n)_{n \geq 0}\) are convergent, which happens when \(u_1^n \to 0\), \(u_2^n \to 0\), and \(u_3^n \to 0\).

(2) Adding the equation in the system (1), one obtains that for every integer \(n \geq 0\), we have

\[
\begin{align*}
  a_n + b_n + c_n + d_n &= a_0 + b_0 + c_0 + d_0 = 4g_0, \\
  \text{where } g_0 &\text{ is the complex coordinates of the centroid } G_0 \text{ of the initial quadrilateral } A_0 B_0 C_0 D_0. \\
\end{align*}
\]

Assume that \(a_n \to a^*, b_n \to b^*, c_n \to c^*,\) and \(d_n \to d^*\). From system (1), we obtain

\[
\begin{align*}
  a^* &= a a^* + (1 - a) b^* \\
  b^* &= a b^* + (1 - a) c^* \\
  c^* &= a c^* + (1 - a) d^* \\
  d^* &= a d^* + (1 - a) a^*. \\
\end{align*}
\]

Because \(\alpha \neq 1\), the only solution of this system is \(a^* = b^* = c^* = d^* = g_0\).

For \(0 < \alpha < 1\), one has \(\alpha \in D_1 \cap D_3\), and moreover, in this case, the vertices \(A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}\) are interior points of the segments \([A_n,B_n],[B_n,C_n],[C_n,D_n],[D_n,A_n]\), respectively. Such an example is depicted in Figure 3.

![Figure 3](image1.png)

Figure 3. Convergent orbits (right) obtained for \(\alpha = 0.25\) (left).

On the other hand, when the parameter \(\alpha \in D_1 \cap D_3\) is not real, the orbit is convergent, but the points are not aligned any more, as illustrated in Figure 4.

![Figure 4](image2.png)

Figure 4. Convergent orbits (right) obtained for \(\alpha = \frac{1}{2} + \frac{\sqrt{3} i}{12}\) (left).
4.2. Periodic Orbits

If \( r_1 = r_2 = r_3 = 1 \), then \( |\alpha - z_1| = |\alpha - z_3| = \frac{\sqrt{2}}{2} \) and \( |\alpha - z_2| = \frac{1}{2} \), which can only happen for \( \alpha \in C_1 \cap C_2 \cap C_3 = \{0, 1\} \).

**Case 1.** \( \alpha = 0 \). From the system (1), for all \( n \geq 0 \), one obtains
\[
a_{n+4} = b_{n+3} = c_{n+2} = d_{n+1} = a_n.
\]

Similarly, \( b_{n+4} = b_n, c_{n+4} = c_n, \) and \( d_{n+4} = d_n \), so the sequence terms satisfy
\[
\begin{align*}
\alpha_n & \equiv \alpha_0, b_0, c_0, d_0, a_0, b_0, c_0, d_0, \ldots \\
b_n & \equiv b_0, c_0, d_0, a_0, b_0, c_0, d_0, \ldots \\
c_n & \equiv c_0, d_0, a_0, b_0, c_0, d_0, a_0, \ldots \\
d_n & \equiv d_0, a_0, b_0, c_0, d_0, a_0, b_0, \ldots .
\end{align*}
\] (21)

**Case 2.** \( \alpha = 1 \). From the system (1), for all \( n \geq 0 \), one obtains
\[
a_{n+1} = a_n, \quad b_{n+1} = b_n, \quad c_{n+1} = c_n, \quad d_{n+1} = d_n,
\]
so, in this case, the sequences are actually constant.

4.3. Divergent Orbits

If \( \max\{r_1, r_3\} > 1 \), then \( \alpha \in \mathbb{R} \setminus (D_1 \cap D_3) \); hence, by (16), either \( u_n^a \) or \( u_n^c \) are divergent. By Formula (10), the sequences \((a_n)_{n \geq 0}, (b_n)_{n \geq 0}, (c_n)_{n \geq 0}, \) and \((d_n)_{n \geq 0}\) are divergent (as long as the corresponding coefficients \( M, N, P \) in (10) are not all vanishing).

Figure 5 shows a divergent iteration. The diagram on the left we plot the position of \( \alpha \), while on the right side we illustrate the polygons \(A_nB_nC_nD_n, n = 0, \ldots, 10\).

4.4. Orbits with a Finite Number of Convergent Subsequences

If \( 0 < \min\{r_1, r_3\} < \max\{r_1, r_3\} = 1 \), then one either has \( \alpha \in C_1 \cap D_3 \) for \( r_1 = 1 \), or \( \alpha \in C_3 \cap D_1 \) for \( r_3 = 1 \). The orbit has a finite number of limit points if the complex argument \( \theta \) of \( u_1 \) if \( r_1 = 1 \) or of \( u_3 \) if \( r_3 = 1 \) is rational.

4.4.1. Upper Arc of \( C_1 \)

First, assume that \( r_1 = \max\{r_1, r_3\} = 1 \), i.e., \( \alpha \) is on the upper arc \( C_1 \cap D_3 \).

As \( \alpha \in C_1 \), there is \( t \in \left[ \frac{1}{8}, \frac{3}{8} \right] \) with \( \alpha = z_1 + \frac{\sqrt{2}}{2} e^{2\pi i t} \), so by (16), we obtain
\[
u_1 = e^{2\pi i \theta_1} = \sqrt{2}(\alpha - z_1)e^{-\pi i} = e^{2\pi i (t - \frac{1}{8})}.
\] (22)

When \( \theta_1 = \frac{p}{q} \) is an irreducible fraction, the orbit has a finite number of convergent subsequences. Therefore, we have the following result.
Theorem 2. If for the integers $0 < p < q$, we have $\theta_1 = \frac{p}{q} \in \left[0, \frac{1}{4}\right]$ is an irreducible fraction, then $u_1 = e^{2\pi i \theta}$ and by Formula (10), the sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$ have subsequences which converge to the vertices of a regular $q$-gon centred at $G_0$ of radius $|M|$.

Proof. In this case, we have $u_n^{q+j} = u_1^j$ for $j = 0, \ldots, q-1$ and $u_2^n \to 0$ and $u_3^n \to 0$, so using the notations of (10) and (11), one obtains the relations

$$\lim_{n \to \infty} a_{nq+j} = \lim_{n \to \infty} \left(g_0 + Mu_1^{nq+j} + Nu_2^{nq+j} + Pu_3^{nq+j}\right) = g_0 + Mu_1^j$$

$$\lim_{n \to \infty} b_{nq+j} = \lim_{n \to \infty} \left(g_0 + (Mi)u_1^{nq+j} + (-N)u_2^{nq+j} + (-Pi)u_3^{nq+j}\right) = g_0 + (Mi)u_1^j$$

$$\lim_{n \to \infty} c_{nq+j} = \lim_{n \to \infty} \left(g_0 + (-M)u_1^{nq+j} + Nu_2^{nq+j} + (-P)u_3^{nq+j}\right) = g_0 + (-M)u_1^j$$

$$\lim_{n \to \infty} d_{nq+j} = \lim_{n \to \infty} \left(g_0 + (-Mi)u_1^{nq+j} + (-N)u_2^{nq+j} + (Pi)u_3^{nq+j}\right) = g_0 + (-Mi)u_1^j,$n

which ends the proof. This case is depicted in Figure 6. The sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$ are plotted in Figure 7. Moreover, one can check that for $\theta_1 = 1/5$ the limit polygon is a pentagon centred at $G_0$, of radius $|M| \simeq 7.81$ (by (19)).

![Figure 6](image1.png)

**Figure 6.** First 200 iterations (right) obtained for $\theta = p/q = 1/5$ where $\alpha = z_1 + \sqrt{2}e^{2\pi i (\frac{1}{5} + \frac{1}{5})}$ (left).

![Figure 7](image2.png)

**Figure 7.** Iterations obtained for $\theta_1 = \frac{1}{5}$. (a) $(a_n)_{n=0}^{199}$; (b) $(b_n)_{n=0}^{199}$; (c) $(c_n)_{n=0}^{199}$; (d) $(d_n)_{n=0}^{199}$. 
4.4.2. Lower Arc of $C_3$

Similarly, if $r_3 = \max\{r_1, r_3\} = 1$, then $\alpha$ is on the arc $C_3 \cap D_1$ defined by (17). Therefore, there is $t \in \left[\frac{5\pi}{8}, \frac{7\pi}{8}\right]$ with $\alpha = z_3 + \sqrt{2}e^{2\pi it}$, and by (16), we obtain

$$u_3 = e^{2\pi it} = \sqrt{2}(\alpha-z_3)e^{\pi i} = e^{2\pi i\left(t + \frac{1}{8}\right)}.$$ (24)

The following result can be proved similarly to Theorem 2.

**Theorem 3.** If for the integers $0 < p < q$, we have $\theta_3 = \frac{p}{q} \in \left[\frac{3}{4}, 1\right]$ is an irreducible fraction, then $u_1 = e^{2\pi i\frac{p}{q}}$ and by Formula (10), the sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$ have $q$ subsequences convergent to the vertices of four regular $q$-gons centred at $G_0$ of radius $|P|$.

The first 200 iterations obtained when $\theta_3 = \frac{5}{6}$ are presented in Figure 8.

![Figure 8. First 200 iterations (right) obtained for $\theta_3 = p/q = 5/6$ where $\alpha = z_3 + \sqrt{2}e^{2\pi i\left(t + \frac{1}{8}\right)}$ (left).](image)

The sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$ are plotted in Figure 9. Similarly to (23), the limit polygon is a hexagon centred at $G$, which has radius $|P| = 1$.

![Figure 9. Iterations obtained for $\theta_3 = \frac{5}{6}$. (a) $(a_n)_{n=0}^{199}$; (b) $(b_n)_{n=0}^{199}$; (c) $(c_n)_{n=0}^{199}$; (d) $(d_n)_{n=0}^{199}$.](image)

4.5. Dense Orbits

When $0 < \min\{r_1, r_3\} < \max\{r_1, r_3\} = 1$ but $\theta_1$ or $\theta_3$ are irrational modulo $2\pi$, the orbits of $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$ are dense within circles.
4.5.1. Upper Arc of $C_1$

First, assume that $0 < r_3 < r_1 = 1$, i.e., $a$ is on the upper arc $C_1 \cap D_3$. Using the notations in (22), the following result can be deduced from Lemma 1 (d).

**Theorem 4.** If $r_1 = 1$ and $\theta_1 \in \left[0, \frac{1}{2}\right]$ is irrational, then the set of limit points for each of the sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$ is the circle centred at $G_0$ of radius $|M|$.

**Proof.** By (10), we have $a_n = g_0 + Mu_1^n + Nu_2^n + Pu_3^n$, with $M$, $N$, and $P$ constants given by (11). Because $|u_2| < 1$, $|u_3| < 1$, we have $a_n = g_0 + Mu_1^n + z_n$, where $\lim_{n \to \infty} z_n = 0$.

Let $z$ be an arbitrary point on the circle of centre $G_0$ and radius $|M|$. If $M = 0$, then $\lim_{n \to \infty} a_n = g_0$. Otherwise, denoting $z' = \frac{z - g_0}{|M|}$, we have $z' \in C(0, 1)$. Because $u_1 = e^{2\pi i \theta_1}$ with $\theta_1$ irrational, by Lemma (1), it follows that there is a subsequence $n_1 < n_2 < \cdots$ such that $\lim_{k \to \infty} u_1^{n_k} = z'$. For $\epsilon > 0$, one can find $K_1(\epsilon)$ and $K_2(\epsilon)$ such that

$$|u_1^{n_k} - z'| < \frac{1}{|M| + 1}\epsilon, \quad k \geq K_1(\epsilon) \text{ and } |z_{n_k}| < \frac{1}{|M| + 1}\epsilon, \quad k \geq K_2(\epsilon),$$

hence, for $k \geq \max\{K_1(\epsilon), K_2(\epsilon)\}$, one obtains

$$|a_{n_k} - z| = |g_0 + Mu_1^{n_k} + z_{n_k} - g_0 - Mz'| \leq |M| \cdot |u_1^{n_k} - z'| + |z_{n_k}| < \epsilon,$$

hence $\lim_{k \to \infty} a_{n_k} = z$. This shows that $z$ is a limit point for the sequence $(a_n)_{n \geq 0}$. Analogously, this is proved for $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$. \qed

Figure 10 illustrates the position of $a$ and the polygons obtained for $n = 10$ iterations, respectively, when $a \in C_1 \cap D_3$. Figure 11 depicts the vertices of the original quadrilateral of affixes $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$, and 200 iterations.

![Figure 10. Orbits for $n = 10$ iterations (right), for $a = z_1 + \frac{\sqrt{3}}{2}(\cos 1 + i \sin 1)$ (left).](image)

![Figure 11. Orbits for $\theta_1 = \frac{1}{2\pi}$. (a) $(a_n)_{n \geq 0}$; (b) $(b_n)_{n \geq 0}$; (c) $(c_n)_{n \geq 0}$; (d) $(d_n)_{n \geq 0}$.](image)
4.5.2. Lower Arc of $C_3$

When $0 < r_1 < r_3 = 1$, $\alpha$ is on the arc $C_3 \cap D_1$ defined by (17), as in Figure 12. Using the notations in (24), we can formulate the following result.

**Theorem 5.** If $r_3 = 1$ and $\theta_3 \in \left[\frac{3}{4}, 1\right]$ is irrational, then the set of limit points for each of the sequences $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$ is the circle centred at $G_0$ of radius $|P|$. 

**Proof.** The proof follows the similar lines as for Theorem 4, but now by (10), one has $a_n = g_0 + Mu_1^n + Nu_2^n + Pu_3^n$. Because $|u_1| < 1$, $|u_2| < 1$, we obtain $a_n = g_0 + z_n + Pu_3^n$, where $\lim_{n \to \infty} z_n = 0$. Figure 12 shows the position of $\alpha$ and the first $n = 10$ iterations, respectively, when $\alpha \in C_3 \cap D_1$. Figure 13 plots the vertices of the original quadrilateral of affixes $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$, $(c_n)_{n \geq 0}$, and $(d_n)_{n \geq 0}$, and 200 iterations. 

\[ \Box \]

**Figure 12.** Dense orbits obtained after $n = 10$ iterations (right), generated for $\alpha = z_3 + e^{2\pi i \left(\frac{3}{4} - \frac{1}{4}\right)}$ (left), when $u_3 = e^{2\pi i \theta_3}$, with $\theta_3 = \frac{3}{4}$.

**Figure 13.** Orbits for $\theta_3 = \frac{3}{4}$. (a) $(a_n)_{n \geq 0}$; (b) $(b_n)_{n \geq 0}$; (c) $(c_n)_{n \geq 0}$; (d) $(d_n)_{n \geq 0}$.

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References