Fuzzy Continuous Mappings on Fuzzy F-spaces

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Abstract: In the present paper, we first introduce different types of fuzzy continuity for mappings between fuzzy F-normed linear spaces and the relations between them are investigated. Secondly, the principles of fuzzy functional analysis are established in the context of fuzzy F-spaces. More precisely, based on the fact that fuzzy continuity and topological continuity are equivalent, we obtain the closed graph theorem and the open mapping theorem. Using Zabreiko’s lemma, we prove the uniform bounded principle and Banach–Steinhaus theorem. Finally, some future research directions are presented.

Keywords: fuzzy pseudo-norm; fuzzy F-spaces; fuzzy continuous mapping; principles of fuzzy functional analysis

MSC: 46S40

1. Introduction

After L.A. Zadeh introduced in his famous paper [1] the notion of fuzzy set, many researchers wanted to extend the classical mathematical knowledge in this new fuzzy context. The approach is both an internal, meaning developing mathematics from within, as well as an external one, namely responding to some needs and challenges coming from other domains, starting with engineering and even reaching economic and social domains. Therefore, in the paper [2], starting from the classic notions of pseudo-norm, F-norm and F-space, we introduced the notions of fuzzy pseudo-norm, fuzzy F-norm and fuzzy F-space. An important result is that a fuzzy F-normed linear space is a metrizable topological linear space.

The goals of this research were established in 2016, in the paper [2]: the study of fuzzy continuous mappings on fuzzy F-spaces and obtaining, in this more general setting, a fuzzy version for principles of functional analysis. The aims of this study are: to introduce different types of fuzzy continuity for mappings between fuzzy F-normed linear spaces and to investigate the relations between them. In this context, an important goal is to compare the newly introduced concepts with the classic continuity from F-normed linear space. We expect to prove that fuzzy continuity and topological continuity are equivalent. Other studied concepts such as strong fuzzy continuity and weak fuzzy continuity will be useful in future papers where we will study the boundedness of linear operators. Last but not least, an important motivation for this approach is the researchers’ interest in these spaces and their generalizations.

Thus, in 2019, Dinda, Ghosh and Samanta introduce the concept of intuitionistic fuzzy pseudo-normed linear space (see [3]), and then they study, within this context, different types of intuitionistic fuzzy continuities and intuitionistic fuzzy boundedness as well as the relationships between them (see [4]). In their paper [5], they study the spectrum and the spectral properties of bounded linear operators in intuitionistic fuzzy pseudo-normed linear spaces.

In 2022, following the notion of fuzzy pseudo-normed linear spaces defined by Nădăban, Wu [6] introduces the notion of fuzzy pseudo-semi-normed linear space, according to general t-norm.
The structure of the paper is as follows: after the preliminary section, in Section 3, we introduce different types of continuity: fuzzy continuity, sequentially fuzzy continuity, and strong fuzzy continuity. We also establish relations between them. Such methods have been carried out by Bag and Samanta [7], Nádabán [8], and Sadeqi and Solaty Kia [9] in the case of fuzzy normed linear spaces and by Dinda, Ghosh and Samanta [4] in the case of intuitionistic fuzzy pseudo-normed linear spaces. As is known, in the functional analysis closed graph theorem, the open mapping theorem and the uniform boundedness principle are of great importance. We will present these theorems in Section 4, in the context of fuzzy F-spaces. These principles were obtained in the frame of fuzzy normed linear space by Bag and Samanta [7] and Sadeqi and Solaty Kia [9]. We will use a result obtained by Zabreiko [10], which proved useful even in the classic case in order to prove many important theorems in functional analysis. We can use Zabreiko’s theorem because in [2] we showed that fuzzy F-spaces are metrizable topological vector spaces.

2. Preliminaries

Let $E$ be a linear space over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

Definition 1 ([11]). A mapping $p : E \to \mathbb{R}$, which satisfies:

(PN1) $p(\xi) \geq 0, (\forall) \xi \in E$;

(PN2) $p(\xi) = 0$ if and only if $\xi = 0$;

(PN3) $p(\alpha \xi) \leq p(\xi), (\forall) \xi \in E, (\forall) \alpha \in \mathbb{K}, |\alpha| \leq 1$;

(PN4) $p(\xi + \eta) \leq p(\xi) + p(\eta), (\forall) \xi, \eta \in E$.

will be called pseudo-norm (shortly p-norm).

Remark 1. If we consider the invariant metric $d(\xi, \eta) = p(\xi - \eta)$, we obtain a topology $\mathcal{T}$ on $E$. If we add two more axioms:

(PN5) $\alpha_n \to 0 \Rightarrow p(\alpha_n \xi) \to 0, (\forall) \xi \in E$,

(PN6) $p(\xi_n) \to 0 \Rightarrow p(a \xi_n) \to 0, (\forall) a \in \mathbb{K}$,

the p-norm is called F-norm. In this case, the topology is compatible with the linear space structure of $E$ and we obtain a topological vector space. If $d$ is complete, we obtain an $F$-space (see [12], 1.8).

Definition 2. [2] A fuzzy set $F$ in $E \times \mathbb{R}$ is called a fuzzy pseudo-norm (shortened to FP-norm) on $E$ if it satisfies:

(F1) $F(\xi, \tau) = 0, (\forall) \xi \in E, (\forall) \tau \leq 0$;

(F2) $F(\xi, \tau) = 1, (\forall) \tau \in \mathbb{R}^*_+, if and only if \xi = 0$;

(F3) $F(\alpha \xi, \tau) \geq F(\xi, \tau), (\forall) \xi \in E, (\forall) \tau \in \mathbb{R}, (\forall) a \in \mathbb{K}, |a| \leq 1$;

(F4) $F(\xi + \eta, \tau + \zeta) \geq \min\{F(\xi, \tau), F(\eta, \zeta)\}, (\forall) \xi, \eta, \zeta \in E, (\forall) \tau, \zeta \in \mathbb{R}$;

(F5) $\lim_{\tau \to +\infty} F(\xi, \tau) = 1, (\forall) \xi \in E$.

(F6) If there exists $\kappa_0 \in (0, 1)$ such that $F(\xi, \tau) > \kappa_0, (\forall) \tau \in \mathbb{R}^*_+$, then $\xi = 0$;

(F7) $(\forall) \xi \in E, F(\xi, \cdot)$ is left continuous on $\mathbb{R}$.

The pair $(E, F)$ is called fuzzy pseudo-normed linear space (shortly $\text{FPNL}$-space).

Theorem 1 ([2]). Let $(E, F)$ be a $\text{FPNL}$-space. Let $p_\alpha : E \to \mathbb{R}$ defined by

$$p_\alpha(\xi) := \inf\{\tau > 0 : F(\xi, \tau) > \alpha\}, \alpha \in (0, 1), \xi \in E$$

Then:

1. $\{p_\alpha\}_{\alpha \in (0, 1)}$ is an ascending family of $p$-norms on $E$.
2. \( F(\xi, p_n(\xi)) \leq \alpha, (\forall) \alpha \in (0,1), (\forall) \xi \in E; \)
3. for \( \xi \in E, \tau > 0, \alpha \in (0,1) \), we have that
   \[ p_n(\xi) < \tau \iff F(\xi, \tau) > \alpha; \]
4. \( \{p_n\}_{n \in (0,1)} \) is a right continuous, namely \( (\forall) \gamma \in (0,1), (\forall) \xi \in E, \) we have
   \[ \lim_{n \to \gamma, \alpha \gamma} p_n(\xi) = p_\gamma(\xi). \]

**Theorem 2** ([2]). Let \((E, F)\) be an FPNL-space. For \( \xi \in E, \alpha \in (0,1), \epsilon > 0, \) we define
   \[ B(\xi, \alpha, \epsilon) := \{ \eta \in E : F(\xi - \eta, \epsilon) > \alpha \}. \]

Let
   \[ T_F = \{ T \subset E : x \in T \iff (\exists) \epsilon > 0, \alpha \in (0,1) \text{ s.t. } B(\xi, \alpha, \epsilon) \subset T \}. \]

Then, \( T_F \) is a topology on \( X. \)

**Definition 3** ([2]). Let \((E, F)\) be a FPNL-space and \((\xi_n)\) be a sequence in \( E. \)
1. The sequence \((\xi_n)\) is called convergent if \((\exists) \xi \in E \) such that \( \lim_{n \to \infty} F(\xi_n - \xi, \tau) = 1, (\forall) \tau > 0. \) In this situation, \( \xi \) is said to be the limit of the sequence \((\xi_n)\) and we denote \( \lim_{n \to \infty} \xi_n = \xi. \)
2. The sequence \((\xi_n)\) is called a Cauchy sequence if
   \[ (\forall) \alpha \in (0,1), (\forall) \epsilon > 0, (\exists) n_0 \in \mathbb{N} : F(\xi_n - \xi_m, \epsilon) > \alpha, (\forall) n, m \geq n_0. \]
3. \((E, F)\) is called complete if any Cauchy sequence is convergent.
4. Let \( \alpha \in (0,1). \) The sequence \((\xi_n)\) is called \( \alpha \)-convergent if there exists \( \xi \in X \) such that
   \[ (\forall) \epsilon > 0, (\exists) k_0 \in \mathbb{N} : F(\xi_n - \xi, \epsilon) > \alpha, (\forall) n \geq k_0. \]
   In this situation, \( \xi \) is called \( \alpha \)-limit of the sequence \((\xi_n)\) and we write \( \xi_n \xrightarrow{\alpha} \xi. \)

**Definition 4** ([2]). A FP-norm \( F \) on \( E \) is called fuzzy F-norm if we have:

(F8) \( a_n \to 0 \implies a_n \xi \to 0, (\forall) \xi \in E; \)

(F9) \( \xi_n \xrightarrow{\alpha} \xi \implies a \xi_n \xrightarrow{\alpha} a \xi, (\forall) \alpha \in \mathbb{K}. \)

The pair \((E, F)\) will be called a fuzzy F-normed linear space (shortly FFNL-space).

**Theorem 3** ([2]). If \((E, F)\) is a FFNL-space, then:
1. The topology \( T_F \) is compatible with the vector space structure of \( E \) and hence \( E \) is a topological vector space;
2. \( \{p_n\}_{n \in (0,1)} \) is a right continuous and an ascending family of F-norms on \( E; \)
3. \( E \) is a metrizable topological vector space.

**Definition 5** ([2]). A complete FFNL-space will be called a fuzzy F-space.

**Lemma 1** ([Zabreiko’s Lemma] [10]). Let \( E \) be a metrizable topological linear space of second category and \( \mu : E \to \mathbb{R}_+ \) satisfying the conditions:
1. \( \mu \) is subadditive, namely \( \mu(\xi + \eta) \leq \mu(\xi) + \mu(\eta), (\forall) \xi, \eta \in E; \)
2. \( \lim_{\tau \to 0} \mu(\tau \xi) = 0, (\forall) \xi \in E; \)
3. for any convergent series \( \sum_{n=1}^{\infty} \xi_n \) in \( E \) we have
   \[ \mu \left( \sum_{n=1}^{\infty} \xi_n \right) \leq \sum_{n=1}^{\infty} \mu(\xi_n). \]

Then, \( \mu \) is continuous.
3. Fuzzy Continuous Mappings

Let \((E_1, F_1)\) and \((E_2, F_2)\) be two FFNL-spaces with corresponding families of F-norms \(\{p_\alpha\}_{\alpha \in (0,1)}\), respectively \(\{q_\alpha\}_{\alpha \in (0,1)}\).

**Definition 6.** Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. A mapping \(f : E_1 \to E_2\) is called fuzzy continuous (FC) with respect to \(F_1\) and \(F_2\) if

\[
(\forall) \xi > 0, (\forall) \gamma \in (0,1), (\exists) \theta = \theta(\xi, \gamma, \xi_0) > 0, (\exists) \omega = \omega(\xi, \gamma, \xi_0) < (0,1)
\]

such that \((\forall) \zeta \in E_1 : f_1(\zeta - \xi_0, \theta) > \omega\) we have that \(F_2(f(\xi) - f(\xi_0), \xi) > \gamma\).

**Definition 7.** Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. A mapping \(f : E_1 \to E_2\) is said to be uniformly fuzzy continuous (uniformly FC) with respect to \(F_1\) and \(F_2\) on a subset \(A\) of \(E_1\) if

\[
(\forall) \xi > 0, (\forall) \gamma \in (0,1), (\exists) \theta = \theta(\xi, \gamma) > 0, (\exists) \omega = \omega(\xi, \gamma) < (0,1)
\]

such that \((\forall) \zeta \in A, (\forall) \eta \in E_1 : f_1(\eta - \xi, \theta) > \omega\) we have that \(F_2(f(\eta) - f(\xi), \xi) > \gamma\).

**Example 1.** Let \(E\) be a vector space and \(p\) be a F-norm on \(E\). Then,

\[
F_1(\zeta, \tau) := \begin{cases} \frac{\tau}{\tau + p(\zeta)} & \text{if } \xi \in E, \tau \in \mathbb{R}, \tau > 0 \\ 0 & \text{if } \xi \in E, \tau \in \mathbb{R}, \tau \leq 0 \end{cases}
\]

is a fuzzy F-norm on \(E\) (see [2]). Similarly,

\[
F_2(\zeta, \tau) := \begin{cases} \frac{\tau}{\tau + \frac{1}{2}p(\zeta)} & \text{if } \xi \in E, \tau \in \mathbb{R}, \tau > 0 \\ 0 & \text{if } \xi \in E, \tau \in \mathbb{R}, \tau \leq 0 \end{cases}
\]

is a fuzzy F-norm on \(E\).

Let \(F : E \to E, f(\xi) = \frac{1}{2} \xi\). We will prove that \(f\) is uniformly FC on \(E\) with respect to \(F_1\) and \(F_2\).

Let \(\xi > 0, \gamma \in (0,1)\). We take \(\theta = \xi, \omega = \gamma\). Let \(\xi, \eta \in E : F_1(\eta - \xi, \theta) > \omega\). Thus, \(F_2(\eta - \xi, \xi) > \gamma\). Hence, \(\xi, \omega = \gamma\). As \(\frac{\xi}{2} \eta \left(\frac{1}{2} - \frac{\xi}{2} \zeta\right) \leq p\left(\frac{1}{2} \eta - \frac{1}{2} \zeta\right) \leq p(\eta - \xi),\) we obtain that \(\frac{\xi}{2} \eta \left(\frac{1}{2} - \frac{\xi}{2} \zeta\right) > \gamma\), namely \(\frac{\xi}{2} \eta \left(\frac{1}{2} - \frac{\xi}{2} \zeta\right) > \gamma\). Thus, \(F_2(f(\eta) - f(\xi), \xi) > \gamma\).

**Remark 2.** If \(f\) is uniformly FC on a subset \(A\) of \(E_1\), then \(f\) is FC on \(A\).

**Theorem 4** (Uniform continuity theorem). Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \(A\) be a compact subset of \(E_1\). If \(f : E_1 \to E_2\) is a FC mapping with respect to \(F_1\) and \(F_2\) on \(A\), then \(f\) is uniformly FC with respect to \(F_1\) and \(F_2\) on \(A\).

**Proof.** Let \(\xi > 0\) and \(\alpha \in (0,1)\). As \(f : E_1 \to E_2\) is FC on \(A\), for all \(x \in A\), there exist \(\beta_x = \beta(\xi, \alpha, x) \in (0,1)\) such that

\[
(\forall) y \in E_1 : F_1(y - x, \beta_x) > \beta_x \Rightarrow F_2(f(y) - f(x), \xi) > \alpha.
\]

We can take \(\gamma_x > \beta_x\).

Since \(A\) is compact and \( \{B(x, \gamma_x, \delta_x)\}_{x \in A} \) is an open covering of \(A\), there exist \(z_1, z_2, \ldots, z_n\) in \(A\) such that \(A \subseteq \bigcup_{i=1}^{n} B(z_i, \gamma_i, \delta_i)\). Let \(\beta = \max\{\gamma_{z_i}\}\) and \(\delta = \min\{\delta_i\}\), for \(i \in \{1, 2, \ldots, n\}\).
Let \( x \in A, y \in E_1 \) be arbitrary, such that \( f_1(y - x, \delta) > \beta \). As \( x \in A \), there exists \( i \in \{1, 2, \ldots, n\} \) such that \( x \in B\left(z_i, \gamma_{z_i}, \frac{\delta_{z_i}}{2}\right) \), namely \( f_1\left(z_i - x, \frac{\delta_{z_i}}{2}\right) > \gamma_{z_i} \). Hence,

\[
F_1(z_i - x, \delta_{z_i}) \geq F_1\left(z_i - x, \frac{\delta_{z_i}}{2}\right) > \gamma_{z_i} > \beta_{z_i}.
\]

Thus,

\[
F_2\left(f(z_i) - f(x), \frac{\epsilon}{2}\right) > \alpha.
\]

We note that

\[
F_1(y - z_i, \delta_{z_i}) \geq \min\left\{F_1\left(y - x, \frac{\delta_{z_i}}{2}\right), F_1\left(x - z_i, \frac{\delta_{z_i}}{2}\right)\right\} \geq \min\left\{F_1(y - x, \delta), F_1\left(x - z_i, \frac{\delta_{z_i}}{2}\right)\right\} > \min\{\beta, \gamma_{z_i}\} = \gamma_{z_i} > \beta_{z_i}.
\]

Thus, \( F_2(f(y) - f(z_i), \frac{\xi}{2}) > \alpha \).

Finally,

\[
F_2(f(y) - f(x), \epsilon) \geq \min\left\{F_2\left(f(y) - f(z_i), \frac{\epsilon}{2}\right), F_2\left(f(z_i) - f(x), \frac{\epsilon}{2}\right)\right\} > \alpha.
\]

Therefore, \( f \) is uniformly FC on \( A \). \( \square \)

**Proposition 1.** Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \( T : E_1 \to E_2 \) be a linear operator. Then, \( T \) is FC with respect to \( F_1 \) and \( F_2 \) on \( E_1 \) if and only if \( T \) is FC with respect to \( F_1 \) and \( F_2 \) at a point \( \zeta_0 \in E_1 \).

**Proof.** \( \Rightarrow \) It is clear.

\( \Leftarrow \) Let \( z \in E_1 \) be arbitrary. We will prove that \( T \) is FC at \( z \). Let \( \epsilon > 0, \gamma \in (0, 1) \). As \( T \) is FC at \( \zeta_0 \), there exist \( \omega \in (0, 1), \theta > 0 \) such that

\[
(\forall)\zeta \in E_1 : F_1(\zeta - \zeta_0, \theta) > \omega \Rightarrow F_2(T(\zeta) - T(\zeta_0), \epsilon) > \gamma.
\]

Replacing \( \zeta \) with \( \zeta + \zeta_0 - z \), we have: \( F_1(\zeta - z, \theta) > \omega \Rightarrow F_2(T(\zeta) - T(z), \epsilon) > \gamma \).

Thus, \( T \) is FC at \( z \in E_1 \). \( \square \)

**Corollary 1.** Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \( T : E_1 \to E_2 \) be a linear operator. Then, \( T \) is FC with respect to \( F_1 \) and \( F_2 \) on \( E_1 \) if and only if

\[
(\forall)\epsilon > 0, (\forall)\gamma \in (0, 1), (\exists)\theta = \theta(\epsilon, \gamma) > 0, (\exists)\omega = \omega(\epsilon, \gamma) \in (0, 1)
\]

such that \((\forall)\zeta \in E_1 : F_1(\zeta, \theta) > \omega \) we have that \( F_2(T(\zeta), \epsilon) > \gamma \).

**Remark 3.** We note that a mapping \( f : E_1 \to E_2 \) is said to be continuous at \( x_0 \in E_1 \), if \( (\forall) V \subset E_2 \) an open neighborhood of \( f(x_0) \) we have that \( f^{-1}(V) \) is an open neighborhood of \( x_0 \) in \( E_1 \). If \( f \) is continuous at each point of \( E_1 \), then \( f \) is called continuous on \( E_1 \).

**Theorem 5.** Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \( f : E_1 \to E_2 \). Then \( f \) is FC with respect to \( F_1 \) and \( F_2 \) at \( x_0 \in E_1 \) if and only if \( f \) is continuous at \( x_0 \).

**Proof.** \( \Rightarrow \) Let \( V \subset E_2 \) be an open neighborhood of \( f(x_0) \). It results that there exists \( B(f(x_0), a, \epsilon) \subset V \). As \( f \) is FC at \( x_0 \in E_1 \) we obtain that there exist \( \delta > 0, \beta \in (0, 1) \) such that

\[
(\forall)x \in E_1 : F_1(x - x_0, \delta) > \beta \text{ we have that } F_2(f(x) - f(x_0), \epsilon) > \alpha.
\]
Thus, \( f(B(x_0, \beta, \delta)) \subseteq B(f(x_0), \alpha, \epsilon) \subseteq V \). Hence \( B(x_0, \beta, \delta) \subseteq f^{-1}(V) \), namely \( f^{-1}(V) \) is an open neighborhood of \( x_0 \) in \( E_1 \). Therefore, \( f \) is continuous in \( x_0 \in E_1 \).

\[ \Leftrightarrow \]

Let \( \epsilon > 0, \alpha \in (0, 1) \) and \( B(f(x_0), \alpha, \epsilon) \) be an open neighborhood of \( f(x_0) \) in \( E_2 \). As \( f \) is continuous at \( x_0 \in E_1 \) it result that \( f^{-1}(B(f(x_0), \alpha, \epsilon)) \) is an open neighborhood of \( x_0 \) in \( E_1 \). Thus, \( \exists \delta > 0, \beta \in (0, 1) \) such that \( B(x_0, \beta, \delta) \subseteq f^{-1}(B(f(x_0), \alpha, \epsilon)), \) i.e., \( f(B(x_0, \beta, \delta)) \subseteq B(f(x_0), \alpha, \epsilon) \). Thus, for \( x \in B(x_0, \beta, \delta) \), we have that \( f(x) \in B(f(x_0), \alpha, \epsilon) \), i.e.,

\[ (\forall) x \in E_1 : F_1(x - x_0, \delta) > \beta \Rightarrow F_2(f(x) - f(x_0), \epsilon) > \alpha. \]

Hence, \( f \) is FC at \( x_0 \).

**Proposition 2.** Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \( T : E_1 \to E_2 \) be a linear operator and \( \mu_a : E_1 \to \mathbb{R}, \mu_a(x) = q_a(T(x)) \). Then \( T \) is continuous at \( 0_{E_1} \) if and only if \( \{ \mu_a \}_{a \in (0, 1)} \) are continuous at \( 0_{E_1} \).

**Proof.** \( \Rightarrow \) Let \( \alpha \in (0, 1) \). We prove that \( \mu_a \) is continuous at \( 0_{E_1} \). Let \( \epsilon > 0 \). As \( T \) is continuous at \( 0_{E_1} \) it result that there exist \( \exists \delta > 0, \beta \in (0, 1) \) such that

\[ (\forall) x \in E_1 : F_1(x, \delta) > \beta \Rightarrow F_2(T(x), \epsilon) > \alpha. \]

Thus, for all \( x \in (B(0, \beta, \delta)) \) we have that \( q_a(T(x)) < \epsilon \), namely \( \mu_a(x) < \epsilon \). Hence \( \mu_a \) is continuous at \( 0_{E_1} \).

\( \Leftarrow \) Let \( \epsilon > 0, \alpha \in (0, 1) \). As \( \mu_a \) is continuous at \( 0_{E_1} \), we obtain that there exist \( B(0, \beta, \delta) \) an open neighborhood of \( 0_{E_1} \) such that \( (\forall) x \in B(0, \beta, \delta) \) we have that \( \mu_a(x) < \epsilon \), namely \( q_a(T(x)) < \epsilon \). Thus \( (\forall) x \in E_1 : F_1(x, \delta) > \beta \) we have that \( F_2(T(x), \epsilon) > \alpha \). Therefore \( T \) is continuous at \( 0_{E_1} \).

**Definition 8.** Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. A mapping \( f : E_1 \to E_2 \) is said to be sequentially fuzzy continuous (sequentially FC) with respect to \( F_1 \) and \( F_2 \) at \( \xi_0 \in E_1 \) if

\[ (\forall) (\xi_n)_{n \in \mathbb{N}} \subseteq E_1, \xi_n \to \xi_0 \Rightarrow f(\xi_n) \to f(\xi_0). \]

**Theorem 6.** Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \( f : E_1 \to E_2 \). Then, \( f \) is FC with respect to \( F_1 \) and \( F_2 \) at \( \xi_0 \in E_1 \) if only if \( f \) is sequentially FC with respect to \( F_1 \) and \( F_2 \) at \( \xi_0 \).

**Proof.** \( \Rightarrow \) Let \( (\xi_n)_{n \in \mathbb{N}} \subseteq E_1, \xi_n \to \xi_0 \), i.e., \( \lim_{n \to \infty} F_1(\xi_n - \xi_0, t) = 1, (\forall) t > 0 \).

Let \( \epsilon > 0, \gamma \in (0, 1) \). As \( f \) is FC at \( \xi_0 \in E_1 \), we find that there exist \( \exists \theta > 0, \omega \in (0, 1) \) such that

\[ (\forall) \zeta \in E_1 : F_1(\zeta - \xi_0, \theta) > \omega \text{ we have that } F_2(f(\zeta) - f(\xi_0), \epsilon) > \gamma. \]

As \( \lim_{n \to \infty} F_1(\xi_n - \xi_0, t) = 1, (\forall) t > 0 \), we obtain that, for \( \theta > 0, \omega \in (0, 1), (\exists) \eta_0 \in \mathbb{N} : F_1(\xi_n - \xi_0, \theta) > \omega, (\forall) n \geq n_0 \). Thus \( F_2(f(\xi_n) - f(\xi_0), \epsilon) > \gamma, (\forall) n \geq n_0 \). Therefore, \( f(\xi_n) \to f(\xi_0) \).

\( \Leftarrow \) We suppose that \( f \) is not FC at \( \xi_0 \in E_1 \). Thus,

\[ (\exists) \epsilon > 0, (\exists) \gamma \in (0, 1) : (\forall) \theta > 0, (\forall) \omega \in (0, 1), (\exists) \xi_{\theta \omega} \in E_1 : F_1(\xi_{\theta \omega} - x_0, \theta) > \omega \text{ and } F_2(f(\xi_{\theta \omega}) - f(\xi_0), \epsilon) \leq \gamma. \]

In particular, for \( \theta = \frac{1}{n}, \omega = 1 - \frac{1}{n}, n \geq 2, (\exists) \xi_n \in E_1 : F_1(\xi_n - \xi_0, \frac{1}{n}) > 1 - \frac{1}{n} F_2(f(\xi_n) - f(\xi_0), \epsilon) \leq \gamma. \)
Then, $T$ is strongly FC with respect to $F$.

Let $\epsilon > 0$. If $\xi_0$ is an arbitrary point in $E$, we find that there exists $\theta > 0$ such that $F_2(f(\xi_n) - f(\xi_0), \epsilon) > F_1(\xi_n - \xi_0, \theta, (\forall)\xi \in E_1)$. We take $\omega = \gamma$. Thus,

$$(\forall)\xi \in E_1 : F_1(\xi_n - \xi_0, \theta) > \omega \Rightarrow \text{we have that } F_2(f(\xi_n) - f(\xi_0), \epsilon) > \gamma.$$ 

Hence, $f$ is FC at $\xi_0 \in E_1$.  

**Proposition 3.** Let $(E_1, F_1), (E_2, F_2)$ be two FFNL-spaces. Let $T : E_1 \rightarrow E_2$ be a linear operator. Then, $T$ is strongly FC with respect to $F_1$ and $F_2$ on $E_1$ if and only if $T$ is strongly FC with respect to $F_1$ and $F_2$ at a point $\xi_0 \in E_1$.

**Proof.** "⇒" It is clear.

"⇐" Let $z \in E_1$ be arbitrary. As $T$ is strongly FC at $\xi_0 \in E_1$, for $\epsilon > 0$:

$$(\exists)\theta = \theta(\epsilon, \xi_0) > 0 \text{ such that } F_2(T(\xi_n) - T(\xi_0), \epsilon) \geq F_1(\xi_n - \xi_0, \theta, (\forall)\xi \in E_1).$$

Replacing $\xi$ with $z + \xi_0 - z$, we have: $F_2(T(\xi_0) - T(z), \epsilon) \geq F_1(\xi - z, \theta, (\forall)\xi \in E_1)$. Thus, $T$ is strongly FC at $z \in E_1$.  

**Corollary 2.** Let $(E_1, F_1), (E_2, F_2)$ be two FFNL-spaces. Let $T : E_1 \rightarrow E_2$ be a linear operator. Then, $T$ is strongly FC with respect to $F_1$ and $F_2$ on $E_1$ if and only if

$$(\forall)\epsilon > 0, (\exists)\theta > 0 : F_2(T(\xi), \epsilon) \geq F_1(\xi, \theta, (\forall)\xi \in E_1).$$

**Example 2.** Clearly the mapping $T : E \rightarrow E, T(\xi) = \frac{1}{4} \xi$ by Example 1 is a linear operator. We will prove that $T$ is strongly FC on $E$. Let $\epsilon > 0$. We take $\theta = \epsilon$. Therefore,

$$F_2(T(\xi), \epsilon) = \frac{\epsilon}{\epsilon + \frac{1}{4} p(T(\xi))} = \frac{\epsilon}{\epsilon + \frac{1}{5} p(\xi)}.$$ 

As $\frac{1}{5} p(\xi) \leq p(\frac{1}{4} \xi) \leq p(\xi)$, we obtain

$$F_2(T(\xi), \epsilon) \geq \frac{\epsilon}{\epsilon + p(\xi)} = F_1(\xi, \epsilon).$$

Hence, $T$ is strongly FC on $E$. 

Theorem 8. Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. A mapping \(f : E_1 \to E_2\) is said to be weakly fuzzy continuous (weakly FC) with respect to \(F_1\) and \(F_2\) at \(\zeta_0 \in E_1\) if \((\forall)\epsilon > 0, (\forall)\gamma \in (0,1), (\exists)\theta = \theta(\epsilon, \gamma, \zeta_0) > 0\) such that
\[
F_1(\zeta - \zeta_0, \theta) > \gamma \Rightarrow F_2(f(\zeta) - f(\zeta_0), \epsilon) > \gamma.
\]

Proposition 4. Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \(T : E_1 \to E_2\) be a linear operator. Then, \(T\) is weakly FC with respect to \(F_1\) and \(F_2\) on \(E_1\) if and only if \(T\) is weakly FC with respect to \(F_1\) and \(F_2\) at a point \(\zeta_0 \in E_1\).

Proof. "\(\Rightarrow\)" It is clear.

"\(\Leftarrow\)" Let \(z \in E_1\) be arbitrary. We will prove that \(T\) is weakly FC at \(z\). Let \(\epsilon > 0, \gamma \in (0,1)\). As \(T\) is weakly FC at \(\zeta_0\), there exists \(\theta > 0\) such that
\[
F_1(\zeta - \zeta_0, \theta) > \gamma \Rightarrow F_2(T(\zeta) - T(\zeta_0), \epsilon) > \gamma.
\]
Replacing \(\zeta\) with \(\zeta + \zeta_0 - z\), we have:
\[
F_1(\zeta - z, \theta) > \gamma \Rightarrow F_2(T(\zeta) - T(z), \epsilon) > \gamma.
\]
Thus, \(T\) is weakly FC at \(z \in E_1\). \(\square\)

Corollary 3. Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \(T : E_1 \to E_2\) be a linear operator. Then, \(T\) is weakly FC with respect to \(F_1\) and \(F_2\) on \(E_1\) if and only if

\[
(\forall)\epsilon > 0, (\forall)\gamma \in (0,1), (\exists)\theta = \theta(\epsilon, \gamma) > 0,
\]

such that \((\forall)\zeta \in E_1 : F_1(\zeta, \theta) > \gamma\) we have that \(F_2(T(\zeta), \epsilon) > \gamma\).

Theorem 8. Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \(f : E_1 \to E_2\). If \(f\) is weakly FC with respect to \(F_1\) and \(F_2\) at a point \(\zeta_0 \in E_1\), then \(f\) is FC with respect to \(F_1\) and \(F_2\) at \(\zeta_0 \in E_1\).

Proof. Let \(\epsilon > 0, \gamma \in (0,1)\). As \(f\) is weakly FC at \(\zeta_0 \in E_1\), it results that there exists \(\theta > 0\) such that \(F_1(\zeta - \zeta_0, \theta) > \gamma \Rightarrow F_2(f(\zeta) - f(\zeta_0), \epsilon) > \gamma\). We take \(\omega = \gamma\). Thus,
\[
(\forall)\zeta \in E_1 : F_1(\zeta - \zeta_0, \theta) > \omega \text{ we have that } F_2(f(\zeta) - f(\zeta_0), \epsilon) > \gamma.
\]

Hence, \(f\) is FC at \(\zeta_0 \in E_1\). \(\square\)

Theorem 9. Let \((E_1, F_1), (E_2, F_2)\) be two FFNL-spaces. Let \(f : E_1 \to E_2\). If \(f\) is strongly FC with respect to \(F_1\) and \(F_2\) at a point \(\zeta_0 \in E_1\), then \(f\) is weakly FC with respect to \(F_1\) and \(F_2\) at \(\zeta_0 \in E_1\).

Proof. Let \(\epsilon > 0, \gamma \in (0,1)\). As \(f\) is strongly FC at \(\zeta_0 \in E_1\), it results that there exists \(\theta > 0\) such that \(F_2(f(\zeta) - f(\zeta_0), \epsilon) \geq F_1(\zeta - \zeta_0, \theta)\). If \(F_1(\zeta - \zeta_0, \theta) > \gamma\), then \(F_2(f(\zeta) - f(\zeta_0), \epsilon) > \gamma\). Hence, \(f\) is weakly FC at \(\zeta_0 \in E_1\). \(\square\)

4. Principles of Fuzzy Functional Analysis

Based on the fact that fuzzy F-spaces are metrizable topological linear spaces and since fuzzy continuity and topological continuity are equivalent, we obtain the closed graph theorem and the open mapping theorem in this context. However, for the beauty of the proof, I made the proof for the next theorem.

Theorem 10 (Closed graph theorem). Let \((E_1, F_1), (E_2, F_2)\) be two fuzzy F-spaces and \(T : E_1 \to E_2\) be a linear operator. Then, \(T\) is FC on \(E_1\) if and only if \(G_T := \{(x, T(x)) : x \in E_1\}\) is closed in \(E_1 \times E_2\).

Proof. "\(\Rightarrow\)" Let \((x_0, y_0) \in G_T\). Let \((x_n, T(x_n))\) be a convergent sequence to \((x_0, y_0)\). Thus, \(x_n \to x_0\) and \(T(x_n) \to y_0\). As \(T\) is FC at \(x_0\), we have that \(T\) is sequentially FC at \(x_0\). Hence, \(T(x_n) \to T(x_0)\). Thus, \(T(x_0) = y_0\). Therefore, \((x_0, y_0) \in G_T\) and \(G_T\) is closed.
Let $\alpha \in (0, 1)$. We define $\mu_\alpha : E_1 \rightarrow \mathbb{R}_+$ by $\mu_\alpha(x) = q_\alpha(T(x))$. We prove that $\mu_\alpha$ satisfies the assumptions from Zabreiko’s Lemma:

1. $\mu_\alpha$ is subadditive. Indeed,

$$
\mu_\alpha(x + y) = q_\alpha(T(x + y)) = q_\alpha(T(x) + T(y)) \leq q_\alpha(T(x)) + q_\alpha(T(y)) = \mu_\alpha(x) + \mu_\alpha(y).
$$

2. 

$$
\lim_{t \to 0} \mu_\alpha(tx) = \lim_{t \to 0} q_\alpha(T(tx)) = \lim_{t \to 0} q_\alpha(tT(x)) = 0,
$$

by (PN5).

3. Let \( \sum x_n = x \) be a convergent series in \( E_1 \). If \( \sum_{n=1}^{\infty} \mu_\alpha(x_n) = \infty \), then \( \mu_\alpha \left( \sum_{n=1}^{\infty} x_n \right) \leq \sum_{n=1}^{\infty} \mu_\alpha(x_n) \). We assume that \( \sum_{n=1}^{\infty} \mu_\alpha(x_n) < \infty \). Hence, \( \sum_{n=1}^{\infty} q_\alpha(T(x_n)) < \infty \). Thus, \( \left\{ \sum_{n=1}^{\infty} T(x_n) \right\}_k \) is a Cauchy sequence in \( E_2 \). As \( E_2 \) is complete, we have that \( \left\{ \sum_{n=1}^{\infty} T(x_n) \right\}_k \) is a convergent sequence in \( E_2 \). Let \( y = \lim_{k \to \infty} \sum_{n=1}^{k} T(x_n) \). As \( x = \lim_{k \to \infty} \sum_{n=1}^{k} x_n \), we obtain that \( (x, y) \in \overline{GR} \). Thus, \( y = T(x) \). Hence, \( \mu_\alpha \left( \sum_{n=1}^{\infty} x_n \right) = q_\alpha \left( T \left( \sum_{n=1}^{\infty} x_n \right) \right) = q_\alpha(y) = 

$$
= \lim_{k \to \infty} q_\alpha \left( \sum_{n=1}^{k} T(x_n) \right) \leq \lim_{k \to \infty} \sum_{n=1}^{k} q_\alpha(T(x_n)) = \sum_{n=1}^{\infty} q_\alpha(T(x_n)) = \sum_{n=1}^{\infty} \mu_\alpha(x_n) \). Thus,

$$
\mu_\alpha \left( \sum_{n=1}^{\infty} x_n \right) \leq \sum_{n=1}^{\infty} \mu_\alpha(x_n).
$$

As $\mu_\alpha$ satisfies the hypotheses of Zabreiko’s lemma, we find that $\mu_\alpha$ is continuous. As \( \{ \mu_\alpha \}_{\alpha \in (0, 1)} \) are continuous at \( 0_{E_1} \), using Proposition 2, it results that $T$ is continuous at \( 0_{E_1} \). Applying Proposition 1, we have that $T$ is continuous on \( E_1 \). \( \square \)

**Theorem 11** (Bounded inverse theorem). Let \( (E_1, F_1), (E_2, F_2) \) be two fuzzy F-spaces. If \( T : E_1 \rightarrow E_2 \) is a bijective FC linear operator, then \( T^{-1} \) is FC.

**Theorem 12** (Open mapping theorem). Let \( (E_1, F_1), (E_2, F_2) \) be two fuzzy F-spaces. If \( T : E_1 \rightarrow E_2 \) is a surjective FC linear operator, then \( T \) is an open mapping.

In the case of the following two theorems, a certain type of uniform boundedness is involved, characteristic of fuzzy F-normed linear spaces and for this reason we will prove them. We will use a result obtained by Zabreiko [10], which, even in the classic case, proved to be useful in order to demonstrate many important theorems in functional analysis.

**Theorem 13** (Uniform bounded principle). Let \( (E_1, F_1) \) be a fuzzy F-spaces, \( (E_2, F_2) \) be a fuzzy F-normed linear spaces and \( \{ T_\gamma \}_{\gamma \in \Gamma} \) be a family of FC linear operators from \( E_1 \) to \( E_2 \). If \( \forall \gamma \in \Gamma \), \( \forall \alpha \in (0, 1) \) we have that \( \sup_{\gamma \in \Gamma} q_\alpha(T_\gamma(x)) < \infty \), then \( \lim_{x \to 0} T_\gamma(x) = 0 \) and the convergence is uniform for \( \gamma \in \Gamma \).

**Proof.** Let \( \alpha \in (0, 1) \) and \( \mu_\alpha(x) = \sup_{\gamma \in \Gamma} q_\alpha(T_\gamma(x)) \). We will prove that $\mu_\alpha$ satisfies the hypotheses of Zabreiko’s lemma:

1. It is obvious that $\mu_\alpha$ is subadditive;
2. Let \( t_n \to 0 \) and \( x \in E_1 \). Then \( T_\gamma(t_n x) = t_n T_\gamma(x) \). Applying (PN5), we obtain \( q_\alpha(t_n T_\gamma(x)) \to 0 \). Hence, \( \mu_\alpha(t_n x) \to 0 \).
3. Let \( \sum_{n=1}^{\infty} x_n \) be an convergent series in \( E_1 \). Let \( x = \sum_{n=1}^{\infty} x_n \). As \( T_{\gamma} \) is linear and an FC operator, we have that \( T_{\gamma}(x) = \sum_{n=1}^{\infty} T_{\gamma}(x_n) \). If \( \sum_{n=1}^{\infty} \mu_a(x_n) = \infty \), the desired inequality takes place. If \( \sum_{n=1}^{\infty} \mu_a(x_n) < \infty \), then \( \sum_{n=1}^{\infty} q_a(T_{\gamma}(x_n)) \leq \sum_{n=1}^{\infty} \mu_a(x_n) < \infty \). Hence,

\[
\mu_a(x) = \mu_a\left(\sum_{n=1}^{\infty} x_n\right) = \sup_{\gamma \in \Gamma} q_a\left(T_{\gamma}\left(\sum_{n=1}^{\infty} x_n\right)\right) = \sup_{\gamma \in \Gamma} \sum_{n=1}^{\infty} q_a(T_{\gamma}(x_n)) \leq \sum_{n=1}^{\infty} \mu_a(x_n).
\]

By Zabreiko’s lemma, we obtain that \( \mu_a \) is continuous in \( 0 \). Hence, \( \lim_{x \to 0} q_a(T_{\gamma}(x)) = 0 \), uniform for \( \gamma \in \Gamma \). Thus, \( \lim_{x \to 0} q_a(T_{\gamma}(x)) = 0 \), uniform for \( \gamma \in \Gamma \), \( (\forall) x \in (0, 1) \). Therefore, \( \lim_{x \to 0} T_{\gamma}(x) = 0 \), uniform for \( \gamma \in \Gamma \).

**Theorem 14** (Banach–Steinhaus). Let \( (E_1, F_1) \) be a fuzzy F-space, \( (E_2, F_2) \) be a fuzzy F-normed linear space and \( T_n : E_1 \to E_2 \) be FC linear operators. If

1. \( (\forall) x \in E_1, (\forall) \alpha \in (0, 1), \text{ we have } \sup_{n \in \mathbb{N}} q_a(T_n(x)) < \infty; \)
2. \( (\exists) M \text{ a dense subset of } E_1 \text{ such that } \{ T_n(x) \} \text{ is a Cauchy sequence, for all } x \in M \).

Then:

1. \( \{ T_n(x) \} \) is a Cauchy sequence, for all \( x \in E_1 \);
2. If \( (E_2, F_2) \) is a fuzzy F-space and \( T \) is defined by \( T(x) = \lim_{n \to \infty} T_n(x) \), then \( T \) is a FC linear operator.

**Proof.** (1). Let \( x \in E_1 \). As \( M \) is dense in \( E_1 \), there exist \( (x_k) \subset M : x_k \to x \). As \( x_k \to x \), by previous theorem we have that \( \lim_{k \to \infty} T_n(x_k - x) = 0 \), uniform for \( n \in \mathbb{N} \). Hence,

\[
(\forall) r \in (0, 1), (\forall) t > 0, (\exists) k_0 = k_0(r, t) : q_r(T_n(x_k - x)) < \frac{t}{3}, (\forall) k \geq k_0.
\]

As \( \{ T_n(x_k) \}_{k \in \mathbb{N}} \) is a Cauchy sequence, we have that

\[
(\exists) n_0 \in \mathbb{N} : q_r(T_n(x_k) - T_m(x_k)) < \frac{t}{3}, (\forall) n \geq n_0.
\]

Thus,

\[
q_r(T_n(x) - T_m(x)) \leq q_r(T_n(x) - T_m(x_k)) + q_r(T_m(x_k) - T_m(x)) < t, (\forall) n, m \geq n_0.
\]

Hence, \( \{ T_n(x) \} \) is a Cauchy sequence, for all \( x \in E_1 \).

(2) As \( E_1 \) is complete, we have that \( \{ T_n(x) \} \) is convergent. Let \( T \) defined by \( T(x) = \lim_{n \to \infty} T_n(x) \). It is obvious that \( T \) is linear. If \( x_k \to 0 \), then \( T_n(x_k) \to 0 \), uniform by \( n \in \mathbb{N} \). Hence \( T(x_k) \to 0 \). Therefore \( T \) is continuous in \( 0_{E_1} \). Thus \( T \) is continuous on \( E_1 \).

5. Conclusions and Further Works

In this paper, first, we introduced different types of continuity in fuzzy F-normed linear spaces and we established the relationships between them. Then, we obtained the principles of fuzzy functional analysis in the context of fuzzy F-spaces.

The present study will be followed by a detailed analysis of different types of boundedness in fuzzy F-normed spaces. Such concepts have been introduced by several authors
in fuzzy normed linear spaces (see [13]). Further on, motivated by the present papers in the context of probabilistic normed spaces (see [14]), of those in the context of fuzzy normed linear space (see [7,9,15,16]) we will study various types of boundedness for linear operator between fuzzy F-spaces.

As is only natural, the first fixed point theorems in fuzzy context were established on fuzzy metric spaces. Interesting results were also obtained in the context of fuzzy normed linear spaces (see [17]). In the framework of extended rectangular fuzzy b-metric space, in the paper [18], the authors introduced the concept of Ćirić-type fuzzy contraction and obtained fixed point theorems for these fuzzy contractions. In the paper [19], the generalization properties of contractive conditions of Ćirić-type in fuzzy metric space were investigated. In the paper [20], some new fixed point results for nonlinear fuzzy set-valued θ-contractions in the context of metric-like spaces are introduced. Our aim is to obtain in future papers fixed point theorems in fuzzy F-normed linear spaces.

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