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Extended Multi-Step Jarratt-like Schemes of High Order for Equations and Systems

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Abstract: The local convergence analysis of multi-step, high-order Jarratt-like schemes is extended for solving Banach space valued systems of equations using the derivative instead of up to the ninth derivative as in previous works. Our idea expands the usage of the scheme in cases not considered earlier and can also be utilized in other schemes, too. Experiments test the theoretical results.

Keywords: multi-step scheme; Jarratt scheme; Banach space; local convergence; derivative-free

MSC: 65H10; 65G99; 49M15

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1. Introduction

Let $\Omega \neq 0$ be an open subset of a Banach X and H a continuous operator mapping Ω into Y, another Banach space.

Numerous problems from diverse disciplines turn into an equation

$$H(x) = 0, \tag{1}$$

by mathematical modeling [1–11].

We want to find a solution γ in closed form, but this is achieved just in special occasions. This is why iterative schemes are developed generating sequences converging to γ under suitable convergence criteria [2,4,12–18].

We study the multi-step Jarratt-like scheme developed for $x_0 \in \Omega$ and all $n = 0, 1, 2, \ldots$ and for $k = 1, 2, \ldots m-3$, $(m \geq 3)$ by

$$\begin{cases} z_1 = H'(x_n)^{-1}H(x_n), \quad y_1 = x_n - z_1, \\ z_2 = H'(x_n)^{-1}H(y_1), \quad y_2 = y_1 - 3z_2, \\ z_3 = H'(x_n)^{-1}H'(y_2)z_2, \quad y_3 = y_1 - \dfrac{7}{4}z_2 + \dfrac{1}{2}z_3 + \dfrac{1}{4}z_4, \\ z_4 = H'(x_n)^{-1}H'(y_2)z_3, \\ z_{2k+3} = H'(x_n)^{-1}H(y_{k+2}), \\ z_{2k+4} = H'(x_n)^{-1}H'(y_2)z_{2k+3}, \\ y_{k+3} = y_{k+2} - 2z_{2k+3} + z_{2k+4}. \end{cases} \tag{2}$$

The $3m-4$ convergence order of (2) is presented in [19] for $X = Y = \mathbb{R}^j$ from Taylor series expansions with conditions reaching the ninth derivative of H (not appearing in scheme (2)). These restrict the application of scheme (2). Consider the simple illustration for $f : \Omega := [-1/2, 3/2] \to \mathbb{R}$ defined by

$$f(s) = \begin{cases} s^3 \ln(s^2) + s^5 - s^3, & s \neq 0 \\ 0, & s = 0. \end{cases}$$

Then, it is easily seen that f''' is unbounded. Therefore, the work in [19] cannot guarantee convergence to a solution γ using scheme (2). Notice that the method (2) is of order at least eight if $m = 4$, which is optimum in the sense of the Kung–Traub conjecture [20]. Moreover, each step increases the convergence order by three [19]. The computational efficiency and computational cost of the method (2) have been given and discussed in detail (see Section 4 in [19]). It is also shown that it is better from the efficiency point of view when compared with other modified Jarratt-type methods. We also refer the reader to the notable relevant papers by Ignatova et al. [21], Kung et al. [20] and Petkovic [17]. Moreover, no upper bounds on $\|x_n - \gamma\|$ or results on the uniqueness of γ are given. Motivated by all these, we develop a technique using only H' (that appears in (2)), which also gives computable upper bounds on $\|x_n - \gamma\|$ and a uniqueness result.

Hence, we extend the applicability of scheme (2). Moreover, the Computational Order of Convergence (COC) and the Approximate Computational Order of Convergence (ACOC) are utilized to compute the convergence order, which does not require usage of higher derivatives or divided differences. This is conducted in Section 2. Numerical experiments are given in Section 3. Finally, the conclusion appears in Section 4.

2. Analysis

We develop the real functions ψ_0, ψ and ψ_1 and real indicators. Next, these functions are connected as majorizing for the operator H' (see the conditions (C_1)–(C_4) that follow). Set $T = [0, +\infty)$.

Suppose that there exist functions:

(a) $\psi_0 : T \to T$ so

$$\psi_0(s) - 1 = 0 \tag{3}$$

has a solution denoted by $\rho_0 \in T - \{0\}$, which is continuous and nondecreasing (CN). Set $T_0 = [0, \rho_0)$.

(b) $\psi : T_0 \to T$, $\psi_1 : T_0 \to T$ CN so that for

$$h_1(s) = \frac{\int_0^1 \psi((1-\zeta)s)\,d\zeta}{1 - \psi_0(s)},$$

$$\begin{aligned} h_2(s) &= \left[h_1(h_1(s)s) + \frac{(\psi_0(h_1(s)s) + \psi_0(s)) \int_0^1 \psi_1(\zeta h_1(s)s)\,d\zeta}{(1-\psi_0(s))(1-\psi_0(h_1(s)s))} \right. \\ &\quad \left. + \frac{2\int_0^1 \psi_1(\zeta h_1(s)s)\,d\zeta}{1-\psi_0(s)} \right] h_1(s), \end{aligned}$$

$$\begin{aligned} h_3(s) &= \left[1 + \frac{7}{4} \frac{\int_0^1 \psi_1(\zeta h_1(s)s)\,d\zeta}{1-\psi_0(s)} + \frac{1}{2} \frac{\psi_1(h_1(s)s) h_2(s)s)}{(1-\psi_0(s))^2} \right. \\ &\quad \left. + \frac{1}{4} \frac{\psi_1(h_1(s)s)\psi_1(h_2(s)s)^2}{(1-\psi_0(s))^3} \right] h_1(s), \end{aligned}$$

for $k = 1, \ldots, n-3$

$$h_{k+3}(s) = \left[1 + \frac{2\int_0^1 \psi_1(\zeta h_{k+2}(s)s)\,d\zeta}{1-\psi_0(s)} + \frac{\psi_1(h_2(s)s)\int_0^1 \psi_1(\zeta h_{k+2}(s)s)\,d\zeta}{(1-\psi_0(s))^2}\right] h_{k+1}(s),$$

and

$$\overline{h}_i(s) = h_i(s) - 1, \quad i = 1, 2, \ldots, n,$$

equations

$$\overline{h}_i(s) = 0 \tag{4}$$

have the least solutions ρ_i, respectively, in $T_0 - \{0\}$.

Then, we show that ρ given by

$$\rho = \min\{\rho_i\} \tag{5}$$

is a convergence radius for scheme (2).

It follows by these definitions that for each $s \in [0, \rho)$

$$0 \le \psi_0(s) < 1, \quad 0 \le \psi_0(h_1(s)s) < 1 \tag{6}$$

and

$$0 \le h_i(s) < 1. \tag{7}$$

Let $B[\gamma, \alpha]$ stand for the closure of ball $B(\gamma, \alpha)$.

The conditions (C) are used to provide the functions ψ as given earlier:

(C_1) Equation $H(x) = 0$ has a simple solution $\gamma \in \Omega$.

(C_2) For all $u \in \Omega$

$$\|H'(\gamma)^{-1}(H'(\gamma) - H'(u))\| \le \psi_0(\|\gamma - u\|).$$

Set $\Omega_0 = \Omega \cap B(\gamma, \rho_0)$.

(C_3) For all $u, v \in \Omega_0$

$$\|H'(\gamma)^{-1}(H'(u) - H'(v))\| \le \psi(\|u - v\|)$$

and

$$\|H'(\gamma)^{-1}H'(v)\| \le \psi_1(\|v - \gamma\|).$$

(C_4) $B[\gamma, \rho] \subset \Omega$.

Next, using conditions (C) together with the developed notation, we develop the local result of scheme (2).

Theorem 1. *Suppose conditions (C) hold. Then, if $x_0 \in B_0 := B(\gamma, \rho) - \{\gamma\}$, iteration $\{x_n\}$ developed by scheme (2) exists, stays in B_0 and converges to γ.*

Proof. Let us choose $u \in B_0$ and $t_n := \|x_n - \gamma\|$. Then, by (5), (6) and (C_2), one has

$$\|H'(\gamma)^{-1}(H'(u) - H'(\gamma))\| \le \psi_0(\|u - \gamma\|) \le \psi_0(\rho) < 1. \tag{8}$$

By estimation (8) with the lemma of Banach on linear inverse mapping [5], we obtain $H'(u)^{-1} \in \mathcal{L}(Y, X)$ with

$$\|H'(u)^{-1}H'(\gamma)\| \le \frac{1}{1 - \psi_0(\|u - \gamma\|)}. \tag{9}$$

By (9) for $u = x_0$, y_i exists. Using (5), (9) (for $u = x_0$ and $n = 0$), (7) (for $i = 1$), scheme (2) and (C_3)

$$\begin{aligned}
\|y_1-\gamma\| &= \|x_0-\gamma-H'(x_0)^{-1}H(x_0)\| \\
&\leq \|H'(x_0)^{-1}H'(\gamma)\|\left\|\int_0^1 H'(\gamma)^{-1}\big(H'(\gamma+\zeta(x_0-\gamma))-H'(x_0)\big)(x_0-\gamma)\,d\zeta\right\| \\
&\leq \frac{\int_0^1 \psi((1-\zeta)t_0)\,d\zeta}{1-\psi_0(t_0)}t_0 \\
&\leq h_1(t_0)t_0 \leq t_0 < \rho, \qquad (10)
\end{aligned}$$

then, $y_1 \in B(\gamma,\rho)$ and $H'(y_1)^{-1} \in \mathcal{L}(Y,X)$.

Using (5)–(7) (for $i=2$), (9) (for $u=y_1$), (C_3), (10) and scheme (2), we obtain in turn

$$\begin{aligned}
\|y_2-\gamma\| &= \|y_1-\gamma-H'(y_1)^{-1}H(y_1) \\
&\quad +H'(y_1)^{-1}\big(H'(x_0)-H'(y_1)\big)H'(x_0)^{-1}H(y_1)-2H'(x_0)^{-1}H(y_1)\| \\
&\leq \Bigg[h_1(\|y_1-\gamma\|)+\frac{(\psi_0(\|y_1-\gamma\|)+\psi_0(t_0))\int_0^1\psi_1(\zeta\|y_1-\gamma\|)\,d\zeta}{(1-\psi_0(\|y_1-\gamma\|))(1-\psi_0(t_0))} \\
&\quad +\frac{2\int_0^1\psi_1(\zeta\|y_1-\gamma\|)\,d\zeta}{1-\psi_0(t_0)}\Bigg]\|y_1-\gamma\| \\
&\leq h_2(t_0)t_0, \qquad (11)
\end{aligned}$$

hence, $y_2 \in B(\gamma,\rho)$. Similarly, by (5), (7) (for $i=3$), (10), (11) and scheme (2), we have in turn that

$$\begin{aligned}
\|y_3-\gamma\| &= \|y_1-\gamma-\frac{7}{4}H'(x_0)^{-1}H(y_1)+\frac{1}{2}H'(x_0)^{-1}H'(y_2)H'(x_0)^{-1}H(y_1) \\
&\quad +\frac{1}{4}H'(x_0)^{-1}H'(y_2)H'(x_0)^{-1}H'(y_2)H'(x_0)^{-1}H(y_1)\| \\
&\leq \Bigg[1+\frac{7}{4}\frac{\int_0^1\psi_1(\zeta\|y_1-\gamma\|)\,d\zeta}{1-\psi_0(t_0)}+\frac{1}{2}\frac{\psi_1(\|y_1-\gamma\|)\psi_1(\|y_2-\gamma\|)}{(1-\psi_0(t_0))^2} \\
&\quad +\frac{1}{4}\frac{\psi_1(\|y_1-\gamma\|)\psi_1(\|y_2-\gamma\|)^2}{(1-\psi_0(t_0))^3}\Bigg]\|y_1-\gamma\| \\
&\leq h_3(t_0)t_0 \leq t_0, \qquad (12)
\end{aligned}$$

therefore, $y_3 \in B(\gamma,\rho)$.

Then, for the rest of the substeps, similarly, we obtain for $k=1,\ldots,m-3$

$$\begin{aligned}
\|y_{k+3}-\gamma\| &= \|y_{k+2}-\gamma-2H'(x_0)^{-1}H(y_{k+2})+H'(x_0)^{-1}H'(y_2)H'(x_0)^{-1}H(y_{k+2})\| \\
&\leq \Bigg[1+\frac{2\int_0^1\psi_1(\zeta\|y_{k+2}-\gamma\|)\,d\zeta}{1-\psi_0(t_0)} \\
&\quad +\frac{\psi_1(\|y_2-\gamma\|)\int_0^1\psi_1(\zeta\|y_{k+2}-\gamma\|)\,d\zeta}{(1-\psi_0(t_0))^2}\Bigg]\|y_{k+2}-\gamma\| \\
&\leq h_{k+3}(t_0)t_0 \leq t_0. \qquad (13)
\end{aligned}$$

By the definition of $x_1 = y_m$ and (10)–(13), we deduce

$$t_1 \leq ct_0, \qquad (14)$$

where

$$c = h_1(t_0)h_2(t_0)\ldots h_m(t_0) \in [0,1).$$

Simply replace x_0 by x_j in the preceding calculations to obtain

$$t_{j+1} \leq ct_j \leq c^{j+1}t_0 < \rho, \tag{15}$$

so, $\lim_{j\to+\infty} x_j = \gamma$ and $x_{j+1} \in B(\gamma, \rho)$. $\square$

We have the uniqueness result:

Proposition 1. *Suppose:*

(i) *Point γ is a simple solution in $B(\gamma, \rho_0) \subset \Omega$ of (1) and*

(ii)

$$\int_0^1 \psi_0(\zeta\bar{\rho})\, d\zeta < 1 \quad \text{for some} \quad \bar{\rho} \geq \rho_0. \tag{16}$$

Then, γ is unique in $\Omega_1 = \Omega \cap \overline{B}(\gamma, \bar{\rho})$ as a solution of (1).

Proof. Let $q \in \Omega_1$ be such that $H(q) = 0$. Define $Q = \int_0^1 H'(\gamma + \zeta(q-\gamma))\, d\zeta$. Then, in view of (C_2) and (16), we obtain in turn

$$\|H'(\gamma)^{-1}(Q - H'(\gamma))\| \leq \int_0^1 \psi_0(\zeta\|\gamma - q\|)\, d\zeta \leq \int_0^1 \psi_0(\zeta\bar{\rho})\, d\zeta < 1,$$

so, $\gamma = q$, since $Q^{-1} \in \mathcal{L}(Y, X)$ and $0 = H(q) - H(\gamma) = Q(q - \gamma)$. $\square$

Definition 1. *The COC is defined as*

$$a = \ln\left(\frac{\|s_{n+1} - \gamma\|}{\|s_n - \gamma\|}\right) / \ln\left(\frac{\|s_n - \gamma\|}{\|s_{n-1} - \gamma\|}\right)$$

and, for $\beta_n := \|s_n - s_{n-1}\|$, the ACOC [16] by

$$b = \ln\left(\frac{\beta_{n+1}}{\beta_n}\right) / \ln\left(\frac{\beta_n}{\beta_{n-1}}\right).$$

3. Applications

We test the conditions (C).

Example 1. *If $X = Y = \Omega = \mathbb{R}$, let function $H(s) = \sin s$. The conditions (C) must be verified. Clearly, $\gamma = 0$ solves the equation $H(s) = 0$. Hence, we choose $\gamma = 0$ in conditions (C). In particular, the choice $\gamma = 0$ satisfies the condition (C_1). Concerning the condition (C_2), we have, in turn, by the definition of H and, since $H'(\gamma) = 1$ and $\|H'(\gamma)^{-1}(H'(\gamma) - H'(u))\| \leq \psi_0(\|\gamma - u\|)$, that*

$$\|H'(\gamma)^{-1}(H'(\gamma) - H'(u))\| = |\cos\gamma - \cos u| \leq \psi_0(|\gamma - u|)$$

provided that $\psi_0(s) = s$, since there exists $c \in \mathbb{R}$ such that

$$|\cos\gamma - cosu| = |\sin c(\gamma - u)| \leq |\sin c||\gamma - u| = |\gamma - u|.$$

That is the condition (C_1) that is satisfied for this choice of ψ_0. Moreover, by solving the equation $\psi_0(s) - 1 = 0$, we obtain $\rho_0 = 1$, and we can set $\Omega_0 = B(0, 1)$.

Similarly, we have

$$\|H'(\gamma)^{-1}(H'(u) - H'(v))\| \leq |\sin d(u - v)| \leq |\sin d||u - v| \leq |u - v|$$

for some, $d \in \mathbb{R}$, so we can choose $\psi(s) = s$. Then, the first condition in (C_2) is satisfied. Concerning the second condition in (C_2), we obtain in turn that

$$\|H'(\gamma)^{-1}H'(v)\| = |1 \sin v| \leq 1,$$

so, we can choose $\psi_1(s) = 1$. Then, we have for $k = 3$ the resutls of Table 1.

Table 1. Radius for Example 1.

ρ	ρ_0	ρ_1	ρ_2	ρ_3
0.306688…	1	0.666667…	0.343549…	0.306688…

Example 2. *If $\Omega = B[0,1]$, and considering the continuous operator $H : \Omega \subset S \to S$ (with the maximum norm) as*

$$H(\psi)(s) = \psi(s) - 5\int_0^1 s\zeta\psi(\zeta)^3\, d\zeta,$$

where space S contains continuous functions defined on the interval $[0,1]$, we have that

$$H'(\psi(\lambda))(s) = \lambda(s) - 15\int_0^1 s\zeta\psi(\zeta)^2\lambda(\zeta)\, d\zeta, \quad \textit{for each} \quad \lambda \in \Omega.$$

Then, we obtain that $\gamma = 0$, and for $k = 3$ we obtain the functions $\psi_0(s) = \psi(s) = \frac{15}{2}s$, $\psi_1(s) = 2$. This way, we obtain the Table 2.

Table 2. Radius for Example 2.

ρ	ρ_0	ρ_1	ρ_2	ρ_3
0.024324…	0.133333…	0.088888…	0.032839…	0.024324…

Example 3. *Let $\Omega = B(0,1)$, $X = Y = \mathbb{R}^3$ and $\gamma = (0,0,0)^T$. Let H defined on Ω be*

$$H(w) = H(w_1, w_2, w_3) = \left(e^{w_1} - 1, \frac{e-1}{2}w_2^2 + w_2, w_3\right)^T.$$

For the point $w = (w_1, w_2, w_3)^T$, the derivative H' is

$$H'(w) = \begin{pmatrix} e^{w_1} & 0 & 0 \\ 0 & (e-1)w_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking into account rows with their max. norm and

$$H'(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for $k = 3$ we obtain through conditions (C) $\psi_0(s) = (e-1)s$, $\psi(s) = e^{\frac{1}{e-1}}s$, $\psi_1(s) = e^{\frac{1}{e-1}}$ and the radius in Table 3.

Table 3. Radius for Example 3.

ρ	ρ_0	ρ_1	ρ_2	ρ_3
0.115169…	0.581977…	0.382692…	0.148762…	0.115169…

Example 4. *Taking into account the academic example in the first section of this work, for $k = 3$, we obtain $\psi_0(s) = \psi(s) = 96.662907s$, $\psi_1(s) = 1.0631$. The corresponding radius are in Table 4.*

Table 4. Radius for Example 4.

ρ	ρ_0	ρ_1	ρ_2	ρ_3
0.003149...	0.010345...	0.006896...	0.003463...	0.003149...

Example 5. *We consider the conventional form of Kepler's equation*

$$G(s) = s - \sigma \sin(s) - \tau = 0,$$

where $0 \leq \tau \leq \pi$ *and* $0 \leq \sigma < 1$. *In [22], we can find, for distinct values of* τ *and* σ, *a numerical study. In this case, we select value* $\tau = 0.1$ *and* $\sigma = 0.27$, *so the solution obtained is* $\gamma \approx 0.136828\ldots$. *Since*

$$G'(s) = 1 - \sigma \cos(s),$$

we obtain

$$\begin{aligned} \|G'(\gamma)^{-1}(G'(s) - G'(t))\| &= \frac{\|\sigma(\cos(s) - \cos(t))\|}{\|1 - \sigma\cos(\gamma)\|} \\ &= \frac{2\sigma\|\sin\left(\frac{s+t}{2}\right)\sin\left(\frac{s-t}{2}\right)\|}{\|1 - \sigma\cos(\gamma)\|} \\ &\leq \frac{\sigma}{\|1 - \sigma\cos(\gamma)\|}\|s - t\| \end{aligned}$$

and

$$\|G'(\gamma)^{-1}G'(s)\| = \frac{\|1 - \sigma\cos(s)\|}{\|1 - \sigma\cos(\gamma)\|} \leq \frac{1 + \sigma}{\|1 - \sigma\cos(\gamma)\|}.$$

Consequently, we have $\psi_0(s) = \psi(s) = 0.3685888t$ *and* $\psi_1(s) = 1.7337327$. *The calculated value parameters for* $k = 3$ *are given in Table 5.*

Table 5. Radius for Example 5.

ρ	ρ_0	ρ_1	ρ_2	ρ_3
0.550956...	2.713050...	1.808700...	0.720969...	0.550956...

Example 6. *We analyze the following system of nonlinear equations*

$$\left\{ \begin{array}{rcl} \nu_2\,\nu_3 + \nu_4\,(\nu_2 + \nu_3) & = & 0 \\ \nu_1\,\nu_3 + \nu_4\,(\nu_1 + \nu_3) & = & 0 \\ \nu_1\,\nu_2 + \nu_4\,(\nu_1 + \nu_2) & = & 0 \\ \nu_1\,\nu_2 + \nu_1\,\nu_3 + \nu_2\,\nu_3 - 1 & = & 0 \end{array} \right. \tag{17}$$

with $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathbb{R}$. *Then, the solution of the previous scheme is showed for* $\gamma = (\nu_1, \nu_2, \nu_3, \nu_4)^T$ *by function* $H := (\nu_1, \nu_2, \nu_3, \nu_4) : \mathbb{R}^4 \to \mathbb{R}^4$ *defined by*

$$H(\nu_1, \nu_2, \nu_3, \nu_4) = \begin{pmatrix} \nu_2\,\nu_3 + \nu_4\,(\nu_2 + \nu_3) \\ \nu_1\,\nu_3 + \nu_4\,(\nu_1 + \nu_3) \\ \nu_1\,\nu_2 + \nu_4\,(\nu_1 + \nu_2) \\ \nu_1\,\nu_2 + \nu_1\,\nu_3 + \nu_2\,\nu_3 - 1 \end{pmatrix}$$

The Fréchet derivative is given by

$$H'(\nu_1, \nu_2, \nu_3, \nu_4) = \begin{pmatrix} 0 & \nu_3 + \nu_4 & \nu_2 + \nu_4 & \nu_2 + \nu_3 \\ \nu_3 + \nu_4 & 0 & \nu_1 + \nu_4 & \nu_1 + \nu_3 \\ \nu_2 + \nu_4 & \nu_1 + \nu_4 & 0 & \nu_1 + \nu_2 \\ \nu_2 + \nu_3 & \nu_1 + \nu_3 & \nu_1 + \nu_2 & 0 \end{pmatrix}$$

Then, we obtain

$$\begin{aligned} \nu_1 &= 0.57735026\ldots \\ \nu_2 &= 0.57735026\ldots \\ \nu_3 &= 0.57735026\ldots \\ \nu_4 &= -0.28867513\ldots \end{aligned} \tag{18}$$

Then, we have that $\psi_0(s) = 0.28867513s$, $\psi(s) = 1.55198152s$, $\psi_1(s) = 1.06466589s$ *and the radii are in Table 6.*

Table 6. Radius for Example 6.

ρ	ρ_0	ρ_1	ρ_2	ρ_3
0.391641...	0.775990...	0.484339...	0.413569...	0.391641...

4. Conclusions

A technique is introduced involving only H', which also gives computable upper error bounds on $\|x_n - \gamma\|$. This way, the method (2) becomes applicable in cases not possible before, since higher hypotheses on H' were required. The COC or ACOC are used to determine the convergence order. The technique is not based on method (2) and is very general. Hence, it can be used to extend the applicability of other methods [11,17,20,21]. This will be the focus of our future work.

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References

1. Amat, S.; Argyros, I.K.; Busquier, S.; Hernández–Verón, M.Á.; Martínez, E. On the local convergence study for an efficient *k*–step iterative method. *JCAM* **2018**, *343*, 753–761. [CrossRef]
2. Amorós, C.; Argyros, I.K.; González, D.; Magreñán, Á.A.; Regmi, S.; Sarría, Í. New Improvement of the Domain of Parameters for Newton's Method. *Mathematics* **2020**, *8*, 103. [CrossRef]
3. Argyros, I.K.; George, S. *Mathematical Modeling for the Solution of Equations and Systems of Equations with Applications*; Nova Publishers: Hauppauge, NY, USA, 2019; Volume IV.
4. Argyros, I.K.; González, D.; Magreñán, Á.A.; Moysi, A.; Regmi, S.; Sicilia, J.A. Convergence and Dynamics of a Higher–Order Method. *Symmetry* **2020**, *12*, 420.
5. Argyros, I.K.; Magreñán, Á.A. *A Contemporary Study of Iterative Methods*; Elsevier: Amsterdam, The Netherlands, 2018.
6. Frontini, M.; Sormani, E. Some variant of Newton's method with third–order convergence. *Appl. Math. Comput.* **2003**, *140*, 419–426. [CrossRef]
7. Grau–Sánchez, M.; Grau, Á.; Noguera, M. Ostrowski type methods for solving systems of nonlinear equations. *Appl. Math. Comput.* **2011**, *218*, 2377–2385. [CrossRef]
8. Ortega, J.M.; Rheinboldt, W.C. *Iterative Solution of Nonlinear Equations in Several Variables*; Academic Press: New York, NY, USA, 1970.
9. Rheinboldt, W.C. An adaptive continuation process for solving systems of nonlinear equations. *Pol. Acad. Sci. Banach Ctr. Publ.* **1978**, *3*, 129–142. [CrossRef]
10. Soleymani, F.; Lotfi, T.; Bakhtiari, P. A multi–step class of iterative methods for nonlinear systems. *Optim. Lett.* **2014**, *8*, 1001–1015. [CrossRef]

11. Traub, J.F. *Iterative Methods for the Solution of Equations*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1964.
12. Cordero, A.; Hueso, J.L.; Martínez, E.; Torregrosa, J.R. A modified Newton–Jarratt's composition. *Numer. Algorithms* **2010**, *55*, 87–99. [CrossRef]
13. Gutiérrez, J.M.; Hernández, M.A. A family of Chebyshev–Halley type methods in Banach spaces. *Bull. Aust. Math. Soc.* **1997**, *55*, 113–130. [CrossRef]
14. Homeier, H.H. A modified Newton method with cubic convergence: The multivariable case. *J. Comput. Appl. Math.* **2004**, *169*, 161–169. [CrossRef]
15. Jarratt, P. Some fourth order multipoint iterative methods for solving equations. *Math. Comp.* **1966**, *20*, 434–437. [CrossRef]
16. Kou, J. A third–order modification of Newton method for systems of nonlinear equations. *Appl. Math. Comput.* **2007**, *191*, 117–121.
17. Petkovic, M.S. On a general class of multipoint root–finding methods of high computational efficiency. *SIAM J. Numer. Anal.* **2011**, *49*, 1317–1319. [CrossRef]
18. Sharma, J.R.; Guha, R.K.; Sharma, R. An efficient fourth order weighted–Newton method for systems of nonlinear equations. *Numer. Algorithms* **2013**, *62*, 307–323. [CrossRef]
19. Ahmad, F.; Tohidi, E.; Ullah, M.Z.; Carrasco, J.A. Higher order multi–step Jarratt–like method for solving systems of nonlinear equations: Application to PDEs and ODEs. *Comp. Math. App.* **2015**, *70*, 624–636. [CrossRef]
20. Kung, H.T.; Traub, J.F. Optimal order of one point and multipoint iteration. *J. AMC* **1974**, *21*, 643–651. [CrossRef]
21. Ignatova, B.; Kyurkchiev, N.; Iliev, A. Multipoint Algorithms Arising from Optimal in the Sense of Kung–Traub Iterative Procedures for Numerical Solution of Nonlinear Equations. *Gen. Math. Notes* **2011**, *6*, 45–79.
22. Danby, J.M.A.; Burkardt, T.M. The solution of Kepler's equation, I. *Celest. Mech.* **1983**, *31*, 95–107. [CrossRef]