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Shapley Mapping and Its Axiomatizations in n-Person Cooperative Interval Games

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Abstract: Interval games are an extension of cooperative coalitional games, in which players are assumed to face payoff uncertainty. Characteristic functions thus assign a closed interval, instead of a real number. In this paper, we first examine the notion of solution mapping, a solution concept applied to interval games, by comparing it with the existing solution concept called the interval solution concept. Then, we define a Shapley mapping as a specific form of the solution mapping. Finally, it is shown that the Shapley mapping can be characterized by two different axiomatizations, both of which employ interval game versions of standard axioms used in the traditional cooperative game analysis such as efficiency, symmetry, null player property, additivity and separability.

Keywords: cooperative interval games; interval uncertainty; Shapley value; solution mapping; axiomatization

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1. Introduction

This paper examines cooperative game theory when players face uncertainty. One of the most familiar representations of cooperative game theory without uncertainties is coalitional games with transferable utility (so-called coalitional games or TU games) proposed by von Neumann and Morgenstern [1]. A coalitional game consists of a set N of players and a characteristic function v that gives a real number $v(S)$ (worth of S) to every subset S of N. For each coalition S, $v(S)$ is the total payoff that S can obtain by itself and divide among its members in any possible way. A solution concept of coalitional games such as the Shapley value (Shapley [2]) and core (Gillies [3]) assigns each game a (possibly empty or singleton) set of outcomes, each of which is represented by an n-dimensional real-valued vector.

In reality, the payoffs a coalition can obtain entail uncertainty. For instance, when a project is to be jointly financed by multiple investors and they need to decide whether to join, they may not know the exact return that would be realized by the project or the additional costs that would be incurred by them in the interim period before the project is completed. Similarly, under the classical "bankruptcy problem", a creditor must decide on the amount of money to be lent, thus facing uncertainty regarding the borrower's future fiscal condition or solvency when the claim is scheduled to be paid back and cleared. Therefore, introducing uncertainty into standard coalitional games is a natural and important extension. The existing literature on cooperative game theory with uncertainty has developed within the following two distinct groups of research. The first group consists of models where the uncertainty appears as a degree of cooperation in coalition formations, e.g., fuzzy games in Aubin [4]. The second group considers models where player cooperation is as in the classical model, that is, using crisp coalitions, but uncertainty appears in the payoffs of the coalitions (e.g., games using the stochastic characteristic function form of Charnes and Granot [5], stochastic payoffs of Suijs et al. [6], random payoffs of Timmer et al. [7]). Of these, the interval games, initially studied by Branzei et al. [8] and Alparslan Gök et al. [9], consider "interval uncertainty" in that the uncertainty regarding coalition payoff is represented by an interval that a characteristic function assigns, rather than stochastic payoffs. (Various solution concepts applied to interval games have been proposed and their properties have been examined subsequently by theoretical studies such as Alparslan Gök et al. [9], Fei et al. [10], Li et al. [11], Liang and Li [12], Meng et al. [13]. As applications of interval games, Palanci et al. [14] introduced uncertainty into classical cooperative transportation games and formalized them as interval games. Alparslan Gök et al. [15] examined the Shapley value and Baker–Thompson rule in the interval game version of the airport game. For more details on the literature, see Alparslan Gök [16], Branzei et al. [17] and Ishihara and Shino [18].)

As in coalitional games, an interval game consists of a set N of players and a characteristic function w. (Throughout our study, w denotes the characteristic function in an interval game to distinguish it from the characteristic function v in a coalitional game.) However, w uniquely provides a coalition with a closed interval of real numbers, called the worth set, rather than a real number as in the characteristic function in a classical coalitional game. Therefore, we can regard interval games as a generalization of coalitional games. Note also that this specification is consistent with social situations in the real world, where the potential rewards or costs are not precisely known but we can estimate the intervals to which they belong.

The analyses of interval games, similarly to those of coalitional games, aim to examine the issues of (i) coalition formation (or which coalition will be made) and (ii) payoff allocation (or how to distribute the total payoff to all players under the grand coalition). In particular, regarding the issue of (ii), coalitional game analyses have proposed various types of solution concepts concerning payoff allocation, such as the core, the von Neumann and Morgenstern stable set and the Shapley value.

An interval game analysis can also examine a desirable payoff allocation under the grand coalition. However, we argue that interval game analyses have a unique subject to be carefully considered. The major existing solution concept for n-person interval games, called the internal solution concept (Alparslan Gök et al. [19,20]), associates with each game a (possibly empty or singleton) set of n-dimensional interval payoff vectors. Here, an element of an interval payoff vector is a closed interval rather than a real number. However, if we consider the underlying social situations that interval games essentially assume, another type of solution concept could be proposed because the interval solution concepts do not consider an important aspect of the situation that the uncertainty of outcomes represented by the intervals will be removed when an outcome is realized and allocated among players. In particular, if the grand coalition is assumed to be formed, a solution concept for interval games should instruct how to allocate a realized outcome represented by a real number in the worth set of the grand coalition $w(N)$ among all players. Therefore, the solution concept defined as a mapping from each realization in $w(N)$ to an n-dimensional real-valued vector, rather than one that directly specifies a set of n-dimensional interval vectors, is worth examining.

In fact, such a mapping has already been proposed in Alparslan Gök et al. [9] as the ψ^{α} value. However, the analysis of ψ^{α} is limited and, to our knowledge, no subsequent analysis has since taken place. (The only exception is in Ishihara and Shino's paper [18], in which a new solution mapping was proposed, focusing on a two-person interval game.) With this argument as our motivation, this study proposes a new solution mapping called the Shapley mapping as a solution concept and applies it to general n-person interval games. As we discuss in the subsequent sections, the Shapley mapping has some advantages. First, because a solution mapping assigns an n-dimensional real-valued vector rather than an n-dimensional interval vector, it does not have to consider "how to handle interval solution concepts" after uncertainties are removed. Second, as we discuss later in the analysis, it is defined without using interval subtraction, and thus no restrictions on the coverage of interval games are required. Finally, this solution concept can be justified by standard axiomatizations.

The rest of this paper is organized as follows. Section 2 reviews coalitional and interval games and solution concepts in interval games. Section 3 examines the existing solution concept and solution mapping by providing some examples. Section 4 defines the Shapley mapping and characterizes the mapping by two different axiomatizations. Section 5 concludes the paper.

2. Coalitional Games and Interval Games

An n-person *coalitional game* or a TU game is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function that associates a real number $v(S) \in \mathbb{R}$ for each set $S \subset N$, with the condition $v(\emptyset) = 0$. Number $v(S)$ is called the worth of S. We refer to S and N as a coalition and grand coalition, respectively. Let CG be the set of all coalitional games with player set N. Define $CG_+ \subset CG$ as $CG_+ = \{v \in CG \mid v(N) > 0\}$. We also denote n-person coalitional games (N, v) simply by v, and use the simpler notation $v(\cdot)$ instead of $v(\{\cdot\})$

Representative solution concepts for the coalitional games include the von Neumann and Morgenstern stable set (von Neumann and Morgenstern [1]), set of imputation, core (Gillies [3]), Shapley value (Shapley [2]), nucleolus (Schmeidler [21]), bargaining sets (Aumann and Maschler [22]) and kernel (Davis and Maschler [23]). In this study, we use the notion of the Shapley value [2]. For an n-person coalitional game $v \in CG$, the Shapley value $\phi(v)$ is defined as the n-dimensional real-valued vector $\phi(v) = (\phi_1(v), ..., \phi_i(v), ..., \phi_n(v))$ with:

$$\phi_i(v) = \sum_{S: i \in S} \frac{(s-1)!(n-s)!}{n!} \{v(S) - v(S \setminus \{i\})\}. \tag{1}$$

Similar to an n-person coalitional game (N, v), an n-person *interval game* is defined as a pair (N, w), where N is a set of players and w is a characteristic function of type $2^N \to I(\mathbb{R})$ with $w(\emptyset) = [0, 0]$, where $I(\mathbb{R})$ is the set of all closed and bounded intervals in $\mathbb{R}$. Therefore, an interval game differs from a coalitional form game in that w assigns a closed interval to each coalition (instead of a real number). Interval $w(S)$ is called the worth set of S and the minimum and the maximum of $w(S)$ are denoted by $\underline{w}(S)$ and $\overline{w}(S)$, respectively, that is, $w(S) = [\underline{w}(S), \overline{w}(S)]$. An interval game (N, w) considers a situation in which the players face "interval uncertainty", in that they know a coalition S could have $\underline{w}(S)$ as the minimal reward and $\overline{w}(S)$ as the maximal one, but do not know *ex ante* which one between them would be realized.

Let IG be the set of all interval games with player set N. We also denote n-person interval games (N, w) simply by w and use the simpler notation $w(\cdot)$ instead of $w(\{\cdot\})$. Players i and j are symmetric in an interval game w if $w(S \cup \{i\}) = w(S \cup \{j\})$ for any coalition $S \subset N \setminus \{i, j\}$. Player i is a null player in w when $w(S \cup \{i\}) = w(S)$ holds for any $S \subset N \setminus \{i\}$.

We provide some interval calculus notations for the following analysis. Let $I = [\underline{I}, \overline{I}]$ and $J = [\underline{J}, \overline{J}]$ be two closed intervals. The sum of I and J, denoted by $I + J$, is given as $I + J = [\underline{I} + \underline{J}, \overline{I} + \overline{J}]$. Next, following Alparslan Gök et al. [20], the partial subtraction operator denoted by "$-$" is defined as $I - J = [\underline{I} - \underline{J}, \overline{I} - \overline{J}]$. Note that the partial subtraction operator is only defined for an ordered interval pair, i.e., $(I, J) \in I(\mathbb{R}) \times I(\mathbb{R})$ satisfying $\overline{J} - \underline{J} \leq \overline{I} - \underline{I}$.

For two different interval games $w', w'' \in IG$, the sum of the interval games $w' + w'' \in IG$ is also an interval game itself, defined by $(w' + w'')(S) = w'(S) + w''(S)$ for every $S \subset N$. For an interval game $w \in IG$, let $v_{\underline{w}} \in CG$ be a coalitional game generated from an interval game w, so that $v_{\underline{w}}(S) = \underline{w}(S)$ for every $S \subset N$, and $v_{\overline{w}} \in CG$ so that $v_{\overline{w}}(S) = \overline{w}(S)$ for every $S \subset N$. For an interval game $w \in IG$, define $v_{\overline{w}-\underline{w}} \in CG$ as

$v_{\overline{w}-\underline{w}}(S) = v_{\overline{w}}(S) - v_{\underline{w}}(S)$ for every $S \subset N$. Finally, for an interval $I = [\underline{I}, \overline{I}]$ and a coalitional game $v \in CG$, $Iv \in IG$ is defined as $Iv(S) = [\underline{I}v(S), \overline{I}v(S)]$ for every $S \subset N$.

Let $\mathbb{I} = (I_1, ..., I_n) \in I(\mathbb{R})^n$ be an n-dimensional closed interval vector, so that $I_i \in I(\mathbb{R})$ for $i \in N$. For $\mathbb{I} \in I(\mathbb{R})^n$, we, respectively, define $\min \mathbb{I} \in \mathbb{R}^n$ and $\max \mathbb{I} \in \mathbb{R}^n$ as follows:

$$\min \mathbb{I} = (\min I_1, ..., \min I_n), \quad \max \mathbb{I} = (\max I_1, ..., \max I_n). \tag{2}$$

Note that, for an interval game $w \in IG$, if all worth sets are singletons, that is, $\underline{w}(S) = \overline{w}(S)$ for every $S \subset N$, w corresponds to the coalitional game $v \in CG$, which is defined as $v(S) = \underline{w}(S) = \overline{w}(S)$. In this case, we say that $w \in IG$ and $v \in CG$ are equivalent or w has its equivalent coalitional game v. Let $EG \subset IG$ be a set of interval games that has its equivalent coalitional game.

3. Solution Concepts in Interval Games

In this section, we first review the existing solution concepts applied to interval games—interval solution concept and interval Shapley value—in Section 3.1. Then, in Section 3.2, we examine an alternative solution concept, called a solution mapping, by comparing it with the interval solution concept.

3.1. Existing Solution Concepts: Interval Solution Concepts and Interval Shapley Value

In the literature on interval games, the most popular solution concept, which has played a central part in analyses, is the interval solution concept. (Another type of solution concept proposed early in the history of interval game analysis is the selection-based solution concept proposed by Alparslan Gök et al. [9].) Interval solution concepts are defined as a (possibly empty or singleton) set of n-dimensional interval vectors. Formally, by letting $I_i \in I(\mathbb{R})$ be the interval payoff of player i and $\mathbb{I} = (I_1, ..., I_n) \in I(\mathbb{R})^n$ be an n-dimensional closed interval vector, an *interval solution concept* in $w \in IG$ assigns a (possibly empty or singleton) set of n-dimensional interval vectors $K \subset I(\mathbb{R})^n$. The interval core, interval stable set (Alparslan Gök et al. [19]) and the interval Shapley value (Alparslan Gök et al. [20]) have been proposed as interval solution concepts. Here, for the following analysis we review the definition of the interval Shapley value.

Let a permutation of N be $\sigma : N \to N$ and the group of all permutations of N be $\pi(N)$. For an interval game $w \in IG$ and a permutation $\sigma \in \pi(N)$, we define player i's marginal contribution in σ of w as:

$$m_i^\sigma(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i)), \tag{3}$$

where $P_\sigma(i)$ is given by $P_\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$, that is, the set of i's predecessors in permutation σ. A marginal contribution vector in σ of w is defined as $m^\sigma(w) = (m_1^\sigma(w), ..., m_n^\sigma(w))$. With this setup, Alparslan Gök et al. [20] defined the *interval Shapley value* $\Phi : IG \to I(\mathbb{R})^n$ as:

$$\Phi(w) = (\Phi_1(w), ..., \Phi_n(w)) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} m^\sigma(w). \tag{4}$$

The following example shows the interval Shapley value in a simple three-person interval game.

Example 1. *Let the interval game w be as follows: $w(1) = [0,0]$, $w(2) = [0,0]$, $w(3) = [0,0]$, $w(12) = [3,5]$, $w(13) = [3,5]$, $w(23) = [3,5]$ and $w(123) = [18,27]$. Player 1's marginal contributions for each permutation are as follows: $m_1^{(123)}(w) = w(1) - w(\emptyset) = [0,0]$, $m_1^{(132)}(w) = [0,0]$, $m_1^{(213)}(w) = [3,5]$, $m_1^{(231)}(w) = [15,22]$, $m_1^{(312)}(w) = [3,5]$ and $m_1^{(321)}(w) = [15,22]$. Given that the sum of all marginal contributions is $[36,54]$ and $3! = 6$, player 1's interval Shapley value is $\Phi_1(w) = [6,9]$. Similarly, we obtain $\Phi_2(w) = [6,9]$ and $\Phi_3(w) = [6,9]$. Consequently, the interval Shapley value of w is of the form:*

$$\Phi(w) = ([6,9],[6,9],[6,9]). \tag{5}$$

It is not difficult to show that the ith component of the interval Shapley value can be rearranged as follows:

$$\Phi_i(w) = \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{w(S) - w(S\setminus\{i\})\}. \tag{6}$$

It is evident from (1) and (6) that the interval Shapley value is a natural extension of the Shapley value.

Note that the interval Shapley value for an interval game is not always definable because the partial subtraction operator is used. The following example illustrates this point.

Example 2 (Han et al. [24])**.** *Let the interval game w be as follows: $w(1) = [0,2]$, $w(2) = [1/2,3/2]$, $w(3) = [1,2]$, $w(12) = [2,3]$, $w(13) = [3,4]$, $w(23) = [4,4]$ and $w(123) = [6,7]$. Player 2's marginal contributions for permutation (123) cannot be computed because $m_2^{(123)}(w) = [2,3] - [0,2]$ but $2 - 0 > 3 - 2$.*

To address the so-called interval subtraction problem, Alparslan Gök et al. [20,25,26] restricted the coverage of interval games to size monotonic interval games. An interval game $w \in IG$ is called *size monotonic* if for every pair of coalitions $S \subset T$,

$$\overline{w}(S) - \underline{w}(S) \leq \overline{w}(T) - \underline{w}(T). \tag{7}$$

When an interval game is size monotonic, a player's marginal contribution is always definable as a closed interval. The interval game in Example 2 is not size monotonic. (A different approach to address the interval subtraction problem is to use Moore's [27] subtraction operator in place of the partial subtraction operator. For more detail, see Han et al. [24].)

Based on these arguments, in the next subsection we examine an alternative solution concept applied to interval games, called solution mapping.

3.2. An Alternative Solution Concept: Solution Mapping

An interval game analysis essentially assumes that players face the following underlying situation. First, players face payoff uncertainties, represented by the worth sets, and negotiate over a "rule" or "protocol" that specifies a way to allocate an outcome among players *ex ante*, before uncertainties are removed. Second, after the uncertainties are removed and one of the outcomes in the worth set is realized, the realized outcome is allocated based on the agreed rule or protocol in the *ex ante* negotiation. (This situation is similar to that in Habis and Herings's [28] TUU game, where uncertainties are introduced into cooperation games using a different approach. Namely, they constructed a two-stage model in which the ex ante first stage has multiple coalitional games, and one of them is realized and played by the players in the second stage.) Specifically, when the grand coalition forms, the realized outcome in $w(N)$ is allocated. Based on this interpretation, a mapping that assigns n-dimensional real-valued vectors to each realization in the worth set of the grand coalition operates as a solution concept in interval games. This is the main idea of the solution mapping that we employ as a solution concept. Note that this solution mapping is different from the existing notion of the interval solution concept that assigns an n-dimensional interval vector.

Formally, for an n-person interval game $w \in IG$, a function $F(w) : w(N) \to \mathbb{R}^n$ is called a *solution mapping*. For any interval game w, $F(w)(t)$ assigns n-dimensional payoff vectors to each realization $t \in w(N)$. Such a mapping was proposed by Alparslan Gök et al. [9] as the ψ^α value at the early stage in the history of interval game analyses. However, their analysis considered only two-person interval games and no subsequent analysis of the solution mapping has been developed. As such, in the next section we propose a solution mapping, called the Shapley mapping, and apply it to general n-person interval games.

4. Shapley Mapping and Its Axiomatizations

In this section, we first define the Shapley mapping as a specific form of the solution mapping in Section 4.1. Then, two different axiomatizations of the Shapley mapping are provided in Sections 4.2 and 4.3, respectively.

4.1. Shapley Mapping

For an n-person interval game $w \in IG$, we define the Shapley mapping as a specific form of solution mapping. First, for the realization of the worth set of the grand coalition $t \in w(N)$, $\alpha \in [0,1]$ satisfying $t = (1-\alpha)\underline{w}(N) + \alpha\overline{w}(N)$ is uniquely determined. (It should be noted that there exists a case for which v_w^α cannot be defined. This occurs when $w(N)$ is singleton and $w(S)$ is not singleton for some $S \subset N \setminus \{i\}$. We exclude such a "degenerate case", because it departs from the underlying situation of interval games, where interval uncertainty exists regarding the realization of the outcome in the grand coalition.) Second, we define a coalitional game v_w^α generated from w by α as:

$$v_w^\alpha(S) = (1-\alpha)\underline{w}(S) + \alpha\overline{w}(S) \quad \textit{for every } S \subset N.$$

Note that $v_w^\alpha(N) = t \in w(N)$. By letting $\phi(v_w^\alpha) = (\phi_1(v_w^\alpha), \ldots, \phi_i(v_w^\alpha), \ldots, \phi_n(v_w^\alpha))$ be the Shapley value in coalitional game v_w^α, the *Shapley mapping* $\sigma^*(w) : w(N) \to \mathbb{R}^n$ is defined as:

$$\sigma^*(w)(t) = \phi(v_w^\alpha). \tag{8}$$

Since the interval subtraction operator is not used in the definition of the Shapley mapping, it is free from the "interval subtraction problem" discussed in Section 3. Moreover, the Shapley mapping is also free from the problem of "how to handle interval solution concepts" because it directly specifies an n-dimensional real-valued vector, rather than an n-dimensional interval vector.

It should also be noted that α is endogenously determined depending on $t \in w(N)$, rather than an exogenous parameter representing such factors as players' risk attitudes. Therefore, our α is not directly related to the Hurwicz criterion recently examined by Mallozzi and Vidal-Puga [29] in the context of interval game analyses. We do not impose any specific risk attitudes on players, and all of the results obtained here hold independent of those attitudes.

Note that $\sigma^*(w)(t)$ can be rearranged as follows:

$$\begin{aligned}
\sigma_i^*(w)(t) &= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{v_w^\alpha(S) - v_w^\alpha(S \setminus \{i\})\} \\
&= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}[(1-\alpha)\{\underline{w}(S) - \underline{w}(S \setminus \{i\})\} + \alpha\{\overline{w}(S) - \overline{w}(S \setminus \{i\})\}] \\
&= (1-\alpha)\phi_i(v_{\underline{w}}) + \alpha\phi_i(v_{\overline{w}}),
\end{aligned} \tag{9}$$

where $\phi_i(\cdot)$ is player i's Shapley value in coalitional games $v_{\underline{w}}$ and $v_{\overline{w}}$.

For the interval game in Example 2, when $6 \in w(123)$ is realized, Shapley mapping σ^* gives the following real-valued vector:

$$\sigma^*(w)(6) = \left(\frac{5}{4}, 2, \frac{11}{4}\right),$$

while the interval Shapley value, as we discussed, cannot be applied to this game because it is not size monotonic. Note that the sum of payoffs is 6, which is equal to the realization of the worth set of the grand coalition. Therefore, the efficiency is satisfied.

Example 3 (Alparslan Gök [25] and Palanci et al. [30]). *$w(1) = [7,7]$, $w(2) = [0,0]$, $w(3) = [0,0]$, $w(12) = [12,17]$, $w(13) = [7,7]$, $w(23) = [0,0]$ and $w(123) = [24,29]$. As this game is size monotonic, the interval Shapley value exists: $\Phi(w) = ([27/2,16], [13/2,9], [4,4])$. The Shapley mapping, on the other hand, gives the following three-dimensional real-valued vectors for the realizations of the worth set of the grand coalition—24, 26.5 and 29 (the minimum, midpoint and maximum value of $w(123)$, respectively):*

$$\sigma^*(w)(24) = \left(\frac{27}{2}, \frac{13}{2}, 4\right), \quad \sigma^*(w)(26.5) = \left(\frac{59}{4}, \frac{31}{4}, 4\right), \quad \sigma^*(w)(29) = (16, 9, 4).$$

Comparing Shapley mapping σ^* with interval Shapley mapping Φ, $\sigma^*(w)(24)$ and $\sigma^*(w)(29)$ are the same as the vectors consisting of the lower and upper bounds of each element in $\Phi(w)$, respectively. Similarly, each element of $\sigma^*(w)(26.5)$ is identical to the midpoint of the corresponding player's interval in $\Phi(w)$. In general, regarding the relationship between the Shapley mapping and the interval Shapley value, the following result holds.

Remark 1. *Let Φ and σ^* be the interval Shapley value and Shapley mapping, respectively. For a realization of the worth set of the grand coalition $t \in w(N)$ in an interval game $w \in IG$, let $\alpha \in [0,1]$ be a number satisfying $t = (1-\alpha)\underline{w}(N) + \alpha\overline{w}(N)$. When w is size monotonic, the following holds:*

$$\sigma^*(w)(t) = (1-\alpha)\min\Phi(w) + \alpha\max\Phi(w) \tag{10}$$

Here, $\min\Phi(w)$ *and* $\max\Phi(w)$ *are defined by (2).*

Proof. When $w \in IG$ is size monotonic, $\Phi(w)$ can be defined, and the following holds for every $i \in N$:

$$\begin{aligned}
\Phi_i(w) &= \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{[\underline{w}(S) - \underline{w}(S\setminus\{i\}),\ \overline{w}(S) - \overline{w}(S\setminus\{i\})]\} \\
&= \left[\sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{\underline{w}(S) - \underline{w}(S\setminus\{i\})\},\ \sum_{S:i\in S} \frac{(s-1)!(n-s)!}{n!}\{\overline{w}(S) - \overline{w}(S\setminus\{i\})\}\right] \\
&= [\phi_i(v_{\underline{w}}),\ \phi_i(v_{\overline{w}})],
\end{aligned}$$

where $\phi_i(\cdot)$ is player i's Shapley value in coalitional games $v_{\underline{w}}$ and $v_{\overline{w}}$. This implies $\min\Phi_i(w) = \phi_i(v_{\underline{w}})$ and $\max\Phi_i(w) = \phi_i(v_{\overline{w}})$. Therefore, the following holds:

$$(1-\alpha)\min\Phi_i(w) + \alpha\max\Phi_i(w) = (1-\alpha)\phi_i(v_{\underline{w}}) + \alpha\phi_i(v_{\overline{w}}) = \sigma_i^*(w)(t). \tag{11}$$

The last equality in (11) holds from (9). □

Remark 1 implies some equivalency between the interval Shapley value and Shapley mapping, such that the allocation for a realization of the worth set of the grand coalition proposed by the Shapley mapping is identical to the real-valued vector obtained by dividing internally each interval in the interval Shapley value by the ratio α associated with the realization of the worth set. As the Shapley mapping can be applied to non-size-monotonic interval games, it can be interpreted as a generalization of the interval Shapley value. Furthermore, the Shapley mapping can be characterized by certain axiomatizations.

4.2. An Axiomatization of the Shapley Mapping

In this subsection, we first propose a set of four axioms for a solution mapping σ: *efficiency*, *symmetry*, *null player property*, and *additivity*. Then, in Theorem 1, we show that these four axioms uniquely determine the Shapley mapping.

- Axiom 1: Efficiency (**EF**)

$$\left(\sum_{i\in N}\sigma_i(w)(t) = t\right)\ \left(\forall w \in IG\right)\ \left(\forall t \in w(N)\right).$$

- Axiom 2: Symmetry (**SYM**)

$$\Big(\sigma_i(w)(t) = \sigma_j(w)(t)\Big) \ \Big(\forall w \in IG \ where \ \ i \ and \ j \ are \ symmetric\Big) \ \Big(\forall t \in w(N)\Big).$$

- Axiom 3: Null player property (**NP**)

$$\Big(\sigma_i(w)(t) = 0\Big) \ \Big(\forall t \in w(N)\Big) \ \Big(if \ w(S) = w(S \cup \{i\}) \ \forall S \subset N \setminus \{i\}\Big).$$

- Axiom 4-1: Additivity-1 (**AD1**)
For an $\alpha \in [0,1]$ and $w', w'' \in IG$, we define $t' \in w'(N)$ and $t'' \in w''(N)$ as:

$$\begin{aligned} t' &= (1-\alpha)\underline{w'}(N) + \alpha\overline{w'}(N) \\ t'' &= (1-\alpha)\underline{w''}(N) + \alpha\overline{w''}(N). \end{aligned}$$

Then,

$$\Big(\sigma_i(w' + w'')(t' + t'') = \sigma_i(w')(t') + \sigma_i(w'')(t'')\Big)$$
$$\Big(\forall w', w'' \in IG\Big) \ \Big(\forall \alpha \in [0,1]\Big) \ \Big(\forall i \in N\Big).$$

Axiom EF asserts that all $t \in w(N)$ is allocated to players in the game and no residual exists. Axiom SYM argues that only what a player can obtain on their own in the game should matter, not its specific name. Axiom NP asserts that a zero payoff should be assigned to a null player.

Axiom AD1 essentially comes from the additivity axiom in Shapley [2], which considers a "sum" interval game $(w' + w'') \in IG$. Specifically, it asserts that, when σ gives $\sigma_i(w')(t')$ to player i for realization $t' \in w'(N)$ in $w' \in IG$ and $\sigma_i(w'')(t'')$ to i for realization $t'' \in w''(N)$ in $w'' \in IG$, in the sum game $(w' + w'') \in IG$, σ should give $\sigma_i(w')(t') + \sigma_i(w'')(t'')$ to player i for realization $t' + t'' \in (w' + w'')(N)$. Here, it should be noted that it imposes restrictions on t' and t'', so that these are generated by a "common factor" $\alpha \in [0,1]$ regarding the worth sets of the grand coalition.

More specifically, an underlying situation that AD1 assumes is as follows. Suppose that, for example, $\alpha = 1$, that is, $t' = \overline{w'}(N)$ and $t^{''} = \overline{w^{''}}(N)$ are realized in w' and $w^{''}$, respectively. This case can be interpreted as the best "common factor", such as the weather (if w represents agricultural productions) or aggregate economic growth (if w represents an individual firm's profit), contributing positively and generating the largest amounts of the worth sets both in w' and $w^{''}$. In such a case, AD1 requires us to evaluate the allocation derived by σ in the sum game $w' + w''$ *under the same aggregate condition of* $\alpha = 1$. Note that when all worth sets are singleton, both in w' and w'', AD1 becomes identical to Shapley's [2] axiom of additivity for the coalitional games equivalent to w' and w''.

The following result of the axiomatization can be regarded as an interval game version of Shapley's axiomatization [2].

Theorem 1. *Shapley mapping σ^*, as defined in (8), is the unique solution mapping that satisfies EF, SYM, NP and AD1.*

The proof of this theorem is preceded by the following five lemmas.

Lemma 1. *Shapley mapping σ^* satisfies EF, SYM, NP and AD1.*

Proof. EF and SYM are obvious from the definition of the Shapley mapping. For NP, letting i be a null player in interval game w, it holds that $\sigma_i^*(w)(t) = \sum_{S:i \in S} \frac{(s-1)!(n-s)!}{n!} \{v_w^\alpha(S) - v_w^\alpha(S \setminus \{i\})\} = 0$. For AD1, from (9), it follows that ($\phi_i(\cdot)$ is player i's Shapley value in a coalitional game):

$$\begin{aligned}\sigma_i^*(w')(t')+\sigma_i^*(w'')(t'') &= (1-\alpha)\{\phi_i(v_{\underline{w'}})+\phi_i(v_{\underline{w''}})\}+\alpha\{\phi_i(v_{\overline{w'}})+\phi_i(v_{\overline{w''}})\}\\ &= (1-\alpha)\phi_i(v_{\underline{w'}}+v_{\underline{w''}})+\alpha\phi_i(v_{\overline{w'}}+v_{\overline{w''}})\\ &= (1-\alpha)\phi_i(v_{\underline{w}})+\alpha\phi_i(v_{\overline{w}}) \ = \ \sigma_i^*(w)(t).\end{aligned}$$

□

Lemma 2. *For a nonempty coalition $R\subset N$, we define a coalitional game $v_R\in CG$ as:*

$$v_R(S)=\begin{cases}1 & \text{if} \quad R\subset S\\ 0 & \text{otherwise.}\end{cases}$$

Then, for an interval game $w\in IG$, there uniquely exist $2(2^n-1)$ real numbers, denoted by $\underline{c_R}$ and $\overline{c_R}$ for a nonempty coalition $R\subset N$ (note: the number of R is 2^n-1), which satisfy

$$v_{\underline{w}}(S)=\sum_{R\subset N}\underline{c_R}v_R(S),\quad v_{\overline{w}}(S)=\sum_{R\subset N}\overline{c_R}v_R(S)\quad \forall S\subset N.$$

$\underline{c_R}$ and $\overline{c_R}$ are determined by:

$$\underline{c_R}=\sum_{T\subset R}(-1)^{r-t}v_{\underline{w}}(T),\quad \overline{c_R}=\sum_{T\subset R}(-1)^{r-t}v_{\overline{w}}(T),$$

where r and t denote the number of players in the coalition of R and T, respectively.

Proof. The proof is essentially identical to Shapley's [2]. □

Lemma 3. *For a nonempty coalition $R\subset N$ in interval game $w\in IG$,*

$$w(S)+\sum_{R:\underline{c_R}>\overline{c_R}}[-\underline{c_R},\ -\overline{c_R}]v_R(S)=\sum_{R:\underline{c_R}\le\overline{c_R}}[\underline{c_R},\ \overline{c_R}]v_R(S)\quad \forall S\subset N. \tag{12}$$

Proof. For a nonempty coalition $R\subset N$ satisfying $\underline{c_R}\le\overline{c_R}$, $[\underline{c_R},\ \overline{c_R}]$ can be defined as a closed interval, and for a nonempty coalition $R\subset N$ satisfying $\underline{c_R}>\overline{c_R}$, $[-\underline{c_R},\ -\overline{c_R}]$ can be defined as a closed interval. From Lemma 2, it holds that:

$$v_{\underline{w}}(S) \ = \ \sum_{R\subset N}\underline{c_R}v_R(S)=\sum_{R:\underline{c_R}\le\overline{c_R}}\underline{c_R}v_R(S)+\sum_{R:\underline{c_R}>\overline{c_R}}\underline{c_R}v_R(S)\quad \forall S\subset N,$$

and the same for the upper bound. Therefore,

$$[v_{\underline{w}}(S),\ v_{\overline{w}}(S)]+\sum_{R:\underline{c_R}>\overline{c_R}}[-\underline{c_R},\ -\overline{c_R}]v_R(S)=\sum_{R:\underline{c_R}\le\overline{c_R}}[\underline{c_R},\ \overline{c_R}]v_R(S)\quad \forall S\subset N.$$

□

Lemma 4. *Let σ be a solution mapping in $w\in IG$ satisfying EF, SYM, NP and AD1. Then, for an interval game of $[\underline{c},\ \overline{c}]v_R\in IG$ and a realization $t=(1-\alpha)\underline{c}+\alpha\overline{c}$:*

$$\sigma_i([\underline{c},\ \overline{c}]v_R)(t)=\begin{cases}\dfrac{t}{r} & \text{if} \quad i\in R\\ 0 & \text{otherwise,}\end{cases} \tag{13}$$

where r is the number of players in the nonempty coalition $R\subset N$.

Proof. First, assume $\underline{c}\ge 0$. If i is not in R, i is a null player in the interval game of $[\underline{c},\ \overline{c}]v_R\in IG$. Therefore, $\sigma_i([\underline{c},\ \overline{c}]v_R)(t)=0$. Moreover, because i and j in R are symmetric in $[\underline{c},\ \overline{c}]v_R\in IG$, $\sigma_i([\underline{c},\ \overline{c}]v_R)(t)=\sigma_j([\underline{c},\ \overline{c}]v_R)(t)$. Meanwhile, EF follows $\sum_{i\in R}\sigma_i([\underline{c},\ \overline{c}]v_R)(t)=t$. Consequently, $\sigma_i([\underline{c},\ \overline{c}]v_R)=t/r$ holds for $i\in R$.

Next, assume $\underline{c} < 0$. Then, the following holds: (i) $t - \underline{c} = (1-\alpha) \cdot 0 + \alpha(\overline{c} - \underline{c})$, (ii) $t = (1-\alpha) \cdot \underline{c} + \alpha \cdot \overline{c}$ and (iii) $-\underline{c} = (1-\alpha) \cdot (-\underline{c}) + \alpha \cdot (-\underline{c})$. Therefore, from AD1, it holds that:

$$\begin{aligned}\sigma_i([0,\ \overline{c} - \underline{c}]v_R)(t - \underline{c}) &= \sigma_i([\underline{c},\ \overline{c}]v_R + [-\underline{c},\ -\underline{c}]v_R)(t + (-\underline{c})) \\ &= \sigma_i([\underline{c},\ \overline{c}]v_R)(t) + \sigma_i([-\underline{c},\ -\underline{c}]v_R)(-\underline{c}).\end{aligned}$$

Because it holds that:

$$\begin{aligned}\sigma_i([0,\ \overline{c} - \underline{c}]v_R)(t - \underline{c}) &= \begin{cases} \dfrac{t - \underline{c}}{r} & \text{if} \quad i \in R \\ 0 & \text{otherwise} \end{cases} \\ \sigma_i([-\underline{c},\ -\underline{c}]v_R)(-\underline{c}) &= \begin{cases} \dfrac{-\underline{c}}{r} & \text{if} \quad i \in R \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

(13) also holds when $\underline{c} < 0$. □

Lemma 5. *Assume $\phi_i(v_{\underline{w}})$ and $\phi_i(v_{\overline{w}})$ are i's Shapley values in coalitional games $v_{\underline{w}}$ and $v_{\overline{w}}$, respectively. Then, the following holds:*

$$\phi_i(v_{\underline{w}}) = \sum_{R:i \in R} \frac{\underline{c_R}}{r}, \quad \phi_i(v_{\overline{w}}) = \sum_{R:i \in R} \frac{\overline{c_R}}{r}.$$

Proof. The proof is essentially identical to Shapley's [2]. □

Finally, we prove Theorem 1 using Lemmas 1–5.

Proof of Theorem 1. From Lemma 1, it suffices to prove the uniqueness of σ^*, that is, letting σ be a solution mapping satisfying EF, SYM, NP and AD1; σ must be σ^*.

For a realization of the grand coalition $t = (1-\alpha)\underline{w}(N) + \alpha\overline{w}(N) \in w(N)$ in $w \in IG$, let $t_R = (1-\alpha)\underline{c_R} + \alpha\overline{c_R}$ be a realization of the grand coalition in $[\underline{c},\ \overline{c}]v_R \in IG$. Then, from Lemma 2, $v_{\underline{w}}(N) = \sum\limits_{R \subset N} \underline{c_R}$ and $v_{\overline{w}}(N) = \sum\limits_{R \subset N} \overline{c_R}$ hold. Therefore, $t = \sum\limits_{R \subset N} t_R$ is true. From Lemma 3 and AD1, it follows that:

$$\sigma_i(w)(t) + \sum_{R:\underline{c_R} > \overline{c_R}} \sigma_i([-\underline{c_R},\ -\overline{c_R}]v_R)(-t_R) = \sum_{R:\underline{c_R} \leq \overline{c_R}} \sigma_i([\underline{c_R},\ \overline{c_R}]v_R)(t_R).$$

From Lemma 4:

$$\sigma_i(w)(t) + \sum_{R \ni i:\underline{c_R} > \overline{c_R}} \frac{-t_R}{r} = \sum_{R \ni i:\underline{c_R} \leq \overline{c_R}} \frac{t_R}{r},$$

$$\sigma_i(w)(t) = \sum_{R:i \in R} \frac{t_R}{r} = \sum_{R:i \in R} \left\{(1-\alpha) \cdot \frac{\underline{c_R}}{r} + \alpha \cdot \frac{\overline{c_R}}{r}\right\} = (1-\alpha) \sum_{R:i \in R} \frac{\underline{c_R}}{r} + \alpha \sum_{R:i \in R} \frac{\overline{c_R}}{r}$$

hold. From Lemma 5, ($\phi_i(\cdot)$ is player i's Shapley value in coalitional games $v_{\underline{w}}$ and $v_{\overline{w}}$):

$$\sigma_i(w)(t) = (1-\alpha)\phi_i(v_{\underline{w}}) + \alpha\phi_i(v_{\overline{w}}) = \sigma_i^*(w)(t).$$

□

4.3. An Alternative Axiomatization of the Shapley Mapping

This subsection shows another axiomatization of the Shapley mapping. Specifically, we additionally consider the following axioms of *additivity-2* and *separability*.

- Axiom 4-2: Additivity-2 (AD2)
  For $w'' \in EG$, let $t'' = \underline{w''}(N) = \overline{w''}(N)$. Then:
  $$\Big(\sigma_i(w' + w'')(t' + t'') = \sigma_i(w')(t') + \sigma_i(w'')(t'')\Big)$$
  $$\Big(\forall w' \in IG\Big)\Big(\forall w'' \in EG\Big)\Big(\forall t' \in w'(N)\Big)\Big(\forall i \in N\Big).$$

- Axiom 5: Separability (SP)
  Define $UG \subset IG$ as $UG = \{w \in IG \mid \exists v \in CG_+,\ \exists U = [\underline{U}, \overline{U}] \in I(\mathbb{R})\ s.t.\ w = Uv\}$. A solution mapping σ satisfies separability if there exists a real-valued function $f_i : CG_+ \longmapsto \mathbb{R}$ such that:
  $$\big(\sigma_i(w)(t) = f_i(v)t\big)\big(\forall(w,\ v,\ U) \in UG \times CG_+ \times I(\mathbb{R})\ with\ w = Uv\big)\big(\forall t \in w(N)\big)\big(\forall i \in N\big).$$

Similar to AD1, AD2 is an interval game version of the additivity that considers the sum game $(w' + w'') \in IG$. AD2 asserts that, if one of these games entails no uncertainty ($w'' \in EG$), then additivity must hold *for every realization of the grand coalition $t' \in w'(N)$ in another game w'*.

Axiom SP considers a case in which an interval game w can be broken down into two components: v and U. v is interpreted as the "basis" of w in that an uncertainty factor is excluded, and U as a "common uncertainty factor". As U is independent of $i \in N$, Axiom SP asserts that, when an interval game w consists of v and U, the following (i) and (ii) on a solution mapping σ should be satisfied: (i) the allotment to player i is proportional to t with a constant allotment ratio, and (ii) this allotment ratio $f_i(v)$ is determined not by "common factor" U, but by v, which is the only source that generates asymmetricity among players. Axiom SP has some similarities to the separability conditions examined in the context of cost allocation games as in Fishburn and Pollac [31]. However, our axiom SP is unique to interval game analyses in that this is based on the decomposition consisting of the basis of an interval game and the common uncertainty factor.

The main result of the second axiomatization using those axioms is as follows:

Theorem 2. *Shapley mapping σ^* is the unique solution mapping that satisfies EF, SYM, NP, AD2 and SP.*

To prove Theorem 2, we show the following Lemma.

Lemma 6. *Shapley mapping σ^* satisfies AD2 and SP.*

Proof. We first show that Shapley mapping σ^* satisfies AD2. As σ^* satisfies AD1 from Lemma 1, it suffices to show that σ^* satisfies AD2 whenever it satisfies AD1. For any $w' \in IG$, $w'' \in EG$, $t' \in w'(N)$ and $t'' \in w''(N)$, there exists $\alpha \in [0,\ 1]$ satisfying $t' = (1 - \alpha)\underline{w'}(N) + \alpha\overline{w'}(N)$. As $t'' = \underline{w''}(N) = \overline{w''}(N)$, it follows that $t'' = (1-\alpha)\underline{w''}(N) + \alpha\overline{w''}(N)$. As σ^* satisfies AD1, it follows that:
$$\sigma_i(w' + w'')(t' + t'') = \sigma_i(w')(t') + \sigma_i(w'')(t'') \quad \forall i \in N.$$

Therefore, σ^* satisfies AD2.

Next, we show that σ^* satisfies SP. Let σ^* be a Shapley mapping, and we consider a combination $(w, v, U) \in UG \times CG_+ \times I(\mathbb{R})$ satisfying $w = Uv$. As $\underline{w}(S) = \underline{U}v(S)$ and $\overline{w}(S) = \overline{U}v(S)$ hold for every $S \subset N$, from the linearity of the Shapley value, the following holds for every $i \in N$:
$$\phi_i(v_{\underline{w}}) = \underline{U}\phi_i(v),\ \ \phi_i(v_{\overline{w}}) = \overline{U}\phi_i(v). \tag{14}$$

(i) Suppose $w \in EG$. As $\underline{U} = \overline{U}$, $\sigma_i^*(w)(t) = \underline{U}\phi_i(v) = \overline{U}\phi_i(v)$ and $t = \underline{U}v(N) = \overline{U}v(N)$. Therefore, letting $f_i : CG_+ \to \mathbb{R}$ be $f_i(v) = \frac{\phi_i(v)}{v(N)}$, $\sigma_i^*(w)(t) = f_i(v)t$ holds.

(ii) Suppose $w \notin EG$, i.e., $\underline{w}(N) < \overline{w}(N)$. Then, note that:

$$\alpha = \frac{t - \underline{w}(N)}{\overline{w}(N) - \underline{w}(N)} = \frac{t - \underline{U}v(N)}{(\overline{U} - \underline{U})v(N)}, \quad 1 - \alpha = \frac{\overline{w}(N) - t}{\overline{w}(N) - \underline{w}(N)} = \frac{\overline{U}v(N) - t}{(\overline{U} - \underline{U})v(N)}.$$

Therefore,

$$\begin{aligned} \sigma_i^*(w)(t) &= (1-\alpha)\phi_i(v_{\underline{w}}) + \alpha\phi_i(v_{\overline{w}}) = \{(1-\alpha)\underline{U} + \alpha\overline{U}\}\phi_i(v) \\ &= \left\{ \frac{\overline{U}\underline{U}v(N) - \underline{U}t}{(\overline{U} - \underline{U})v(N)} + \frac{\overline{U}t - \overline{U}\underline{U}v(N)}{(\overline{U} - \underline{U})v(N)} \right\}\phi_i(v) = \frac{\phi_i(v)}{v(N)} \cdot t. \end{aligned} \tag{15}$$

Similar to (i), letting $f_i : CG_+ \to \mathbb{R}$ be $f_i(v) = \frac{\phi_i(v)}{v(N)}$, (15) implies $\sigma_i^*(w)(t) = f_i(v)t$. □

Proof of Theorem 2. From Lemma 1 and 6, it suffices to prove the uniqueness of σ^*. Suppose a solution mapping σ satisfies EF, SYM, NP, AD2 and SP. We show σ must be σ^*.
(i) Suppose $w \in EG$. In this case, $t = v_{\underline{w}}(N)$ for any $t \in w(N)$. Therefore, from Lemma 2,

$$w(S) = \sum_{R \subset N} [\underline{c_R}, \underline{c_R}] v_R(S), \quad t = \sum_{R \subset N} \underline{c_R}.$$

Furthermore, it can be shown that, if a solution mapping σ satisfies EF, SYM and NP, then the following holds (the proof is essentially the same as that of Lemma 4):

$$\sigma_i([\underline{c_R}, \underline{c_R}]v_R)(\underline{c_R}) = \begin{cases} \frac{\underline{c_R}}{r} & \textit{if } i \in R \\ 0 & \textit{otherwise.} \end{cases}$$

Note that $[\underline{c_R}, \underline{c_R}]v_R \in EG$. By applying AD2, it holds that:

$$\sigma_i(w)(t) = \sigma_i\left(\sum_{R \subset N} [\underline{c_R}, \underline{c_R}]v_R\right)\left(\sum_{R \subset N} \underline{c_R}\right) = \sum_{R \subset N} \sigma_i([\underline{c_R}, \underline{c_R}]v_R)(\underline{c_R}) = \sum_{R: i \in R} \frac{\underline{c_R}}{r}.$$

From Lemma 5, $\sigma_i(w)(t) = \phi_i(v_{\underline{w}})$. As $\sigma_i^*(w)(t) = \phi_i(v_{\underline{w}})$, $\sigma_i(w)(t) = \sigma_i^*(w)(t)$.
(ii) Suppose $w \notin EG$, i.e., $\underline{w}(N) < \overline{w}(N)$. Define $w' \in IG$ and $w', w'' \in EG$ as, for every $S \subset N$,

$$w'(S) = [0, \overline{w}(S) - \underline{w}(S)], w''(S) = [\underline{w}(S), \underline{w}(S)], w'''(S) = [\overline{w}(S) - \underline{w}(S), \overline{w}(S) - \underline{w}(S)].$$

As $w'', w''' \in EG$, similar to (i), it holds that:

$$\begin{aligned} \sigma_i(w'')(t'') &= \sum_{R: i \in R} \frac{\underline{c_R}}{r} = \phi_i(v_{\underline{w}}) \ \forall t'' \in w''(N) \\ \sigma_i(w''')(t''') &= \sum_{R: i \in R} \frac{\overline{c_R} - \underline{c_R}}{r} = \phi_i(v_{\overline{w}}) - \phi_i(v_{\underline{w}}) \quad \forall t''' \in w'''(N). \end{aligned}$$

As $w''' = [1, 1]v_{\overline{w} - \underline{w}}$ and from SP, there exists a real-valued function $f_i : CG_+ \longmapsto \mathbb{R}$ satisfying $\sigma_i(w''')(t''') = f_i(v_{\overline{w} - \underline{w}})t'''$ for every $t''' \in w'''(N)$ and $i \in N$. As $t''' = \overline{w}(N) - \underline{w}(N)$, $f_i(v_{\overline{w} - \underline{w}}) = \frac{\phi_i(v_{\overline{w}}) - \phi_i(v_{\underline{w}})}{\overline{w}(N) - \underline{w}(N)}$. Similarly, because $w' = [0, 1]v_{\overline{w} - \underline{w}}$, there exists f_i satisfying $\sigma_i(w')(t') = f_i(v_{\overline{w} - \underline{w}})t'$, which implies $\sigma_i(w')(t') = \frac{\phi_i(v_{\overline{w}}) - \phi_i(v_{\underline{w}})}{\overline{w}(N) - \underline{w}(N)}t'$. From AD2,

$$\begin{aligned} \sigma_i(w)(t) &= \sigma(w' + w'')(t - \underline{w}(N) + \underline{w}(N)) = \sigma(w')(t - \underline{w}(N)) + \sigma(w'')(\underline{w}(N)) \\ &= \frac{\phi_i(v_{\overline{w}}) - \phi_i(v_{\underline{w}})}{\overline{w}(N) - \underline{w}(N)}(t - \underline{w}(N)) + \phi_i(v_{\underline{w}}) = \frac{\overline{w}(N) - t}{\overline{w}(N) - \underline{w}(N)}\phi_i(v_{\underline{w}}) + \frac{t - \underline{w}(N)}{\overline{w}(N) - \underline{w}(N)}\phi_i(v_{\overline{w}}) \\ &= (1-\alpha)\phi_i(v_{\underline{w}}) + \alpha\phi_i(v_{\overline{w}}) = \sigma_i^*(w)(t). \end{aligned}$$

□

5. Conclusions

This paper studied cooperative interval games in which the payoff uncertainty that players face is expressed as a closed interval. We compared the notion of solution mapping as a solution concept applied to interval games with the existing interval solution concept. Then, we defined the Shapley mapping as a specific form of the solution mapping and showed that the Shapley mapping could be characterized by two different axiomatizations, both of which employed interval game versions of standard axioms used in the traditional cooperative game analysis.

We conclude the analysis by identifying some topics for further research. First, other axiomatizations examined in coalitional games such as Young's [32] strong monotonicity would enhance the validity of the Shapley mapping. Second, whether the properties of the Shapley value in coalitional games are preserved in interval game analyses is also an intriguing topic. For example, whether the Shapley mapping has the consistency property as analyzed by Hart and Mas-Colell's [33] potential approach is worth examining. Third, solution mappings could be applied to actual cooperative game situations with uncertainty such as the bankruptcy problem and cost allocation games. For example, a classical game-theoretic analysis of the bankruptcy problem essentially entails the uncertainties that creditors face regarding a debtor's future solvency. Here, new insights could be obtained by reconstructing the bankruptcy problem as an interval game and applying the solution mapping to those games. (Branzei et al. [8] applied interval games to the bankruptcy problem taking the credit amount as a source of uncertainty. Debtors' fiscal conditions can also be a source of uncertainty for debt contracts.) Finally, as a limitation of our study, it should be noted that in this paper we restricted our attention to singleton set solution concepts such as the Shapley mapping. In this regard, nonsingleton set solution concepts such as the core and stable sets may also be redefined as solution mappings in interval games. Once those nonsingleton-type solutions are defined as solution mappings, it is possible to examine the relationships between them. For example, whether a real-valued payoff vector assigned by the Shapley mapping is included in the core mapping for every realization of the worth set of the grand coalition or which class of the interval games satisfies this property may be worth examining.

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