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On $QTAG$-Modules Having All N-High Submodules h-Pure

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Abstract: The paper is concerned with h-pure-N-high submodules of $QTAG$-modules. Here, we characterize the submodules N of an h-reduced $QTAG$-module for which all h-pure-N-high submodules are bounded. We also discuss some interesting properties of subsocles and consequently give a characterization of the direct sum of uniserial modules.

Keywords: $QTAG$-modules; h-pure submodules; subsocles

MSC: 16K20; 13C12; 13C13

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1. Introduction and Backgrounds

Many authors interested in module theory have worked on generalizing the theory of abelian groups. In fact, the theory of torsion abelian groups is one of the principal motives of new research in module theory. Among many generalizations of torsion abelian groups, the notion of TAG-modules and their related properties have attracted considerable attention since 1976 (see, for example, [1,2]). A module M over a ring R is called a TAG-module [3] if it satisfies the following two conditions while the rings are associated with unity.

(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.
(ii) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module M over a ring R satisfying only condition (i) is called a $QTAG$-module (see [4]). Since then, many papers have been written investigating the various notions and structures of $QTAG$-modules. It was seen that many of the developments of these modules very closely paralleled the theory of torsion abelian groups. One of the problems of detecting finite direct sums of uniserial modules was considered in ([5], Theorem 4). This problem has been explored in several papers (see, for instance, [6,7]). Some generalizations in this theme for other important sorts of $QTAG$-modules and related results have recently been established in [8,9]. The purpose of the present work is to generalize, in this direction, some new results of abelian p-groups to obtain a parallel theory for $QTAG$-modules.

Some basic definitions used in this paper have already appeared in one of the co-authors' previous works from [8,10], which is necessary for our successful presentation because of its significance to the mentioned topic here.

"Throughout our discussion all the rings R here are associative with unity $(1 \neq 0)$ and modules M are unital $QTAG$-modules. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. An element x in M is called uniform if xR is a non-zero uniform (hence uniserial) module. For any module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For any uniform element x of M, its exponent $e(x)$ is defined to be equal to the decomposition length $d(xR)$.

For any $0 \neq x \in M$, $H_M(x)$ (the height of x in M) is defined by $H_M(x) = \sup\{d(yR/xR) : y \in M,\ x \in yR$ and y uniform$\}$. For $k \geq 0$, $H_k(M) = \{x \in M |\ H_M(x) \geq k\}$ denotes the submodule of M generated by the elements of height at least k and for some submodule N of M, $H^k(M) = \{x \in M|\ d(xR/(xR \cap N)) \leq k\}$ is the submodule of M generated by the elements of exponents at most k. The set of modules $\{H_k(M)\}_{k=0,1,\ldots,\infty}$ forms a base for the neighborhood system of zero. This gives rise to a topology known as h-topology. The closure of a submodule $N \subset M$ is defined as $\overline{N} = \cap_{k=0}^{\infty}(N + H_k(M))$ and it is closed with respect to the h-topology if $\overline{N} = N$."

"The module M is h-divisible if $M = M^1 = \cap_{k=0}^{\infty} H_k(M)$, where M^1 is the submodule of M generated by uniform elements of M of infinite height, and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is called separable if $M^1 = 0$. Moreover, M is said to be bounded if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule B of M is called a basic submodule of M, if B is an h-pure submodule of M, B is a direct sum of uniserial modules and M/B is h-divisible. The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$ and a submodule S of $Soc(M)$ is called a subsocle of M. A module M is called h-pure-complete, if for every subsocle of M supports an h-pure submodule of M."

It is worthwhile noticing that several results that hold for TAG-modules are also valid for $QTAG$-modules [11]. Many results stated in the present paper are clearly generalizations from the papers [12,13]. For a better understanding of the topic mentioned here, one must go through reference [14]. The terminologies and notations are well-known and followed by [15,16]; for the specific ones, we refer the readers to [17].

2. *h*-Pure-*N*-High Submodules

Though this section contains a discussion of the subclass of $QTAG$-modules named h-pure-N-high submodules, we pause for a few general observations on N-high submodules from [18,19], respectively.

"If N is a submodule of a $QTAG$-module M, then a submodule K of M is N-high in M, if it is maximal with the property of being disjoint from N. It is well-known that all N-high submodules of M are bounded if and only if there exists $k \in Z^+$ such that $(N + Soc(H_k(M)))/N$ is finitely generated and N contains the socle of the h-divisible submodule of M.

All N-high submodules are bounded if and only if for every h-pure submodule L of an h-reduced $QTAG$-modules M containing N, M/L is a direct sum of bounded modules."

Our aim here is to give a complete description of some important assertions of h-pure-N-high submodules with the aid of certain submodules and to use them to study the direct sum of uniserial modules.

We begin with the following concept.

Definition 1. *A submodule T of a $QTAG$-module M is h-pure-N-high in M if it is maximal among the h-pure submodules disjoint from N for some submodules N of M.*

Remark 1. *Clearly, h-pure-N-high submodule is contained in an N-high submodule.*

The following theorem establishes the connection between h-pure-N-high and N-high submodules of a $QTAG$-module.

Theorem 1. *Suppose M is a $QTAG$-module. In $QTAG$-modules, h-pure-N-high submodules are N-high submodules for some submodules N of M.*

Proof. If N is a submodule of M such that $Soc(N) \neq Soc(M)$, then there is a non-zero h-pure submodule of M disjoint from N. We have two cases to consider:

In the first case, all elements of $Soc(M)$ are of infinite height. Then M is a h-divisible module, and so the N-high submodule of M is h-pure. As for the second case, there exists a non-zero uniform element u in $Soc(M)$ such that $H_M(u) < \infty$. Thus, for $0 \neq v \in Soc(M) \setminus Soc(N)$ such that $H_M(v) = \infty$. This, in turn, implies that $u + v \in Soc(M) \setminus Soc(N)$, and hence $H_M(u + v) = H_M(u) < \infty$. This means that v generates the non-zero socle of an h-pure submodule disjoint from N.

Let T be an h-pure-N-high submodule of M. In order to show that T is N-high in M, we need only show that $Soc(T) \oplus Soc(N) = Soc(M)$. In virtue of ([18], Theorem 2.3), we have $Soc((N \oplus T)/T) = Soc(M/T)$. Then there exists an h-pure submodule L/T disjoint form $(N \oplus T)/T$. It follows that L is h-pure in M, which contradicts the maximality of T. Since T is h-pure, we have $Soc(T) \oplus Soc(N) = Soc(M)$. Consequently, T is N-high in M. □

As a direct consequence, we have the following corollary.

Corollary 1. *Every h-pure submodule disjoint from a submodule N of a QTAG-module can be extended to an h-pure-N-high submodule of M.*

Motivated by h-pure-N-high submodules, we make the following definition.

Definition 2. *Let M be a QTAG-module. A submodule N of M is said to be an h-pure-absolute summand if for every h-pure-N-high submodule T of M, $M = N \oplus T$.*

Therefore, we come to the following theorem.

Theorem 2. *Suppose M is a QTAG-module. In QTAG-modules, h-pure-absolute summands are absolute summands for some submodules N of M.*

Proof. If N is an h-pure-absolute summand of M, we obtain that N is a summand of M from the utilization of Theorem 1. If $Soc(N) \subset M^1$, then N is h-divisible in M, and therefore, N is an absolute summand of M. If $Soc(N) \nsubseteq M^1$, then there exists $k \in Z^+$, such that $Soc(H_{k+1}(M)) \subset Soc(N) \subset Soc(H_k(M))$.

Let k be the least positive integer such that $Soc(N) \subset Soc(H_k(M))$ and $Soc(N) \nsubseteq Soc(H_{k+1}(M))$. Such a k exists; otherwise, $Soc(N)$ would not be contained in the submodule of M generated by uniform elements of M of infinite height. Then there exists a uniform element x in $Soc(N)$ such that $H_M(x) = k$. Let $y \in Soc(H_{k+1}(M))$ such that $y \notin Soc(N)$. Then $x + y \notin Soc(N)$ and $H_M(x + y) = k$. Therefore, there exists an h-pure submodule L of M containing $x + y$, and so L is N-high in M. This gives $M = N \oplus L$.

Furthermore, since $x + y \in L$ where $x \in N$ and $H_M(x + y) = H_M(x) = k$, it follows that $H_M(x + y - x) = k$. This is a contradiction. Hence, $y \in Soc(N)$ and $Soc(H_{k+1}(M)) \subset Soc(N) \subset Soc(H_k(M))$. Therefore, N is an absolute summand of M. □

It is good to note that we have argued in ([18], Theorem 2.1) that if some N-high submodule K of an h-reduced $QTAG$-module M is not bounded, then there is a submodule P of K such that $M/P = (K/P) \oplus (L/P)$, where P is a basic submodule of K, and L can be taken to contain N. It follows that L is an h-pure submodule of M. Thus, if L is the smallest h-pure submodule of M containing N, we see that all N-high submodules of M are bounded. This can be strengthened by the following result.

Theorem 3. *Let M be an h-reduced QTAG-module and S be a subsocle of M. Then all h-pure-N-high submodules of M are bounded if and only if $(Soc(H_k(M)) + S)/S$ is finitely generated for some $k \in Z^+$.*

Proof. Note that if $(Soc(H_k(M)) + S)/S$ is finitely generated for some $k \in Z^+$, then it readily follows that $Soc(H_k(T))$ is finitely generated for every h-pure-N-high submodule T of M. Since M is h-reduced, we also have that T is bounded.

Next, assume that $(Soc(H_k(M)) + S)/S$ is infinite for every $k \in Z^+$. Now we want to construct inductively a sequence of elements of $Soc(M)$ whose heights are strictly increasing. Since M is h-reduced, we can find $a_1 \in Soc(M) \setminus S$ such that $H_M(a_1) = t_1 < \infty$. Otherwise, $Soc(N)$ would be contained in the submodule of M generated by uniform elements of M of infinite height, and it is plainly observed that M is h-divisible.

Therefore, to finish off the induction, we have constructed $a_1, a_2, \ldots, a_k$ in $Soc(M)$ such that $H_M(a_i) = t_i$ and $t_i < t_{i+1}, i = 1, 2, \ldots, k-1$. Let $T^k \cap S = 0$, where $T^k = \langle a_1, a_2, \ldots, a_k \rangle$. It is fair to verify that T^k is finitely generated and implies that

$$(T^k + S) \cap Soc(H_{t_k+1}(M)) \subsetneq Soc(H_{t_k+1}(M)).$$

Therefore, there exists $a_{k+1} \in Soc(H_{t_k+1}(M))$ such that $a_{k+1} \notin T^k + S$ and $H_M(a_{k+1}) = t_{k+1} < \infty$. Otherwise, $Soc(H_{t_k+1}(M)) \subset M^1$ and M would not be h-reduced. In addition, $t_{k+1} \geq t_k + 1 > t_k$ and $T^{k+1} = T^k + \langle a_{k+1} \rangle$ is disjoint from S.

Choose $b_i \in M$ such that $d(b_iR/a_iR) = t_i$ and let $T = \langle \{b_i\}_{i=1}^{\infty} \rangle$. Then T is an h-pure unbounded submodule of M disjoint from S, which can be extended to an h-pure unbounded S-high submodule of M. The proof is finished. □

An interesting consequence of the last statement is the following.

Corollary 2. *Let M be a QTAG-module and S be a subsocle of M. Then all h-pure-S-high submodules of M are bounded if and only if $(Soc(H_k(M)) + S)/S$ is finitely generated for some $k \in Z^+$ and S contains the socle of the h-divisible submodule of M.*

Before proceeding by proving another motivation for Theorem 3, we need the following useful observation.

Lemma 1. *Suppose N is a submodule of a QTAG-module M. If $\mathcal{F}$ is the family of h-pure-N-high submodules of M such that $N \neq \phi$, then $\cap_T \mathcal{F} = \phi$ for some h-pure-N-high submodules T of M.*

Proof. Let x be a uniform element in $Soc(T)$, for some h-pure-N-high submodules T of M. In order to prove the result, we shall show that for every uniform element of $Soc(T)$, there exists an h-pure-N-high submodule of M, which does not contain x. We have two cases to consider: either there exists $a \in N$ such that $H_M(x + a) < \infty$ or $H_M(x + a) = \infty$ for all $a \in N$.

In the first case, $x + a$ can be embedded in an h-pure submodule L of M [9] disjoint from N. Consequently, L can be extended to an h-pure-N-high submodule P of M, which does not contain x.

As for the second case, we have $H_M(x) = H_M(a)$ for all $a \in N$. If now $H_M(x) = \infty$, it is plainly seen that $N \subset M^1$, and all N-high submodules of M are h-pure. Likewise, if $H_M(x) < \infty$, we obtain $N = Soc(K)$ for some h-pure submodules K of M. Therefore, since K is bounded, we obtain that $M = K \oplus Q$ for some submodules Q of M. It follows that $Q \supset M^1$ and thus $x + a \in Q$ for all $a \in N$. Henceforth, Q is an h-pure-N-high submodule of M, which does not contain x. □

Therefore, we are able to demonstrate the truthfulness of the following theorem.

Theorem 4. *Let M be a QTAG-module and S be a subsocle of M. Then all h-pure-S-high submodules T of M are closed in the h-topology of M if and only if M is separable and T is bounded for all T.*

Proof. Assume that all h-pure-S-high submodules T of M are closed in the h-topology of M. Then M^1, the closure of 0, is contained in the intersection of the family of h-pure-S-high submodules of M. Consequently, referring to Lemma 1, we yield that $M^1 = 0$, that is, M is separable. On the other hand, suppose that T is not bounded. Then there exists a basic submodule B of T such that $(M/B)^1 \neq 0$. Knowing this, with the aid of ([18], Theorem 2.1), we observe that there exists an h-pure-$(S \oplus B)/B$-high submodule L/B of M/B, which is not closed in the h-topology of M/B, i.e., $(M/B)/(L/B)^1 \neq 0$. Certainly, $(M/L)^1 \neq 0$ and L is not closed in the h-topology of M. Now L is h-pure submodule of M, since B is h-pure, and it is S-high in M, which is a contradiction. Therefore, all h-pure-S-high submodules are bounded, as desired.

The converse implication is obvious. □

Regarding the above theorem, the following immediately follows.

Corollary 3. *Let S be a subsocle of a QTAG-module M, and L be an h-pure submodule of M disjoint from S. Then all h-pure-S-high submodules T of M containing L are closed in the h-topology of M if and only if $(M/L)^1 = 0$ and T/L is bounded for all T.*

Proof. As we have noted earlier, an h-pure-S-high submodule T of M contains an h-pure submodule L of M if and only if T/L is an h-pure-$(S \oplus L)/L$-high submodule of M/L and T is closed in M if and only if T/L is closed in M/L. The proof is complete.

□

3. Role of Direct Sum of Uniserial Modules

Before stating and proving our main attainments, to make this section more nearly self-contained, we shall summarize some known principal results in this theme.

"Singh [3] proved that a $QTAG$-module M is a direct sum of uniserial modules if and only if M is the union of an ascending chain of bounded submodules. This indicates that M is a direct sum of uniserial modules if and only if $Soc(M) = \bigoplus_{k \in \omega} S_k$ and $H_M(x) = k$ for every $x \in S_k$".

In [20], it is easily checked that the separable direct sums of countably generated modules are known to be a direct sum of uniserial modules. Although the direct sum of uniserial modules can also be identified in the h-purity sense. Specifically, the following excellent criterion for h-pure submodules of $QTAG$-modules is fulfilled.

Theorem 5 ([21], Theorem 1)**.** *Let M be a QTAG-module with an h-pure submodule L of M and suppose that N is a submodule of L containing $Soc(L)$ such that $Soc(M) = C \oplus Soc(N)$ and $(C \oplus N)/N = Soc(K/N)$ with K/N is an h-pure submodule of M/N, which is a direct sum of uniserial modules. Then L is a summand of M, and M/L is a direct sum of uniserial modules.*

Note that the hypothesis of Theorem 5 is satisfied in the case where M/N is a direct sum of uniserial modules for some submodule N of M such that $Soc(L) \subset N \subset L$. A special case of this gives the following technicality.

Theorem 6 ([21], Theorem 2)**.** *Let M be a QTAG-module with an h-pure submodule L of M such that M/K is a direct sum of uniserial modules, where K is a submodule of L generated by uniform elements of exponents at most k for some positive integer k. Then M is a direct sum of uniserial modules.*

In particular, when $k = 1$, the following affirmation was obtained.

Corollary 4 ([21], Corollary 3)**.** *Let M be a QTAG-module and S be a subsocle of M with $S = Soc(N)$ for some h-pure submodules N of M. If M/S is a direct sum of uniserial modules, then so is M.*

It is an analog of Theorem 1 or Corollary 7 from [22]. For more detailed information about this result, we refer the interested reader to [8,20].

Now, we obtain a new simple but useful reformulation of the last statement to the following.

Theorem 7. *Let S be a subsocle of a QTAG-module M such that M/L is a direct sum of uniserial modules for all h-pure-S-high submodules T of M. Then M is is a direct sum of uniserial modules.*

Proof. Since M/T is a direct sum of uniserial modules for all h-pure-S-high submodules T of M, it is straightforward that all h-pure-S-high submodules of M are closed in the h-topology. Thus, by what we have just shown above, all h-pure-S-high submodules of M are bounded. Note that $M = L \oplus T$, for some h-pure submodules L of M, if T is a bounded direct sum uniserial module. Consequently, M is a direct sum of uniserial modules. □

The following lemma is of some interest.

Lemma 2. *Let M be a QTAG-module with an h-pure submodule L of M such that $Soc(M) = C \oplus Soc(L)$. Then*

(i) *for every submodule N of M with $Soc(N) = Soc(L)$, $H_{M/N}(u + N) \geq H_{M/L}(u + L)$, where u is uniform in C.*

(ii) *for h-pure submodule N of M with $N \subset L$, $H_{M/N}(u + N) = H_{M/L}(u + L)$ for all $u \in L$.*

Proof. (i) If $u \in C$, then there exists $v \in M$ such that $u + L = v' + L$ where $d(vR/v'R) = k$, and $k \in Z^+$. Thus $u - v' \in L$ where $d(vR/v'R) = k$, and $v' \in L$ where $d(vR/v'R) = k + 1$. Since L is an h-pure submodule of M, there exists $w \in L$ such that $d(vR/wR) = k + 1$. This shows that $u - x \in Soc(N) = Soc(L)$ where $x = v - w$ and $d(xR/x'R) = k$. Therefore, $H_{M/N}(u + N) \geq H_{M/L}(u + L)$.

(ii) If $N \subset L$, then there exists a homomorphism $\phi : M/N \to M/L$ such that $\phi(u + N) = u + L$ for all $u \in L$. Since the homomorphism does not decrease height, we have $H_{M/N}(u + N) \leq H_{M/L}(u + L)$ in conjunction with ([21], Proposition 1). Therefore, $H_{M/N}(u + N) = H_{M/L}(u + L)$. □

Therefore, we have all the instruments necessary to prove the following criterion for a direct sum of uniserial modules.

Theorem 8 (The Characterization Theorem)**.** *Let M be a QTAG-module and S be a subsocle of M such that M is a direct sum of uniserial modules. Then the following are equivalent.*

(i) *M/S is a direct sum of uniserial modules;*

(ii) *M/S is h-pure-complete;*

(iii) *S supports a summand of M.*

Proof. $(i) \Rightarrow (ii)$. If M/S is a direct sum of uniserial modules, then h-pure-completeness follows via the simple fact that in a direct sum of uniserial modules, every subsocle supports an h-pure submodule.

$(ii) \Rightarrow (i)$. If M/S is h-pure-complete, then $Soc(M)/S$ supports an h-pure submodule of M/S. However, $(M/S)/(Soc(M)/S) \simeq M/Soc(M)$ is a direct sum of uniserial modules, and hence, in view of Theorem 6, M/S is a direct sum of uniserial modules.

$(i) \Rightarrow (iii)$. Since M is h-pure-complete, there exists an h-pure submodule L of M such that $S = Soc(L)$. Henceforth, according to Theorem 5, L is a summand of M.

$(iii) \Rightarrow (i)$. If L is a summand of M, then $M = L \oplus N$ for some submodule N of M, and we obtain $M/Soc(L) = (L/Soc(L)) \oplus ((N \oplus Soc(L))/Soc(L))$ as a direct sum of uniserial modules. □

4. Open Problems

In closing, we shall state some left-open problems that still elude us.

Problem 1. *Describe the properties of those QTAG-modules M for which there exist h-pure-S-high submodules T of M such that $(M/T)^1$ is finitely generated for some subsocles S of M.*

Problem 2. *Does it follow that an h-pure-N-high submodule of the direct sum of a separable module and a countably generated module is a direct sum of uniserial modules?*

Problem 3. *Characterize those QTAG-modules $M = M_1 / M_2$ such that for all h-pure-complete modules M_1 and all h-pure-absolute summands M_2, it follows that M_1 is an absolute summand.*

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