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Existence of Common Fixed Points of Generalized Δ-Implicit Locally Contractive Mappings on Closed Ball in Multiplicative G-Metric Spaces with Applications

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Abstract: In this paper, we introduce a generalized Δ-implicit locally contractive condition and give some examples to support it and show its significance in fixed point theory. We prove that the mappings satisfying the generalized Δ-implicit locally contractive condition admit a common fixed point, where the ordered multiplicative G_M-metric space is chosen as the underlying space. The obtained fixed point theorems generalize many earlier fixed point theorems on implicit locally contractive mappings. In addition, some nontrivial and interesting examples are provided to support our findings. To demonstrate the originality of our new main result, we apply it to show the existence of solutions to a system of nonlinear—Volterra type—integral equations.

Keywords: ordered complete multiplicative G_M-metric space; closed ball; integral equations; locally generalized Δ-implicit contraction

MSC: 47H09; 47H10; 54H25

1. Introduction

In the subject of functional analysis, fixed point theory (FPT) plays a vibrant, fascinating and vital role. Banach (1922) [1] provided a foundational principle that has become a significant instrument in the field of metric fixed point theory to ensure the existence and uniqueness of the fixed point (FP). The Banach fixed point theorem (also known as contraction mapping theorem) is the core principle in the metric fixed point theory. Because of its benefits, numerous authors have demonstrated various improvements and expansions of this theorem in diverse distance spaces (see [1–22]).

Bashirov et al. [4] presented the concept of multiplicative calculus and proved its foundational theorem with certain fundamental features. Multiplicative calculus has a vast area of applications and it deals with only positive functions as opposed to the calculus of Newton and Leibniz. Bashirov et al. showed that multiplicative calculus became an important mathematical tool for economics and finance because of the interpretation given to the multiplicative derivative. Furthermore, they proved multiplicative differential and multiplicative integral equations by using the notion of a multiplicative distance space. The research work on the properties of multiplicative metric space was done in [23–26].

In 2012, Özavsar et al. [27] came up with the definition of multiplicative contraction mappings on multiplicative metric space by using the multiplicative triangle inequality instead
of the usual triangular inequality and obtained different existence results of fixed points as well as various topological characteristics of multiplicative metric space. For other examples of fixed point theorems in multiplicative metric space, see weak commutative mappings, locally contractive mappings, \(\mathcal{E}_A\)-property, compatible-type mappings and generalized contraction mappings with cyclic \((\alpha, \beta)\)-admissible mapping ([(2,11,28–31)]). In 2016, Nagpal et al. [32] introduced the concept of multiplicative generalized metric space and studied the notion of weakly commuting compatible maps and its variants by using \((CLR)\) and \((\mathcal{E}_A)\) properties in a multiplicative metric space.

Rasham et al. [33] recently presented fixed point results for a pair of dominated fuzzy maps in multiplicative metric space on a closed ball and discussed relevant applications to graph theory, integral equations and functional equations. For additional information on the obtained fixed point results in Section 2, moreover, some numerical examples are given.

In Section 2, we state basic notions related to fixed point theorems and multiplicative metric equation and fixed point theorem should be investigated in a most general metric space so that corresponding results can be derived as special cases. The paper is organized as follows. In Section 2, we present two applications of the obtained fixed point results on a closed ball in an ordered multiplicative graph theory, integral equations and functional equations. For additional information on these results, see weak commutative and \((\mathcal{E}_A)\)-implicit contraction. To support new results, we present various nontrivial examples and an application for nonlinear—Volterra type—integral equations.

The choice of the multiplicative \(\delta_M\)-metric is based on the concept of generality. The corresponding results in a multiplicative metric space are special cases of the obtained results in a multiplicative \(\delta_M\)-metric space. We think that any new idea regarding contraction and fixed point theorem should be investigated in a most general metric space so that corresponding results can be derived as special cases. The paper is organized as follows. In Section 2, we state basic notions related to fixed point theorems and multiplicative metric spaces. In Section 3, we present many fixed point theorems and related corollaries and examples as explanations of the stated results. In Section 4, we present two applications of the obtained fixed point results in Section 2, moreover, some numerical examples are given.

2. Preliminaries

Now, we recall some well-known notations and definitions that are used in our subsequent discussion.

**Definition 1** ([4]). Consider a nonempty set \(\mathbb{R}\) and let \(\mathcal{L}_M : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+\) be a function satisfying the following properties:

\[
\begin{align*}
(L_1) \quad & \mathcal{L}_M(\xi, \zeta) \geq 1, \quad \forall \xi, \zeta \in \mathbb{R}; \\
(L_2) \quad & \mathcal{L}_M(\xi, \zeta) = 1 \quad \text{if and only if} \quad \xi = \zeta; \\
(L_3) \quad & \mathcal{L}_M(\xi, \zeta) = \mathcal{L}_M(\zeta, \xi) \quad \text{(symmetry);} \\
(L_4) \quad & \mathcal{L}_M(\xi, \zeta) \leq \mathcal{L}_M(\xi, \eta) \mathcal{L}_M(\eta, \zeta) \quad \forall \xi, \zeta, \eta \in \mathbb{R} \quad \text{(multiplicative triangle inequality).}
\end{align*}
\]

Then, \(\mathcal{L}_M\) is a multiplicative metric on \(\mathbb{R}\) and the pair \((\mathbb{R}, \mathcal{L}_M)\) is a multiplicative metric space.

**Definition 2** ([37]). Let \(\mathbb{R}\) be a nonempty set and the function \(\mathcal{L} : \mathbb{R}^3 \rightarrow [0, +\infty)\) satisfies the following conditions:

\[
\begin{align*}
(1) \quad & \mathcal{L}(\bar{u}, \bar{v}, \bar{z}) = 0 \text{ iff } \bar{u} = \bar{v} = \bar{z}; \\
(2) \quad & 0 < \mathcal{L}(\bar{u}, \bar{v}, \bar{w}) \text{ for all } \bar{u}, \bar{v}, \bar{w} \in \mathbb{R} \text{ with } \bar{u} = \bar{v}; \\
(3) \quad & \mathcal{L}(\bar{u}, \bar{v}, \bar{w}) = \mathcal{L}(\bar{w}, \bar{v}, \bar{u}) \quad \text{for all } \bar{u}, \bar{v}, \bar{w} \in \mathbb{R} \text{ with } \bar{v} = \bar{w}; \\
(4) \quad & \mathcal{L}(\bar{u}, \bar{v}, \bar{w}) = \mathcal{L}(\bar{v}, \bar{u}, \bar{w}) = \mathcal{L}(\bar{w}, \bar{v}, \bar{u}) \quad \text{for all } \bar{u}, \bar{v}, \bar{w} \in \mathbb{R}; \\
(5) \quad & \mathcal{L}(\bar{u}, \bar{v}, \bar{w}) = \mathcal{L}(\bar{u}, \bar{a}, \bar{v}) + \mathcal{L}(\bar{a}, \bar{v}, \bar{w}) \text{ for all } \bar{u}, \bar{v}, \bar{a}, \bar{w} \in \mathbb{R}.
\end{align*}
\]

Then, \(\mathcal{L}\) is said to be an \(\mathcal{L}\)-metric on \(\mathbb{R}\) and the pair \((\mathbb{R}, \mathcal{L})\) is called a \(\mathcal{L}\)-metric space.

**Definition 3** ([32]). Suppose that \(\mathbb{R}\) is a nonempty set and \(\delta_M : \mathbb{R}^3 \rightarrow \mathbb{R}^+\) is a function satisfying the following conditions:

\[
\begin{align*}
(\delta_{M_1}) \quad & \delta_M(\xi, \zeta, \eta) = 1 \text{ if } \xi = \zeta = \eta; \\
(\delta_{M_2}) \quad & 1 < \delta_M(\xi, \zeta) \quad \forall \xi, \zeta \in \mathbb{R} \text{ with } \xi = \zeta; \\
(\delta_{M_3}) \quad & \delta_M(\xi, \zeta, \eta) \leq \delta_M(\xi, \zeta, \eta) \quad \forall \xi, \zeta, \eta \in \mathbb{R} \text{ with } \xi \neq \eta; \\
(\delta_{M_4}) \quad & \delta_M(\xi, \zeta, \eta) = \delta_M(\zeta, \xi, \eta) = \delta_M(\eta, \zeta, \xi) = \ldots \quad \text{(symmetry);} \\
\end{align*}
\]
(δ_M) δ_M(ξ,η) ≤ δ_M(ξ,ε) δ_M(ε,η) ∀ ξ, η ∈ ℜ, (rectangular inequality).

Then, the function δ_M is called a multiplicative generalized metric or, more accurately, multiplicative δ_M-metric on ℜ and the pair (ℜ,δ_M) is called a multiplicative δ_M-metric space.

We note that δ_M(ξ,η) = e^{c(ξ,η)} ∀ ξ, η ∈ ℜ. The δ_M-ball with centre 0 and radius γ > 0 is defined by

\[ \Box_{\gamma}(0,\gamma) = \{ q ∈ ℜ : δ_M(0,q) ≤ \gamma \}. \]

Assume that (ℜ, d) is a usual metric space and δ_M: ℜ^3 → ℜ^+ is defined by

δ_M(ξ,η) = d(ξ,η) ∀ ξ, η ∈ ℜ, where a > 1 is any fixed real number. Then, for each a, δ_M is a multiplicative δ_M-metric on ℜ and (ℜ,δ_M) is called a multiplicative δ_M-metric space. Note that a multiplicative δ_M-metric is not a multiplicative metric space nor a ℓ-metric space. Moreover, a multiplicative metric space is usually different from a metric space (see [33]).

Lemma 1 ([32]). Let (ℜ, δ_M) be a multiplicative δ_M-metric space. Then, for all 0, 1, 2 and all 0, then, there exists a point 1 such that 0 ≤ 1, 2 ≤ 0, that is, every three elements in 0, 2 has a lower bound (LB), then, the point 1 is unique in 0, 2.

Proof. Let 0 be any arbitrary point in 0 and 1, 2 = 0 for all n ∈ N ∪ {0}. From inequality (2), we get

δ_M(0,1,2) ≤ (1 − γ) γ ≤ γ,
thereby implying \( \epsilon_1 \in \odot_\gamma (\bar{\epsilon}_0, \gamma) \). By the multiplicative triangle inequality, we have

\[
\sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_2, \bar{\epsilon}_q)} \leq \sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_1, \bar{\epsilon}_1)} \cdot \sqrt[n]{\delta_M(\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_2)} \\
= \sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_2)} \cdot \sqrt[n]{\delta_M(\bar{\epsilon}_2, \bar{\epsilon}_1)} \\
\leq \left[ \sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_1)} \right]^{1 + \eta},
\]

that is,

\[
\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_2, \bar{\epsilon}_q) \leq \left[ \delta_M(\bar{\epsilon}_0, \bar{\epsilon}_2, \bar{\epsilon}_2) \right] \leq \left[ (1 - \eta) \gamma \right]^{1 + \eta} \leq \gamma.
\]

Then, \( \hat{\epsilon}_2 \in \odot_\gamma (\bar{\epsilon}_0, \gamma) \). Consider \( \hat{\epsilon}_3, \hat{\epsilon}_4, \ldots, \hat{\epsilon}_q \) for every \( q \in N \). Taking (1) in consideration, we obtain

\[
\sqrt[n]{\delta_M(\bar{\epsilon}_q, \bar{\epsilon}_{q+1}, \bar{\epsilon}_{q+1})} = \sqrt[n]{\delta_M(\bar{\epsilon}_q, \bar{\epsilon}_{q+1}, \bar{\epsilon}_{q+1})} \\
\leq \left[ \sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_1)} \right]^{1 + \eta + \ldots + \eta^q}.
\]

Using (1) and (3), we find

\[
\sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_{q+1}, \bar{\epsilon}_{q+1})} \leq \sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_1, \bar{\epsilon}_1)} \cdot \sqrt[n]{\delta_M(\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_2)} \cdot \sqrt[n]{\delta_M(\bar{\epsilon}_q, \bar{\epsilon}_{q+1}, \bar{\epsilon}_{q+1})} \\
\leq \left[ \sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_1)} \right]^{1 + \eta + \ldots + \eta^q},
\]

which becomes

\[
\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_{q+1}) \leq \left[ \delta_M(\bar{\epsilon}_0, \bar{\epsilon}_0, \bar{\epsilon}_0) \right]^{1 - \eta^{q+1}} \\
\leq \left[ (1 - \eta) \gamma \right]^{1 - \eta^{q+1}} \leq \gamma.
\]

Hence, \( \hat{\epsilon}_{q+1} \in \odot_\gamma (\bar{\epsilon}_0, \gamma) \). Thus, \( \hat{\epsilon}_j \in \odot_\gamma (\bar{\epsilon}_0, \gamma) \) for all \( j \in N \). Consequently, (3) converts to

\[
\sqrt[n]{\delta_M(\bar{\epsilon}_j, \bar{\epsilon}_{j+1}, \bar{\epsilon}_{j+1})} \leq \left[ \sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_1, \bar{\epsilon}_1)} \right]^{\eta^j}.
\]

From inequality (4), we have

\[
\sqrt[n]{\delta_M(\bar{\epsilon}_j, \bar{\epsilon}_{j+k}, \bar{\epsilon}_{j+k})} \\
\leq \sqrt[n]{\delta_M(\bar{\epsilon}_j, \bar{\epsilon}_{j+1}, \bar{\epsilon}_{j+1})} \cdot \sqrt[n]{\delta_M(\bar{\epsilon}_{j+1}, \bar{\epsilon}_{j+2}, \bar{\epsilon}_{j+2})} \cdot \sqrt[n]{\delta_M(\bar{\epsilon}_{j+k-1}, \bar{\epsilon}_{j+k}, \bar{\epsilon}_{j+k})} \\
\leq \left[ \sqrt[n]{\delta_M(\bar{\epsilon}_0, \bar{\epsilon}_1, \bar{\epsilon}_1)} \right]^{\eta^j} \rightarrow 1, \quad j \rightarrow +\infty.
\]
This means that the sequence \( \{ \varepsilon_j \} \) is a \( M^0 \delta_M - C^* \) sequence in \( (\bar{\omega}_\gamma(\varepsilon_0, \gamma), \delta_M) \). Furthermore, there exists \( \tilde{\varepsilon}^* \in \bar{\omega}_\gamma(\varepsilon_0, \gamma) \) with
\[
\lim_{j \to +\infty} \sqrt[n]{\delta_M(\tilde{\varepsilon}_j, \tilde{\varepsilon}^*, \tilde{\varepsilon}^*)} = \lim_{j \to +\infty} \sqrt[n]{\delta_M(\tilde{\varepsilon}_j, \tilde{\varepsilon}_j, \tilde{\varepsilon}_j)} = 1. \tag{5}
\]

Now, assume that \( \tilde{\varepsilon}^* \preceq \varepsilon_j \preceq \tilde{\varepsilon}_{j-1} \),
\[
\sqrt[n]{\delta_M(\tilde{\varepsilon}^*, \tilde{\varepsilon}^*, \tilde{\varepsilon}^*)} \leq \sqrt[n]{\delta_M(\tilde{\varepsilon}_j, \tilde{\varepsilon}_j, \tilde{\varepsilon}_j)} \leq \sqrt[n]{\delta_M(\tilde{\varepsilon}_j, \tilde{\varepsilon}_{j-1}, \tilde{\varepsilon}_{j-1})} \leq \sqrt[n]{\delta_M(\tilde{\varepsilon}_j, \tilde{\varepsilon}_j, \tilde{\varepsilon}_j)} \leq \lim_{j \to +\infty} \left( \sqrt[n]{\delta_M(\tilde{\varepsilon}_j, \tilde{\varepsilon}_j, \tilde{\varepsilon}_j)} \right)^\eta = 1,
\]
which is a contradiction. Then, \( \tilde{\varepsilon}^* = \exists \tilde{\varepsilon}^* \). By a similar method, \( \delta_M(\exists \tilde{\varepsilon}^*, \exists \tilde{\varepsilon}^*, \tilde{\varepsilon}^*) = 1 \) and hence \( \exists \tilde{\varepsilon}^* = \tilde{\varepsilon}^* \). Now,
\[
\sqrt[n]{\delta_M(\tilde{\varepsilon}_j, \tilde{\varepsilon}_j, \tilde{\varepsilon}_j)} = \sqrt[n]{\delta_M(\exists \tilde{\varepsilon}^*, \exists \tilde{\varepsilon}^*, \tilde{\varepsilon}^*)} \leq \left( \sqrt[n]{\delta_M(\exists \tilde{\varepsilon}^*, \exists \tilde{\varepsilon}^*, \tilde{\varepsilon}^*)} \right)^\eta, \]
which is a contradiction, since \( \eta \in [0, 1) \). Thus, \( \delta_M(\tilde{\varepsilon}_j, \tilde{\varepsilon}_j, \tilde{\varepsilon}_j) = 1 \). \( \square \)

**Uniqueness:**
Consider \( \zeta^* \) as another point in \( \bar{\omega}_\gamma(\varepsilon_0, \gamma) \) such that \( \zeta^* = F\zeta^* \). If \( \tilde{\varepsilon}^* \) and \( \zeta^* \) are comparable, then
\[
\sqrt[n]{\delta_M(\tilde{\varepsilon}_j, \zeta^*, \zeta^*)} = \sqrt[n]{\delta_M(\exists \tilde{\varepsilon}_j^*, \exists \zeta^*, \zeta^*)} \leq \left( \sqrt[n]{\delta_M(\exists \tilde{\varepsilon}_j^*, \exists \zeta^*, \zeta^*)} \right)^\eta,
\]
which is a contradiction and thus,
\[
\delta_M(\tilde{\varepsilon}_j, \zeta^*, \zeta^*) = 1 \implies \tilde{\varepsilon}^* = \zeta^*.
\]
Similarly, we can prove \( \delta_M(\zeta^*, \zeta^*, \tilde{\varepsilon}_j) = 1 \).

On the other hand, if \( \tilde{\varepsilon}^* \) and \( \zeta^* \) are not comparable, then there is a point \( \tilde{u} \in \bar{\omega}_\gamma(\varepsilon_0, \gamma) \) which is the lower bound of \( \tilde{\varepsilon}^* \) and \( \zeta^* \), that is, \( \tilde{u} \preceq \tilde{\varepsilon}^* \) and \( \tilde{u} \preceq \zeta^* \). Furthermore, by the same argument, \( \tilde{\varepsilon}^* \preceq \tilde{u} \preceq \varepsilon_n = \ldots \preceq \varepsilon_0 \).
\[
\sqrt[n]{\delta_M(\tilde{u}, \exists \tilde{u}, \exists \tilde{u})} \leq \sqrt[n]{\delta_M(\tilde{u}, \exists \tilde{u}, \exists \tilde{u})}, \sqrt[n]{\delta_M(\exists \tilde{u}, \exists \tilde{u}, \exists \tilde{u})} \leq \sqrt[n]{\delta_M(\exists \tilde{u}, \exists \tilde{u}, \exists \tilde{u})} \leq \left( \sqrt[n]{\delta_M(\exists \tilde{u}, \exists \tilde{u}, \exists \tilde{u})} \right)^\eta,
\]
that is,
\[
\delta_M(\tilde{u}, \exists \tilde{u}, \exists \tilde{u}) \leq \delta_M(\tilde{u}, \exists \tilde{u}, \exists \tilde{u}) \leq \left( 1 - \eta \right) \gamma \left( 1 - \eta \right) \gamma \leq \gamma \text{ (by (1) and (2))}
\]
where \( \varepsilon_0, \tilde{u} \in \bar{\omega}_\gamma(\varepsilon_0, \gamma) \) and this means that \( \exists \tilde{u} \in \bar{\omega}_\gamma(\varepsilon_0, \gamma) \).

Now, we show that \( \exists \tilde{u} \in \bar{\omega}_\gamma(\varepsilon_0, \gamma) \) by using mathematical induction.

Suppose that \( \exists \tilde{u}, \exists \tilde{u}, \ldots, \exists \tilde{u} = \bar{\omega}_\gamma(\varepsilon_0, \gamma) \) for all \( q \in \mathbb{N} \). As \( \exists \tilde{u} \preceq \exists \tilde{u} \preceq \ldots \leq \tilde{u} \preceq \tilde{\varepsilon}^* \leq \varepsilon_n \leq \ldots \leq \varepsilon_0 \), then
\[ \sqrt[n]{\delta_M(\bar{e}_{q+1}, \mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u})} = \sqrt[n]{\delta_M(\mathfrak{A} \bar{e}_q, \mathfrak{A}(\mathfrak{A}^{q} \bar{u}), \mathfrak{A}(\mathfrak{A}^{q} \bar{u}))} \]

\[ \leq \left[ \sqrt[n]{\delta_M(\bar{e}_q, \mathfrak{A}^{q} \bar{u}, \mathfrak{A}^{q} \bar{u})} \right]^\eta \leq \ldots \leq \left[ \sqrt[n]{\delta_M(\bar{e}_q, \mathfrak{A}^{q} \bar{u}, \mathfrak{A}^{q} \bar{u})} \right]^{\eta^{q+1}}. \]

It follows that
\[ \delta_M(\bar{e}_{q+1}, \mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \leq \left[ \delta_M(\bar{e}_0, \bar{u}, \bar{u}) \right]^{\eta^{q+1}}. \] \hspace{1cm} (6)

Now,
\[ \delta_M(\bar{e}_0, \mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \leq \delta_M(\bar{e}_0, \bar{e}_1, \bar{e}_1) \ldots \delta_M(\bar{e}_q, \bar{e}_{q+1}, \bar{e}_{q+1}) \delta_M(\bar{e}_{q+1}, \mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \]
\[ \leq \delta_M(\bar{e}_0, \bar{e}_1, \bar{e}_1) \ldots \left[ \delta_M(\bar{e}_0, \bar{e}_1, \bar{e}_1) \right]^\eta \left[ \delta_M(\bar{e}_0, \bar{u}, \bar{u}) \right]^{\eta^{q+1}} \]
\[ \leq \left[ \delta_M(\bar{e}_0, \bar{e}_1, \bar{e}_1) \right]^{1 + \eta + \ldots + \eta^{q+1}} \left[ \delta_M(\bar{e}_0, \bar{u}, \bar{u}) \right]^{\eta^{q+1}} \]
\[ \leq \left[ (1 - \eta) \gamma \right]^{1 - \eta^{q+1}} \left[ (1 - \eta) \gamma \right]^{\eta^{q+1}} \]
\[ \leq \left[ (1 - \eta) \gamma \right]^{1 - \eta^{q+2}} \leq \gamma. \]

It means that \( \mathfrak{A}^{q+1} \bar{u} \in \mathfrak{A}^{\gamma}(\bar{e}_0, \gamma) \) and so \( \mathfrak{A}^{q+1} \bar{u} \in \mathfrak{A}^{\gamma}(\bar{e}_0, \gamma) \) for every \( j \in \mathbb{N} \). Further,
\[ \delta_M(\bar{e}^*, \bar{e}^*, \bar{e}^*) \leq \delta_M(\mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \delta_M(\mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \delta_M(\mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \]
\[ = \delta_M(\mathfrak{A}(\mathfrak{A}^{q+1} \bar{e}), \mathfrak{A}(\mathfrak{A}^{q+1} \bar{u}), \mathfrak{A}(\mathfrak{A}^{q+1} \bar{u})) \delta_M(\mathfrak{A}(\mathfrak{A}^{q+1} \bar{u}), \mathfrak{A}(\mathfrak{A}^{q+1} \bar{u})) \]
\[ \leq \left[ \delta_M(\mathfrak{A}^{q+1} \bar{e}, \mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \right]^{\eta} \left[ \delta_M(\mathfrak{A}(\mathfrak{A}^{q+1} \bar{u}), \mathfrak{A}(\mathfrak{A}^{q+1} \bar{u})) \right]^{\eta} \]
\[ \vdots \]
\[ \leq \left[ \delta_M(\mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \right]^{\eta} \left[ \delta_M(\mathfrak{A}^{q+1} \bar{u}, \mathfrak{A}^{q+1} \bar{u}) \right]^{\eta} \rightarrow 1, \text{ when } j \rightarrow +\infty. \]

Hence, \( \delta_M(\bar{e}^*, \bar{e}^*, \bar{e}^*) = 1 \implies \bar{e}^* = \bar{e}^* \). By a similar method
\[ \delta_M(\bar{e}^*, \bar{e}^*, \bar{e}^*) = 1 \text{ implies } \bar{e}^* = \bar{e}^*. \]

Therefore, a point \( \bar{e}^* \) is unique in \( \mathfrak{C} \).

**Corollary 1.** Let \( (\mathfrak{C}, \leq, \delta_M) \) be an ordered complete multiplicative \( \delta_M \) metric space. Suppose the mapping \( \mathfrak{A} : \mathfrak{C} \rightarrow \mathfrak{C} \) with \( \eta \in [0, 1) \) and \( \gamma > 0 \) satisfies the following,
\[ \delta_M(\mathfrak{A} \bar{u}, \mathfrak{A} \bar{u}, \mathfrak{A} \bar{u}) \leq \left[ G_M(\bar{e}, \bar{h}) \right]^{\eta}, \] \hspace{1cm} (7)

for \( \bar{e}, \bar{h}, \bar{h} \in \mathfrak{A}(\mathfrak{C}(\bar{e}, \gamma)) \), with the condition (2).

If for a nonincreasing sequence \( \{ \bar{e}_n \} \rightarrow s \in \mathfrak{A}(\mathfrak{C}(\bar{e}, \gamma)) \) such that \( s \leq \bar{e}_n \), then there exists a point \( \bar{e}^* \) in \( \mathfrak{A}(\mathfrak{C}(\bar{e}, \gamma)) \) so that \( \bar{e}^* = \mathfrak{A} \bar{e}^* \) and \( \delta_M(\bar{e}^*, \bar{e}^*, \bar{e}^*) = 1 \). Moreover, if for any three points \( \bar{e}, \bar{h}, \bar{h} \) in \( \mathfrak{A}(\mathfrak{C}(\bar{e}, \gamma)) \), there exists a point \( \bar{u} \in \mathfrak{A}(\mathfrak{C}(\bar{e}, \gamma)) \) such that \( \bar{u} \leq \bar{h}, \bar{u} \leq \bar{e} \) and \( \bar{u} \leq \bar{h} \), that is, every two points in \( \mathfrak{A}(\mathfrak{C}(\bar{e}, \gamma)) \) has a lower bound, then the point \( \bar{e}^* \) is unique.

**Example 1.** Let \( \mathfrak{C} \) be a set of non-negative rationals with \( \delta_M : \mathfrak{C}^3 \rightarrow \mathfrak{C} \) a multiplicative \( \delta_M \)-metric on \( \mathfrak{C} \) defined as follows:
\[ \delta_M(\bar{e}, \bar{e}, \bar{h}) = e^{\bar{e} - \bar{h}} + |\bar{e} - \bar{h}| + |\bar{h} - \bar{e}|. \]

Furthermore, let \( \mathfrak{A} : \mathfrak{C} \rightarrow \mathfrak{C} \) be defined as
we get the following corollaries.

Corollary 3. Consider 
\[
\exists \tilde{e} = \begin{cases} 
\frac{\tilde{e}}{4} & \text{if } \tilde{e} \in \left[0, \frac{1}{3}\right); \\
\tilde{e} - \frac{1}{3} & \text{if } \tilde{e} \in \left[\frac{1}{3}, \infty\right). 
\end{cases}
\]

For \[\tilde{e}_0 = \frac{1}{3}, \gamma = \frac{11}{2}, \eta = \frac{5}{8}\] and \(\overline{\gamma}(\tilde{e}_0, \gamma) = \left[0, \frac{11}{2}\right]\), we have
\[
(1 - \eta) \gamma = \frac{3}{8} \cdot \frac{11}{2} = \frac{33}{16} = 2.0625,
\]
and
\[
\delta_M(\tilde{e}_0, \exists \tilde{e}_0, \exists \tilde{e}_0) = \delta_M\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \delta_M\left(\frac{1}{3}, 0, 0\right) = e^{2/3} = 1.9477 
\]
\[
\leq (1 - \eta) \gamma.
\]

Step 1: (when the points are in a closed ball). If \(\tilde{e}, \gamma, \bar{h} \in \left[0, \frac{1}{3}\right) \subseteq \overline{\gamma}(\tilde{e}_0, \gamma) = \left[0, \frac{11}{2}\right]\), we get
\[
\delta_M(\exists \tilde{e}, \exists \gamma, \exists \bar{h}) = e^{\frac{1}{2}(|\tilde{e} - \gamma| + |\gamma - \bar{h}| + |\bar{h} - \bar{e}|)} 
\]
\[
\leq e^{\frac{1}{2}(|\tilde{e} - \gamma| + |\gamma - \bar{h}| + |\bar{h} - \bar{e}|)} = \left[\delta_M(x, y, z)\right]^{\eta}.
\]

Step 2: (when the points are not in a closed ball). If \(\tilde{e}, \gamma, \bar{h} \in \left(\frac{1}{3}, \infty\right), \) we have
\[
\delta_M(\exists \tilde{e}, \exists \gamma, \exists \bar{h}) = e^{\tilde{e}}(|\tilde{e} - \gamma| + |\gamma - \bar{h}| + |\bar{h} - \bar{e}|) 
\]
\[
\geq e^{\tilde{e}}(|\tilde{e} - \gamma| + |\gamma - \bar{h}| + |\bar{h} - \bar{e}|) = \left[\delta_M(\tilde{e}, \gamma, \bar{h})\right]^{\eta}.
\]

Clearly, the contractive condition is not satisfied in \(\tilde{e}\) and is satisfied in \(\overline{\gamma}(\tilde{e}_0, \gamma)\). Hence, all the conditions of Corollary 1 are verified in the case of \(\tilde{e}, \gamma, \bar{h} \in \overline{\gamma}(\tilde{e}_0, \gamma)\).

Since every multiplicative \(\delta_M\)-metric space generates a multiplicative \(d_M\)-metric space, we get the following corollaries.

Corollary 2. Let \((\overline{\tilde{e}}, \leq, d_M)\) be an ordered complete multiplicative \(d_M\)-metric space. Suppose the mapping \(\exists: \tilde{e} \rightarrow \tilde{e}\) with \(\eta \in [0, 1)\) and \(\gamma > 0\) satisfies the following,
\[
\left(d_M(\exists \tilde{e}, \exists \gamma)\right)^{\eta} \leq \left(d_M(\tilde{e}, \gamma)\right)^{\eta},
\]
(8)
and
\[
d_M(\exists \tilde{e}_0, \exists \gamma) \leq (1 - \eta) \gamma,
\]
(9)
for \(\tilde{e}, \gamma \in \overline{\gamma}(\tilde{e}_0, \gamma)\). If for a nonincreasing sequence \(\{\tilde{e}_n\} \rightarrow s \in \overline{\gamma}(\tilde{e}_0, \gamma)\) such that \(s \leq \tilde{e}_n\), then, there exists a point \(\tilde{e}^* \in \overline{\gamma}(\tilde{e}_0, \gamma)\) so that \(\tilde{e}^* = \exists \tilde{e}^*\) and \(d_M(\tilde{e}^*, \tilde{e}^*) = 1\). Moreover, if for any two points \(\tilde{e}, \gamma \in \overline{\gamma}(\tilde{e}_0, \gamma)\), there exists a point \(\tilde{u} \in \overline{\gamma}(\tilde{e}_0, \gamma)\) such that \(\tilde{u} \geq \tilde{e}\) and \(\tilde{u} \geq \gamma\), that is, every two points in \(\overline{\gamma}(\tilde{e}_0, \gamma)\) has a lower bound, then \(\tilde{e}^*\) is the unique point in \(\overline{\gamma}(\tilde{e}_0, \gamma)\).

Corollary 3. Consider \((\overline{\tilde{e}}, \leq, d_M)\) as an ordered complete multiplicative \(d_M\) metric space. Suppose that the mapping \(\exists: \tilde{e} \rightarrow \tilde{e}\) with \(\eta \in [0, 1)\) and \(\gamma > 0\) satisfies the following,
\[
\left(d_M(\exists \tilde{e}, \exists \gamma)\right)^{\eta} \leq \left[d_M(\tilde{e}, \gamma)\right]^{\eta},
\]
(10)
for \(\tilde{e}, \gamma \in \overline{\gamma}(\tilde{e}_0, \gamma),\) with condition (9).
If for a nonincreasing sequence \( \{ \xi_n \} \rightarrow s \in \overline{\gamma}(\xi_0, \gamma) \) implies that \( s \geq \xi_n \), then there is a point \( \xi^* \) in \( \overline{\gamma}(\xi_0, \gamma) \) such that \( \xi^* = \mathfrak{v} \epsilon^* \) and \( \delta_M(\xi^*, \mathfrak{v}^*) = 1 \). Moreover, if for any two points \( \xi, \zeta \) in \( \overline{\gamma}(\xi_0, \gamma) \), there exists a point \( \overline{\zeta} \in \overline{\gamma}(\xi_0, \gamma) \) such that \( \overline{\zeta} \leq \phi \) and \( \overline{\zeta} \geq \zeta \), that is, every two points in \( \overline{\gamma}(\xi_0, \gamma) \) has a lower bound, then a fixed point \( \xi^* \) is unique in \( \overline{\gamma}(\xi_0, \gamma) \).

**Theorem 2.** Let \( (\xi, \leq, \delta_M) \) be an ordered complete multiplicative \( \delta_M \)-metric space. Suppose that the mapping \( \mathfrak{v} : \xi \rightarrow \xi \) with \( \eta \in [0,1) \) and \( \gamma > 0 \) satisfies the following,

\[
\sqrt[\eta]{\delta_M(\mathfrak{v} \xi, \mathfrak{v} \zeta, \mathfrak{v} h)} \leq \mathcal{M},
\]

since

\[
\mathcal{M} = \max \left\{ \frac{\sqrt{\delta_M(\mathfrak{v} \xi, \mathfrak{v} \zeta, \mathfrak{v} h)} + \sqrt{\delta_M(\mathfrak{v} \xi, \mathfrak{v} \zeta, \mathfrak{v} h)}}{\sqrt{\delta_M(\mathfrak{v} \xi, \mathfrak{v} \zeta, \mathfrak{v} h)}} \right\} \eta,
\]

and

\[
\mathcal{G}_M(\mathfrak{v} \xi_0, \mathfrak{v} \xi_0, \mathfrak{v} \xi_0) \leq (1 - \eta) \gamma,
\]

for \( \mathfrak{v}, \mathfrak{z}, \mathfrak{h} \in \overline{\gamma}(\xi_0, \gamma) \). If for a nonincreasing sequence \( \{ \xi_n \} \) in \( \overline{\gamma}(\xi_0, \gamma) \), \( \{ \xi_n \} \rightarrow \mathfrak{v} \in \overline{\gamma}(\xi_0, \gamma) \) so that \( \mathfrak{v} \leq \xi_n \), then there exists a unique fixed point \( \xi^* \) such that \( \delta_M(\xi^*, \xi^*, \xi^*) = 1 \) and \( \xi^* = \mathfrak{v} \xi^* \).

**Proof.** Consider an arbitrary point \( \xi_0 \) in \( \xi \) and \( \xi_{j+1} = \mathfrak{v} \xi_j \leq \xi_j \) for all \( n \in N \cup \{ 0 \} \). From inequality (12), we find

\[
\delta_M(\mathfrak{v} \xi_0, \mathfrak{v} \xi_1, \mathfrak{v} \xi_1) \leq (1 - \eta) \gamma \leq \gamma,
\]

for all \( j \in N \cup \{ 0 \} \). Now, from inequality (12), we obtain \( \delta_M(\mathfrak{v} \xi_0, \mathfrak{v} \xi_1, \mathfrak{v} \xi_1) \leq \gamma \) and \( \delta_M(\mathfrak{v} \xi_1, \mathfrak{v} \xi_2, \mathfrak{v} \xi_2) \leq \gamma \), which yields \( \xi_1, \xi_2, \ldots, \xi_q \in \overline{\gamma}(\xi_0, \gamma) \). Similarly \( \xi_3, \ldots, \xi_q \in \overline{\gamma}(\xi_0, \gamma) \) for all \( q \in N \). Now,

\[
\sqrt[\eta]{\delta_M(\xi_q, \xi_q, \xi_q)} \leq \max \left\{ \frac{\sqrt[\eta]{\delta_M(\xi_q, \xi_q, \xi_q)} + \sqrt[\eta]{\delta_M(\xi_q, \xi_q, \xi_q)}}{\sqrt[\eta]{\delta_M(\xi_q, \xi_q, \xi_q)}} \right\} \eta.
\]
thereby implying,
\[
\sqrt[\eta]{\delta_M(\hat{\epsilon}_{q+1}, \hat{\epsilon}_{q+1})} \leq \left[ \sqrt[\eta]{\delta_M(\hat{\epsilon}_{q-1}, \hat{\epsilon}_{q}, \hat{\epsilon}_{q})} \cdot \sqrt[\eta]{\delta_M(\hat{\epsilon}_{q+1}, \hat{\epsilon}_{q+1})} \right]^\eta,
\]
that is,
\[
\delta_M(\hat{\epsilon}_{q}, \hat{\epsilon}_{q+1}) \leq \left[ \delta_M(\hat{\epsilon}_{q-1}, \hat{\epsilon}_{q}) \right]^\mu + \delta_M(\hat{\epsilon}_{q-2}, \hat{\epsilon}_{q-1}) \right]^\mu^2 + \cdots + \left[ \delta_M(\hat{\epsilon}_{0}, \hat{\epsilon}_{1}) \right]^{\mu^q},
\]
where \( 0 < \mu = \frac{\eta}{1-\eta} < \frac{1}{2} \). Taking (11) and (12) in consideration, we get
\[
\delta_M(\hat{\epsilon}_{0}, \hat{\epsilon}_{q+1}) \leq \delta_M(\hat{\epsilon}_{0}, \hat{\epsilon}_{1}) \delta_M(\hat{\epsilon}_{1}, \hat{\epsilon}_{2}) \cdots \delta_M(\hat{\epsilon}_{q}, \hat{\epsilon}_{q+1}) \leq \left[ \delta_M(\hat{\epsilon}_{0}, \hat{\epsilon}_{1}) \right]^{1 - \mu^{q+1}} \leq \left( 1 - \eta \right) \gamma \leq \gamma.
\]
Then, \( \hat{\epsilon}_{q+1} \in \overline{\mathbb{G}(\hat{\epsilon}_0, \gamma)} \). Thus, \( \hat{\epsilon}_j \in \overline{\mathbb{G}(\hat{\epsilon}_0, \gamma)} \) for every \( j \in N \). Now, inequality (13) becomes
\[
\delta_M(\hat{\epsilon}_{j}, \hat{\epsilon}_{j+1}) \leq \left[ \delta_M(\hat{\epsilon}_{0}, \hat{\epsilon}_{1}) \right]^{\mu^j}.
\]
(14)
From inequality (14), we find
\[
\delta_M(\hat{\epsilon}_{j}, \hat{\epsilon}_{j+k}) \leq \delta_M(\hat{\epsilon}_{j}, \hat{\epsilon}_{j+1}) \delta_M(\hat{\epsilon}_{j+1}, \hat{\epsilon}_{j+2}) \cdots \delta_M(\hat{\epsilon}_{j+k-1}, \hat{\epsilon}_{j+k}) \leq \left[ \delta_M(\hat{\epsilon}_{0}, \hat{\epsilon}_{1}) \right]^{\mu^j} \leq \frac{1 - \mu^k}{1 - \mu} \to 1, \quad j \to +\infty.
\]
This shows that the sequence \( \{ \hat{\epsilon}_j \} \) is a \( M^\ast \delta_M - C^\ast \) sequence in \( \overline{\mathbb{G}(\hat{\epsilon}_0, \gamma)} \). Then, there exists \( \hat{\epsilon}^\ast \in \overline{\mathbb{G}(\hat{\epsilon}_0, \gamma)} \) with (5) verified.
Now, suppose that \( \hat{\epsilon}^\ast \leq \hat{\epsilon}_j \leq \hat{\epsilon}_{j-1} \),
\[
\sqrt[\eta]{\delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast)} \leq \sqrt[\eta]{\delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}_j, \hat{\epsilon}_j)} \sqrt[\eta]{\delta_M(\hat{\epsilon}_j, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast)} = \sqrt[\eta]{\delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}_j, \hat{\epsilon}_j)} \sqrt[\eta]{\delta_M(\hat{\epsilon}_j-1, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast)} \leq \sqrt[\eta]{\delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}_j, \hat{\epsilon}_j)} \left[ \sqrt[\eta]{\delta_M(\hat{\epsilon}_j-1, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast)} \right]^{\eta} \leq \lim_{j \to +\infty} \left( \sqrt[\eta]{\delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}_j, \hat{\epsilon}_j)} \left[ \sqrt[\eta]{\delta_M(\hat{\epsilon}_j-1, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast)} \right]^{\eta} \right) = 1,
\]
which is a contradiction. Then, \( \hat{\epsilon}^\ast = \hat{\epsilon}^\ast \). By a similar method, \( \delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast) = 1 \) and hence \( \hat{\epsilon}^\ast = \hat{\epsilon}^\ast \). Now,
\[
\sqrt[\eta]{\delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast)} = \sqrt[\eta]{\delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast)} \leq \left[ \sqrt[\eta]{\delta_M(\hat{\epsilon}^\ast, \hat{\epsilon}^\ast, \hat{\epsilon}^\ast)} \right]^{\eta}.
which is a contradiction, since $\eta \in [0,1)$. Thus, $\delta_M(\bar{e}^*, \bar{e}^*, \bar{e}^*) = 1$. \hfill \Box

Uniqueness:

Let $\zeta^*$ be another point in $\overline{\gamma}((\delta_0, \gamma))$ such that $\zeta^* = \exists \zeta^*$. If $\bar{e}^*$ and $\zeta^*$ are comparable, then

$$\sqrt[\eta]{\delta_M(\bar{e}^*, \zeta^*, \zeta^*)} = \sqrt[\eta]{\delta_M(\exists \bar{e}^*, \exists \zeta^*, \exists \zeta^*)} \leq \sqrt[\eta]{\delta_M(\bar{e}^*, \zeta^*, \zeta^*)}$$

which is a contradiction that leads us to

$$\delta_M(\bar{e}^*, \zeta^*, \zeta^*) = 1 \quad \text{implies} \quad \bar{e}^* = \zeta^*. $$

Similarly, we can prove $\delta_M(\zeta^*, \zeta^*, \bar{e}^*) = 1$.

On the other hand, if $\bar{e}^*$ and $\zeta^*$ are not comparable, then there exists a point $\bar{u} \in \overline{\gamma}((\delta_0, \gamma))$ which is the lower bound of $\bar{e}^*$ and $\zeta^*$, that is, $\bar{u} \preceq \bar{e}^*$ and $\bar{u} \preceq \zeta^*$. Furthermore, $\bar{e}^* \preceq \bar{u}$ as $\bar{e}_n \rightarrow \bar{e}^*$, $\bar{u} \preceq \bar{e}^* \preceq \bar{e}_n \preceq \ldots \preceq \delta_0$. Therefore,

$$\sqrt[\eta]{\delta_M(\delta_0, \exists \bar{u}, \exists \bar{u})} \leq \sqrt[\eta]{\delta_M(\delta_0, \bar{e}_1, \bar{e}_1)} \cdot \sqrt[\eta]{\delta_M(\bar{e}_1, \exists \bar{u}, \exists \bar{u})} = \sqrt[\eta]{\delta_M(\delta_0, \exists \delta_0, \exists \delta_0)} \cdot \sqrt[\eta]{\delta_M(\exists \delta_0, \exists \bar{u}, \exists \bar{u})} \leq \sqrt[\eta]{\delta_M(\delta_0, \exists \delta_0, \exists \delta_0)} \cdot \left[ \sqrt[\eta]{\delta_M(\delta_0, \bar{u}, \bar{u})} \right]^{\eta},$$

that is,

$$\delta_M(\delta_0, \exists \bar{u}, \exists \bar{u}) \leq \delta_M(\delta_0, \exists \delta_0, \exists \delta_0) \cdot \left[ \delta_M(\delta_0, \bar{u}, \bar{u}) \right]^{\eta} \leq (1 - \eta) \gamma \left[ 1 - \eta \right]^{\eta} \gamma \leq \gamma \text{ (by (12))}$$

where $\delta_0, \bar{u} \in \overline{\gamma}((\delta_0, \gamma))$ and this means that $\exists \bar{u} \in \overline{\gamma}((\delta_0, \gamma))$.

Now, we prove that $\exists^q \bar{u} \in \overline{\gamma}((\delta_0, \gamma))$ by using mathematical induction. Suppose $\exists^q \bar{u}, \exists^{q+1} \bar{u}, \ldots, \exists^q \bar{u} \in \overline{\gamma}((\delta_0, \gamma))$ for all $q \in N$. As $\exists^q \bar{u} \preceq \exists^{q+1} \bar{u} \preceq \ldots \preceq \bar{u} \preceq \bar{e}^* \preceq \ldots \preceq \delta_0$, then

$$\sqrt[\eta]{\delta_M(\delta_q, \exists^{q+1} \bar{u}, \exists^{q+1} \bar{u})} = \sqrt[\eta]{\delta_M(\exists \delta_q, \exists (\exists^q \bar{u}), \exists (\exists^q \bar{u}))} \leq \left[ \sqrt[\eta]{\delta_M(\delta_q, \exists^q \bar{u}, \exists^q \bar{u})} \right]^{\eta} \leq \ldots \leq \left[ \sqrt[\eta]{\delta_M(\delta_q, \exists^{q+1} \bar{u}, \exists^{q+1} \bar{u})} \right]^{\eta^{q+1}},$$

it follows that

$$\delta_M(\delta_q, \exists^{q+1} \bar{u}, \exists^{q+1} \bar{u}) \leq \left[ \delta_M(\delta_q, \bar{u}, \bar{u}) \right]^{\eta^{q+1}}. \quad (15)$$

Now,

$$\delta_M(\delta_0, \exists^{q+1} \bar{u}, \exists^{q+1} \bar{u}) \leq \delta_M(\delta_0, \bar{e}_1, \bar{e}_1) \ldots \delta_M(\delta_q, \bar{e}_{q+1}, \bar{e}_{q+1}). \delta_M(\delta_{q+1}, \exists^{q+1} \bar{u}, \exists^{q+1} \bar{u}) \leq \delta_M(\delta_0, \bar{e}_1, \bar{e}_1) \ldots \left[ \delta_M(\delta_0, \bar{e}_1, \bar{e}_1) \right]^{\eta q} \left[ \delta_M(\delta_0, \bar{u}, \bar{u}) \right]^{\eta^{q+1}} \leq \left[ \delta_M(\delta_0, \bar{e}_1, \bar{e}_1) \right]^{1 + \eta + \ldots + \eta^{q+1}} \left[ \delta_M(\delta_0, \bar{u}, \bar{u}) \right]^{\eta^{q+1}} \leq \left[ (1 - \eta) \gamma \right]^{1 - \eta^{q+1}} \left[ (1 - \eta) \gamma \right]^{\eta^{q+1}} \leq \left[ (1 - \eta) \gamma \right]^{1 - \eta^{q+2}} \leq \gamma.$$
It follows that $\mathcal{V}^j u \in \mathcal{O}_\eta(\mathcal{E}_0, \gamma)$ and so $\mathcal{V}^j u \in \mathcal{O}_\eta(\mathcal{E}_0, \gamma)$ for every $j \in \mathbb{N}$. Furthermore
\[
\delta_M(\mathcal{E}^*, \mathcal{E}^*, \mathcal{E}^*) \leq \delta_M(\mathcal{V}^j \mathcal{E}^*, \mathcal{V}^j \mathcal{E}^*, \mathcal{V}^j \mathcal{E}^*), \delta_M(\mathcal{V}^j \mathcal{E}^*, \mathcal{V}^j \mathcal{E}^*, \mathcal{V}^j \mathcal{E}^*) \\
= \delta_M(\mathcal{V}(\mathcal{E}^*), \mathcal{V}(\mathcal{E}^*), \mathcal{V}(\mathcal{E}^*)) \delta_M(\mathcal{V}(\mathcal{E}^*), \mathcal{V}(\mathcal{E}^*), \mathcal{V}(\mathcal{E}^*)) \\
\leq \left[ \delta_M(\mathcal{V}^j \mathcal{E}^*, \mathcal{V}^j \mathcal{E}^*, \mathcal{V}^j \mathcal{E}^*) \right]^\eta \left[ \delta_M(\mathcal{V}^j \mathcal{E}^*, \mathcal{V}^j \mathcal{E}^*, \mathcal{V}^j \mathcal{E}^*) \right]^\eta \\
\leq \left[ \delta_M(\mathcal{E}^*, \mathcal{E}^*, \mathcal{E}^*) \right]^\eta \left[ \delta_M(\mathcal{E}^*, \mathcal{E}^*, \mathcal{E}^*) \right]^\eta \\
\rightarrow 1, \text{ where } j \rightarrow +\infty.
\]

Hence, $\delta_M(\mathcal{E}^*, \mathcal{E}^*, \mathcal{E}^*) = 1 \implies \mathcal{E}^* = \mathcal{E}^*$. Similarly,

$\delta_M(\mathcal{E}^*, \mathcal{E}^*, \mathcal{E}^*) = 1 \implies \mathcal{E}^* = \mathcal{E}^*$.

Therefore, a point $\mathcal{E}^*$ is unique in $\mathcal{E}$.

As illustrated, Theorem 1 is a corollary to Theorem 6.

**Example 2.** Consider $\mathcal{E} = R^+ \cup \{0\}$ with $\delta_M: \mathcal{E}^3 \rightarrow \mathcal{E}$ a multiplicative $G_M$-metric on $\mathcal{E}$ defined by

$\delta_M(\mathcal{E}, \mathcal{E}, \mathcal{E}) = e^{\mathcal{E} - \mathcal{E}^*} + |\mathcal{E} - h| + |\mathcal{E}^* - \mathcal{E}^*|.$

Furthermore, let the mapping $\mathfrak{M}: \mathcal{E} \rightarrow \mathcal{E}$ be defined as

$$
\mathfrak{M} = \left\{ \begin{array}{ll}
\frac{\mathcal{E}}{6} & \text{if } \mathcal{E} \in \left(0, \frac{1}{3}\right) \cap \mathcal{E}; \\
\mathcal{E} - \frac{1}{8} & \text{if } \mathcal{E} \in \left[\frac{1}{3}, \infty\right) \cap \mathcal{E},
\end{array} \right.
$$

and

$$
\mathcal{M} = \max \left\{ \sqrt[\eta]{\delta_M(\mathcal{E}, \mathcal{E}, \mathcal{E})}, \sqrt[\eta]{\mathcal{G}_M(\mathcal{E}, \mathfrak{M}, \mathfrak{M})}, \sqrt[\eta]{\delta_M(\mathcal{E}, \mathcal{E}, \mathcal{E})}, \sqrt[\eta]{\mathcal{G}_M(\mathcal{E}, \mathfrak{M}, \mathfrak{M})} \right\}.
$$

For $\mathcal{E}_0 = \frac{1}{4}, \gamma = \frac{13}{2}, \eta = \frac{2}{3}$ and $\mathcal{O}_\eta(\mathcal{E}_0, \gamma) = \left[0, \frac{13}{2}\right]$, we have

$$(1 - \eta) \gamma = \frac{1}{3} \frac{13}{2} = \frac{13}{6} = 2.16,$$

and

$$
\delta_M(\mathcal{E}_0, \mathfrak{M}(\mathcal{E}_0), \mathfrak{M}(\mathcal{E}_0)) = \delta_M(\mathcal{E}_0, \mathfrak{M}(\mathcal{E}_0), \mathfrak{M}(\mathcal{E}_0)) = \delta_M(\mathcal{E}_0, \mathfrak{M}(\mathcal{E}_0), \mathfrak{M}(\mathcal{E}_0)) = e^{6/16} = 1.4533
$$

or

$$
\leq (1 - \eta) \gamma.
$$
Step 1: If \( \xi, \zeta, \eta \in \left(0, \frac{1}{3}\right) \cap \zeta \subseteq \overline{\mathcal{M}(\xi_0, \gamma)} = \left[0, \frac{13}{2}\right] \), we obtain

\[
\sqrt[n]{\delta_M(3\xi, 3\zeta, 3\eta)} = \sqrt[n]{e^{\frac{1}{2}(|\xi-\zeta| + |\xi-\eta| + |\eta-\zeta|)}} \\
\leq \left[ \max \left\{ \begin{array}{c} \sqrt[n]{e^{\frac{1}{2}(|\xi-\zeta| + |\xi-\eta| + |\eta-\zeta|)}, \sqrt[n]{e^{|\xi+\zeta|}}, \sqrt[n]{e^{|\xi-\zeta| + |\xi-\eta| + |\eta-\zeta|}} \end{array} \right\} \right]^{\eta} \\
= \left[ \sqrt[n]{\delta_M(\xi, \zeta, \eta)} \right]^{\eta} .
\]

Step 2: If \( \xi, \zeta, \eta \in \left[\frac{1}{3}, \infty\right) \cap \zeta \), we have

\[
\sqrt[n]{\delta_M(3\xi, 3\zeta, 3\eta)} = \sqrt[n]{e^{\frac{1}{2}(|\xi-\zeta| + |\xi-\eta| + |\eta-\zeta|)}} \\
\geq \left[ \max \left\{ \begin{array}{c} \sqrt[n]{e^{\frac{1}{2}(|\xi-\zeta| + |\xi-\eta| + |\eta-\zeta|)}, \sqrt[n]{e^{|\xi+\zeta|}}, \sqrt[n]{e^{|\xi-\zeta| + |\xi-\eta| + |\eta-\zeta|}} \end{array} \right\} \right]^{\eta} \\
= \left[ \sqrt[n]{\delta_M(\xi, \zeta, \eta)} \right]^{\eta} .
\]

Clearly, the contractive condition is not verified in \( \left[\frac{1}{3}, \infty\right) \cap \zeta \) and is verified in \( \overline{\mathcal{M}(\xi_0, \gamma)} \). Hence, all the assertions of Theorem 6 are satisfied in the case of \( \xi, \zeta, \eta \in \overline{\mathcal{M}(\xi_0, \gamma)} \).

4. Application for Nonlinear Volterra Type Integral Equations

Clearly, many researchers have justified many kinds of linear and nonlinear Volterra and Fredholm type integral equations by using various contraction principles. Rasham et al. [38] proved an expressive fixed point results for sufficient conditions to solve two systems of nonlinear integral equations. For further fixed point results with applications related to integral equations, (see [39–44]).

Theorem 3. Let \( (\xi, \leq, \delta_M) \) be an ordered complete multiplicative \( \delta_M \)-metric space. Suppose the mapping \( \mathfrak{M} : \xi \rightarrow \xi \) with \( \eta \in [0,1) \) and \( \gamma > 0 \) satisfies the following,

\[
\sqrt[n]{\delta_M(3\xi, 3\zeta, 3\eta)} \leq \left[ \sqrt[n]{\delta_M(\xi, \zeta, \eta)} \right]^{\eta} .
\]
Then, every nonincreasing sequence \( \{ \xi_n \} \) in a multiplicative \( \delta_M \)-metric space converges to \( \xi^* \). Moreover, \( \xi^* \) is the fixed point of the mapping \( \mathcal{M} \).

**Proof.** The proof of Theorem 3 is similar to that of Theorem 1. Consider the nonlinear Volterra type integral equations as follows:

\[
\begin{align*}
\xi(\bar{u}) &= \int_0^a \mathcal{H}_1(\bar{u}, h, \xi) \, dh, \\
\zeta(\bar{u}) &= \int_0^a \mathcal{H}_2(\bar{u}, h, \zeta) \, dh, \\
h(\bar{u}) &= \int_0^a \mathcal{H}_3(\bar{u}, h, h) \, dh,
\end{align*}
\]

for all \( \bar{u} \in [0, 1] \), and \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 : [0, 1] \times [0, 1] \times C([0, 1], R_+) \rightarrow R_+ \). We prove the existence of the solution of (16)–(18). For \( \tilde{\mathcal{M}} \in C([0, 1], R_+) \), define its norm as:

\[
\| \tilde{\mathcal{M}} \|_\tau = \sup_{\bar{u} \in [0, 1]} \{ e^{\| \tilde{\mathcal{M}}(\bar{u}) \|} \}.
\]

Then, define

\[
\delta_M(\xi, \zeta, h) = \left[ \sup_{\bar{u} \in [0, 1]} \left\{ e^{\left| h - \xi \right| + \left| \zeta - h \right| + \left| h - \bar{u} \right|} \right\} \right] = e^{\left\| \xi - \zeta \right\| + \left\| \zeta - h \right\| + \left\| h - \bar{u} \right\|},
\]

where \( \tau > 0 \), for all \( \xi, \zeta \) and \( h \in C([0, 1], R_+) \). With these settings, \( C([0, 1], R_+), \delta_M \) becomes a complete multiplicative \( \delta_M \)-metric space. \( \square \)

Now, we prove the following theorem to show the existence of the solution to integral equations.

**Theorem 4.** Suppose the following conditions are satisfied:

(i) \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 : [0, 1] \times [0, 1] \times C([0, 1], R_+) \rightarrow R_+ \);

(ii) Define

\[
\begin{align*}
(\exists \xi)(\bar{u}) &= \int_0^a \mathcal{H}_1(\bar{u}, h, \xi) \, dh; \\
(\exists \zeta)(\bar{u}) &= \int_0^a \mathcal{H}_2(\bar{u}, h, \zeta) \, dh; \\
(\exists h)(\bar{u}) &= \int_0^a \mathcal{H}_3(\bar{u}, h, h) \, dh.
\end{align*}
\]

If there exists

\[
\left( e \sqrt{\int_0^a \left( |\mathcal{H}_1(\bar{u}, h, \xi) - \mathcal{H}_2(\bar{u}, h, \zeta) + |\mathcal{H}_2(\bar{u}, h, \xi) - \mathcal{H}_3(\bar{u}, h, h) + |\mathcal{H}_3(\bar{u}, h, h) - \mathcal{H}_1(\bar{u}, h, \xi)| \right) \, dh \right)^\eta \leq \int_0^a \left( e^{\sqrt{\left| \xi - \zeta \right| + \left| \zeta - h \right| + \left| h - \bar{u} \right|}} \right)^\eta \, dh.
\]

for every \( \bar{u}, h \in [0, 1] \) and \( \xi, \zeta, h \in C([0, 1], R_+) \), then the integral Equations (16)–(18) have one solution in \( C([0, 1], R_+) \).
Proof. By (ii),
\[ \sqrt[n]{\delta_M(3\varepsilon, 3\zeta, 3h)} \]
\[ = \int_0^a \left( e^{-\int \sqrt[n]{\delta_M(3\varepsilon - 3\zeta + 3\zeta - 3h + 3h - 3\varepsilon) \, dh} \right) \eta \]
\[ \leq \int_0^a \left( e^{\sqrt[n]{\delta_M(3\zeta, 3\zeta) \, dh} \right) \eta \]
\[ \leq \left( \sqrt[n]{\delta_M(3\varepsilon, 3\zeta, 3h)} \right) \eta . \]

So, all the conditions of Theorem 3 are satisfied. Hence, the integral Equations (16)–(18) have a unique solution. □

**Example 3.** Take \( E = [0, 1] \). If we put \( \bar{u} = 1 \) in (16)–(18), then we get the following integral equations
\[ (3\varepsilon)(\bar{u}) = \int_0^a \mathcal{H}_1(\bar{u}, h, \varepsilon) dh = \int_0^a \frac{4}{9(\bar{u} + 1 + \varepsilon(h))} dh \]
\[ (3\zeta)(\bar{u}) = \int_0^a \mathcal{H}_2(\bar{u}, h, \zeta) dh = \int_0^a \frac{4}{9(\bar{u} + 1 + \zeta(h))} dh \]
\[ (3h)(\bar{u}) = \int_0^a \mathcal{H}_3(\bar{u}, h, h) dh = \int_0^a \frac{4}{9(\bar{u} + 1 + h(h))} dh. \]

Equations (19)–(21) are the special case of (16)–(18), respectively, where \( \bar{u} \in [0, 1] \).
It follows that

\[
\sqrt[n]{\delta_M(\mathcal{M}, \mathcal{N}, \mathcal{H})} \leq \sqrt[n]{\delta_M(\mathcal{M}, \mathcal{N}, \mathcal{H})}.
\]

Hence, all the conditions of Theorem 16 hold. The integral Equations (19)–(21) have a unique solution by using Theorem 16.

5. Conclusions

We provided some novel fixed point results in an ordered complete multiplicative \(\delta_M\)-metric space that satisfied a generalized locally \(A\)-implicit contractive mapping. In these spaces, some new definitions and examples were presented. Furthermore, we provided examples to support our new findings. To demonstrate the originality of our main theorems, we apply them to show the existence of the solutions to a system of nonlinear integral equations. The obtained results improved and generalized the corresponding results in the ordered metric space, ordered dislocated metric space, ordered \(G\)-metric space, dislocated \(G\)-metric space, ordered partial metric space, multiplicative metric space, ordered multiplicative metric space and multiplicative \(D\)-metric space. The research work done in this paper will set a direction to work on multivalued mappings, fuzzy mappings, bipolar fuzzy mappings, \(L\)-fuzzy mappings and intuitionistic fuzzy mappings.


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