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# The Partial Inverse Spectral and Nodal Problems for Sturm–Liouville Operators on a Star-Shaped Graph

**Xian-Biao Wei [1,*], Yan-Hsiou Cheng [2] and Yu-Ping Wang [3]**

1 Department of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China
2 Department of Mathematics and Information Education, National Taipei University of Education, Taipei City 106, Taiwan
3 Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China
* Correspondence: xbwei@ahjzu.edu.cn

**Abstract:** We firstly prove the Horváth-type theorem for Sturm–Liouville operators on a star-shaped graph and then solve a new partial inverse nodal problem for this operator. We give some algorithms to recover this operator from a dense nodal subset and prove uniqueness theorems from paired-dense nodal subsets in interior subintervals having a central vertex. In particular, we obtain some uniqueness theorems by replacing the information of nodal data on some fixed edge with part of the eigenvalues under some conditions.

**Keywords:** partial inverse spectral problem; partial inverse nodal problem; boundary value problem; graph; paired-dense nodal subset

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## 1. Introduction

Consider the following boundary value problem $B := B(q, \alpha)$, $q(x) := \{q_l(x)\}_{l=1}^p$, $\alpha = \{\alpha_l\}_{l=1}^p$ on a star-shaped graph with $p$ edges of identical length 1, defined as follows:

$$-y_l'' + q_l(x)y_l = \lambda y_l, \quad x \in (0,1), \quad l = \overline{1,p}, \tag{1}$$

associated with the separated boundary conditions at the pendant vertices 0

$$y_l(0,\lambda)\cos\alpha_l + y_l'(0,\lambda)\sin\alpha_l = 0, \quad l = \overline{1,p}, \tag{2}$$

and the standard matching conditions at the central vertex 1

$$y_1(1,\lambda) = y_l(1,\lambda), \quad l = \overline{2,p}, \quad \sum_{l=1}^{p} y_l'(1,\lambda) = 0, \tag{3}$$

where $\lambda$ is the spectral parameter, $\alpha_l \in [0,\pi)$ and $q_l(x)$, and $l = \overline{1,p}$ is called the potential and is an integrable real-valued function on the $l$-th edge. The differential operators on quantum graphs have many applications in chemistry, mathematics, networks, spider webs, and so on (see [1–17] and the references therein).

The problem $B$ is a natural extension of the classical Sturm–Liouville problem on the finite interval. The inverse nodal problems for the classical Sturm–Liouville operators are to recover the potential and boundary conditions by using its nodal data [18–23]. McLaughlin [22] firstly studied the inverse nodal problem for the classical Sturm–Liouville operator and showed that one set of nodal points can determine the Sturm–Liouville operators uniquely. The solution of the potential function to this problem was given by Hald and McLaughlin [19]. The uniqueness results show that the inverse nodal problem is overdetermined. Later on, there was much study focus on how to use less information of nodal data to recover the potential. The uniqueness theorems and the reconstruction formulae are given by using twin-dense nodal subset [15,20,21,24–26], dense nodal subset [23,27,28], and partial nodal data [15,29–31]. Guo and Wei [30] presented a sharp condition on the nodal subset and proved the uniqueness for the classical Sturm–Liouville operator with a paired-dense nodal subset in interior subintervals under some conditions based on the Gesztesy–Simon theorem in [32]. In addition, the theory on dynamic Sturm–Liouville boundary value problems via variational methods was found in [33,34].

Beginning in 2002, Kuchment [5–8] studied quantum graphs and investigated the spectral properties of periodic boundary value problems for a carbon atom in graphene. In [12,13], Pivovarchik studied inverse spectral problems with Dirichlet boundary conditions for a star-shaped graph with $p$ edges. He gave the asymptotic expansion of eigenvalues and showed that there are $p$ sequences of eigenvalues where one sequence is simple while the others might not be. In particular, Law and Pivovarchik [35] discussed the multiplicity of the eigenvalues and interlacing properties between two spectral sets of the Sturm–Liouville problems defined on a tree. Recently, Luo, Jatulan and Law [36] gave a complete classification of Archimedean tilings for the periodic quantum graphs and investigated the sufficient conditions for point spectrum and continuous spectrum. Bondarenko [2] showed that if all components $q_l(x)$, $l = \overline{1,p}$ but one on the graph are given a priori, the remaining component can be uniquely determined by two sequences of chosen eigenvalues and provided a constructive algorithm for the solution of the partial inverse problem. In [37], Wang and Shieh generalized Bondarenko's theorem by the methods in [38]. In this paper, we are going to solve the following partial inverse spectral problem for $B$:

**IP1:** (Inverse Problem 1) If $q_l(x) = \widetilde{q}_l(x)$ on $[0,1]$, $\alpha_l$ for $l \neq i_0$ and $q_{i_0}(x) = \widetilde{q}_{i_0}(x)$ on $[a_0, 1]$ for some $a_0$, $a_0 \in (0,1]$ given a priori, recover $q_{i_0}(x)$ and $\alpha_{i_0}$ from part of the eigenvalues.

On the other hand, the inverse nodal problems on quantum graphs have been studied. In 2007, Currie and Watson [39] studied the inverse nodal problems on general graphs and showed that, for $q_i \in L^\infty$, one set of eigenvalues and nodal positions is sufficient to reconstruct the potentials $q_i's$. In 2008, Yurko [40] discussed the inverse nodal problem for $B$ with $\alpha_l = 0$, $l = \overline{1,p}$ and proved that each component $q_l(x)$ can be uniquely determined up to a constant by a dense nodal set. Later on, Cheng [41] derived the asymptotics of eigenvalues of $B$ with $q_l \in L^1$, $l = \overline{1,p}$ and presented direct and explicit formulae on recovering the potentials using a twin-dense nodal subset. Wang and Shieh [31] investigated the partial inverse nodal problem for $B$ with Dirichlet boundary conditions from a twin-dense nodal subset in interior subintervals under some conditions. Therefore, we are going to solve the following partial inverse nodal problem for $B$ with less nodal information:

**IP2:** (Inverse Problem 2) Recover the component $q_l(x)$, $l = \overline{1,p}$ from given paired-dense nodal subsets on subintervals having a central vertex.

We firstly prove the Horváth-type theorem for $B$ and extend Horváth's method in [38] for the classical Sturm–Liouville operator to $B$, which is also the theoretical basis for the solution of the partial inverse nodal problem for $B$. Then, we show that the components $\{q_l(x)\}_{l=1}^p$ for $B$ can be uniquely determined up to a constant by a dense nodal subset corresponding to the first eigenvalue sequence in $[0,1]$; see Theorem 2. We also give algorithms to reconstruct $\{q_l(x)\}_{l=1}^p$ and $\{\alpha_l\}_{l=1}^p$ from a dense nodal subset. In Theorem 3, combined with the Horváth-type theorem for $B$, we show that if there is a paired-dense nodal subset corresponding to the first eigenvalue sequence in a interior subinterval, then, with a sufficiently large counting number corresponding to the first eigenvalue sequence, we can uniquely determine the components $\{q_l(x)\}_{l=1}^p$ up to a constant on the whole graph. Finally, in Theorem 4, without any nodal data on some $i_0$-th edge but with part of the eigenvalues, we can also uniquely determine components $\{q_l(x)\}_{l=1}^p$ up to a constant on the whole graph from a paired-dense nodal subset corresponding to the first eigenvalue sequence and sufficiently large counting numbers. We extend Guo-Wei's method in [30] for the classical Sturm–Liouville operator to $B$.

This article is organized as follows. In Section 2, we present preliminaries. We give the asymptotic formulae of nodal points. We present solutions to **IP1** in Section 3 and **IP2** in Section 4, respectively.

## 2. Preliminaries

Let $S_l(x,\lambda),\ C_l(x,\lambda)$, and $\varphi_l(x,\lambda)$ be solutions of (1) for each $l=\overline{1,p}$ associated with the initial conditions:

$$S_l(0,\lambda)=0,\ S_l'(0,\lambda)=1,\ C_l(0,\lambda)=1,\ C_l'(0,\lambda)=0,$$
$$\varphi_l(0,\lambda)=\sin\alpha_l,\ \varphi_l'(0,\lambda)=-\cos\alpha_l.$$

Moreover, we have

$$\varphi_l(x,\lambda)=\sin\alpha_l C_l(x,\lambda)-\cos\alpha_l S_l(x,\lambda).$$

By the results in [42], we obtain the asymptotic formulae:

(a) If $\alpha_l=0$,

$$\begin{cases} \varphi_l(x,\lambda)=\dfrac{\sin\rho x}{\rho}-\dfrac{Q_l(x)\cos\rho x}{\rho^2}+\mathrm{o}\left(\dfrac{\mathrm{e}^{\tau x}}{\rho^2}\right), & 0<x<1,\\ \varphi_l'(x,\lambda)=\cos\rho x+\dfrac{Q_l(x)\sin\rho x}{\rho}+\mathrm{o}\left(\dfrac{\mathrm{e}^{\tau x}}{\rho}\right), & 0<x<1, \end{cases} \tag{4}$$

(b) If $0<\alpha_l<\pi$,

$$\begin{cases} \varphi_l(x,\lambda)=\sin\alpha_l\cos\rho x+(Q_l(x)\sin\alpha_l-\cos\alpha_l)\dfrac{\sin\rho x}{\rho}+\mathrm{o}\left(\frac{\mathrm{e}^{\tau x}}{\rho}\right), & 0<x<1,\\ \varphi_l'(x,\lambda)=-\rho\sin\alpha_l\sin\rho x+(Q_l(x)\sin\alpha_l-\cos\alpha_l)\cos\rho x+\mathrm{o}(\mathrm{e}^{\tau x}), & 0<x<1 \end{cases} \tag{5}$$

for $|\lambda|\to\infty$, where $\rho=\sqrt{\lambda}$, $\tau=|\mathrm{Im}\rho|$, and

$$Q_l(x):=\frac{1}{2}\int_0^x q_l(t)\mathrm{d}t,\quad l=\overline{1,p}.$$

The characteristic function $\Delta(\lambda)$ of $B$ is defined by

$$\Delta(\lambda):=\sum_{l=1}^{p}\varphi_l'(1,\lambda)\prod_{k=1,k\neq l}^{p}\varphi_k(1,\lambda), \tag{6}$$

which is an entire function in $\lambda$ of order $1/2$, where all zeros of $\Delta(\lambda)$ coincide with the eigenvalues of $B$. Denote $\sigma(B):=\cup_{m=1}^{p}M_m$ asthe eigenvalue set of $B$ (counting with their multiplicities) where $M_m=\{\lambda_{m,n}\}_{n\in\mathbb{N}}$ and $\rho_{m,n}:=\sqrt{\lambda_{m,n}}$. We shall find the asymptotic formulae of nodal points separately corresponding to the three cases:

$$\begin{aligned} &\text{(I)}\ \ \alpha_l=0,\quad l=\overline{1,p};\\ &\text{(II)}\ \ \alpha_l\in(0,\pi),\quad l=\overline{1,p};\\ &\text{(III)}\ \alpha_l=0,\quad l=\overline{1,T},\quad \text{and}\quad \alpha_l\in(0,\pi),\quad l=\overline{T+1,p},\quad 1\le T\le p-1. \end{aligned} \tag{7}$$

By (Theorem 2.1 [41]), all eigenvalues are real. For the case I, there exist $p$ sequences of eigenvalues $\lambda_{m,n}$ with the asymptotic formulae:

$$\begin{cases} \rho_{1,n}=\left(n-\dfrac{1}{2}\right)\pi+\dfrac{1}{2p(n-\frac{1}{2})\pi}\displaystyle\int_0^1(1-\cos((2n-1)\pi t))\left(\sum_{l=1}^{p}q_l(t)\right)\mathrm{d}t+\mathrm{O}\left(\dfrac{1}{n^2}\right),\\ \rho_{m,n}=n\pi+\dfrac{\Lambda_{m,n,1}}{n\pi}+\mathrm{O}\left(\dfrac{1}{n^2}\right),\ m=\overline{2,p}, \end{cases} \tag{8}$$

for $n \gg 1$, where $\Lambda_{m,n,1}$ is the $(m-1)$-th, $m=\overline{2,p}$ zero of the polynomial $p_1(\Lambda)$ of degree $(p-1)$

$$p_1(\Lambda) := \sum_{l=1}^{p} \prod_{i\neq l} \left[ \Lambda - \frac{1}{2}\int_0^1 (1-\cos((2n-1)\pi t)) q_i(t) \mathrm{d}t \right]. \tag{9}$$

For case II, there exist $p$ sequences of eigenvalues $\lambda_{m,n}$ with the asymptotic formulae

$$\begin{cases} \rho_{1,n} = (n-1)\pi + \dfrac{1}{2(n-1)p\pi}\left( -2A_1 + \displaystyle\int_0^1 (1+\cos(2(n-1)\pi t)\left(\sum_{l=1}^{p} q_l(t)\right)\right)\mathrm{d}t + \mathrm{O}\left(\dfrac{1}{n^2}\right), \\ \rho_{m,n} = n\pi + \dfrac{\Lambda_{m,n,2}}{n\pi} + \mathrm{O}\left(\dfrac{1}{n^2}\right), \ m=\overline{2,p}, \end{cases} \tag{10}$$

for $n \gg 1$, where

$$A_1 = \sum_{l=1}^{p} \cot\alpha_l, \tag{11}$$

and $\Lambda_{m,n,2}$ is the $(m-1)$-th, $m=\overline{2,p}$ zero of the polynomial $p_2(\Lambda)$ of degree $(p-1)$

$$p_2(\Lambda) := \sum_{l=1}^{p} \prod_{i\neq l} \left[ \Lambda - \cot\alpha_i + \frac{1}{2}\int_0^1 (1+\cos(2(n-1)\pi t)) q_i(t) \mathrm{d}t \right], \tag{12}$$

and for the case III, there exist $p$ sequences of eigenvalues $\lambda_{m,n}$ with the asymptotic formulae

$$\begin{cases} \rho_{m,n} = n\pi + (-1)^m d_1 + \dfrac{\omega_1}{2n\pi} + \mathrm{o}\left(\dfrac{1}{n}\right), \quad m=1,2, \\ \rho_{m,n} = n\pi + \dfrac{\Lambda_{m,n,3}}{n\pi} + \mathrm{O}\left(\dfrac{1}{n^2}\right), \quad m=\overline{3,T+1}, \\ \rho_{m,n} = \left(n-\frac{1}{2}\right)\pi + \dfrac{\Lambda_{m,n,4}}{\left(n-\frac{1}{2}\right)\pi} + \mathrm{O}\left(\dfrac{1}{n^2}\right), \quad m=\overline{T+2,p}, \end{cases} \tag{13}$$

for $n \gg 1$, where

$$d_1 := \arcsin\sqrt{\frac{T}{p}},\ A_2 := \sum_{l=T+1}^{p} \cot\alpha_l,\ \omega_1 := \frac{1}{p}\left( (p-T)\sum_{l=1}^{T} Q_l(1) + T\sum_{l=T+1}^{p} Q_l(1) - TA_2 \right), \tag{14}$$

$\Lambda_{m,n,3}$ is the $(m-2)$-th, $m=\overline{3,T+1}$ root of the polynomial $p_3(\Lambda)$ of degree $(T-1)$

$$p_3(\Lambda) := \sum_{l=1}^{T} \prod_{i\neq l, i=1}^{T} \left( \Lambda - \frac{1}{2}\int_0^1 (1-\cos(2n\pi t)) q_i(t) \mathrm{d}t \right), \tag{15}$$

and $\Lambda_{m,n,4}$ is the $(m-T-1)$-th, $m=\overline{T+2,p}$ root of the polynomial $p_4(\Lambda)$ of degree $(p-T-1)$

$$p_4(\Lambda) := \sum_{l=T+1}^{p} \prod_{i\neq l, i=T+1}^{p} \left( \Lambda - \cot\alpha_i + \frac{1}{2}\int_0^1 (1+\cos((2n-1)\pi t)) q_i(t) \mathrm{d}t \right). \tag{16}$$

The function

$$m_l(x,\lambda) := -\frac{\varphi_l'(x,\lambda)}{\varphi_l(x,\lambda)}, \quad x\in(0,1], \quad l=\overline{1,p},$$

is called the Weyl $m$-function of $B_l$, where the problem $B_l$ is defined by by (1), (2) and $\varphi_l(1,\lambda)=0$. Applying the same arguments as the proof of Marchenko's theorem in [43], one shows that the Weyl $m$-function $m_l(a,\lambda)$ uniquely determines $q_l(x)$ on $[0,a]$ with

$0 < a \le 1$ and $\alpha_l$. The eigenfunction $y(x, \lambda_{m,n})$ corresponding to the eigenvalue $\lambda_{m,n}$ of $B$ is of the form:

$$y(x, \lambda_{m,n}) = \{c_l(\lambda_{m,n})\varphi_l(x, \lambda_{m,n})\}_{l=1}^{p},$$

where $c_l(\lambda_{m,n}), l = \overline{1,p}$ are constant, do not depend on $x$, and are not all zeros. The function $\varphi_l(x, \lambda_{1,n})$ is called the $l$-th component of $y(x, \lambda_{1,n})$. Let $x_{l,1,n}^j$ be the $j$-th nodal point of the $l$-th component $\varphi_l(x, \lambda_{1,n})$ corresponding to the eigenvalue $\lambda_{1,n}$, i.e., $\varphi_l(x_{l,1,n}^j, \lambda_{1,n}) = 0,\ l = \overline{1,p}$. The $l$-th component $\varphi_l(x, \lambda_{1,n})$ has exactly $n-1$ (simple) zeros inside the interval $(0,1)$, and

$$0 < x_{l,1,n}^1 < x_{l,1,n}^2 < \cdots < x_{l,1,n}^j < \cdots < x_{l,1,n}^{n-1} < 1.$$

For $l = \overline{1,p}$, let $X_{l,1} := \{x_{l,1,n}^j\}$ be the nodal set of the $l$-th component $\varphi_l(x, \lambda_{1,n})$ corresponding to $M_1$. Then, $X_{l,1}$ is dense on $[0,1]$ (see below for Lemma 1). Since we can only obtain the same nodal information from the same eigenvalues, we assume that $I_1 := \{n_{1,k}\}_{k=K_0}^{\infty}$ is a strictly increasing subsequence in $\mathbb{N}$ (where $K_0$ is defined in Lemma 2) such that

$$M_{1,0} := \left\{\lambda_{1,n_{1,k}} : \lambda_{1,n_{1,k_1}} < \lambda_{1,n_{1,k_2}} \text{ for any } n_{1,k_1} < n_{1,k_2},\ n_{1,k_1}, n_{1,k_2} \in I_1\right\}.$$

Next, we shall give the definition of a paired-dense nodal subset on a finite interval.

**Definition 1.** *For each $l = \overline{1,p}$, denote $W_{I_1}([a_l, b_l]) \subseteq X_{l,1} \bigcap [a_l, b_l]$ with $0 \le a_l < b_l \le 1$ on the $l$-th edge. The nodal subset $W_{I_1}([a_l, b_l])$ is called a paired-dense nodal subset on $[a_l, b_l]$ corresponding to $I_1$ if the following conditions hold:*

1. *For each $n_{1,k} \in I_1$, there exist some $j_k, r_k \ge 1$, $r_k \in \mathbb{N}$, such that $x_{l,1,n_{1,k}}^{j_k}, x_{l,1,n_{1,k}}^{j_k+r_k} \in W_{I_1}([a_l, b_l])$.*
2. *$\overline{W_{I_1}([a_l, b_l])} = [a_l, b_l]$.*

The definition of the paired-dense nodal subset was given in [30]. Clearly, the twin-dense nodal subset is a special case of the paired-dense nodal subset. Denote

$$\omega_0 = \frac{1}{p}\sum_{l=1}^{p} Q_l(1), \quad \alpha_n^j = \begin{cases} \dfrac{j}{n-\frac{1}{2}}, & \text{for I}, \\ \dfrac{j-\frac{1}{2}}{n-1}, & \text{for II}, \\ \dfrac{j}{n}, \quad l = \overline{1,T}, & \text{for III}, \\ \dfrac{j-\frac{1}{2}}{n}, \quad l = \overline{T+1,p}, & \text{for III}, \end{cases}$$

By the asymptotic behavior of $\lambda_{1,n}$ and $\varphi_l(x, \lambda_{1,n})$, one can easily obtain asymptotic behavior of nodal points. We omit the proof.

**Lemma 1.** *For three cases, the nodal points $x_{l,1,n}^j$ of the $l$-th component $\varphi(x, \lambda_{1,n})$ corresponding to the eigenvalue $\lambda_{1,n}$ have the asymptotic formulae:*

$$x_{l,1,n}^j = \alpha_n^j + \frac{1}{2(n-\frac{1}{2})^2\pi^2}\left(\int_0^{\alpha_n^j} q_l(t)\mathrm{d}t - 2\omega_0\alpha_n^j\right) + \mathrm{o}\left(\frac{1}{n^2}\right), \quad \text{for I}, \tag{17}$$

$$x_{l,1,n}^j = \alpha_n^j - \frac{\cot\alpha_l}{(n-1)^2\pi^2} + \frac{1}{2(n-1)^2\pi^2}\int_0^{\alpha_n^j} q_l(t)\mathrm{d}t$$

$$+\frac{1}{2(n-1)^2\pi^2}\left(\frac{2A_1}{p}-\omega_0\right)\alpha_n^j+\mathrm{o}\left(\frac{1}{n^2}\right),\ \text{for}\quad \text{II}, \tag{18}$$

$$x_{l,1,n}^j=\alpha_n^j+\frac{d_1\alpha_n^j}{n\pi}+\frac{1}{2(n\pi)^2}\int_0^{\alpha_n^j}q_l(t)\mathrm{d}t$$
$$+\frac{2d_1^2-\omega_1}{2(n\pi)^2}\alpha_n^j+\mathrm{o}\left(\frac{1}{n^2}\right),\ l=\overline{1,T},\quad \text{for}\quad \text{III}, \tag{19}$$

$$x_{l,1,n}^j=\alpha_n^j+\frac{\alpha_n^j}{n\pi}-\frac{\cot\alpha_l}{n^2\pi^2}+\frac{1}{2(n\pi)^2}\int_0^{\alpha_n^j}q_l(t)\mathrm{d}t$$
$$+\frac{2d_1^2-\omega_1}{2(n\pi)^2}\alpha_n^j+\mathrm{o}\left(\frac{1}{n^2}\right),\quad l=\overline{T+1,p},\quad \text{for}\quad \text{III} \tag{20}$$

*for $n\gg 1$ uniformly in $j$, where $\omega_1$, $d_1$, and $A_1$ are defined in* (11) *and* (14).

## 3. Partial Inverse Spectral Problems

In this section, we shall study the partial inverse spectral problem for $B$. Let the boundary value problem $\widetilde{B}$ have the same form as $B$ but with different coefficients. If a certain symbol $\gamma$ denotes an object related to $B$, then the corresponding symbol $\widetilde{\gamma}$ with a tilde denotes the analogous object related to $\widetilde{B}$. Let $\widehat{\gamma}=\gamma-\widetilde{\gamma}$.

For $m=\overline{2,p}$, let $I_m:=\{n_{m,k}\}_{k=1}^{\infty}$ be a strictly increasing subsequence in $\mathbb{N}$, and denote $M_{m,0}:=\{\lambda_{m,n_{m,k}}:\ n_{m,k}\in I_m\}\subseteq M_m$. For each $m=\overline{1,p}$, the counting function corresponding to $M_{m,0}$ is defined by

$$N_{M_{m,0}}(t):=\sum_{\rho_{m,n_{m,k}}<t,\lambda_{m,n_{m,k}}\in M_{m,0}}1,\quad t\in\mathbb{R}^+.$$

By (6), this yields

$$\Delta(\lambda)=\gamma_1(\lambda)\varphi_{i_0}'(1,\lambda)+\gamma_2(\lambda)\varphi_{i_0}(1,\lambda), \tag{21}$$

where

$$\gamma_1(\lambda)=\prod_{l\neq i_0}\varphi_l(1,\lambda),\quad\text{and}\quad \gamma_2(\lambda)=\sum_{l\neq i_0}^{p}\varphi_l'(1,\lambda)\prod_{k\neq i_0,k\neq l}^{p}\varphi_k(1,\lambda).$$

Clearly the entire functions $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ in $\lambda$ are only dependent on $q_l(x)$ and $\alpha_l, l\neq i_0$. If $\gamma_1(\lambda_{m,n})=\gamma_2(\lambda_{m,n})=0$, then we cannot obtain any information about the component $q_{i_0}(x)$ from the eigenvalue $\lambda_{m,n}$ by (21). Hence, we add the following Assumption 1:

**Assumption 1.** *For each $\lambda_{m,n_{m,k}}\in M_{m,0}$, such that*

$$\gamma_1^2(\lambda_{m,n_{m,k}})+\gamma_2^2(\lambda_{m,n_{m,k}})\neq 0,\quad m=\overline{1,p}.$$

*We shall prove the following Horváth type-theorem for B, which is a solution to* ***IP1****:*

**Theorem 1.** *Let $q_l(x)=\widetilde{q}_l(x)$ on $[0,1]$, $\alpha_l=\widetilde{\alpha}_l$ for $l\neq i_0$ and $q_{i_0}(x)=\widetilde{q}_{i_0}(x)$ on $[a_0,1]$ for some $a_0$, $a_0\in(0,1]$ be given a priori. If $M_{k,0}=\widetilde{M}_{k,0}$ with Assumption 1 satisfied, and there exist $t_0>0$, $0\le\kappa_1\le 1$, $\delta_1>0$ such that*

$$\sum_{m=1}^{p}N_{M_{m,0}}(t)\ge 2a_0\left\{\kappa_1\left[\frac{t}{\pi}+\frac{1}{2}\right]+(1-\kappa_1)\left(\left[\frac{t}{\pi}\right]+\frac{1}{2}\right)-1+\kappa_1+\mathrm{O}(t^{-\delta_1})\right\},\ \text{if } \alpha_{i_0}=0; \tag{22}$$

$$\sum_{m=1}^{p}N_{M_{m,0}}(t)\ge 2a_0\left\{\kappa_1\left[\frac{t}{\pi}+\frac{1}{2}\right]+(1-\kappa_1)\left(\left[\frac{t}{\pi}\right]+\frac{1}{2}\right)+\kappa_1+\mathrm{O}(t^{-\delta_1})\right\},\ \text{if } \alpha_{i_0}\neq 0 \tag{23}$$

*for sufficiently large $t \geq t_0$, and*

$$\lim_{t\to\infty} \frac{\sum_{m=1}^{p} N_{M_{m,0}}(t)}{t} = \frac{2a_0}{\pi},$$

*where $[x]$ denotes the largest integer less than or equal to $x$, then*

$$q_{i_0}(x) \overset{a.e.}{=} \widetilde{q}_{i_0}(x) \quad on \quad [0,1] \quad and \quad \alpha_{i_0} = \widetilde{\alpha}_{i_0}. \tag{24}$$

**Proof.** It follows from (1) for $l = i_0$

$$\int_0^1 \widehat{q}_{i_0}(x)\varphi_{i_0}(x,\lambda_{m,n_{m,k}})\widetilde{\varphi}_{i_0}(x,\lambda_{m,n_{m,k}})\mathrm{d}x =< \varphi_{i_0}, \widetilde{\varphi}_{i_0} > (1,\lambda_{m,n_{m,k}}) - < \varphi_{i_0}, \widetilde{\varphi}_{i_0} > (0,\lambda_{m,n_{m,k}}),$$

where $< \varphi_l, \widetilde{\varphi}_l > (x,\lambda) := \varphi_l(x,\lambda)\widetilde{\varphi}_l'(x,\lambda) - \varphi_l'(x,\lambda)\widetilde{\varphi}_l(x,\lambda), l = \overline{1,p}$, is called the Wronskian of $\varphi_l(x,\lambda)$ and $\widetilde{\varphi}_l(x,\lambda)$. This implies

$$< \varphi_{i_0}, \widetilde{\varphi}_{i_0} > (1,\lambda_{m,n_{m,k}}) = \int_0^1 \widehat{q}_{i_0}(x)\varphi_{i_0}(x,\lambda_{m,n_{m,k}})\widetilde{\varphi}_{i_0}(x,\lambda_{m,n_{m,k}})\mathrm{d}x - \sin\widehat{\alpha}_{i_0}. \tag{25}$$

The assumption $q_l(x) = \widetilde{q}_l(x)$ on $[0,1]$ and $\alpha_l = \widetilde{\alpha}_l$ for all $l \neq i_0$ together with the initial conditions $\varphi_l(0,\lambda) = \widetilde{\varphi}_l(0,\lambda) = \sin\alpha_l,\ \varphi_l'(0,\lambda) = \widetilde{\varphi}_l'(0,\lambda) = -\cos\alpha_l$ show that

$$\varphi_l(1,\lambda_{m,n_{m,k}}) = \widetilde{\varphi}_l(1,\lambda_{m,n_{m,k}}) \quad \text{and} \quad \varphi_l'(1,\lambda_{m,n_{m,k}}) = \widetilde{\varphi}_l'(1,\lambda_{m,n_{m,k}}) \tag{26}$$

for all $m = \overline{1,p}$. From (21) and (26), it is clear that

$$\gamma_1(\lambda_{m,n_{m,k}}) = \widetilde{\gamma}_1(\lambda_{m,n_{m,k}}) \quad \text{and} \quad \gamma_2(\lambda_{m,n_{m,k}}) = \widetilde{\gamma}_2(\lambda_{m,n_{m,k}}) \tag{27}$$

By Assumption 1 and (21), we have

$$\begin{cases} \varphi_{i_0}'(1,\lambda_{m,n_{m,k}}) = -\dfrac{\gamma_2(\lambda_{m,n_{m,k}})}{\gamma_1(\lambda_{m,n_{m,k}})}\varphi_{i_0}(1,\lambda_{m,n_{m,k}}), \\ \widetilde{\varphi}_{i_0}'(1,\lambda_{m,n_{m,k}}) = -\dfrac{\gamma_2(\lambda_{m,n_{m,k}})}{\gamma_1(\lambda_{m,n_{m,k}})}\widetilde{\varphi}_{i_0}(1,\lambda_{m,n_{m,k}}), \end{cases} \quad \text{if} \quad \gamma_1(\lambda_{m,n_{m,k}}) \neq 0, \tag{28}$$

or

$$\begin{cases} \varphi_{i_0}(1,\lambda_{m,n_{m,k}}) = -\dfrac{\gamma_1(\lambda_{m,n_{m,k}})}{\gamma_2(\lambda_{m,n_{m,k}})}\varphi_{i_0}'(1,\lambda_{m,n_{m,k}}), \\ \widetilde{\varphi}_{i_0}(1,\lambda_{m,n_{m,k}}) = -\dfrac{\gamma_1(\lambda_{m,n_{m,k}})}{\gamma_2(\lambda_{m,n_{m,k}})}\widetilde{\varphi}_{i_0}'(1,\lambda_{m,n_{m,k}}), \end{cases} \quad \text{if} \quad \gamma_2(\lambda_{m,n_{m,k}}) \neq 0. \tag{29}$$

It follows from (25) and (27)–(29) that

$$< \varphi_{i_0}, \widetilde{\varphi}_{i_0} > (1,\lambda_{m,n_{m,k}}) = 0, \quad \forall n_{m,k} \in I_m, \quad m = \overline{1,p}. \tag{30}$$

By $q_{i_0}(x) = \widetilde{q}_{i_0}(x)$ on $[a_0,1]$, this yields

$$< \varphi_{i_0}, \widetilde{\varphi}_{i_0} > (a_0,\lambda_{m,n_{m,k}}) = 0, \quad \forall n_{m,k} \in I_m, \quad m = \overline{1,p}. \tag{31}$$

Define the function $K_{i_0}(\lambda)$ by

$$K_{i_0}(\lambda) := \frac{< \varphi_{i_0}, \widetilde{\varphi}_{0,i_0} > (a_0,\lambda)}{\prod_{m=1}^{p} F_m(\lambda)}, \tag{32}$$

where

$$F_m(\lambda) := \prod_{\lambda_{m,n_{m,k}} \in M_{m,0}} \left(1 - \frac{\lambda}{\lambda_{m,n_{m,k}}}\right), \quad m = \overline{1,p}. \tag{33}$$

If $\lambda_{m,n_{m,k}} = 0$, we substitute $1 - \dfrac{\lambda}{\lambda_{m,n_{m,k}}}$ by $\lambda$ in (33). If the eigenvalue $\lambda_{m,n_{m,k}} \in \cup_{m=1}^{p} M_{m0}$ is simple, then (31) guarantees that the function $K_{i_0}(\lambda)$ is analytical at $\lambda = \lambda_{m,n_{m,k}}$. By (8), (10) and (13), we see that the multiplicity of each eigenvalue can be only finite. Assume that the multiplicity of the eigenvalue $\lambda_{m,n_{m,k}} := \lambda_0 \in \cup_{m=1}^{p} M_{m,0}$ is $k_0, k_0 \geq 2$. Then,

$$\Delta(\lambda_0) = \frac{\mathrm{d}\Delta(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} = \cdots = \frac{\mathrm{d}^{k_0-1}\Delta(\lambda)}{\mathrm{d}\lambda^{k_0-1}}\Big|_{\lambda=\lambda_0} = 0. \tag{34}$$

Consequently, (21) and (34) show that

$$\begin{cases} \gamma_1(\lambda_0)\varphi'_{i_0}(1,\lambda_0) + \gamma_2(\lambda_0)\varphi_{i_0}(1,\lambda_0) = 0, \\ \gamma_1(\lambda_0)\frac{\mathrm{d}\varphi'_{i_0}(1,\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} + \gamma_2(\lambda_0)\frac{\mathrm{d}\varphi_{i_0}(1,\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} \\ +\dfrac{\mathrm{d}\gamma_1(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}\varphi'_{i_0}(1,\lambda_0) + \dfrac{\mathrm{d}\gamma_2(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}\varphi_{i_0}(1,\lambda_0) = 0, \\ \qquad \vdots \\ \sum_{k=0}^{k_0-1} C_{k_0-1}^{k}\dfrac{\mathrm{d}^k\gamma_1(\lambda)}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0}\dfrac{\mathrm{d}^{k_0-1-k}\varphi'_{i_0}(1,\lambda)}{\mathrm{d}\lambda^{k_0-1-k}}\Big|_{\lambda=\lambda_0} \\ +\sum_{k=0}^{k_0-1} C_{k_0-1}^{k}\dfrac{\mathrm{d}^k\gamma_2(\lambda)}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0}\dfrac{\mathrm{d}^{k_0-1-k}\varphi_{i_0}(1,\lambda)}{\mathrm{d}\lambda^{k_0-1-k}}\Big|_{\lambda=\lambda_0} = 0, \end{cases} \tag{35}$$

Similar to (35), we have

$$\begin{cases} \gamma_1(\lambda_0)\widetilde{\varphi}'_{i_0}(1,\lambda_0) + \gamma_2(\lambda_0)\widetilde{\varphi}_{i_0}(1,\lambda_0) = 0, \\ \gamma_1(\lambda_0)\frac{\mathrm{d}\widetilde{\varphi}'_{i_0}(1,\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} + \gamma_2(\lambda_0)\frac{\mathrm{d}\widetilde{\varphi}_{i_0}(1,\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} \\ +\dfrac{\mathrm{d}\gamma_1(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}\widetilde{\varphi}'_{i_0}(1,\lambda_0) + \dfrac{\mathrm{d}\gamma_2(\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0}\widetilde{\varphi}_{i_0}(1,\lambda_0) = 0, \\ \qquad \vdots \\ \sum_{k=0}^{k_0-1} C_{k_0-1}^{k}\dfrac{\mathrm{d}^k\gamma_1(\lambda)}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0}\dfrac{\mathrm{d}^{k_0-1-k}\widetilde{\varphi}'_{i_0}(1,\lambda)}{\mathrm{d}\lambda^{k_0-1-k}}\Big|_{\lambda=\lambda_0} \\ +\sum_{k=0}^{k_0-1} C_{k_0-1}^{k}\dfrac{\mathrm{d}^k\gamma_2(\lambda)}{\mathrm{d}\lambda^k}\Big|_{\lambda=\lambda_0}\dfrac{\mathrm{d}^{k_0-1-k}\widetilde{\varphi}_{i_0}(1,\lambda)}{\mathrm{d}\lambda^{k_0-1-k}}\Big|_{\lambda=\lambda_0} = 0. \end{cases} \tag{36}$$

It follows from (35) and (36) that

$$< \varphi_{i_0}, \widetilde{\varphi}_{0,i_0} > (1,\lambda_0) = \frac{\mathrm{d} < \varphi_{i_0}, \widetilde{\varphi}_{0,i_0} > (1,\lambda)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_0} = \cdots = \frac{\mathrm{d}^{k_0-1} < \varphi_{i_0}, \widetilde{\varphi}_{0,i_0} > (1,\lambda)}{\mathrm{d}\lambda^{k_0-1}}\Big|_{\lambda=\lambda_0} = 0.$$

This implies that the function $< \varphi_{i_0}, \widetilde{\varphi}_{0,i_0} > (1,\lambda)$ has zeros at $\lambda_0$ of at least $k_0$. Moreover, the function $< \varphi_{i_0}, \widetilde{\varphi}_{0,i_0} > (a_0,\lambda)$ has zeros at $\lambda_0$ of at least $k_0$. Thus, the function $K_{i_0}(\lambda)$ is analytical at $\lambda = \lambda_{m,n_{m,k}}$. Note that (4) and (5) show that

$$|< \varphi_{i_0}, \widetilde{\varphi}_{i_0} > (a_0,\lambda)| = \begin{cases} \mathrm{O}\left(\dfrac{\mathrm{e}^{2a_0\tau}}{|\rho|^2}\right), & \text{if} \quad \alpha_{i_0} = 0, \quad (37) \\ \mathrm{O}(\mathrm{e}^{2a_0\tau}), & \text{if} \quad \alpha_{i_0} \neq 0, \quad (38) \end{cases}$$

for $|\lambda| \to \infty$. By the results on the Weyl $m$-functions $m_l(x,\lambda)$ and $\widetilde{m}_l(x,\lambda)$ in [32], we have

$$\left|m_{i_0}(x,\lambda) - \widetilde{m}_{i_0}(x,\lambda)\right| = |i\rho + \mathrm{o}(1) - (i\rho + \mathrm{o}(1))| = \mathrm{o}(1) \tag{39}$$

uniformly in $x \in [\delta, 1]$ for $|\lambda| \to \infty$ in any sector $\varepsilon_0 < \arg\lambda < \pi - \varepsilon_0$ for $\varepsilon_0 > 0$, where $\delta \in (0,1]$ (for details, see [32]). Consequently, it follows from (4), (5) and (39) that

$$|< \varphi_{i_0}, \widetilde{\varphi}_{i_0} > (a_0,\lambda)| = \begin{cases} \mathrm{o}\left(\dfrac{\mathrm{e}^{2a_0\tau}}{|\rho|^2}\right), & \text{if} \quad \alpha_l = 0, \qquad (40) \\ \mathrm{o}(\mathrm{e}^{2a_0\tau}), & \text{if} \quad \alpha_l \neq 0, \qquad (41) \end{cases}$$

for $|\lambda| \to \infty$ in any sector $\varepsilon_0 < \arg\lambda < \pi - \varepsilon_0$. By Levinson's estimate (see [44]), then the first formula of (8), or (10), or (13) and (22), or (23) imply that there exists a constant $c_m$ such that

$$\frac{1}{\left|\prod_{m=1}^{p} F_m(\lambda)\right|} = \mathrm{O}\left(\mathrm{e}^{-2a_0\tau + \varepsilon\sqrt{|\lambda|}}\right), \quad \forall\lambda \in \bigcap_{m=1}^{p} D_{m,c_m} \tag{42}$$

for sufficiently large $|\lambda|$, where

$$D_{m,c_m} := \left\{\lambda : |\rho - \rho_{m,n_{m,k}}| \geq \frac{1}{8}c_m, \quad \lambda_{m,n_{m,k}} \in M_{m,0}\right\}.$$

Thus (37), (38), and (42) show that

$$|K_{i_0}(\lambda)| = \mathrm{O}\left(\mathrm{e}^{2\varepsilon\sqrt{|\lambda|}}\right), \quad \forall\lambda \in \bigcap_{m=1}^{p} D_{m,c_m} \tag{43}$$

for sufficiently large $|\lambda|$. Consequently, (43) and the maximum modulus principle show that the entire function $K_{i_0}(\lambda)$ is of the zero-exponential type, i.e., for arbitrary $\varepsilon > 0$, then

$$|K_{i_0}(\lambda)| \leq c\mathrm{e}^{2\varepsilon\sqrt{|\lambda|}}, \quad \lambda \in \mathbb{C} \tag{44}$$

for sufficiently large $|\lambda|$, where $c$ is constant. Noting that

$$\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{n^2\pi^2}\right),$$

we obtain

$$\begin{aligned} \int_1^{\infty} \frac{\left[\frac{t}{\pi}\right]}{t} \frac{y^2}{y^2+t^2}\mathrm{d}t &= \ln\left|\prod_{n=1}^{\infty}\left(1 - \frac{iy}{n^2\pi^2}\right)\right| + \mathrm{O}(1) = \ln\left|\frac{\sin\sqrt{iy}}{\sqrt{iy}}\right| + \mathrm{O}(1) \\ &= \sqrt{\frac{|y|}{2}} - \frac{1}{2}\ln|y| + \mathrm{O}(1), \end{aligned} \tag{45}$$

$$\begin{aligned} \int_1^{\infty} \frac{\left[\frac{t}{\pi} + \frac{1}{2}\right]}{t} \frac{y^2}{y^2+t^2}\mathrm{d}t &= \ln\left|\prod_{n=1}^{\infty}\left(1 - \frac{iy}{\left(n - \frac{1}{2}\right)^2\pi^2}\right)\right| + \mathrm{O}(1) = \ln\left|\cos\sqrt{iy}\right| + \mathrm{O}(1) \\ &= \sqrt{\frac{|y|}{2}} + \mathrm{O}(1). \end{aligned} \tag{46}$$

$$\int_1^\infty \frac{1}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t = \int_1^\infty \left(\frac{1}{t} - \frac{t}{y^2+t^2}\right)\mathrm{d}t = \frac{1}{2}\ln(y^2+1) = \ln|y| + \mathrm{O}(1). \tag{47}$$

Next, we shall prove by two cases

$$|K_{i_0}(iy)| \le \begin{cases} \frac{c}{|y|^{1-a_0(1-\kappa_1)}}, & \text{if} \quad \alpha_{i_0} = 0; \\ \frac{c}{|y|^{a_0\kappa_1}}, & \text{if} \quad \alpha_{i_0} \ne 0 \end{cases} \tag{48}$$

for sufficiently large $y > 0$. Here and below, we use the symbol $c$ to represent a positive constant that may vary from one formula to another.

**Case (1)**: All eigenvalues $\lambda_{m,n_{m,k}} \ge 1$. It follows from (45), (46), (47), (22) and (23) that

$$\begin{aligned} \ln\left|\prod_{m=1}^p F_m(iy)\right| &= \int_1^\infty \frac{\sum_{k=1}^p N_{k,0}(t)}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t + \mathrm{O}(1) \\ &\ge 2a_0 \begin{cases} \int_1^\infty \frac{\kappa_1\left[\frac{t}{\pi}+\frac{1}{2}\right]+(1-\kappa_1)\left(\left[\frac{t}{\pi}\right]+\frac{1}{2}\right)-1+\kappa_1}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t + \mathrm{O}(1), & \text{if} \quad \alpha_{i_0} = 0; \\ \int_1^\infty \frac{\kappa_1\left[\frac{t}{\pi}+\frac{1}{2}\right]+(1-\kappa_1)\left(\left[\frac{t}{\pi}\right]+\frac{1}{2}\right)+\kappa_1}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t + \mathrm{O}(1), & \text{if} \quad \alpha_{i_0} \ne 0 \end{cases} \\ &= \begin{cases} 2a_0\sqrt{\frac{|y|}{2}} - a_0(1-\kappa_1)\ln|y| + \mathrm{O}(1), & \text{if} \quad \alpha_{i_0} = 0; \\ 2a_0\sqrt{\frac{|y|}{2}} + a_0\kappa_1\ln|y| + \mathrm{O}(1), & \text{if} \quad \alpha_{i_0} \ne 0. \end{cases} \end{aligned}$$

This implies

$$\begin{cases} \left|\prod_{m=1}^p F_m(iy)\right| \ge c\frac{\mathrm{e}^{2a_0\sqrt{|y|/2}}}{|y|^{a_0(1+\kappa_1)}}, & \text{if} \quad \alpha_{i_0} = 0; \\ \left|\prod_{m=1}^p F_m(iy)\right| \ge c|y|^{a_0(1-\kappa_1)}\mathrm{e}^{2a_0\sqrt{|y|/2}}, & \text{if} \quad \alpha_{i_0} \ne 0 \end{cases} \tag{49}$$

for a $y \in \mathrm{R}^+$ that is sufficiently large. By (40), (41), and (49), we obtain (48).

**Case (2)**: There exist $k_0 \ge 1$ eigenvalues such that $\lambda_{m,n_{m,k_m}} < 1, m = \overline{1,p}, k_m = \overline{1,k_{m,0}}$, where there may exist some $k_{m,0}$ such that $k_{m,0} = 0$. If $k_{m,0} = 0$, then $\lambda_{m,n_{m,k}} \ge 1$ for all $n_{m,k}$. Without loss of generality, we assume

$$\lambda_{m,n_{m,k_m}} < 1, \quad m = \overline{1,p}, \quad k_m = \overline{1,k_{m,0}}, \quad \sum_{m=1}^p k_{m,0} = k_0.$$

Let

$$\mu_{m,n_{m,k_m}} = n_{m,k_m}\pi > 1, \quad m = \overline{1,p}, \quad S_{m,0} := \{\lambda_{m,n_{m,k_m}}\}_{k_m=1}^{k_{m,0}}, \quad k_m = \overline{1,k_{m,0}}$$

and

$$F_{1,m}(\lambda) = \prod_{k_m=1}^{k_{m,0}}\left(1 - \frac{\lambda}{\mu_{m,n_{m,k_m}}}\right) \times \prod_{\lambda_{m,n_{m,k}} \in M_{m,0}\setminus S_{m,0}}\left(1 - \frac{\lambda}{\lambda_{m,n_{m,k}}}\right), \quad m = \overline{1,p}. \tag{50}$$

Since

$$\lim_{y\to+\infty}\prod_{m=1}^p\prod_{k_m=1}^{k_{m,0}}\frac{1-\frac{iy}{\lambda_{m,n_{m,k_m}}}}{1-\frac{iy}{\mu_{m,n_{m,k_m}}}} = 1,$$

then there exists a sufficiently large $Y_0$ such that

$$\left| \prod_{m=1}^{p} \prod_{k_m=1}^{k_{m,0}} \frac{1 - \frac{iy}{\lambda_{m,n_{m,k_m}}}}{1 - \frac{iy}{\mu_{m,n_{m,k_m}}}} \right| \geq \frac{1}{2} \tag{51}$$

for $y > Y_0$. Note that

$$\prod_{m=1}^{p} F_m(\lambda) = \prod_{m=1}^{p} \prod_{k_m=1}^{k_{m,0}} \frac{1 - \frac{\lambda}{\lambda_{m,n_{m,k_m}}}}{1 - \frac{\lambda}{\mu_{m,n_{m,k_m}}}} \times \prod_{m=1}^{p} F_{1,m}(\lambda).$$

By (49), (50) and (51), we also have (48). It follows from (48)

$$\lim_{y \to \infty} K_{i_0}(iy) = 0. \tag{52}$$

By the Phragmén-Lindelöf-type result in [20] together with (44) and (52), we obtain

$$K_{i_0}(\lambda) \equiv 0, \quad \lambda \in \mathbb{C}. \tag{53}$$

It follows from (53) that

$$< \varphi_{i_0}, \widetilde{\varphi}_{i_0} > (a_0, \lambda) = 0, \quad \forall \lambda \in \mathbb{C}, \quad l = \overline{1, p}.$$

Consequently,

$$m_{i_0}(a_0, \lambda) = \widetilde{m}_{i_0}(a_0, \lambda), \quad \forall \lambda \in \mathbb{C}, \quad l = \overline{1, p}. \tag{54}$$

By Marchenko's result in [21] together with (54), we have

$$\widehat{q}_{i_0}(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [0, a_0], \quad \text{and} \quad \alpha_{i_0} = \widetilde{\alpha}_{i_0}. \tag{55}$$

The proof of Theorem 1 is completed. □

## 4. Partial Inverse Nodal Problems

In this section, we shall study the partial inverse nodal problem for $B$ from a paired-dense nodal subset in an interior subinterval having a central vertex. For $l = \overline{1, p}$, we say $W_{I_1}([a_l, b_l]) = \widetilde{W}_{\widetilde{I}_1}([a_l, b_l])$ if for any $n_{1,k} \in I_1$ there exist $j_k, r_k, \widetilde{n}_{1,k}, \widetilde{j}_k \in \mathbb{N}$ such that $x_{l,1,n_{1,k}}^{j_k}, x_{l,1,n_{1,k}}^{j_k+r_k} \in W_{I_1}([a_l, b_l]), \widetilde{x}_{l,1,\widetilde{n}_{1,k}}^{\widetilde{j}_k}, \widetilde{x}_{l,1,\widetilde{n}_{1,k}}^{\widetilde{j}_k+r_k} \in \widetilde{W}_{\widetilde{I}_1}([a_l, b_l])$ and

$$x_{l,1,n_{1,k}}^{j_k} = \widetilde{x}_{l,1,\widetilde{n}_{1,k}}^{\widetilde{j}_k}, \quad x_{l,1,n_{1,k}}^{j_k+r_k} = \widetilde{x}_{l,1,\widetilde{n}_{1,k}}^{\widetilde{j}_k+r_k}.$$

We obtain the following three uniqueness theorems for $B$.

**Theorem 2.** *For each $l = \overline{1, p}$, let $X_{l,1,0} \subseteq X_{l,1}$ be a dense nodal subset on $[0, 1]$; then*

$$q_l(x) - \widetilde{q}_l(x) \overset{a.e.}{=} 2\widehat{\omega}_0 \quad \text{on} \quad [0, 1], \quad \alpha_l = \widetilde{\alpha}_l \quad \text{for} \quad l = \overline{1, p}, \quad \text{I} \quad \text{or} \quad \text{II}$$
$$q_l(x) - \widetilde{q}_l(x) \overset{a.e.}{=} \widehat{\omega}_1 \quad \text{on} \quad [0, 1], \quad \alpha_l = \widetilde{\alpha}_l \quad \text{for} \quad l = \overline{1, p}, \quad \text{III}$$

Denote

$$C_1 = 2\widehat{\omega}_0, \quad \text{for} \quad \text{I}; \quad C_2 = 2\widehat{\omega}_0 - \frac{2\widehat{A}_1}{p}, \quad \text{for} \quad \text{II}; \quad C_3 = \widehat{\omega}_1, \quad \text{for} \quad \text{III}.$$

We need the following lemma to prove our main results in this paper.

**Lemma 2.** *Let $0 \le a_l < 1$ for $l = \overline{1,p}$. If $W_{I_1}([a_l, 1]) = \widetilde{W}_{\widetilde{I}_1}([a_l, 1])$, then there exists a large number $K_0$ such that*

$$\lambda_{1,n_{1,k}} - \widetilde{\lambda}_{1,\widetilde{n}_{1,k}} = C_\nu, \quad \forall n_{1,k} \in \mathrm{I}_1, \quad \text{for} \quad \nu = 1,2,3, \tag{56}$$

By Theorems 1, 2 and Lemma 2, we prove Theorems 3 and 4, which are solutions to **IP2**.

**Theorem 3.** *Let $0 \le a_l < 1/2$ for $l = \overline{1,p}$ and $0 \le \beta_1 = \max_{1 \le l \le p}\{a_l\} < 1/2$. If $W_{I_1}([a_l, 1]) = \widetilde{W}_{\widetilde{I}_1}([a_l, 1])$, and there exist $t_0 > 0$, $0 \le \kappa_1 \le 1$, and $\delta_1 > 0$ such that*

$$N_{M_{1,0}}(t) \ge 2\beta_1 \left\{ \kappa_1 \left[ \frac{t}{\pi} + \frac{1}{2} \right] + (1-\kappa_1)\left( \left[ \frac{t}{\pi} \right] + \frac{1}{2} \right) \right\} - 1 + \kappa_1 + O(t^{-\delta_1}) \ \text{ for I}, \tag{57}$$

$$N_{M_{1,0}}(t) \ge 2\beta_1 \left\{ \kappa_1 \left[ \frac{t}{\pi} + \frac{1}{2} \right] + (1-\kappa_1)\left( \left[ \frac{t}{\pi} \right] + \frac{1}{2} \right) + \kappa_1 \right\} + O(t^{-\delta_1}) \ \text{ for II, III} \tag{58}$$

*for sufficiently large $t \ge t_0$, and*

$$\lim_{t \to \infty} \frac{N_{M_{1,0}}(t)}{t} = \frac{2\beta_1}{\pi},$$

*then*

$$q_l(x) - \widetilde{q}_l(x) \overset{a.e.}{=} C_\mu \quad \text{on} \quad [0,1], \quad \alpha_l = \widetilde{\alpha}_l \quad \text{for} \quad l = \overline{1,p}, \quad \mu = 1,2,3, \tag{59}$$

**Remark 1.** *We can only study the partial inverse nodal problems for the cases $0 \le a_l < 1/2$, $l = \overline{1,p}$. The general cases $0 \le a_l < 1$, $l = \overline{1,p}$ require a separate investigation.*

Without any nodal data on the component $q_{i_0}(x)$, we have Theorem 4 from Theorems 3 and 1.

**Theorem 4.** *Let $0 \le a_l < 1/2$ for $l \ne i_0$ and $\beta_1 = \max_{l \ne i_0}\{a_l\}$. Suppose that $W_{I_1}([a_l, 1]) = \widetilde{W}_{\widetilde{I}_1}([a_l, 1])$ for $l \ne i_0$, and $M_{m,0} = \widetilde{M}_{m,0}$ for $m \ne 1$, $M_{m,0}$ for $m = \overline{1,p}$ satisfying the assumption* **(A)***, and there exist $t_0 > 0$, $0 \le \kappa_\xi \le 1$, $\delta_\xi > 0$, $\xi = 0,1$, such that*

$$\begin{cases} N_{M_{1,0}}(t) \ge 2\beta_1 \left\{ \kappa_0 \left[ \frac{t}{\pi} + \frac{1}{2} \right] + (1-\kappa_0)\left( \left[ \frac{t}{\pi} \right] + \frac{1}{2} \right) - 1 + \kappa_0 + O(t^{-\delta_0}) \right\}, \\ \sum_{m=1}^{p} N_{M_{m,0}}(t) \ge 2 \left\{ \kappa_1 \left[ \frac{t}{\pi} + \frac{1}{2} \right] + (1-\kappa_1)\left( \left[ \frac{t}{\pi} \right] + \frac{1}{2} \right) - 1 + \kappa_1 + O(t^{-\delta_1}) \right\}, \end{cases} \text{for I;} \tag{60}$$

$$\begin{cases} N_{M_{1,0}}(t) \ge 2\beta_1 \left\{ \kappa_0 \left[ \frac{t}{\pi} + \frac{1}{2} \right] + (1-\kappa_0)\left( \left[ \frac{t}{\pi} \right] + \frac{1}{2} \right) + \kappa_0 + O(t^{-\delta_0}) \right\}, \\ \sum_{m=1}^{p} N_{M_{m,0}}(t) \ge 2 \left\{ \kappa_1 \left[ \frac{t}{\pi} + \frac{1}{2} \right] + (1-\kappa_1)\left( \left[ \frac{t}{\pi} \right] + \frac{1}{2} \right) + \kappa_1 + O(t^{-\delta_1}) \right\}, \end{cases} \text{for II, III} \tag{61}$$

*for sufficiently large $t \ge t_0$;*

$$\lim_{t \to \infty} \frac{N_{M_{1,0}}(t)}{t} = \frac{2\beta_1}{\pi}, \quad \lim_{t \to \infty} \frac{\sum_{m=1}^{p} N_{M_{m,0}}(t)}{t} = \frac{2}{\pi},$$

*then (59) holds.*

Next, we present proofs of Lemma 2 and Theorems 2–4.

**Proof of Theorem 2.** For each fixed $x \in [0,1]$ and $l = \overline{1,p}$, we choose $x^{j_k}_{l,1,n_{1,k}} \in X_{l,1,0}$ such that $\lim\limits_{k\to\infty} x^{j_k}_{l,1,n_{1,k}} = x$. This implies $\lim\limits_{k\to\infty} \alpha^{j_k}_{n_{1,k}} = x$. By (17)–(20) and the Riemann–Lebesgue lemma, we have

$$
\begin{aligned}
f_{l,1,1}(x) :=& \lim_{k\to\infty} 2(n_{1,k} - \frac{1}{2})^2\pi^2\left(x^{j_k}_{l,1,n_{1,k}} - \alpha^{j_k}_{n_{1,k}}\right)\\
=& \lim_{k\to\infty}\left(\int_0^{\alpha^{j_k}_{n_{1,k}}} q_l(t)\mathrm{d}t - 2\omega_0\alpha^{j_k}_{n_{1,k}} + \mathrm{o}(1)\right)\\
=& \int_0^x q_l(t)\mathrm{d}t - 2\omega_0 x, \quad x \in [0,1] \quad \text{for} \quad \mathrm{I}; \qquad (62)\\
f_{l,1,2}(x) :=& \lim_{k\to\infty} 2(n_{1,k} - 1)^2\pi^2\left(x^{j_k}_{l,1,n_{1,k}} - \alpha^{j_k}_{n_{1,k}}\right)\\
=& \lim_{k\to\infty}\left(\int_0^{\alpha^{j_k}_{n_{1,k}}} q_l(t)\mathrm{d}t + \left(\frac{2A_1}{p} - 2\omega_0\right)\alpha^{j_k}_{n_{1,k}} + \mathrm{o}(1)\right)\\
=& -2\cot\alpha_l + \int_0^x q_l(t)\mathrm{d}t + \left(\frac{2A_1}{p} - 2\omega_0\right)x, \quad x \in [0,1] \quad \text{for} \quad \mathrm{II}; \qquad (63)\\
h_{l,1}(x) :=& \lim_{k\to\infty} (n_{1,k}\pi)\left(x^{j_k}_{l,1,n_{1,k}} - \alpha^{j_k}_{n_{1,k}}\right)\\
=& \lim_{k\to\infty}\left(d_1\alpha^{j_k}_{n_{1,k}} + \mathrm{o}(1)\right)\\
=& d_1 x, \quad x \in [0,1], \quad l = \overline{1,T}, \quad \text{for} \quad \mathrm{III}, \qquad (64)\\
f_{l,1,3}(x) :=& \lim_{k\to\infty} 2(n_{1,k}\pi)^2\left(x^{j_k}_{l,1,n_{1,k}} - \alpha^{j_k}_{n_{1,k}} - \frac{d_1\alpha^{j_k}_{n_{1,k}}}{n_{1,k}\pi}\right)\\
=& \lim_{k\to\infty}\left(\int_0^{\alpha^{j_k}_{n_{1,k}}} q_l(t)\mathrm{d}t - (\omega_1 - 2d_1^2)\alpha^{j_k}_{n_{1,k}} + \mathrm{o}(1)\right)\\
=& \int_0^x q_l(t)\mathrm{d}t - (\omega_1 - 2d_1^2)x, \quad x \in [0,1], \quad l = \overline{1,T}, \quad \text{for} \quad \mathrm{III}, \qquad (65)\\
f_{l,1,4}(x) :=& \lim_{k\to\infty} 2(n_{1,k}\pi)^2\left(x^{j_k}_{l,1,n_{1,k}} - \alpha^{j_k}_{n_{1,k}} - \frac{d_1\alpha^{j_k}_{n_{1,k}}}{n_{1,k}\pi}\right)\\
=& \lim_{k\to\infty}\left(-2\cot\alpha_l + \int_0^{\alpha^{j_k}_{n_{1,k}}} q_l(t)\mathrm{d}t - (\omega_1 - 2d_1^2)\alpha^{j_k}_{n_{1,k}} + \mathrm{o}(1)\right)\\
=& -2\cot\alpha_l + \int_0^x q_l(t)\mathrm{d}t - (\omega_1 - 2d_1^2)x, \quad x \in [0,1], \quad l = \overline{T+1,p}, \quad \text{for} \quad \mathrm{III}. \qquad (66)
\end{aligned}
$$

By taking derivatives with respect to $x$ in (62)–(66), we obtain

$$
\begin{cases}
f'_{l,1,1}(x) \overset{a.e.}{=} q_l(x) - 2\omega_0, \quad x \in [0,1], \quad l = \overline{1,p}, & (67)\\
f'_{l,1,2}(x) \overset{a.e.}{=} q_l(x) + \left(\dfrac{2A_1}{p} - 2\omega_0\right), \quad x \in [0,1], \quad l = \overline{1,p}, & (68)\\
h'_{l,1}(x) = d_1 = \arcsin\sqrt{\dfrac{T}{p}}, & (69)\\
f'_{l,1,3}(x) = f'_{l,1,4}(x) \overset{a.e.}{=} q_l(x) - \omega_1 + 2d_1^2, \quad x \in [0,1], \quad l = \overline{1,p}. & (70)
\end{cases}
$$

It follows from the assumption $X_{l,1,0} \overset{a.e.}{=} \widetilde{X}_{l,1,0}$ that

$$
h_{l,1}(x) = \widetilde{h}_{l,1}(x) \quad \text{and} \quad f_{l,1,\nu}(x) = \widetilde{f}_{l,1,\nu}(x), \quad x \in [0,1], \quad l = \overline{1,p}, \quad \nu = \overline{1,4}. \qquad (71)
$$

By (69), we find $T$ by

$$T = p\sin^2 h'_{l,1}(x). \tag{72}$$

For cases II and III with $l = \overline{T+1,p}$, letting $x = 0$ in (63) and (66), we obtain

$$\alpha_l = \widetilde{\alpha}_l = \begin{cases} \operatorname{arccot}\dfrac{-f_{l,1,2}(0)}{2}, & l = \overline{1,p}, \quad \text{for} \quad \text{II}, & (73) \\ \operatorname{arccot}\dfrac{-f_{l,1,4}(0)}{2}, & l = \overline{T+1,p}, \quad \text{for} \quad \text{III}. & (74) \end{cases}$$

Furthermore, it follows from (71) that

$$f'_{l,1,\nu}(x) \overset{a.e.}{=} \widetilde{f}'_{l,1,\nu}(x), \quad x \in [0,1], \quad l = \overline{1,p}, \quad \nu = \overline{1,4}. \tag{75}$$

Consequently, (67)–(75) imply that

$$\widehat{q}_l(x) := q_l(x) - \widetilde{q}_l(x) \overset{a.e.}{=} C_\mu, \quad x \in [0,1], \quad l = \overline{1,p}, \quad \mu = 1,2,3. \tag{76}$$

This completes the proof of Theorem 2. □

The proof of Theorem 2 is constructive. We reconstruct the potential $q_l(x)$ up to a constant on the equilateral graph with the dense nodal subset $X_{l,1,0}$ on the $l$-th edge, $l = \overline{1,p}$, by the following algorithms:

**Algorithm 1:** For case I, reconstruct the potential $q_l(x)$ up to a constant by the following two steps:

(1) Find $f_{l,1,1}(x)$ by (62) for each $l = \overline{1,p}$.
(2) Reconstruct $q_l(x) - 2\omega_0$ on $(0,1)$ by (67).

**Algorithm 2:** For case II, reconstruct the potential $q_l(x)$ up to a constant by the following three steps:

(1) Find $f_{l,1,2}(x)$ by (63) for each $l = \overline{1,p}$.
(2) Reconstruct $\alpha_l$ for each $l = \overline{1,p}$ by (73), and then find $A_1$.
(3) Recover $q_l(x) - 2\omega_0$ on $(0,1)$ for each $l = \overline{1,p}$ by (68).

**Algorithm 3:** For case III, reconstruct the potential $q_l(x)$ up to a constant by the following four steps:

(1) Find $h_{l,1}(x)$ by (64) for each $l = \overline{1,p}$; reconstruct $T$ by (72).
(2) Find $f_{l,1,3}(x)$ by (65) for each $l = \overline{1,T}$ and find $f_{l,1,4}(x)$ by (66) for each $l = \overline{T+1,p}$.
(3) Reconstruct $\alpha_l$ for each $l = \overline{T+1,p}$ by (74), and then find $A_2$.
(4) Recover $q_l(x) - \omega_1$ on $(0,1)$ for each $l = \overline{1,p}$ by (70).

**Proof of Lemma 2.** By suitably modifying the proof of Theorem 2, we obtain

$$q_l(x) - \widetilde{q}_l(x) \overset{a.e.}{=} C_\nu, \quad x \in [a_l,1], \quad l = \overline{1,p}, \quad \nu = 1,2,3. \tag{77}$$

It follows from (17)–(20) that as $k \to \infty$

$$L_{l,1,n_{1,k}} := x^{j_k+r_k}_{l,1,n_{1,k}} - x^{j_k}_{l,1,n_{1,k}} = \begin{cases} \dfrac{r_k}{n_{1,k} - \dfrac{1}{2}} + o\left(\dfrac{1}{n_{1,k}^2}\right), & \text{for I}, \\ \dfrac{r_k}{n_{1,k} - 1} + o\left(\dfrac{1}{n_{1,k}^2}\right), & \text{for II}, \\ \dfrac{r_k}{n_{1,k}} + \dfrac{d_1 r_k}{n_{1,k}^2\pi} + o\left(\dfrac{1}{n_{1,k}^2}\right), & \text{for III}, \ l = \overline{1,T}, \\ \dfrac{r_k}{n_{1,k}} + \dfrac{r_k}{n_{1,k}^2\pi} + o\left(\dfrac{1}{n_{1,k}^2}\right), & \text{for III}, \ l = \overline{T+1,p}. \end{cases} \tag{78}$$

If the problems $B$ and $\widetilde{B}$ belong to the same subcase in (7), say, case I, it follows from the first formula of (78) and the assumption that

$$L_{l,1,n_{1,k}} = \frac{r_k}{n_{1,k} - \frac{1}{2}} + o\left(\frac{1}{n_{1,k}^2}\right) = \frac{r_k}{\widetilde{n}_{1,k} - \frac{1}{2}} + o\left(\frac{1}{\widetilde{n}_{1,k}^2}\right) = \widetilde{L}_{l,1,n_{1,k}}, \quad \text{for} \quad k \gg 1.$$

Without loss of generality, we assume $\widetilde{n}_{1,k} \geq n_{1,k}$ here and below. This implies

$$\frac{r_k(n_{1,k} - \widetilde{n}_{1,k})}{\left(n_{1,k} - \frac{1}{2}\right)\left(\widetilde{n}_{1,k} - \frac{1}{2}\right)} = o\left(\frac{1}{n_{1,k}^2}\right) \quad \text{for} \quad k \gg 1. \tag{79}$$

It follows from (79) that

$$n_{1,k} = \widetilde{n}_{1,k} \quad \text{for} \quad k \gg 1. \tag{80}$$

If the problem $B$ belongs to case II, while the problem $\widetilde{B}$ belongs to case III, then it follows from the second and third formulae of (78) and the assumption that

$$\frac{r_k}{n_{1,k} - 1} + o\left(\frac{1}{n_{1,k}^2}\right) = \frac{r_k}{\widetilde{n}_{1,k}} + \frac{d_1 r_k}{\widetilde{n}_{1,k}^2 \pi} + o\left(\frac{1}{\widetilde{n}_{1,k}^2}\right) \quad \text{for} \quad k \gg 1. \tag{81}$$

By virtue of (81), this yields

$$\frac{r_k}{n_{1,k} - 1} + o\left(\frac{1}{n_{1,k}^2}\right) = \frac{r_k}{\widetilde{n}_{1,k}} + \frac{d_1 r_k}{\widetilde{n}_{1,k}^2 \pi} + o\left(\frac{1}{\widetilde{n}_{1,k}^2}\right) \quad \text{for} \quad k \gg 1. \tag{82}$$

In particular, we have

$$\frac{r_k \widetilde{n}_{1,k}}{n_{1,k} - 1} - r_k - \frac{d_1 r_k}{\widetilde{n}_{1,k} \pi} = o\left(\frac{\widetilde{n}_{1,k}}{n_{1,k}^2}\right) + o\left(\frac{1}{\widetilde{n}_{1,k}}\right),$$

and hence

$$\lim_{k \to \infty} \frac{\widetilde{n}_{1,k}}{n_{1,k}} = 1.$$

By (82), we obtain

$$\frac{\widetilde{n}_{1,k} - n_{1,k} + 1 - \frac{d_1}{\pi}}{\widetilde{n}_{1,k}} = o\left(\frac{1}{n_{1,k}}\right) \quad \text{for} \quad k \gg 1.$$

This implies

$$\widetilde{n}_{1,k} - n_{1,k} + 1 - \frac{d_1}{\pi} = 0 \quad \text{for} \quad k \gg 1, \tag{83}$$

which is impossible by $0 < \frac{1}{\pi} \arcsin \sqrt{\frac{T}{p}} < \frac{1}{2}$ and $\widetilde{n}_{1,k} - n_{1,k} + 1 \geq 1$. Therefore, the problems $B$ and $\widetilde{B}$ belong to the same subcase, and other cases can be treated similarly. This implies that (80) is valid for $k \gg 1$. Next, we only consider the problems $B$ and $\widetilde{B}$ belonging to case III and $l \in \{T+1, \cdots, p\}$. For each $l = \overline{T+1, p}$, consider two Dirichlet boundary value problems defined on the interval $[x_{l,1,n_{1,k}}^{j_k}, x_{l,1,n_{1,k}}^{j_k + r_k}]$,

$$\begin{cases} -\varphi_l''(x,\lambda_{1,n_{1,k}}) + q_l(x)\varphi_l(x,\lambda_{1,n_{1,k}}) = \lambda_{1,n_{1,k}}\varphi_l(x,\lambda_{1,n_{1,k}}), & (84) \\ \varphi_l(x^{j_k}_{l,1,n_{1,k}},\lambda_{1,n_{1,k}}) = \varphi_l(x^{j_k+r_k}_{l,1,n_{1,k}},\lambda_{1,n_{1,k}}) = 0, & (85) \end{cases}$$

and

$$\begin{cases} -\widetilde{\varphi}_l''(x,\widetilde{\lambda}_{1,\widetilde{n}_{1,k}}) + \widetilde{q}_l(x)\widetilde{\varphi}_l(x,\widetilde{\lambda}_{1,\widetilde{n}_{1,k}}) = \widetilde{\lambda}_{1,\widetilde{n}_{1,k}}\widetilde{\varphi}_l(x,\widetilde{\lambda}_{1,\widetilde{n}_{1,k}}), & (86) \\ \widetilde{\varphi}_l(x^{j_k}_{l,1,n_{1,k}},\widetilde{\lambda}_{1,\widetilde{n}_{1,k}}) = \widetilde{\varphi}_l(x^{j_k+r_k}_{l,1,n_{1,k}},\widetilde{\lambda}_{1,\widetilde{n}_{1,k}}) = 0. & (87) \end{cases}$$

It follows from the first formula of (13) and (20) that

$$\rho_{1,n_{1,k}}x^{j_k}_{l,1,n_{1,k}} = \left(j_k - \frac{1}{2}\right)(d_1+\pi) + \frac{d_1}{2n_{1,k}\pi} + \mathrm{o}\left(\frac{1}{n_{1,k}}\right), \tag{88}$$

$$b_1 := -(2j_k-1)(d_1+\pi)d_1 - 2\pi\cot\alpha_l + \alpha^{j_k}_{n_{1,k}}(\omega_1+d_1^2) + \int_0^{\alpha^{j_k}_{n_{1,k}}} q_l(t)\mathrm{d}t;$$

$$\rho_{1,n_{1,k}}x^{j_k+r_k}_{l,1,n_{1,k}} = \left(j_k + r_k - \frac{1}{2}\right)(d_1+\pi) + \frac{d_2}{2n_{1,k}\pi} + \mathrm{o}\left(\frac{1}{n_{1,k}}\right), \tag{89}$$

$$b_2 := -(2j_k+r_k-1)(d_1+\pi)d_1 - 2\pi\cot\alpha_l + \alpha^{j_k}_{n_{1,k}}(\omega_1+d_1^2) + \int_0^{\alpha^{j_k+r_k}_{1,n_{1,k}}} q_l(t)\mathrm{d}t;$$

$$\widetilde{\rho}_{1,\widetilde{n}_{1,k}}x^{j_k}_{l,1,n_{1,k}} = \left(\widetilde{j}_k - \frac{1}{2}\right)(d_1+\pi) + \frac{\widetilde{d}_1}{2\widetilde{n}_{1,k}\pi} + \mathrm{o}\left(\frac{1}{n_{1,k}}\right), \tag{90}$$

$$\widetilde{b}_1 := -(2\widetilde{j}_k-1)(d_1+\pi)d_1 - 2\pi\cot\widetilde{\alpha}_l + \alpha^{j_k}_{n_{1,k}}(\widetilde{\omega}_1+d_1^2) + \int_0^{\alpha^{j_k}_{n_{1,k}}} \widetilde{q}_l(t)\mathrm{d}t;$$

$$\widetilde{\rho}_{1,n_{1,k}}x^{j_k+r_k}_{l,1,n_{1,k}} = \left(\widetilde{j}_k + r_k - \frac{1}{2}\right)(d_1+\pi) + \frac{\widetilde{d}_2}{2\widetilde{n}_{1,k}\pi} + \mathrm{o}\left(\frac{1}{n_{1,k}}\right), \tag{91}$$

$$\widetilde{b}_2 := -(2\widetilde{j}_k+r_k-1)(d_1+\pi)d_1 - 2\pi\cot\widetilde{\alpha}_l + \alpha^{j_k}_{n_{1,k}}(\widetilde{\omega}_1+d_1^2) + \int_0^{\alpha^{j_k+r_k}_{1,n_{1,k}}} \widetilde{q}_l(t)\mathrm{d}t.$$

By (84)–(87) and the integrations, we easily obtain

$$\int_{x^{j_k}_{l,1,n_{1,k}}}^{x^{j_k+r_k}_{l,1,n_{1,k}}} ((q_l(x)-\widetilde{q}_l(x)) - (\lambda_{1,n_{1,k}} - \widetilde{\lambda}_{1,\widetilde{n}_{1,k}}))\varphi_l(x,\lambda_{1,n_{1,k}})\widetilde{\varphi}_l(x,\widetilde{\lambda}_{1,\widetilde{n}_{1,k}})\mathrm{d}x = 0. \tag{92}$$

By virtue of (92) and $q_l(x) - \widetilde{q}_l(x) \overset{a.e.}{=} C_\nu$ on $[a_l, 1]$, we have

$$(C_\nu - (\lambda_{1,n_{1,k}} - \widetilde{\lambda}_{1,\widetilde{n}_{1,k}}))\int_{x^{j_k}_{l,1,n_{1,k}}}^{x^{j_k+r_k}_{l,1,n_{1,k}}} \varphi_l(x,\lambda_{1,n_{1,k}})\widetilde{\varphi}_l(x,\widetilde{\lambda}_{1,\widetilde{n}_{1,k}})\mathrm{d}x = 0. \tag{93}$$

On the other hand, it follows from (5)

$$\varphi_l(x,\lambda_{1,n_{1,k}})\widetilde{\varphi}_l(x,\widetilde{\lambda}_{1,\widetilde{n}_{1,k}}) = \sin\alpha_l\sin\widetilde{\alpha}_l\cos\rho_{1,n_{1,k}}x\cos\widetilde{\rho}_{1,\widetilde{n}_{1,k}}x + \mathrm{O}\left(\frac{1}{n_{1,k}}\right). \tag{94}$$

By virtue of (94), this yields

$$\int_{x^{j_k}_{l,1,n_{1,k}}}^{x^{j_k+r_k}_{l,1,n_{1,k}}} \varphi_l(x,\lambda_{1,n_{1,k}})\widetilde{\varphi}_l(x,\widetilde{\lambda}_{1,\widetilde{n}_{1,k}})\mathrm{d}x$$

$$
\begin{aligned}
&= \sin\alpha_l \sin\widetilde{\alpha}_l \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \cos\rho_{1,n_{1,k}} x \cos\widetilde{\rho}_{1,\widetilde{n}_{1,k}} x \mathrm{d}x + \mathrm{O}\left(\frac{1}{n_{1,k}^2}\right) \\
&= \frac{\sin\alpha_l \sin\widetilde{\alpha}_l}{2} \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \Big(\cos(\rho_{1,n_{1,k}} + \widetilde{\rho}_{1,\widetilde{n}_{1,k}})x + \cos(\rho_{1,n_{1,k}} - \widetilde{\rho}_{1,\widetilde{n}_{1,k}})x\Big) \mathrm{d}x + \mathrm{O}\left(\frac{1}{n_{1,k}^2}\right).
\end{aligned} \tag{95}
$$

For $k \gg 1$, (80) shows that

$$
\begin{aligned}
&\frac{1}{2L_{l,1,n_{1,k}}} \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \cos(\rho_{1,n_{1,k}} + \widetilde{\rho}_{1,\widetilde{n}_{1,k}}) x \mathrm{d}x \\
=& \frac{1}{2(\rho_{1,n_{1,k}} + \widetilde{\rho}_{1,n_{1,k}}) L_{l,1,n_{1,k}}} \Big( \sin(\rho_{1,n_{1,k}} + \widetilde{\rho}_{1,n_{1,k}}) x_{l,1,n_{1,k}}^{j_k+r_k} - \sin(\rho_{1,n_{1,k}} + \widetilde{\rho}_{1,n_{1,k}}) x_{l,1,n_{1,k}}^{j_k} \Big) \\
=& \frac{(-1)^{j_k+\widetilde{j}_k-1}}{2r_k(d_1+\pi)(1+\frac{1}{2n_{1,k}}) + \mathrm{o}\left(\frac{1}{n_{1,k}}\right)} \Bigg( \sin\left( (j_k+\widetilde{j}_k+2r_k-1)d_1 + \frac{b_1+\widetilde{b}_1}{2n_{1,k}\pi} + \mathrm{o}\left(\frac{1}{n_{1,k}}\right) \right) \\
& - \sin\left( (j_k+\widetilde{j}_k-1)d_1 + \frac{b_2+\widetilde{b}_2}{2n_{1,k}\pi} + \mathrm{o}\left(\frac{1}{n_{1,k}}\right) \right) \Bigg) \\
=& \frac{(-1)^{j_k+\widetilde{j}_k-1}\Big(\sin(j_k+\widetilde{j}_k+2r_k-1)d_1 - \sin(j_k+\widetilde{j}_k-1)d_1\Big)}{2r_k(d_1+\pi)} + \mathrm{O}\left(\frac{1}{n_{1,k}}\right).
\end{aligned} \tag{96}
$$

It follows from the first formula of (13) and (80) that

$$
\begin{aligned}
&\frac{1}{2L_{l,1,n_{1,k}}} \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \cos(\rho_{1,n_{1,k}} - \widetilde{\rho}_{1,\widetilde{n}_{1,k}}) x \mathrm{d}x \\
=& \frac{1}{2L_{l,1,n_{1,k}}} \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \cos\left( \frac{\omega_1 x}{n_{1,k}} + \mathrm{o}\left(\frac{1}{n_{1,k}}\right) \right) \mathrm{d}x \\
=& \frac{1}{2L_{l,1,n_{1,k}}} \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \left[ 1 - 2\sin^2\left( \frac{\omega_1 x}{2n_{1,k}} + \mathrm{o}\left(\frac{1}{n_{1,k}}\right) \right) \right] \mathrm{d}x \\
=& \frac{1}{2} + \mathrm{O}\left(\frac{1}{n_{1,k}^2}\right).
\end{aligned} \tag{97}
$$

By (80), (96) and (97), there exists a sufficiently large constant $K_0$ such that

$$
\int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \cos(\rho_{1,n_{1,k}} - \widetilde{\rho}_{1,\widetilde{n}_{1,k}}) x \mathrm{d}x > \left| \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \cos(\rho_{1,n_{1,k}} + \widetilde{\rho}_{1,\widetilde{n}_{1,k}}) x \mathrm{d}x \right| \tag{98}
$$

for all $k \geq K_0$. It follows from (92) and (98) that

$$
\begin{aligned}
&\left| \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \varphi(x, \lambda_{n_k}) \widetilde{\varphi}(x, \widetilde{\lambda}_{\widetilde{n}_k}) \mathrm{d}x \right| \\
\geq& \sin\alpha \sin\widetilde{\alpha} \left( \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \cos(\rho_{1,n_{1,k}} - \widetilde{\rho}_{1,\widetilde{n}_{1,k}}) x \mathrm{d}x - \left| \int_{x_{l,1,n_{1,k}}^{j_k}}^{x_{l,1,n_{1,k}}^{j_k+r_k}} \cos(\rho_{1,n_{1,k}} + \widetilde{\rho}_{1,\widetilde{n}_{1,k}}) x \mathrm{d}x \right| + \mathrm{O}\left(\frac{1}{n_{1,k}^2}\right) \right) \\
>& 0.
\end{aligned} \tag{99}
$$

Therefore, (93) and (99) imply that

$$
\widehat{\lambda}_{n_{1,k}} = \lambda_{n_{1,k}} - \widetilde{\lambda}_{\widetilde{n}_{1,k}} = C_\nu
$$

for all $n_{1,k} \in I_1$. This completes the proof of Lemma 2. $\square$

Next, we prove Theorem 3.

**Proof of Theorem 3.** By the assumption of Theorem 3 together with Lemma 2, we have

$$\begin{cases} q_l(x) - \widetilde{q}_l(x) \overset{a.e.}{=} C_\mu, \quad x \in [a_l, 1], \quad l = \overline{1,p}, \quad \mu = 1,2,3, & (100)\\ \lambda_{1,n_{1,k}} - \widetilde{\lambda}_{1,\widetilde{n}_{1,k}} = C_\mu, \quad \mu = 1,2,3, \quad \forall n_{1,k} \in \mathrm{I}_1. & (101)\end{cases}$$

Let $\widetilde{q}_{0,l}(x) := \widetilde{q}_l(x) + C_\mu$, $\widehat{q}_{0,l}(x) := q_l(x) - \widetilde{q}_{0,l}(x)$, and $\widetilde{\varphi}_{0,l}(x,\lambda)$ be the solution of

$$\begin{cases} u''(x,\lambda) + (\lambda - \widetilde{q}_{0,l}(x))u(x,\lambda) = 0, \quad 0 < x < 1, \\ u(0,\lambda) = \sin\alpha_l, \ u'(0,\lambda) = -\cos\alpha_l. \end{cases} \tag{102}$$

By a shift of the spectrum to the constant $C_\mu$, then (100) and (101) imply

$$\begin{cases} \widehat{q}_{0,l}(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [a_l, 1], \quad l = \overline{1,p}, & (103)\\ \lambda_{1,n_{1,k}} - \widetilde{\lambda}_{01,\widetilde{n}_{1,k}} = 0, \quad \forall n_{1,k} \in I_1. & (104)\end{cases}$$

Next, we prove

$$\widehat{q}_{0,l}(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [0, a_l], \quad \text{and} \quad \alpha_l = \widetilde{\alpha}_l, \quad l = \overline{1,p}.$$

For each $\lambda_{1,n_{1,k}}$, (103) and (104) show that

$$< \varphi_l, \widetilde{\varphi}_{0,l} > (a_l, \lambda_{1,n_{1,k}}) = 0, \quad \forall\, n_{1,k} \in I_1. \tag{105}$$

It follows from (4) and (5) that

$$\left| < \varphi_l, \widetilde{\varphi}_{0,l} > (a_l,\lambda)\right| = \begin{cases} \mathrm{O}\left(\dfrac{\mathrm{e}^{2a_l\tau}}{|\rho|^2}\right), \quad \text{if} \quad \alpha_l = 0, & (106)\\ \mathrm{O}(\mathrm{e}^{2a_l\tau}), \quad \text{if} \quad \alpha_l \neq 0, & (107)\end{cases}$$

for $|\lambda| \to \infty$. Consequently, it follows from (4), (5) and (39) that

$$\left| < \varphi_l, \widetilde{\varphi}_{0,l} > (a_l,\lambda)\right| = \begin{cases} \mathrm{o}\left(\dfrac{\mathrm{e}^{2a_l\tau}}{|\rho|^2}\right), \quad \text{if} \quad \alpha_l = 0, & (108)\\ \mathrm{o}(\mathrm{e}^{2a_l\tau}), \quad \text{if} \quad \alpha_l \neq 0, & (109)\end{cases}$$

for $|\lambda| \to \infty$ in any sector $\varepsilon_0 < \arg\lambda < \pi - \varepsilon_0$. Define the function $K_{l,1}(\lambda)$ by

$$K_{l,1}(\lambda) := \frac{< \varphi_l, \widetilde{\varphi}_{0,l} > (a_l,\lambda)}{F_1(\lambda)}, \quad l = \overline{1,p},$$

Therefore, (105) together with the assumption on $M_{1,0}$ show that $K_{l,1}(\lambda)$ is an entire function in $\lambda$. By Levinson's estimate (see [44]), the first formula of (8), or (10), or (13) and (58) imply that there exists a constant $c_1$ such that

$$\frac{1}{|F_1(\lambda)|} = \mathrm{O}\left(\mathrm{e}^{-2\alpha_1\tau + \varepsilon\sqrt{|\lambda|}}\right), \quad \forall\lambda \in D_{1,c_1} \tag{110}$$

for sufficiently large $|\lambda|$. Thus (106), (107) and (110) for $m = 1$ show that

$$|K_{l,1}(\lambda)| = \mathrm{O}\left(\mathrm{e}^{-2(\alpha_1 - a_l)\tau + 2\varepsilon\sqrt{|\lambda|}}\right), \quad \forall\lambda \in D_{1,c_1} \tag{111}$$

for sufficiently large $|\lambda|$. Consequently, it follows from $a_l \le \alpha_1 \le \frac{1}{2}$, (111) and the maximum modulus principle that the entire function $K_{l,1}(\lambda)$ is of the zero-exponential type, and then for arbitrary $\varepsilon > 0$,

$$|K_{l,1}(\lambda)| \le c e^{2\varepsilon\sqrt{|\lambda|}}, \quad \lambda \in \mathbb{C} \tag{112}$$

for sufficiently large $|\lambda|$. By calculations, we have

$$\begin{cases} |F_1(iy)| \ge c\dfrac{e^{2\alpha_1\sqrt{|y|/2}}}{|y|^{\alpha_1(1+\kappa_0)}}, & \text{for} \quad \text{I}; \\ |F_1(iy)| \ge c|y|^{\alpha_1(1-\kappa_0)} e^{2\alpha_1\sqrt{|y|/2}}, & \text{for} \quad \text{II}, \quad \text{III} \end{cases} \tag{113}$$

for a $y \in \mathrm{R}^+$ that is sufficiently large. It follows from (108), (109) and (113) that

$$|K_{l,1}(iy)| = \mathrm{o}(1)$$

for a sufficiently large $y > 0$. This implies

$$\lim_{y\to\infty} K_{l,1}(iy) = 0. \tag{114}$$

By the Phragmén-Lindelöf-type result in [20] together with (112) and (114) again, we obtain

$$K_{l,1}(\lambda) \equiv 0, \quad \lambda \in \mathbb{C}. \tag{115}$$

It follows from (115) that

$$< \varphi_l, \widetilde{\varphi}_{0,l} > (a_l, \lambda) = 0, \quad \forall\lambda \in \mathbb{C}, \quad l = \overline{1,p}.$$

Consequently,

$$m_l(a_l, \lambda) = \widetilde{m}_{0,l}(a_l, \lambda), \quad \forall\lambda \in \mathbb{C}, \quad l = \overline{1,p}. \tag{116}$$

By (116) together with Marchenko's result in [21], we obtain

$$\widehat{q}_{0,l}(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [0, a_l], \quad \text{and} \quad \alpha_l = \widetilde{\alpha}_l, \quad l = \overline{1,p}. \tag{117}$$

Hence, (103) and (117) show that (59) holds. The proof of Theorem 3 is completed. □

**Proof of Theorem 4.** We use the same symbols as these in Theorem 3. Applying the same arguments as the proof of Theorem 3, we have

$$\begin{cases} \widehat{q}_{0,l}(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [0,1], \quad \text{and} \quad \alpha_l = \widetilde{\alpha}_l, \quad l \neq i_0, & (118) \\ \lambda_{1,n_{1,k}} - \widetilde{\lambda}_{01,\widetilde{n}_{1,k}} = 0, \quad \forall n_{1,k} \in I_1. & (119) \end{cases}$$

Next, we prove

$$\widehat{q}_{0,i_0}(x) \overset{a.e.}{=} 0 \quad \text{on} \quad [0,1], \quad \text{and} \quad \alpha_{i_0} = \widetilde{\alpha}_{i_0}. \tag{120}$$

It follows from (1) and (102) for $l = i_0$ that

$$< \varphi_{i_0}, \widetilde{\varphi}_{0,i_0} > (1, \lambda_{m,n_{m,k}}) = \int_0^1 \widehat{q}_{0,i_0}(x)\varphi_{i_0}(x, \lambda_{m,n_{m,k}})\widetilde{\varphi}_{0,i_0}(x, \lambda_{m,n_{m,k}})\mathrm{d}x - \sin\widehat{\alpha}_{i_0}. \tag{121}$$

Similar to the argument as the proof of Theorem 1, one can complete the remaining proof of Theorem 4. □

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