Application of Mixed Generalized Quasi-Einstein Spacetimes in General Relativity

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Abstract: In the present article, some geometric and physical properties of $MG(QE)_n$ were investigated. Moreover, general relativistic viscous fluid $MG(QE)_4$ spacetimes with some physical applications were studied. Finally, through a non-trivial example of $MG(QE)_4$ spacetime, we proved its existence.

Keywords: Einstein manifold; mixed generalized quasi-Einstein manifold; Einstein’s field equation; energy–momentum tensor; general relativistic viscous fluid

MSC: 53C25; 53Z05

1. Introduction

A Riemannian or a semi-Riemannian manifold $(M^n, g)$ of dimension $n(> 2)$ is termed as an Einstein manifold if its $(0, 2)$-type Ricci tensor $Ric(\neq 0)$ satisfies $Ric = \frac{1}{n} g$, where $r$ stands for the scalar curvature [1]. In addition to Riemannian geometry, Einstein manifolds also have a vital contribution to the general theory of relativity (GTR).

Approximately two decades ago, Chaki and Maity introduced and studied quasi-Einstein manifolds [2]. An $(M^n, g), (n > 2)$ is said to be a quasi-Einstein manifold $(QE)_n$ if its $Ric(\neq 0)$ realizes the following condition:

$$Ric(U_1, U_2) = ag(U_1, U_2) + bA(U_1)A(U_2), \quad (1)$$

where $a, b \in \mathbb{R}$ such that $b \neq 0$ and $A(\neq 0)$ is the 1-form such that

$$g(U_1, \rho) = A(U_1), \quad g(\rho, \rho) = A(\rho) = 1, \quad (2)$$

for any vector field $U_1$, and a unit vector field $\rho$ called the generator of $(M^n, g)$. In addition, $A$ is named the associated 1-form. Einstein manifolds form a natural subclass of the class of $(QE)_n$.

Under the study of exact solutions of the Einstein field equations, as well as under the consideration of quasi-umbilical hypersurfaces of semi-Euclidean spaces, $(QE)_n$ came into existence. For instance, the Robertson–Walker spacetimes are $(QE)_n$. Thus, $(QE)_n$ have great importance in GTR.

An $(M^n, g), (n \geq 2)$ is said to be a generalized quasi-Einstein manifold $G(QE)_n$ [3] if its $Ric(\neq 0)$ realizes the following condition:

$$Ric(U_1, U_2) = ag(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2), \quad (3)$$
where $a, b, c$ are non-zero scalars and $A, B$ are two non-zero 1-forms such that

$$g(U_1, \rho) = A(U_1), \quad g(U_1, \sigma) = B(U_1),$$

(4)

where $\rho$ and $\sigma$ are mutually orthogonal unit vector fields, i.e., $g(\rho, \sigma) = 0$. The vector fields $\rho$ and $\sigma$ are called the generators of the manifold. If $c = 0$, then the manifold reduces to a quasi-Einstein manifold.

In 2007, Bhattacharya, De and Debnath [4] introduced the notion of a mixed generalized quasi-Einstein manifold. A non-flat Riemannian or semi-Riemannian manifold is said to be of constant curvature $k$ if the manifold is said to be of constant curvature $k$, then the manifold is said to be of constant curvature $k$.

Further, we know that if the Riemannian curvature tensor $\bar{R}$ of type $(0, 4)$ has the form

$$\bar{R}(U_1, U_2, U_3, U_4) = kg(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4),$$

(9)

then the manifold is said to be of constant curvature $k$. The generalization of this manifold is the manifold of quasi-constant curvature and, in this case, the curvature tensor has the following form:

$$\bar{R}(U_1, U_2, U_3, U_4) = f_1[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)]$$

$$+ f_2[g(U_2, U_3)A(U_1)A(U_4) - g(U_2, U_4)A(U_1)A(U_3)]$$

$$+ g(U_1, U_4)A(U_2)A(U_3) - g(U_1, U_3)A(U_2)A(U_4)],$$

(10)

where $g(K(U_1, U_2)U_3, U_4) = \bar{R}(U_1, U_2, U_3, U_4), K$ is the curvature tensor of type $(1, 3)$ and $f_1, f_2$ are scalars, and $\rho$ is a unit vector field defined by

$$g(U_1, \rho) = A(U_1).$$

It can be easily seen that, if the curvature tensor $\bar{R}$ is of the form (10), then the manifold is conformally flat [3]. Thus, a Riemannian or semi-Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor $\bar{R}$ satisfies the relation (10); we denote such a manifold of dimension $n$ by $(QC)_n$. 

A non-flat Riemannian or semi-Riemannian manifold \((M^n, g) \ (n \geq 3)\) is said to be a manifold of generalized quasi-constant curvature if the curvature tensor \(\mathcal{K}\) of type \((0, 4)\) satisfies the condition \([3]\)

\[
\mathcal{K}(U_1, U_2, U_3, U_4) = f_1[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)]
+ f_2[g(U_1, U_4)A(U_2)A(U_3) - g(U_2, U_4)A(U_1)A(U_3)]
+ g(U_2, U_3)A(U_1)A(U_4) - g(U_1, U_3)A(U_2)A(U_4)
+ f_3[g(U_1, U_4)B(U_2)B(U_3) - g(U_2, U_4)B(U_1)B(U_3)]
+ g(U_2, U_3)B(U_1)B(U_4) - g(U_1, U_3)B(U_2)B(U_4)].
\]

where \(f_1, f_2, f_3\) are scalars and \(A, B\) are two non-zero 1-forms. \(\rho\) and \(\sigma\) are orthonormal unit vectors corresponding to \(A\) and \(B\) such that \(g(U_1, \rho) = A(X), \ g(U_1, \sigma) = B(X)\) and \(g(\rho, \sigma) = 0\). Such a manifold is denoted by \(G(QC)_n\).

In [9], Bhattacharya and De introduced the notion of mixed generalized quasi-constant curvature. A non-flat Riemannian or semi-Riemannian manifold \((M^n, g) \ (n \geq 3)\) is said to be a manifold of mixed generalized quasi-constant curvature if the curvature tensor \(\mathcal{K}\) of type \((0, 4)\) satisfies the condition

\[
\mathcal{K}(U_1, U_2, U_3, U_4) = f_1[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)]
+ f_2[g(U_1, U_4)A(U_2)A(U_3) - g(U_2, U_4)A(U_1)A(U_3)]
+ g(U_2, U_3)A(U_1)A(U_4) - g(U_1, U_3)A(U_2)A(U_4)
+ f_3[g(U_1, U_4)B(U_2)B(U_3) - g(U_2, U_4)B(U_1)B(U_3)]
+ g(U_2, U_3)B(U_1)B(U_4) - g(U_1, U_3)B(U_2)B(U_4)].
\]

where \(f_1, f_2, f_3, f_4\) are scalars. \(A, B\) are two non-zero 1-forms. \(\rho\) and \(\sigma\) are orthonormal unit vectors corresponding to \(A\) and \(B\) such that \(g(U_1, \rho) = A(X), \ g(U_1, \sigma) = B(X)\) and \(g(\rho, \sigma) = 0\). Such a manifold is denoted by \(MG(QC)_n\).

The spacetime of general relativity and cosmology is regarded as a connected four-dimensional semi-Riemannian manifold \((M^4, g)\) with Lorentzian metric \(g\) with signature \((- , + , + , + )\). The geometry of the Lorentz manifold begins with the study of a causal character of vectors of the manifold. Due to this causality, the Lorentz manifold becomes a convenient choice for the study of general relativity. Spacetimes have been studied by various authors in several ways, such as [10–14] and many others.

2. **\(MG(QE)_n\)** Admitting the Generators \(\rho\) and \(\sigma\) as Recurrent Vector Fields

Let us consider the generators \(\rho\) and \(\sigma\) corresponding to the associated recurrent 1-forms \(A\) and \(B\). Then, we have

\[
(DU_1 A)(U_2) = \eta(U_1)A(U_2),
\]

\[
(DU_1 B)(U_2) = \varphi(U_1)B(U_2),
\]

where \(\eta\) and \(\varphi\) are non-zero 1-forms.

A non-flat Riemannian or semi-Riemannian manifold \((M^n, g) \ (n > 2)\) is said to be Ricci-recurrent \([15,16]\) if its \(Ric(\neq 0)\) satisfies the following condition:

\[
(DU_1 Ric)(U_2, U_3) = a(U_1)Ric(U_2, U_3),
\]

(13)
where $\alpha$ is a non-zero 1-form. Since we know that
\[(D_{U_1}Ric)(U_2, U_3) = U_1Ric(U_2, U_3) - Ric(D_{U_1}U_2, U_3) - Ric(U_2, D_{U_1}U_3), \tag{14}\]
using (14) in (13), it follows that
\[\alpha(U_1)Ric(U_2, U_3) = U_1Ric(U_2, U_3) - Ric(D_{U_1}U_2, U_3) - Ric(U_2, D_{U_1}U_3). \tag{15}\]

Using (5) in (15), we obtain
\[
\begin{align*}
\alpha(U_1)[ag(U_2, U_3) + bA(U_2)A(U_3) + cB(U_2)B(U_3) &\]
\ + d\{A(U_2)B(U_3) + A(U_3)B(U_2)\} = U_1[ag(U_2, U_3) + bA(U_2)A(U_3)
\ + cB(U_2)B(U_3) + d\{A(U_3)B(U_2) + A(U_2)B(U_3)\}]
\ - [ag(D_{U_1}U_2, U_3) + bA(D_{U_1}U_2)A(U_3) + cB(D_{U_1}U_2)B(U_3)
\ + d\{A(D_{U_1}U_3)B(U_3) + A(U_3)B(D_{U_1}U_2)\}]
\ - [ag(U_2, D_{U_1}U_3) + bA(U_2)A(D_{U_1}U_3) + cB(U_2)B(D_{U_1}U_3)
\ + d\{A(U_2)B(D_{U_1}U_3) + A(D_{U_1}U_3)B(U_2)\}]. \tag{16}\end{align*}
\]

Putting $U_2 = U_3 = \rho$ in (16), we obtain
\[
U_1(a + b) - \alpha(U_1)(a + b) = 2(a + b)A(D_{U_1}\rho) + 2dB(D_{U_1}\rho). \tag{17}\]
By using the fact that $A(D_{U_1}\rho) = 0$ and (6) in (17), we have
\[
U_1(a + b) - \alpha(U_1)(a + b) = 2dg(D_{U_1}\rho, \sigma), \tag{18}\]
which can be written as
\[
U_1(a + b) - \alpha(U_1)(a + b) = -2dA(D_{U_1}\sigma). \]

Thus, we have $A(D_{U_1}\sigma) = 0$ if and only if $U_1(a + b) - \alpha(U_1)(a + b) = 0$. This implies that either $D_{U_1}\sigma \perp \rho$ or $\sigma$ is a parallel vector field.

Again, putting $U_2 = U_3 = \sigma$ in (16), we have
\[
U_1(a + b) - \alpha(U_1)(a + b) = 2(a + c)B(D_{U_1}\rho) + 2dA(D_{U_1}\sigma). \tag{19}\]

Again, using the fact that $B(D_{U_1}\sigma) = 0$ and (6) in (19), we have
\[
U_1(a + b) - \alpha(U_1)(a + b) = 2dg(D_{U_1}\sigma, \rho), \tag{20}\]
or
\[
U_1(a + b) - \alpha(U_1)(a + b) = -2dB(D_{U_1}\rho). \]
Thus, we have $B(D_{U_1}\rho) = 0$ if and only if $U_1(a + b) - \alpha(U_1)(a + b) = 0$. This implies that either $D_{U_1}\rho \perp \sigma$ or $\rho$ is a parallel vector field. Hence, we can state the following theorem:

**Theorem 1.** Let a mixed generalized quasi-Einstein manifold $\text{MG(QE)}_\alpha$ be Ricci-recurrent; then, the following statements are equivalent:
(i) $\rho$ and $\sigma$ are parallel vector fields;
(ii) $U_1(a + b) - \alpha(U_1)(a + b) = 0$ if and only if $D_{U_1}\sigma \perp \rho$;
(iii) $U_1(a + b) - \alpha(U_1)(a + b) = 0$ if and only if $D_{U_1}\rho \perp \sigma$. 

3. **$MG(QE)_n$ Admitting the Generators $\rho$ and $\sigma$ as Concurrent Vector Fields**

A vector field $\pi$ is said to be concurrent if it satisfies the following condition [17,18]:

$$D_{U_1} \pi = \xi U_1,$$

where $\xi$ is constant.

Let us consider the generators $\rho$ and $\sigma$ corresponding to the associated concurrent 1-forms $A$ and $B$. Then, we have

$$(D_{U_1} A)(U_2) = \lambda g(U_1, U_2),$$

and

$$(D_{U_1} B)(U_2) = \mu g(U_1, U_2),$$

where $\lambda$ and $\mu$ are non-zero constants.

Taking the covariant derivative of (5) with respect to $U_3$, we obtain

$$(D_{U_1} Ric)U_2 = b[(D_{U_1} A)(U_1) A(U_2) + A(U_1)(D_{U_1} A)(U_2)]$$

$$+ c[(D_{U_1} B)(U_1) B(U_2) + B(U_1)(D_{U_1} B)(U_2)]$$

$$+ d[(D_{U_1} A)(U_1) B(U_2) + A(U_1)(D_{U_1} B)(U_2)]$$

$$+ (D_{U_1} B)(U_1) A(U_2) + B(U_1)(D_{U_1} A)(U_2)].$$

Using (22) and (23) in (24), it follows that

$$(D_{U_1} Ric)(U_1, U_2) = b[\lambda g(U_1, U_3) A(U_2) + \lambda g(U_2, U_3) A(U_1)]$$

$$+ c[\mu g(U_1, U_3) B(U_2) + \mu g(U_2, U_3) B(U_1)]$$

$$+ d[\lambda g(U_1, U_3) B(U_2) + \mu g(U_1, U_3) A(U_2)]$$

$$+ \lambda g(U_2, U_3) B(U_1) + \mu g(U_2, U_3) A(U_1)].$$

Contracting (25) over $U_1$ and $U_2$ leads to

$$\partial r(U_3) = A(U_3)[2b\lambda + 2d\mu] + B(U_3)[2c\mu + 2d\lambda].$$

From (7), it follows that

$$\partial r(U_1) = 0.$$  

(27)

In view of (27), (26) turns to

$$A(U_3)[2b\lambda + 2d\mu] + B(U_3)[2c\mu + 2d\lambda] = 0.$$  

(28)

Thus, by virtue of (28), (5) takes the form

$$Ric(U_1, U_2) = ag(U_1, U_2) + \left[ b + c \left( \frac{b\lambda + d\mu}{c\mu + d\lambda} \right)^2 - 2d \left( \frac{b\lambda + d\mu}{c\mu + d\lambda} \right) \right] A(U_1) A(U_2)$$

(29)

which is a quasi-Einstein manifold. Thus, we can state the following theorem:

**Theorem 2.** Let $MG(QE)_n$ be a mixed generalized quasi-Einstein manifold. If the associated vector fields of $MG(QE)_n$ are concurrent and the associated scalars are constants, then the manifold reduces to a quasi-Einstein manifold.

4. **$MG(QE)_n$ Admitting Einstein’s Field Equations**

The Einstein’s field equations with and without cosmological constants are given by

$$Ric(U_1, U_2) - \frac{r}{2} g(U_1, U_2) + \lambda g(U_1, U_2) = \kappa T(U_1, U_2),$$

(30)
and
\[ \text{Ric}(U_1, U_2) - \frac{r}{2}g(U_1, U_2) = \kappa T(U_1, U_2), \] (31)
respectively; \( \kappa \) is a gravitational constant, \( \lambda \) is a cosmological constant, and \( T \) is the energy–momentum tensor.

Using (6) in (31), it follows that
\[ \left( a - \frac{r}{2} \right)g(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2) \]
\[ + d[A(U_1)B(U_2) + A(U_2)B(U_1)] = \kappa T(U_1, U_2). \] (32)

Now, taking the covariant derivative of (32) with respect to \( U_3 \), we arrive at
\[ b[[D_{U_3}A](U_1)A(U_2) + A(U_1)(D_{U_3}A)(U_2)] \]
\[ + c[[D_{U_3}B](U_1)B(U_2) + B(U_1)(D_{U_3}B)(U_2)] \]
\[ + d[[D_{U_3}A](U_1)B(U_2) + A(U_1)(D_{U_3}B)(U_2) \]
\[ + (D_{U_3}B)(U_1)A(U_2) + B(U_1)(D_{U_3}A)(U_2)] = \kappa (D_{U_3}T)(U_1, U_2). \] (33)

Thus, we have a result.

**Theorem 3.** Let \( \text{MG}(QE)_n \) admit Einstein’s field equation without a cosmological constant. If the associated 1-forms \( A \) and \( B \) are covariantly constant, then the energy–momentum tensor is also covariantly constant.

5. \( MG(QE)_4 \) Spacetime Admitting Space–Matter Tensor

In 1969, Petrov [19] introduced and studied the space–matter tensor \( \overrightarrow{P} \) of type (0, 4) and defined by
\[ \overrightarrow{P} = \overrightarrow{K} + \frac{\kappa}{2}g \wedge T - \nu G, \] (34)
where \( \overrightarrow{K} \) is the curvature tensor of type (0, 4), \( T \) is the energy–momentum tensor of type (0, 2), \( \kappa \) is the gravitational constant, and \( \nu \) is the energy density. Furthermore, \( G \) and \( g \wedge T \) are, respectively, defined by
\[ G(U_1, U_2, U_3, U_4) = g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4), \] (35)
and
\[ (g \wedge T)(U_1, U_2, U_3, U_4) = g(U_2, U_3)T(U_1, U_4) + g(U_1, U_4)T(U_2, U_3) \]
\[ - g(U_1, U_3)T(U_2, U_4) - g(U_2, U_4)T(U_1, U_3), \] (36)
for all \( U_1, U_2, U_3, U_4 \) on \( M \).

Using (35) and (36) in (34), it follows that
\[ \overrightarrow{P}(U_1, U_2, U_3, U_4) = \overrightarrow{K}(U_1, U_2, U_3, U_4) + \frac{\kappa}{2}g(U_2, U_3)T(U_1, U_4) \]
\[ + g(U_1, U_4)T(U_2, U_3) - g(U_1, U_3)T(U_2, U_4) \]
\[ - g(U_2, U_4)T(U_1, U_3) - \nu g(U_2, U_3)g(U_1, U_4) \]
\[ - g(U_1, U_3)g(U_2, U_4). \] (37)

If \( \overrightarrow{P} = 0 \), then (37) gives
\[ \overrightarrow{K}(U_1, U_2, U_3, U_4) = - \frac{\kappa}{2}g(U_2, U_3)T(U_1, U_4) + g(U_1, U_4)T(U_2, U_3) \]
\[ - g(U_1, U_3)T(U_2, U_4) - g(U_2, U_4)T(U_1, U_3) \]
\[ + \nu g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4). \] (38)
In view of (5), from (31), it follows that
\[ \kappa T(U_1, U_2) = \left( a - \frac{r}{2} \right) g(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2) + d[A(U_1)B(U_2) + A(U_2)B(U_1)]. \]  
(39)

Thus, from (38) and (39), we obtain
\[ \mathcal{K}(U_1, U_2, U_3, U_4) = f_1[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)] \\
+ f_2[g(U_1, U_4)A(U_2)A(U_3) - g(U_2, U_4)A(U_1)A(U_3)] \\
+ g(U_2, U_3)A(U_1)A(U_4) - g(U_1, U_3)A(U_2)A(U_4) \\
+ f_3[g(U_1, U_4)B(U_2)B(U_3) - g(U_2, U_4)B(U_1)B(U_3)] \\
+ g(U_2, U_3)B(U_1)B(U_4) - g(U_1, U_3)B(U_2)B(U_4)] \\
+ f_4[g(U_1, U_4)\{ A(U_2)B(U_3) + B(U_2)A(U_3) \} \\
- g(U_2, U_4)\{ A(U_1)B(U_3) + B(U_1)A(U_3) \} \\
+ g(U_2, U_3)\{ A(U_1)B(U_4) + B(U_1)A(U_4) \} \\
- g(U_1, U_3)\{ A(U_2)B(U_4) + B(U_2)A(U_4) \}], \]

where \( f_1 = (\nu - a + \frac{\rho}{2}) \), \( f_2 = -\frac{b}{2} \), \( f_3 = -\frac{c}{2} \), \( f_4 = -\frac{d}{2} \). Thus, we can state the following theorem:

**Theorem 4.** For a vanishing space–matter tensor, \( M\!G(QE)_4 \) spacetime satisfying Einstein’s field equation without a cosmological constant is a \( M\!G(QC)_4 \) spacetime.

Next, we investigate the existence of a sufficient condition under which \( M\!G(QE)_4 \) can be a divergence-free space–matter tensor.

From (31) and (37), we obtain
\[ (\text{div}\mathcal{P})(U_1, U_2, U_3) = (\text{div}\mathcal{K})(U_1, U_2, U_3) + \frac{1}{2} [(\text{div} \mathcal{K})(U_1, U_2, U_3) \\
- (D_{U_2} \text{Ric})(U_1, U_3)] - g(U_2, U_3)\left[ \frac{1}{4} \partial_r(U_1) + \partial_\nu(U_1) \right] \]  
(41)

By using \( (\text{div} \mathcal{K})(U_1, U_2, U_3) = (D_{U_1} \text{Ric})(U_2, U_3) - (D_{U_2} \text{Ric})(U_1, U_3) \) in (41), we obtain
\[ (\text{div}\mathcal{P})(U_1, U_2, U_3) = \frac{3}{2} [(D_{U_1} \text{Ric})(U_2, U_3) - (D_{U_2} \text{Ric})(U_1, U_3)] \\
- g(U_2, U_3)\left[ \frac{1}{4} \partial_r(U_1) + \partial_\nu(U_1) \right] \]  
(42)

Let \( (\text{div}\mathcal{P})(U_1, U_2, U_3) = 0 \); then, contracting (42) over \( U_2 \) and \( U_3 \), we obtain \( \partial_\nu(U_1) = 0 \), where (27) is used. Hence, we can state the following theorem:

**Theorem 5.** For a divergence-free space–matter tensor, the energy density in \( M\!G(QE)_4 \) spacetime satisfying Einstein’s field equation without a cosmological constant is constant.

Now, by using (5) in (42), we obtain
By assuming that \( v, a, b, c, \) and \( d \) are constants and the generator \( \rho \) is a parallel vector field, i.e., \( D_{U_1}\rho = 0 \), we obtain
\[
\partial r(U_1) = 0, \quad \partial v(U_1) = 0, \quad (D_{U_1}A)(U_2) = 0. \tag{44}
\]

In view of (44), we derive
\[
a + b = 0, \quad c = 0, \quad d = 0. \tag{45}
\]

Using (44) and (45), (43) reduces to
\[
(d\text{iv}\mathcal{P})(U_1, U_2, U_3) = 0.
\]

Thus, we can state the following theorem:

**Theorem 6.** In \( MG(QE)_4 \) spacetimes admitting parallel vector field \( \rho \) satisfying Einstein’s field equation without a cosmological constant, if the energy density and associated scalars constant are constants, then the divergence of the space–matter tensor vanishes.

6. **\( MG(QE)_4 \) Spacetime Admitting General Relativistic Viscous Fluid**

Ellis [20] defined the energy–momentum tensor for a perfect fluid distribution with heat conduction as
\[
T(U_1, U_2) = \omega g(U_1, U_2) + (\nu + \omega) A(U_1)A(U_2) + B(U_1)B(U_2) + A(U_1)B(U_2) + A(U_2)B(U_1),
\]
where \( g(U_1, \rho) = A(U_1), g(U_1, \sigma) = B(U_1), A(\rho) = -1, B(\sigma) > 0, g(\rho, \sigma) = 0, \) and \( \nu, \omega \) are called the isotropic pressure and the energy density, respectively. \( \sigma \) is the heat conduction vector field perpendicular to the velocity vector field \( \rho \). Assuming a mixed generalized quasi-Einstein spacetime satisfying Einstein’s field equation without a cosmological con-
Theorem 7. If MG\(_{(QE)_4}\) spacetime admitting viscous fluid satisfies Einstein’s field equation without a cosmological constant, then the square of the length of Ricci operator is \(\kappa^2(v^3\omega^2 + v + \omega - 3)\).

7. Example of MG\((QE)_{4}\) Spacetime

In this section, we constructed a non-trivial concrete example to prove the existence of a MG\((QE)_{4}\) spacetime.
We assume a Lorentzian manifold \((M^4, g)\) endowed with the Lorentzian metric \(g\) given by
\[
d s^2 = g_{ij} du^i du^j = (1 + 2p)(du^1)^2 + (du^2)^2 + (du^3)^2 - (du^4)^2,
\]
where \(u^1, u^2, u^3, u^4\) are standard coordinates of \(M^4\), \(i, j = 1, 2, 3, 4\), and \(p = e^{\nu} k^{-2}\), and \(k\) is a non-zero constant. Here, the signature of \(g\) is \((+, +, +, -)\), which is Lorentzian. Then, the only non-vanishing components of the Christoffel symbols and the curvature tensors are
\[
\begin{align*}
\{1\}^{11} = \{1\}^{44} &= \{2\}^{12} = \{3\}^{13} = \{4\}^{14} = p \quad \text{and} \quad \{1\}^{22} = \{1\}^{33} = \{1\}^{44} = \frac{-p}{1 + 2p}, \\
K_{1212} = K_{1313} &= \frac{-p}{1 + 2p}, \quad K_{1414} = \frac{p}{1 + 2p}, \\
K_{3333} = K_{4444} &= \frac{-p^2}{1 + 2p}, \quad K_{4242} = K_{4343} = \frac{p^2}{1 + 2p}
\end{align*}
\]
and the components are obtained by the symmetry properties.

The non-vanishing components of the Ricci tensors are
\[
R_{11} = \frac{3p}{(1 + 2p)^2}, \quad R_{22} = R_{33} = \frac{p}{(1 + 2p)^2}, \quad R_{44} = \frac{-p}{(1 + 2p)^2}.
\]

Thus, the scalar curvature \(r\) is \(\frac{6p(1 + \nu)}{(1 + 2p)^2}\).

Let us consider the associated scalars \(a, b, c, d\) defined by
\[
a = \frac{p}{(1 + 2p)^2}, \quad b = \frac{1}{(1 + 2p)^2}, \quad c = \frac{-1}{(1 + 2p)^2}, \quad d = \frac{-p}{(1 + 2p)^2}
\]
and the 1-forms are defined by
\[
A_1 = B_1 = \sqrt{1 + 2p}, \quad A_i = B_i = 0 \quad \forall \quad i = 2, 3, 4,
\]
where the generators are unit vector fields; then, from (5), we have
\[
\begin{align*}
R_{11} &= g_{11} + bA_1A_1 + cB_1B_1 + d(A_1B_1 + A_1B_1), \\
R_{22} &= g_{22} + bA_2A_2 + cB_2B_2 + d(A_2B_2 + A_2B_2), \\
R_{33} &= g_{33} + bA_3A_3 + cB_3B_3 + d(A_3B_3 + A_3B_3), \\
R_{44} &= g_{44} + bA_4A_4 + cB_4B_4 + d(A_4B_4 + A_4B_4).
\end{align*}
\]

Now, R.H.S. of (58) = \(\frac{3p}{(1 + 2p)^2}\) = \(R_{11}\) = L.H.S. of (58).

Similarly, it can easily be shown that (59), (60), and (61) are also true. Hence, \((\mathbb{R}^4, g)\) is a MG(QE) \(_4\).

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