Review
Survey on Mathematical Models and Methods of Complex Logical Dynamical Systems

Xiangshan Kong, Qilong Sun and Haitao Li *

School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China
* Correspondence: haitaoli09@gmail.com

Abstract: Logical dynamical systems (LDSs) have wide applications in gene regulation, game theory, digital circuits, and so on. In LDSs, phenomena such as impulsive effect, time delays, and asynchronous behavior are not negligible, which generate complex LDSs. This paper presents a detailed survey on models and methods of investigating LDSs. Firstly, some preliminary results on LDSs and semi-tensor product (STP) method are presented. Secondly, some new developments on modeling complex LDSs are summarized, including switched LDSs, probabilistic LDSs, delayed LDSs, LDSs with impulsive effects, asynchronous LDSs, constrained LDSs, and implicit LDSs. Finally, the control design techniques of LDSs are reviewed, including reachable set approach, sampled-data control, event-triggered control, and control Lyapunov function method.

Keywords: complex logical dynamical system; mathematical model; control design; semi-tensor product of matrices

MSC: 93C29

1. Introduction

From the mathematical point of view, logical dynamical systems (LDSs) are a class of discrete-time nonlinear systems whose states, inputs, and outputs are quantized on a finite set, and the dynamical equations are represented by multi-valued/mix-valued logical functions [1,2]. In general, the dynamic model of LDSs is

\[
\begin{cases}
  x_1(s + 1) = \varphi_1(x_1(s), \ldots, x_m(s), v_1(s), \ldots, v_n(s)), \\
  \vdots \\
  x_m(s + 1) = \varphi_m(x_1(s), \ldots, x_m(s), v_1(s), \ldots, v_n(s)), \\
  y_p(s) = \psi_p(x_1(s), \ldots, x_m(s)), \quad p = 1, \ldots, q,
\end{cases}
\]

(1)

where \(x_i, i = 1, \ldots, m\) are state variables, \(v_j, j = 1, \ldots, n\) are control inputs, \(y_p, p = 1, \ldots, q\) are outputs, \(\varphi_i, i = 1, \ldots, m\) are logical functions determining the state evolution of system (1), and \(\psi_p, p = 1, \ldots, q\) are logical functions determining the output evolution of system (1). Furthermore,

(i) if \(x_i, v_j, y_p \in \mathcal{D} := \{1, 0\}, \varphi_i: \mathcal{D} \times \cdots \times \mathcal{D} \to \mathcal{D}, \) and \(\psi_p: \mathcal{D} \times \cdots \times \mathcal{D} \to \mathcal{D},\) system (1) is called Boolean networks [3–5];

(ii) if \(x_i, v_j, y_p \in \mathcal{D}_k := \{0, \cdots, k - 1\}, \varphi_i: \mathcal{D}_k \times \cdots \times \mathcal{D}_k \to \mathcal{D}_k,\) and \(\psi_p: \mathcal{D}_k \times \cdots \times \mathcal{D}_k \to \mathcal{D}_k,\) system (1) is called k-valued logical networks [6,7];

(iii) if \(x_i \in \mathcal{D}_{k_i}, v_j \in \mathcal{D}_{\lambda_j}, y_p \in \mathcal{D}_{\mu_p}, \varphi_i: \mathcal{D}_{k_1} \times \cdots \times \mathcal{D}_{k_m} \times \mathcal{D}_{\lambda_1} \times \cdots \times \mathcal{D}_{\lambda_n} \to \mathcal{D}_{k_i},\) and \(\psi_p: \mathcal{D}_{k_1} \times \cdots \times \mathcal{D}_{k_m} \to \mathcal{D}_{\mu_p},\) system (1) is called mix-valued logical networks [8–10].
Boolean networks were firstly proposed by Kauffman [3] to study genetic regulatory networks. In order to solve practical problems in computers and engineering, $k$-valued logical networks were proposed. When describing the dynamic game between machine and human, the model can be described as a network, in which each node may adopt different finite strategies. This network can be described by mix-valued logical networks [11,12]. Since LDSs are parameter free, they can be used to model large-scale systems [13]. In the past half a century, LDSs have been successfully applied to many fields, such as gene regulation [3], game theory [14], fuzzy control [15], finite automata [16], digital circuits [17], information security [18], and so on. Indeed, LDSs have become a frontier research direction of interdisciplinary intersection. Many analytical and/or numerical methods have been recently developed to study the topological structure [19–22] and dynamic characteristics [23–26] of LDSs.

Because the lack of mathematical tools for logical process, it becomes very inconvenient to study the control problems and some other theoretical issues of LDSs [2,13]. The semitensor product (STP) of $M \in \mathbb{R}^{p \times q}$ and $N \in \mathbb{R}^{r \times m}$ is defined as where $\alpha$ is the least common multiple of $q$ and $r$, and “$\otimes$” is the Kronecker product [27]. For the specific properties of STP, please refer to [1,28–30]. Due to these special properties of STP, it has brought great convenience to the investigation of LDSs [1,31,32]. Based on the STP, an algebraic state space representation framework was established for the analysis and control of LDSs [33,34]. One can use the algebraic state space representation framework to study LDSs via the classic control theory [35,36].

We recall the algebraic state space representation framework below. Let $\Delta_m := \{ \delta^s_m : s = 1, \ldots, m \}$, where $\delta^s_m$ is the $s$-th column of the $m$-dimensional identity matrix $I_m$. An $m \times e$ matrix $A$ is called a logical matrix, if $\text{Col}(A) \subseteq \Delta_m$, where $\text{Col}(A)$ denotes the set of all columns of $A$. All $p \times q$ logical matrices in the form of $\delta^j_1 \delta^j_2 \cdots \delta^j_q$ with $j_i \in \{1, \ldots, p\}$, $i = 1, \ldots, q$ form a set, denoted by $\mathcal{L}_{p \times q}$. Without losing generality, take Boolean networks as an example. Identifying $1 \sim \delta^2_1$ and $0 \sim \delta^2_2$, we have $\Delta = \{ \delta^2_1, \delta^2_2 \}$. For system (1), there exists a unique matrix $M_{\varphi_i} \in \mathcal{L}_{2 \times 2^m+m}$ satisfying

$$
\varphi_i(x_1(s), \ldots, x_m(s), v_1(s), \ldots, v_n(s)) = M_{\varphi_i} \cdot \otimes_{j=1}^n v_j \cdot \otimes_{i=1}^m x_i, x_i, v_j \in \Delta.
$$

We call $M_{\varphi_i}$ the structural matrix of $\varphi_i$ [2]. From (2), the algebraic state space representation of system (1) can be obtained as

$$
\begin{align*}
\{ & \begin{array}{l}
x(s+1) = Mv(s)x(s),
y(s) = Lx(s),
\end{array} \\
& \text{where } x(s) = \otimes_{i=1}^m x_i(s) \in \Delta_{2^m}, v(s) = \otimes_{j=1}^n v_j(s) \in \Delta_{2^n}, y(s) = \otimes_{i=1}^m y_i(s) \in \Delta_{2^n}, M = M_{\varphi_1} \cdots M_{\varphi_n} \in \mathcal{L}_{2^m \times 2^m+m}, L \in \mathcal{L}_{2^m \times 2^n}, \text{ and } “\otimes” \text{ denotes the Khatri–Rao product.}
\end{align*}
$$

In the past decade, based on the algebraic state space representation approach, many theoretic problems of LDSs have been achieved, including controllability [37–41], observability [42–46], stability [47–51], stabilization [52–56], tracking control [57–61], disturbance decoupling [62–66], input–output decoupling [67–71], optimal control [12,72–74], and so on. In some recent studies, the above results have been extended to various complex LDSs. For example, switched LDSs [75], probabilistic LDSs [76], delayed LDSs [77], LDSs with impulsive effects [78], asynchronous LDSs [79], constrained LDSs [80], singular LDSs [81], and so on. Based on the algebraic state space representation technique, several efficient control design techniques such as a reach set approach [52], control Lyapunov function approach [51], event-triggered control technique [64], and sampled-data control technique [82] have been introduced to solve the control problems of complex LDSs. This paper aims to present a detailed survey on these complex LDSs and control design techniques. Lu et al. [13] presented a survey on recent development of fundamental works on LDSs and briefly introduced research works on generalised Boolean (control) networks. Compared with [13], this paper presents a detailed survey on complex LDSs, and it is easy to see that
the general LDSs and generalised Boolean (control) networks considered in [13] are special cases of the complex LDSs considered in this paper.

The remainder of this paper is organized as follows: Section 2 presents a comprehensive survey on the mathematical models of complex LDSs. Some control design techniques of complex LDSs are summarized in Section 3, which is followed by a conclusion in Section 4.

2. Mathematical Models of Complex LDSs

2.1. LDSs with Switching Structures

Switched systems are hybrid systems in which several subsystems controlled by a switching law. Recently, switched systems have received intensive attention [83]. Switched LDSs are actually a nonlinear switched system [84,85], in which the switching mode is generated by asynchronous updating and external disturbances. The dynamic model of LDS (1) with switching structures is

\[
\begin{align*}
    x_1(s+1) &= \varphi_1^{(s)}(x_1(s), \ldots, x_m(s), v_1(s), \ldots, v_n(s)), \\
    \vdots \\
    x_m(s+1) &= \varphi_m^{(s)}(x_1(s), \ldots, x_m(s), v_1(s), \ldots, v_n(s)),
\end{align*}
\]

(4)

where \( \varphi : \mathbb{N} \to S := \{1, 2, \ldots, \epsilon\} \) is the switching signal, and \( \mathbb{N} \) denotes the set of nonnegative integers.

The model of switched Boolean control networks (BCNs) was firstly proposed in [75] to study the reachability and controllability. Since then, several fundamental results of switched LDSs have been developed. In [86–89], stability of switched Boolean networks was discussed. In [59,90–94], stabilization and control design of switched Boolean networks were investigated. Complete synchronization problem for the drive-response switched Boolean networks under arbitrary switching signals was investigated in [95]. Stability of switched \( k \)-valued logical networks was investigated in [96,97]. Stabilization and control design of switched \( k \)-valued logical networks were investigated in [6,98,99]. Controllability of switched mix-valued logical networks with constraints was investigated in [100].

2.2. Probabilistic LDSs

In the real world, stochasticity is very common. Therefore, extending LDSs to probabilistic LDSs is reasonable. Profiting from this kind of nondeterministic systems, the uncertainty in both data and model selection can be well dealt with [101]. The dynamic model of probabilistic LDSs is

\[
\begin{align*}
    x_1(s+1) &= \varphi_1^{(s)}(x_1(s), \ldots, x_m(s), v_1(s), \ldots, v_n(s)), \\
    \vdots \\
    x_m(s+1) &= \varphi_m^{(s)}(x_1(s), \ldots, x_m(s), v_1(s), \ldots, v_n(s)),
\end{align*}
\]

(5)

where \( \varphi : \mathbb{N} \to S := \{1, 2, \ldots, \epsilon\} \) is the switching signal, and the switching signal \( \varphi(s) \) is a stochastic sequence, which is independently and identically distributed with the probability distribution \( P \{ \varphi(s) = i \} = p_i > 0, i \in S, \sum_{i=1}^\epsilon p_i = 1 \).

Probabilistic LDSs were firstly proposed in [102]. Stability of probabilistic Boolean networks was discussed in [103–105]. Controllability and observability of probabilistic Boolean networks were investigated in [76,106,107]. Stabilization of probabilistic Boolean networks was investigated in [108–112]. The synchronization problem of master–slave probabilistic Boolean networks was analyzed in [113]. Controllability and stabilizability of probabilistic logical control networks were investigated in [114]. Liu et al. [115] considered two kinds of optimal control problems for probabilistic mix-valued LDSs. Some further results on the finite-time stability of probabilistic logical networks were presented in [101].
2.3. LDSs with Time Delays

As a source of instability, the phenomenon of time delays could occur under some circumstances, such as the delay of long-distance information transmission, chemical processes, drug action, and slow processes of gene transcription and translation [116–119]. Thus, considering the influence of time delays could predict the dynamics of models more accurately. Three kinds of time delays including constant time delay, time-variant delay, and state-dependent delay are commonly considered [120]. LDSs with time delays are called delayed LDSs. The dynamic model of LDS (1) with constant time delay is

\[
\begin{aligned}
x_1(s + 1) &= \varphi_1 (X(s - \tau + 1), \ldots, X(s), V(s)), \\
\vdots \\
x_m(s + 1) &= \varphi_m (X(s - \tau + 1), \ldots, X(s), V(s)),
\end{aligned}
\]

where \( \tau \in \mathbb{Z}_+ \) is the state time delay, \( \mathbb{Z}_+ \) denotes the set of positive integers, \( X(i) := (x_1(i), \ldots, x_n(i)), i = s - \tau + 1, \ldots, s \), and \( V(s) := (v_1(s), \ldots, v_m(s)) \).

The study of delayed LDSs has aroused many scholars’ research interests. The delayed LDSs were firstly studied by using the STP of matrices in [1]. Observability and controllability of Boolean networks with constant time delays were investigated in [77, 121–123]. Stabilization and set stabilization of BCNs with constant time delays were investigated in [124, 125]. Control design for output tracking of BCNs with constant time delays was investigated in [60]. Mu et al. [126] investigated controllability and reachability of \( k \)-valued LDSs with constant time delays. Controllability and observability of BCNs with time-variant delays were investigated in [127–129]. In [130, 131], stability and stabilization of Boolean networks with stochastic time delays were studied. Considering state-dependent delay, the set stability of Boolean networks was investigated in [132]. Stability and uniform sampled-data stabilization of constrained Boolean networks with state-dependent delays were discussed in [120].

2.4. LDSs with Impulsive Effects

Evolutionary processes may experience abrupt changes of states, which may occur at prescribed time instants and/or triggered by specified events. When mathematically modeling the evolution of processes with a short-time perturbation, the perturbation is assumed to be instantaneous and the duration is neglected, that is, in the form of impulse. In view of the important influence of impulse in biological networks, impulsive effects are introduced into the study of gene regulatory networks [133, 134]. The dynamic model of LDS (1) with impulsive effects is

\[
\begin{aligned}
x_1(s + 1) &= \varphi_1 (x_1(s-1), \ldots, x_m(s), v_1(s), \ldots, v_h(s)), \\
\vdots \\
x_m(s + 1) &= \varphi_m (x_1(s-1), \ldots, x_m(s), v_1(s), \ldots, v_h(s)), \\
x_1(s_h) &= f_1 (x_1(s_h - 1), \ldots, x_m(s_h - 1)), \\
\vdots \\
x_m(s_h) &= f_m (x_1(s_h - 1), \ldots, x_m(s_h - 1)), \quad s \in \mathbb{N} \setminus Y,
\end{aligned}
\]

where \( s_h, h \in \mathbb{Z}_+ \) is the impulsive instant satisfying \( 0 < s_1 < \cdots < s_h < \cdots \), and \( Y := \{s_h - 1 : h \in \mathbb{Z}_+ \} \).

LDSs with impulsive effects were firstly proposed in [135] to study observability. Stability and stabilization of Boolean network with impulsive effects were studied in [78, 136–139]. Controllability of BCNs with impulsive effects was investigated in [140, 141]. The optimal control problem of BCNs with impulsive effects was investigated in [80, 142]. Considering impulsive effects, the robust set stabilization and the output tracking problem of BCNs were addressed [143, 144]. Impulsive control for the output tracking of probabilistic BCNs was investigated in [61]. The bisimulations of BCNs with impulsive effects was addressed in [145].
2.5. Asynchronous LDSs

For classical LDSs such as (1), based on the assumption that the update scheme is independent of the dynamical behaviors of the network, it is generally assumed that all nodes on the network are updated in parallel. However, in reality, it is difficult to find synchronous clocks in biological systems. For example, factors such as mRNA and protein transport, degradation and synthesis time mean that the system is full of different degrees of delays, and genes are activated or suppressed in a basically asynchronous manner [146]. Therefore, it is necessary to discuss LDSs under the asynchronous updating rule. In general, a deterministic asynchronous scheme and random asynchronous scheme are two kinds of asynchronous schemes usually considered [93]. Now, we consider the deterministic asynchronous LDSs. \( \tau_i \in \{0, \ldots, \varsigma_i - 1\} \) and \( \varsigma_i \) denotes the initial updating time and the updating period of node \( i \), respectively, \( i = 1, \cdots, m \). The dynamic model of LDS (1) with asynchronous updating rule is given as follows:

\[
\begin{align*}
&x_1(s + 1) = \begin{cases}
\varphi_1(x_1(s), \cdots, x_m(s), v_1(s), \cdots, v_n(s)), & \text{if } g_1(s + 1) = 1, \\
x_1(s), & \text{if } g_1(s + 1) = 0,
\end{cases} \\
&\vdots \\
&x_m(s + 1) = \begin{cases}
\varphi_m(x_1(s), \cdots, x_m(s), v_1(s), \cdots, v_n(s)), & \text{if } g_m(s + 1) = 1, \\
x_m(s), & \text{if } g_m(s + 1) = 0,
\end{cases}
\end{align*}
\]

(8)

where

\[
g_i(s) = \begin{cases}
1, & \text{if } s \bmod \varsigma_i = \tau_i, \\
0, & \text{else,}
\end{cases} \quad i = 1, \cdots, m.
\]

(9)

For node \( i \), if \( g_i(s + 1) = 1 \), the value of \( x_i(s + 1) \) is determined by the updating rule \( \varphi_i \); otherwise, the value of \( x_i(s + 1) \) remains unchanged.

Asynchronous LDSs were firstly proposed in [147]. In [148], the dynamics of asynchronous \( k \)-valued logical networks were investigated based on the linear representation. In [79], the dynamics of asynchronous Boolean networks is investigated via algebraic approach, where at each time step a random number of nodes can be updated. In [149], the controllability of asynchronous BCNs was studied. By virtue of the results obtained in [149], the controllability of BCNs with delays and asynchronous stochastic update was considered [150]. Controllability of asynchronous BCNs with constant time delay was studied in [151]. The complete synchronization for asynchronous switched Boolean networks was discussed in [152]. In [153], a new linear approach was proposed to model the dynamics of asynchronous Boolean networks. In [93], the dynamics of deterministic asynchronous BCNs were converted into the form of periodic switching BCNs, and the time-variant state feedback stabilization problem of deterministic asynchronous BCNs was solved. In [154], the asynchronous event-triggered control mechanism was introduced to the set stabilization problem of \( k \)-valued logical control networks.

2.6. Constrained LDSs

As we all know, constraints often play an important role in both linear systems and nonlinear systems [155–157]. Similarly, in the gene regulatory network, because some gene states may cause serious diseases, or some treatment schemes may have dangerous effects, it is necessary to impose constraints on the states and controls. LDSs with state constraints were introduced in [39]. LDSs with state or control constraints are called constrained LDSs. Considering the constraints of system (3), the state constraint set and control constraint set are generally defined as

\[
C_x = \{ s_{2m}^1, \cdots, s_{2m}^n \} \subseteq \Delta_{2m},
\]

(10)

and

\[
C_v = \{ s_{2v}^1, \cdots, s_{2v}^n \} \subseteq \Delta_{2v},
\]

(11)
respectively, where \( i_1 < \cdots < i_n, j_1 < \cdots < j_\beta \).

In the past decade, many excellent results on constrained LDSs have been proposed, including observability [158], controllability [32,80,91,100], stabilization [125,155,159,160], and so on.

2.7. Implicit LDSs

When the dynamic equations of systems are constrained, singular systems, which are also referred to as implicit systems, differential algebraic equations or descriptor systems, are often much more natural and convenient than standard models in describing some scientific and engineering systems, and such systems have been widely used in many fields, including biological systems, power networks, flexible arm control of robots and aircraft attitude control [161]. Inspired by this, singular LDSs were proposed [81,162], and the more general model is implicit LDSs [163]. The dynamic model of implicit LDSs is

\[
\begin{align*}
\phi_1(x_1(s), \cdots, x_m(s), x_1(s+1), \cdots, x_m(s+1)) &= 1, \\
\vdots \\
\phi_m(x_1(s), \cdots, x_m(s), x_1(s+1), \cdots, x_m(s+1)) &= 1.
\end{align*}
\] (12)

It should be noted that the existence of solutions in implicit LDSs is not necessarily unique.

Singular LDSs [81] were firstly introduced in [162] and called dynamic-algebraic LDSs. In [164], the properties of singular LDSs were studied in detail by using the STP method. Since then, many excellent results about singular LDSs have been established such as controllability and observability [100,165,166], stability [167], optimal control [168], disturbance decoupling [169], and function perturbations [170]. As the more general model of singular LDSs, implicit LDSs were proposed for the first time in [163]. In [163], some criteria were proposed to equivalently convert implicit LDSs into classic or restricted LDSs. Then, in order to determine the topological structure of singular LDSs, an improved method was presented, based on which transformation relations between implicit LDSs and singular LDSs were given.

3. Control Design Techniques for LDSs

3.1. Reachable Set Approach

Control design for stabilization is one of the fundamental issues of LDSs, and many excellent results have been obtained [1,48,52,78,171]. Among them, the reachable set approach for studying the state feedback stabilization of LDSs proposed in [48,52] provides a convenient approach for constructing stabilizers of LDSs.

The objective of state feedback stabilization is to find a state feedback control of the form [52]

\[
\begin{align*}
\nu_1(s) &= g_1(x_1(s), \cdots, x_m(s)), \\
\vdots \\
\nu_n(s) &= g_n(x_1(s), \cdots, x_m(s)),
\end{align*}
\] (13)

where \( g_i, i = 1, \cdots, n \) are logical functions that stabilizes system (3) to a given equilibrium point \( x^*_e = \delta^{i_n}_{i_m} \). From the algebraic state space representation framework, the state feedback control (13) can be converted into the following algebraic form:

\[
\nu(s) = Gx(s),
\] (14)

where \( G \) is said to be the state feedback gain matrix.
For the given equilibrium point $x_e = \delta_{\text{en}}^0$, the reachable set is defined as follows:

$$E_r(\theta) = \left\{ x_0 \in \Delta_{\text{en}} : \text{there are } v(0), \ldots, v(r - 1) \in \Delta_{\text{en}} \text{ such that } x(r; x_0; v(0), \ldots, v(r - 1)) = \delta_{\text{en}}^0 \right\},$$

where $E_r(\theta)$ represents a set of initial states that reach $x_e$ in $r$ steps, and the control design approach based on $E_r(\theta)$ is called a reachable set approach.

Following [48,52], many valuable results on LDSs have been obtained by the reachable set approach, including stabilization [82,125,155,159,172], set stabilization [111,173–175], synchronization [176–178], output tracking [179,180], output regulation [181–183], and so on.

### 3.2. Sampled-Data Control

Sampled-data control, which can decrease the controller update frequency and reduce the computational burden, is a commonly used technique to decrease the control costs [184]. In many research fields such as nonlinear systems [185], neural networks [186] and fuzzy systems [187], sampled-data control theory has been well developed. The sampled-data control technique was also introduced to the control of LDSs [82,188], switched LDSs [189] and probabilistic LDSs [110,111]. Accordingly, some fundamental results were proposed for the sampled-data controllability [190], synchronization [191], and stabilization [192].

**Definition 1.** Given a set of sampling points $\{s_h : h \in \mathbb{N}\}$ with $s_0 = 0$, $(V(0), V(1), \cdots)$ is said to be a uniform sampled-data control, if

$$V(s) = V(s_h), s \in [s_h, s_{h+1}) \subseteq Z, s_{h+1} - s_h = c, \forall h \in \mathbb{N},$$

where $c \in \mathbb{Z}_+$ is called the sampling period, and $[s_h, s_{h+1}) \subseteq \mathbb{Z} = \{s_h, s_h + 1, \cdots, s_{h+1} - 1\}$.

Sampled-data control with the intervals between sampling points being time-variant is called nonuniform sampled-data control. From Definition 1, it can be seen that the sampling points are fixed. For BCNs, uniform sampled-data state feedback control was first used to investigate the stabilization problem in [82]. In [193], the uniform sampled-data state feedback control problem of mix-valued logical control networks was studied. By virtue of the STP method, the robust uniform sampled-data control invariance of BCNs were investigated in [188]. Controllability and observability of uniform sampled-data BCNs were considered in [194]. The uniform sampled-data reachability and stabilization of constrained $k$-valued logical control networks were investigated in [155]. In [110,111], considering uniform sampled-data state feedback controllers, stabilization and set stabilization of probabilistic BCNs were addressed. In [191], the general partial synchronization of BCNs was studied by the uniform sampled-data feedback controller. Under the nonuniform sampled-data control, the time-variant state feedback stabilization of constrained BCNs with time delays was investigated in [125].

**Definition 2.** $\{V(s) : s \in \mathbb{N}\}$ is said to be an aperiodic sampled-data control, if

$$V(s) = V(s_h), s \in [s_h, s_{h+1}) \subseteq Z, s_{h+1} - s_h = \tau_h, \forall h \in \mathbb{N},$$

where $s_h, h \in \mathbb{N}$ are sampling points, the interval length $\tau_h \in \Omega : = \{l_1, \cdots, l_p\} \subseteq \mathbb{Z}_+, l_1 < l_2 < \cdots < l_p$, and $s_0 = 0$.

From Definition 2, it can be seen that the sampling points are aperiodic. In [192], under the aperiodic sampled-data control, the global stability analysis of BCNs was considered via a novel technique, and the key is that the BCNs under aperiodic sampled-data control was converted into a switched Boolean network. In [195], the global stability of BCNs with
the aperiodic sampled-data control was further studied. Controllability and stabilizability of BCNs under the aperiodic sampled-data control was investigated in [190].

**Definition 3.** \( \{ V(s) : s \in \mathbb{N} \} \) is said to be a nonuniform sampled-data control, if

\[
V(s) = V\left(s_h, s \in [s_{h}, s_{h+1}] |_{\mathbb{Z}}, s_{h+1} - s_h = \tau, \forall h \in \mathbb{N}, \right.
\]

where \( s_h, h \in \mathbb{N} \) are sampling points, the sampling period \( \tau \in \{ \tau_1, \tau_2 \} \) with probability distribution \( \{ p, 1 - p \} \), and \( s_0 = 0 \).

From Definition 3, it can be seen that the sampling points are determined by the probability distribution. Under nonuniform sampled-data control, the output regulation problem of BCNs was investigated [196]. Wang et al. [197] discussed sampled-data state feedback control with stochastic sampling periods for BCNs. Sun et al. [198] introduced a novel technique for the global stochastic stability of aperiodic sampled-data BCNs.

### 3.3. Event-Triggered Control

Event-triggered control was firstly introduced in [199], and it has been well developed in nonlinear systems and networked control systems [200–202]. Event-triggered control can reduce control execution times and computation costs. The event-triggered control approach to LDSs was firstly introduced in [64], and the disturbance decoupling problem of BCNs was considered. An event-triggered control scheme was developed for the robust set stabilization of disturbed LDSs [203].

Generally speaking, event-triggered control consists of two basic elements as feedback control and event-triggered condition [204]. Now, we show the two basic elements of the event-triggered control by referring to the results in [203]. Let an initial state \( x(0) = d^0 \), a nonempty set \( P \subseteq \mathbb{R}^n \), and a time-variant state feedback control \( v(s) = \Phi(s, x(0))x(s) \) be given. The event-triggered condition was formulated as

\[
d_H(Y(s + 1), P) > 0,
\]

where \( Y(s + 1) = \text{Col}(\text{Blk}_\theta(x^0_{\tau})) [L\Phi(i, x(0)) R_{k_i}^M]) \), \( \text{Blk}_\theta(P) \) denotes the \( \theta \)-th \( n \times n \) square block of an \( n \times nm \) matrix \( P \), and \( d_H(Y(s + 1), P) \) represents the Hausdorff distance between \( Y(s + 1) \) and \( P \). Denote the sequence of triggering time by \( s_1 < s_2 < \cdots < s_\tau < \cdots \). Correspondingly, one can obtain a sequence of state feedback control updates as \( \Phi(s_1, x(0)), \Phi(s_2, x(0)), \cdots, \Phi(s_\tau, x(0)), \cdots \). Thus, the event-triggered controllers can be designed as

\[
v(s) = \Phi(s_i, x(0))x(s), s \in [s_i, s_{i+1}) \cap \mathbb{N},
\]

where \( i = 0, 1, \cdots, \tau, \cdots \).

Recently, by virtue of an event-triggered control approach, several excellent results on LDSs have been proposed, including synchronization [98,178], stabilization [205–207], set stabilization [154,208], disturbance decoupling [209], output regulation [210] and so on[211], and the corresponding event-triggered mechanisms and techniques for designing state feedback controllers were proposed.

### 3.4. Control Lyapunov Function

Lyapunov theory plays an important role in the stability analysis and control synthesis of nonlinear dynamic systems. In the past few decades, a series of construction methods of control Lyapunov functions have been proposed and used to design the state feedback stabilizers in nonlinear control systems [212,213]. Control Lyapunov functions were generalized to finite evolutionary games in [214]. In [51], a framework of Lyapunov stability theory for LDSs by the STP method was first established. Then, a control Lyapunov approach was proposed in [215] to investigate the feedback stabilization, where all stabilizers and corresponding control Lyapunov functions were designed.
The Lyapunov theory of LDSs was developed in [51], where the Lyapunov function of LDS (1) is a pseudo-Boolean function in the form of
\[
P(x_1, \ldots, x_m) = a_0 + a_1 x_1 + \cdots + a_m x_m + a_{m+1} x_1 x_2 + \cdots + a_{2^m-1} x_1 \cdots x_m, \tag{18}
\]
where \(a_i, i = 0, 1, \ldots, 2^m - 1\) are real coefficients. At the same time, \(P(x_1, \ldots, x_m)\) meets
(i) \(P(x_1, \ldots, x_m) > 0, \forall (x_1, \ldots, x_m) \in \mathcal{D}^m \setminus \mathcal{S}\), and \(P(x_1, \ldots, x_m) = 0, \forall (x_1, \ldots, x_m) \in \mathcal{S}\);
(ii) along the trajectories of the system (1), \(\Delta P(x_1(s), \ldots, x_m(s)) = P(x_1(s+1), \ldots, x_m(s+1) - P(x_1(s), \ldots, x_m(s)) < 0\) holds for any \((x_1(s), \ldots, x_m(s)) \notin \mathcal{S}\), and \(\Delta P(x_1(s), \ldots, x_m(s)) = 0\) holds for any \((x_1(s), \ldots, x_m(s)) \in \mathcal{S}\),

where \(\mathcal{S}\) denotes the set of fixed points, and \(\mathcal{S}\) denotes the set of attractors.

Using the STP method, the pseudo-Boolean function (18) can be expressed as
\[
P(x) = M_P x, \tag{19}
\]
where \(M_P \in \mathbb{R}^{1 \times 2^m}\) is unique. Conversely, \(M_P\) determines a unique pseudo-Boolean function \(P(x_1, \ldots, x_m)\).

Recently, some excellent results on LDSs via the control Lyapunov function approach have been proposed. In [192], in order to derive sufficient conditions for the global stability of BCNs with aperiodic sampled-data control, switching-based Lyapunov function techniques and average dwell time method were established. Lyapunov functions for the set stability of Boolean networks and control Lyapunov functions for the feedback set stabilization and synchronization of BCNs were proposed in [216]. By virtue of the control Lyapunov function method, the partial stabilization of probabilistic BCNs with sample-data state-feedback control was investigated in [109]. In [160], the stabilization for delayed BCNs with state constraints was studied by using the barrier Lyapunov function.

**Remark 1.** In the past few decades, neural networks have received extensive attention and have been applied to various fields such as signal processing, fault diagnosis, and industrial automation. It is worth noting that neural networks with time delays were also investigated via sampled-data techniques [217–219]. The main difference between delayed neural networks and delayed LDSs lies in the fact that the states of neural networks take values from set of real numbers while that of LDSs taking values from finite sets.

**4. Conclusions**

In this survey, we have reviewed new developments for several generalized forms of LDSs, including switched LDSs, probabilistic LDSs, delayed LDSs, LDSs with impulsive effects, asynchronous LDSs, constrained LDSs, and implicit LDSs (see Table 1). Furthermore, we have summarized some control design techniques of LDSs, including the reachable set approach, sampled-data control technique, event-triggered control technique, and control Lyapunov function approach. In the future, one may apply the model and control theory of LDSs to the modeling and control of engineering devices such as unmanned aerial vehicles and hybrid electric vehicles.
Table 1. A classification for the considered models and methods.

<table>
<thead>
<tr>
<th>Models</th>
<th>Methods</th>
<th>Reachable Set</th>
<th>SAMPLED-Data</th>
<th>Event-Triggered</th>
<th>Lyapunov Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Switched LDSs</td>
<td></td>
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<td>✓ 173</td>
<td>✓ 189</td>
<td>✓ 98</td>
</tr>
<tr>
<td>Probabilistic LDSs</td>
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<td></td>
<td>✓ 110</td>
<td>✓ 205</td>
</tr>
<tr>
<td>Delayed LDSs</td>
<td></td>
<td>✓ 172</td>
<td>✓ 125</td>
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<td>✓ 160</td>
</tr>
<tr>
<td>Impulsive LDSs</td>
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<td>✓ 143</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Asynchronous LDSs</td>
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<td>✓ 176</td>
<td></td>
<td>✓ 178</td>
<td></td>
</tr>
<tr>
<td>Constrained LDSs</td>
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<td></td>
</tr>
<tr>
<td>Implicit LDSs</td>
<td></td>
<td>✓ 166</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

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