Extended Multi-Step Jarratt-like Schemes of High Order for Equations and Systems

Ioannis K. Argyros 1, Christopher Argyros 2, Michael Argyros 3, Johan Ceballos 4* and Daniel González 4,*

Abstract: The local convergence analysis of multi-step, high-order Jarratt-like schemes is extended for solving Banach space valued systems of equations using the derivative instead of up to the ninth derivative as in previous works. Our idea expands the usage of the scheme in cases not considered earlier and can also be utilized in other schemes, too. Experiments test the theoretical results.

Keywords: multi-step scheme; Jarratt scheme; Banach space; local convergence; derivative-free

MSC: 65H10; 65G99; 49M15

1. Introduction

Let Ω ≠ 0 be an open subset of a Banach X and H a continuous operator mapping Ω into Y, another Banach space.

Numerous problems from diverse disciplines turn into an equation

\[ H(x) = 0, \]  

by mathematical modeling [1–11].

We want to find a solution γ in closed form, but this is achieved just in special occasions. This is why iterative schemes are developed generating sequences converging to γ under suitable convergence criteria [2,4,12–18].

We study the multi-step Jarratt-like scheme developed for \( x_0 \in \Omega \) and all \( n = 0, 1, 2, \ldots \) and for \( k = 1, 2, \ldots m - 3, (m \geq 3) \) by

\[
\begin{align*}
z_1 &= H'(x_n)^{-1}H(x_n), & y_1 &= x_n - z_1, \\
z_2 &= H'(x_n)^{-1}H(y_1), & y_2 &= y_1 - 3z_2, \\
z_3 &= H'(x_n)^{-1}H'(y_2)z_2, & y_3 &= y_1 - \frac{7}{4}z_2 + \frac{1}{2}z_3 + \frac{1}{4}z_4, \\
z_4 &= H'(x_n)^{-1}H'(y_2)z_3, \\
z_{2k+3} &= H'(x_n)^{-1}H(y_{k+2}), \\
z_{2k+4} &= H'(x_n)^{-1}H'(y_2)z_{2k+3}, \\
y_{k+3} &= y_{k+2} - 2z_{2k+3} + z_{2k+4}.
\end{align*}
\]

The \( 3m - 4 \) convergence order of (2) is presented in [19] for \( X = Y = \mathbb{R}^j \) from Taylor series expansions with conditions reaching the ninth derivative of \( H \) (not appearing in
scheme (2)). These restrict the application of scheme (2). Consider the simple illustration for \( f : \Omega := [-1/2, 3/2] \to \mathbb{R} \) defined by

\[
f(s) = \begin{cases} 
    s^3 \ln (s^2) + s^5 - s^3, & s \neq 0 \\
    0, & s = 0.
\end{cases}
\]

Then, it is easily seen that \( f''' \) is unbounded. Therefore, the work in [19] cannot guarantee convergence to a solution \( \gamma \) using scheme (2). Notice that the method (2) is of order at least eight if \( m = 4 \), which is optimum in the sense of the Kung–Traub conjecture [20]. Moreover, each step increases the convergence order by three [19]. The computational efficiency and computational cost of the method (2) have been given and discussed in detail (see Section 4 in [19]). It is also shown that it is better from the efficiency point of view when compared with other modified Jarratt-type methods. We also refer the reader to the notable relevant papers by Ignatova et al. [21], Kung et al. [20] and Petkovic [17]. Moreover, no upper bounds on \( \|x_n - \gamma\| \) or results on the uniqueness of \( \gamma \) are given. Motivated by all these, we develop a technique using only \( H' \) (that appears in (2)), which also gives computable upper bounds on \( \|x_n - \gamma\| \) and a uniqueness result.

Hence, we extend the applicability of scheme (2). Moreover, the Computational Order of Convergence (COC) and the Approximate Computational Order of Convergence (ACOC) are utilized to compute the convergence order, which does not require usage of higher derivatives or divided differences. This is conducted in Section 2. Numerical experiments are given in Section 3. Finally, the conclusion appears in Section 4.

2. Analysis

We develop the real functions \( \psi_0, \psi \) and \( \psi_1 \) and real indicators. Next, these functions are connected as majorizing for the operator \( H' \) (see the conditions \((C_1)-(C_4)\) that follow). Set \( T = [0, +\infty) \).

Suppose that there exist functions:

(a) \( \psi_0 : T \to T \) so

\[
\psi_0(s) - 1 = 0 \quad (3)
\]

has a solution denoted by \( \rho_0 \in T - \{0\} \), which is continuous and nondecreasing (CN). Set \( T_0 = [0, \rho_0) \).

(b) \( \psi : T_0 \to T, \psi_1 : T_0 \to T \) CN so that for

\[
h_1(s) = \int_0^1 \frac{\psi((1 - \xi)s) d\xi}{1 - \psi_0(s)},
\]

\[
h_2(s) = \left[ h_1(h_1(s)) + \frac{(\psi_0(h_1(s)s) + \psi_0(s)) \int_0^1 \psi_1(\xi h_1(s)s) d\xi}{(1 - \psi_0(s))(1 - \psi_0(h_1(s)s))} \right] h_1(s),
\]

\[
h_3(s) = \left[ 1 + \frac{7}{4} \int_0^1 \psi_1(\xi h_1(s)s) d\xi - \frac{1}{2} \frac{\psi_1(h_1(s)s)h_2(s)s}{(1 - \psi_0(s))^2} \right] h_1(s),
\]

\[
- \frac{1}{4} \frac{\psi_1(h_1(s)s)\psi_2(h_2(s)s)^2}{(1 - \psi_0(s))^3} h_1(s),
\]

\]

\[
\]
Suppose conditions

\textbf{Theorem 1.}

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\[(1)\quad \frac{1}{1 - \psi_0(s)} \frac{\int_0^1 \psi_1(\zeta h_{k+2}(s) s) d\zeta}{1 - \psi_0(s)} h_{k+1}(s),
\]

and

\[\bar{h}_i(s) = h_i(s) - 1, \quad i = 1, 2, \ldots, n,\]

equations

\[\bar{h}_i(s) = 0\]

have the least solutions \(\rho_i\), respectively, in \(T_0 = \{0\}\).

Then, we show that \(\rho\) given by

\[\rho = \min\{\rho_i\}\]

is a convergence radius for scheme (2).

It follows by these definitions that for each \(s \in [0, \rho]\)

\[0 \leq \psi_0(s) < 1, \quad 0 \leq \psi_0(h_1(s) s) < 1\]

and

\[0 \leq h_i(s) < 1.\]  \hspace{1cm} (7)

Let \(B[\gamma, a]\) stand for the closure of ball \(B(\gamma, a)\).

The conditions \((C)\) are used to provide the functions \(\gamma\) as given earlier:

\((C_1)\) Equation \(H(x) = 0\) has a simple solution \(\gamma \in \Omega\).

\((C_2)\) For all \(u \in \Omega\)

\[\|H'(\gamma)^{-1}(H'(\gamma) - H'(u))\| \leq \psi_0(\|\gamma - u\|).\]

Set \(\Omega_0 = \Omega \cap B(\gamma, \rho_0)\).

\((C_3)\) For all \(u, v \in \Omega_0\)

\[\|H'(\gamma)^{-1}(H'(u) - H'(v))\| \leq \psi_1(\|u - v\|)
\]

and

\[\|H'(\gamma)^{-1}H'(v)\| \leq \psi_1(\|v - \gamma\|).\]

\((C_4)\) \(B[\gamma, \rho] \subset \Omega\).

Next, using conditions \((C)\) together with the developed notation, we develop the local result of scheme (2).

\textbf{Theorem 1.} Suppose conditions \((C)\) hold. Then, if \(x_0 \in B_0 := B(\gamma, \rho) - \{\gamma\}\), iteration \(\{x_n\}\) developed by scheme (2) exists, stays in \(B_0\) and converges to \(\gamma\).

\textbf{Proof.} Let us choose \(u \in B_0\) and \(t_u := \|x_n - \gamma\|\). Then, by \((5)\), \((6)\) and \((C_2)\), one has

\[\|H'(\gamma)^{-1}(H'(u) - H'(\gamma))\| \leq \psi_0(\|u - \gamma\|) \leq \psi_0(\rho) < 1.\]  \hspace{1cm} (8)

By estimation \((8)\) with the lemma of Banach on linear inverse mapping \([5]\), we obtain \(H'(u)^{-1} \in \mathcal{L}(Y, X)\) with

\[\|H'(u)^{-1}H'(\gamma)\| \leq \frac{1}{1 - \psi_0(\|u - \gamma\|)}.\]  \hspace{1cm} (9)

By \((9)\) for \(u = x_0, y_1\) exists. Using \((5)\), \((9)\) \((u = x_0 \text{ and } n = 0)\), \((7)\) \((i = 1)\), scheme (2) and \((C_3)\).
\[
\|y_1 - \gamma\| = \|x_0 - \gamma - H'(x_0)^{-1}H(x_0)\| \\
\leq \|H'(x_0)^{-1}H'(\gamma)\| \left\| \int_0^1 H'(\gamma)^{-1}(H'(\gamma + \zeta(x_0 - \gamma)) - H'(x_0))(x_0 - \gamma) \, d\zeta \right\| \\
\leq \frac{\int_0^1 \psi((1 - \zeta)t_0) \, d\zeta}{1 - \psi_0(t_0)} t_0 \\
\leq h_1(t_0) t_0 \leq t_0 < \rho,
\]

(10)

then, \(y_1 \in B(\gamma, \rho)\) and \(H'(y_1)^{-1} \in \mathcal{L}(Y, X)\).

Using (5)–(7) (for \(i = 2\), (9) (for \(u = y_1\)), (C_3), (10) and scheme (2), we obtain in turn

\[
\|y_2 - \gamma\| = \|y_1 - \gamma - H'(y_1)^{-1}H(y_1) \\
+ H'(y_1)^{-1}(H'(x_0) - H'(y_1))H'(x_0)^{-1}H(y_1) - 2H'(x_0)^{-1}H(y_1)\| \\
\leq \left[ h_1(\|y_1 - \gamma\|) + \frac{\psi_0(\|y_1 - \gamma\|) + \psi_0(t_0) \int_0^1 \psi_1(\zeta \|y_1 - \gamma\|) \, d\zeta}{(1 - \psi_0(\|y_1 - \gamma\|))(1 - \psi_0(t_0))} \\
+ \frac{\int_0^1 \psi_1(\zeta \|y_1 - \gamma\|) \, d\zeta}{1 - \psi_0(t_0)} \right] \|y_1 - \gamma\| \\
\leq h_2(t_0) t_0,
\]

(11)

hence, \(y_2 \in B(\gamma, \rho)\). Similarly, by (5), (7) (for \(i = 3\), (10), (11) and scheme (2), we have in turn that

\[
\|y_3 - \gamma\| = \|y_1 - \gamma - \frac{7}{4} H'(x_0)^{-1}H(y_1) + \frac{1}{2} H'(x_0)^{-1}H'(y_2)H'(x_0)^{-1}H(y_1) \\
+ \frac{1}{4} H'(x_0)^{-1}H'(y_2)H'(x_0)^{-1}H'(y_2)(x_0)^{-1}H(y_1)\| \\
\leq \left[ 1 + \frac{7}{4} \int_0^1 \psi_1(\zeta \|y_1 - \gamma\|) \, d\zeta + \frac{1}{2} \psi_1(\|y_1 - \gamma\|) \psi_1(\|y_2 - \gamma\|) \right] \|y_1 - \gamma\| \\
\leq h_3(t_0) t_0 \leq t_0,
\]

(12)

therefore, \(y_3 \in B(\gamma, \rho)\).

Then, for the rest of the substeps, similarly, we obtain for \(k = 1, \ldots, m \leq 3\)

\[
\|y_{k+3} - \gamma\| = \|y_{k+2} - \gamma - 2H'(x_0)^{-1}H(y_{k+2}) + H'(x_0)^{-1}H'(y_2)H'(x_0)^{-1}H(y_{k+2})\| \\
\leq \left[ 1 + 2 \int_0^1 \psi_1(\zeta \|y_{k+2} - \gamma\|) \, d\zeta \right] \|y_{k+2} - \gamma\| \\
\leq \frac{\psi_1(\|y_2 - \gamma\|) \int_0^1 \psi_1(\zeta \|y_{k+2} - \gamma\|) \, d\zeta}{(1 - \psi_0(t_0))^2} \|y_{k+2} - \gamma\| \\
\leq h_{k+3}(t_0) t_0 \leq t_0.
\]

(13)

By the definition of \(x_1 = y_m\) and (10)–(13), we deduce

\[t_1 \leq c t_0,\]

(14)

where

\[c = h_1(t_0) h_2(t_0) \ldots h_m(t_0) \in [0, 1).\]
Simply replace $x_0$ by $x_j$ in the preceding calculations to obtain
\[ t_{j+1} \leq c t_j \leq c^{j+1} t_0 < \rho, \]
so, $\lim_{j \to +\infty} x_j = \gamma$ and $x_{j+1} \in B(\gamma, \rho)$. □

We have the uniqueness result:

**Proposition 1.** Suppose:
(i) Point $\gamma$ is a simple solution in $B(\gamma, \rho_0) \subset \Omega$ of (1) and
(ii) $\int_0^1 \psi_0(\zeta \bar{p}) \, d\zeta < 1$ for some $\bar{p} \geq \rho_0$.

Then, $\gamma$ is unique in $\Omega_1 = \Omega \cap B(\gamma, \bar{p})$ as a solution of (1).

**Proof.** Let $q \in \Omega_1$ be such that $H(q) = 0$. Define $Q = \int_0^1 H'(\gamma + \zeta(q - \gamma)) \, d\zeta$. Then, in view of (C2) and (16), we obtain in turn
\[ \|H'(\gamma)^{-1}(Q - H'(\gamma))\| \leq \int_0^1 \psi_0(\zeta \|\gamma - q\|) \, d\zeta \leq \int_0^1 \psi_0(\zeta \bar{p}) \, d\zeta < 1, \]
so, $\gamma = q$, since $Q^{-1} \in L(Y, X)$ and $0 = H(q) - H(\gamma) = Q(q - \gamma)$. □

**Definition 1.** The COC is defined as
\[ a = \ln \left( \frac{\|s_{n+1} - \gamma\|}{\|s_n - \gamma\|} \right) / \ln \left( \frac{\|s_n - \gamma\|}{\|s_{n-1} - \gamma\|} \right) \]
and, for $\beta_n := \|s_n - s_{n-1}\|$, the ACOC [16] by
\[ b = \ln \left( \frac{\beta_{n+1}}{\beta_n} \right) / \ln \left( \frac{\beta_n}{\beta_{n-1}} \right). \]

3. Applications

We test the conditions (C).

**Example 1.** If $X = Y = \Omega = \mathbb{R}$, let function $H(s) = \sin s$. The conditions (C) must be verified. Clearly, $\gamma = 0$ solves the equation $H(s) = 0$. Hence, we choose $\gamma = 0$ in conditions (C). In particular, the choice $\gamma = 0$ satisfies the condition (C1). Concerning the condition (C2), we have, in turn, by the definition of $H$ and, since $H'(\gamma) = 1$ and $\|H'(\gamma)^{-1}(H'(\gamma) - H'(u))\| \leq \psi_0(\|\gamma - u\|)$, that
\[ \|H'(\gamma)^{-1}(H'(\gamma) - H'(u))\| = |\cos \gamma - \cos u| \leq \psi_0(\|\gamma - u\|) \]
provided that $\psi_0(\gamma) = s$, since there exists $c \in \mathbb{R}$ such that
\[ |\cos \gamma - \cos u| = |\sin c(\gamma - u)| \leq |\sin c| |\gamma - u| = |\gamma - u|. \]
That is the condition (C1) that is satisfied for this choice of $\psi_0$. Moreover, by solving the equation $\psi_0(\gamma) - 1 = 0$, we obtain $\rho_0 = 1$, and we can set $\Omega_0 = B(0, 1)$.

Similarly, we have
\[ \|H'(\gamma)^{-1}(H'(u) - H'(v))\| \leq |\sin d(u - v)| \leq |\sin d| |u - v| \leq |u - v| \]
for some, \( d \in \mathbb{R} \), so we can choose \( \psi(s) = s \). Then, the first condition in \((C_2)\) is satisfied. Concerning the second condition in \((C_2)\), we obtain in turn that

\[
\|H'(\gamma)^{-1}H'(v)\| = |1 \sin v| \leq 1,
\]

so, we can choose \( \psi_1(s) = 1 \). Then, we have for \( k = 3 \) the results of Table 1.

**Table 1.** Radius for Example 1.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \rho_0 )</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \rho_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.306688...</td>
<td>1</td>
<td>0.666667...</td>
<td>0.343549...</td>
<td>0.306688...</td>
</tr>
</tbody>
</table>

**Example 2.** If \( \Omega = B[0, 1] \), and considering the continuous operator \( H : \Omega \subset S \rightarrow S \) (with the maximum norm) as

\[
H(\psi)(s) = \psi(s) - 5 \int_0^1 s \zeta \psi(\zeta)^3 d\zeta,
\]

where space \( S \) contains continuous functions defined on the interval \([0, 1]\), we have that

\[
H'(\psi(\lambda))(s) = \lambda(s) - 15 \int_0^1 s \zeta \psi(\zeta)^2 \lambda(\zeta) d\zeta,
\]

for each \( \lambda \in \Omega \).

Then, we obtain that \( \gamma = 0 \), and for \( k = 3 \) we obtain the functions \( \psi_0(s) = \psi(s) = \frac{15}{2} s, \psi_1(s) = 2 \).

This way, we obtain the Table 2.

**Table 2.** Radius for Example 2.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \rho_0 )</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \rho_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.024324...</td>
<td>0.133333...</td>
<td>0.088888...</td>
<td>0.032839...</td>
<td>0.024324...</td>
</tr>
</tbody>
</table>

**Example 3.** Let \( \Omega = B(0, 1), X = Y = \mathbb{R}^3 \) and \( \gamma = (0, 0, 0)^T \). Let \( H \) defined on \( \Omega \) be

\[
H(w) = H(w_1, w_2, w_3) = \left( e^{w_1} - 1, \frac{e - 1}{2} w_2^2 + w_2, w_3 \right)^T.
\]

For the point \( w = (w_1, w_2, w_3)^T \), the derivative \( H' \) is

\[
H'(w) = \begin{pmatrix}
e^{w_1} & 0 & 0 \\
0 & (e - 1)w_2 + 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Taking into account rows with their max. norm and

\[
H'(\gamma) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

for \( k = 3 \) we obtain through conditions \((C)\) \( \psi_0(s) = (e - 1)s, \psi(s) = e^{\frac{1}{2}} s, \psi_1(s) = e^{\frac{1}{2}} \) and the radius in Table 3.

**Table 3.** Radius for Example 3.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \rho_0 )</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \rho_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.115169...</td>
<td>0.581977...</td>
<td>0.382692...</td>
<td>0.148762...</td>
<td>0.115169...</td>
</tr>
</tbody>
</table>

**Example 4.** Taking into account the academic example in the first section of this work, for \( k = 3 \), we obtain \( \psi_0(s) = \psi(s) = 96.662907s, \psi_1(s) = 1.0631 \). The corresponding radius are in Table 4.
The Fréchet derivative is given by

\[ H'(v_1, v_2, v_3, v_4) = \begin{pmatrix}
0 & v_3 + v_4 & v_2 + v_4 & v_2 + v_3 \\
v_3 + v_4 & 0 & v_1 + v_4 & v_1 + v_3 \\
v_2 + v_4 & v_1 + v_4 & 0 & v_1 + v_2 \\
v_2 + v_3 & v_1 + v_3 & v_1 + v_2 & 0
\end{pmatrix} \]

Table 4. Radius for Example 4.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( p_0 )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.003149...</td>
<td>0.010345...</td>
<td>0.006896...</td>
<td>0.003463...</td>
<td>0.003149...</td>
</tr>
</tbody>
</table>

Example 5. We consider the conventional form of Kepler’s equation

\[ G(s) = s - \sigma \sin(s) - \tau = 0, \]

where \( 0 \leq \tau \leq \pi \) and \( 0 \leq \sigma < 1 \). In [22], we can find, for distinct values of \( \tau \) and \( \sigma \), a numerical study. In this case, we select value \( \tau = 0.1 \) and \( \sigma = 0.27 \), so the solution obtained is \( \gamma \approx 0.136828 \ldots \). Since

\[ G'(s) = 1 - \sigma \cos(s), \]

we obtain

\[
\| G'(\gamma)^{-1} (G'(s) - G'(t)) \| = \frac{\| \sigma (\cos(s) - \cos(t)) \|}{\| 1 - \sigma \cos(\gamma) \|}
\]

\[
= \frac{2\sigma \| \sin(\frac{s + t}{2}) \sin(\frac{s - t}{2}) \|}{\| 1 - \sigma \cos(\gamma) \|}
\]

\[
\leq \frac{\sigma}{\| 1 - \sigma \cos(\gamma) \|} \| s - t \|
\]

and

\[
\| G'(\gamma)^{-1} G'(s) \| = \frac{\| 1 - \sigma \cos(s) \|}{\| 1 - \sigma \cos(\gamma) \|} \leq \frac{1 + \sigma}{\| 1 - \sigma \cos(\gamma) \|}
\]

Consequently, we have \( \psi_0(s) = \psi(s) = 0.3685888t \) and \( \psi_1(s) = 1.7337327 \). The calculated value parameters for \( k = 3 \) are given in Table 5.

Table 5. Radius for Example 5.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( p_0 )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.550956...</td>
<td>2.713050...</td>
<td>1.808700...</td>
<td>0.720969...</td>
<td>0.550956...</td>
</tr>
</tbody>
</table>

Example 6. We analyze the following system of nonlinear equations

\[
\begin{align*}
v_2 v_3 + v_4 (v_2 + v_3) & = 0 \\
v_1 v_3 + v_4 (v_1 + v_3) & = 0 \\
v_1 v_2 + v_4 (v_1 + v_2) & = 0 \\
v_1 v_2 + v_1 v_3 + v_2 v_3 - 1 & = 0
\end{align*}
\]

with \( v_1, v_2, v_3, v_4 \in \mathbb{R} \). Then, the solution of the previous scheme is showed for \( \gamma = (v_1, v_2, v_3, v_4)^T \) by function \( H := (v_1, v_2, v_3, v_4) : \mathbb{R}^4 \to \mathbb{R}^4 \) defined by

\[
H(v_1, v_2, v_3, v_4) = \begin{pmatrix}
v_2 v_3 + v_4 (v_2 + v_3) \\
v_1 v_3 + v_4 (v_1 + v_3) \\
v_1 v_2 + v_4 (v_1 + v_2) \\
v_1 v_2 + v_1 v_3 + v_2 v_3 - 1
\end{pmatrix}
\]

The Fréchet derivative is given by

\[
H'(v_1, v_2, v_3, v_4) = \begin{pmatrix}
0 & v_3 + v_4 & v_2 + v_4 & v_2 + v_3 \\
v_3 + v_4 & 0 & v_1 + v_4 & v_1 + v_3 \\
v_2 + v_4 & v_1 + v_4 & 0 & v_1 + v_2 \\
v_2 + v_3 & v_1 + v_3 & v_1 + v_2 & 0
\end{pmatrix}
\]
Then, we obtain

\[ \nu_1 = 0.57735026 \ldots \]
\[ \nu_2 = 0.57735026 \ldots \]
\[ \nu_3 = 0.57735026 \ldots \]
\[ \nu_4 = -0.28867513 \ldots \]

(18)

Then, we have that \( \psi_0(s) = 0.28867513s \), \( \psi(s) = 1.55198152s \), \( \psi_1(s) = 1.06466589s \) and the radii are in Table 6.

Table 6. Radius for Example 6.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \rho_0 )</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( \rho_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.391641…</td>
<td>0.775990…</td>
<td>0.484339…</td>
<td>0.413569…</td>
<td>0.391641…</td>
</tr>
</tbody>
</table>

4. Conclusions

A technique is introduced involving only \( H' \), which also gives computable upper error bounds on \( |x_n - \gamma| \). This way, the method (2) becomes applicable in cases not possible before, since higher hypotheses on \( H' \) were required. The COC or ACOC are used to determine the convergence order. The technique is not based on method (2) and is very general. Hence, it can be used to extend the applicability of other methods [11,17,20,21]. This will be the focus of our future work.


Funding: This research was funded by Universidad de Las Américas, Quito, Ecuador, grant number FGE.DGS.20.15.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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