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On a Lyapunov-Type Inequality for Control of a ψ-Model Thermostat and the Existence of Its Solutions

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Abstract: In this paper, a new structure of an applied model of thermostat is defined using the generalized ψ-operators with three-point boundary conditions. Some useful properties of the relevant Green's function are established, and based on these properties, the Lyapunov-type inequality is constructed for the given extended ψ-model thermostat with the help of Jensen's inequality. By defining mild solutions for such an extended system, the existence and non-existence conditions are discussed.

Keywords: boundary value problem; Lyapunov inequality; generalized fractional operator; thermostat model; non-existence

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1. Introduction

Inequalities in their various forms play a vital role in mathematics. In particular, their effective operation can be seen in ordinary and partial differential equations (ODEs and PDEs) that lead to various standard formulas in different applications. In this direction, in recent years, mathematicians have introduced many important inequalities by considering various assumptions on the given functions and using operators with singular and non-singular kernels. One of the most famous of these inequalities is the Lyapunov inequality. To investigate the spectral properties of ODEs, the inequality of the Lyapuov type is a helpful tool [1–3]. Moreover, eigenvalue problems, disconjugacy, and oscillation theory are other fields in which this type of inequality is useful [4].

As a starting point in this area, Lyapunov [5] formulated the Lyapunov inequality for a second-order boundary value problem (BVP) for the first time. In fact, by assuming the existence of a non-trivial solution for the following linear BVP,

$$\begin{cases} v''(s) + \psi(s)v(s) = 0, & s \in (m,n), \\ v(m) = v(n) = 0, \end{cases} \tag{1}$$

Lyapunov derived an inequality as

$$\int_m^n |\psi(q)|\mathrm{d}q > \frac{4}{n-m}, \quad m,n \in \mathbb{R},$$

so that ψ is a continuous function on $[m,n]$ with real values. After that, some researchers such as Yang et al. [6] and Agarwal et al. [7] extended this inequality to higher-order systems. The investigation of Lyapunov inequalities was initiated in the context of standard integer-order ODEs, and then other generalized versions of it were introduced by defining fractional operators. One can consider the first conducted research on the fractional type of the Lyapunov inequality in a paper from Ferreira [8]. In fact, Ferreira extended the linear BVP (1) to a fractional BVP with the Riemann–Liouville derivative given by

$$\begin{cases} D_m^p v(s) + \psi(s)v(s) = 0, \quad s \in (m,n), \\ v(m) = v(n) = 0, \end{cases} \tag{2}$$

with $1 < p \leq 2$ and established the following inequality:

$$\int_m^n |\psi(q)| \mathrm{d}q > \Gamma(p) \left[\frac{4}{n-m} \right]^{p-1}, \quad m,n \in \mathbb{R},$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t$ is the Gamma function.

One year later, Ferreira [9], in another research, conducted a similar analysis with the Caputo fractional derivative and obtained the following inequality:

$$\int_m^n |\psi(q)| \mathrm{d}q > \frac{p^p \Gamma(p)}{(p-1)^{p-1}(n-m)^{p-1}}, \quad m,n \in \mathbb{R}.$$

Due to the importance of such inequalities in different applied areas, various versions of Lyapunov-type inequalities have been obtained by some other researchers. For instance, Jleli et al. [10] studied the corresponding inequality with the help of q-difference operators. Additionally, Ma and Han [11] implemented a similar study with q-operators on the Schrodinger equation with Woods–Saxon potential. In 2018, Pathak [12] generalized Lyapunov-type inequality when the derivative of the given BVP is of the Hilfer type. For more details, see [13–16].

By developing practical concepts in the theory of fractional calculus, mathematicians became eager to design various mathematical models with the help of various mathematical tools such as mathematical operators with singular or non-singular kernels. The power of simulation and analysis of fractional and fractal-fractional operators compared to classical operators has caused us to see the publication of various articles in the field of modeling phenomena every day. For instances about the analytical and numerical studies, the readers can find new advanced models via fractional and fractal-fractional operators, such as [17–29].

In 2006, a second-order model of thermostat was formulated by Infante and Webb [30], which is insulated at $s = 0$ under the controller at $s = 1$, and it has the following formulation:

$$\begin{cases} -v''(s) = \varphi(s, v(s)), \quad (s \in \mathbb{I} := [0,1]), \\ v'(0) = 0, \quad v(p) + \mu v'(1) = 0, \end{cases} \tag{3}$$

with the real constant $p \in \mathbb{I}$ and parameter $\mu > 0$, and continuous nonlinear function $\varphi : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$. By the structure of such a second-order model, the addition or discharging of heat under the performance of a thermostat depends on the temperature assessed by the sensor at $s = p$. From the mathematical point of view, Infante and his colleague continued their study on the existence results using fixed-point index theory in the context of integral Hammerstein equations. Further, Nieto and his colleague Pimentel [31] discussed and turned to analysis on the properties of existence for solutions of the fractional version of BVP (3) by substituting fractional derivatives of order q instead of classical derivative, in which $q \in (1,2]$ stands for the order of the Caputo fractional derivative. Some years later, Cabrera, Rocha and Sadarangani [32] presented some new structures of Lyapunov-type inequalities in relation to the aforemantioned fractional thermostat BVP under nonlocal boundary conditions.

In this paper, we focus on this target in which a Lyapunov-type inequality is obtained for a generalized fractional model of thermostat control involving generalized ψ-operators given by

$$\begin{cases} -{}^{c}D^{\psi;q}v(s) = K(s, v(s)), \quad (s \in \mathbb{I} := [a_1, a_2],\ a_1, a_2 \in \mathbb{R}), \\ {}^{c}D^{\psi;1}v(a_1) = 0, \quad v(p) + \mu\, {}^{c}D^{\psi;q-1}v(a_2) = 0, \end{cases} \tag{4}$$

with some hypotheses such as $q \in (1, 2]$, $p \in (a_1, a_2)$, $\mu > 0$, and ${}^{c}D^{\psi;1} = \frac{1}{\psi'(s)}\frac{\mathrm{d}}{\mathrm{d}s}$, which is the same generalized ψ-Caputo derivative of order one. Along with these, the given function $K : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ is continuous, and ${}^{c}D^{\psi;\gamma}$ denotes the generalized ψ-Caputo fractional derivative of order $\gamma \in \{1, q, q-1\}$. It is natural that by assuming $\psi(s) = s$ and $q = 2$ and $[a_1, a_2] = [0, 1]$, the fractional ψ-model of thermostat control (4) reduces to the standard second-order model (3) of thermostat control.

In the present study, we concentrate on the establishment of the Lyapunov-type inequality for a new extended ψ-model of thermostat with generalized ψ-operators. As far as we know, the Lyapunov-type inequality for this version of thermostat ψ-model has seldom been studied up to now. Additionally, some required conditions guaranteeing the existence and non-existence of solutions are investigated in the sequel via some established properties of the relevant Green's function. The insights of the present manuscript can be specified as follows. First, we provide several properties of fractional ψ-integrals and derivatives (Section 2). Then, we obtain the Green's function and investigate some important properties of it (Section 3). By considering the concavity and increasing properties of some functions, the Lyapunov-type inequality is constructed for the ψ-thermostat model (4) (Section 4). After that, non-existence and existence theorems are stated for our applied ψ-model of thermostat (Section 5). Finally, the conclusion section is provided (Section 6).

2. Basic Notions

In this section, we state and recall some basic and fundamental notations and definitions, which will be used later. Let $[a_1, a_2]$ $(0 < a_1 < a_2 < \infty)$ be an interval and $\psi : [a_1, a_2] \to \mathbb{R}$ be a function such that $\psi'(s) > 0$ for every $s \in [a_1, a_2]$. Now, by these assumptions, we provide some properties from ψ-fractional calculus.

Definition 1 ([33]). *Let $q > 0$. The ψ-RL-fractional integral (Riemann–Liouville) of order q for an integrable function $v : [a_1, a_2] \to \mathbb{R}$ with respect to the function $\psi : [a_1, a_2] \to \mathbb{R}$ is defined by*

$$I^{\psi;q}_{a_1^+}v(s) = \frac{1}{\Gamma(q)}\int_{a_1}^{s} \psi'(r)(\psi(s) - \psi(r))^{q-1}v(r)\mathrm{d}r, \tag{5}$$

with $\Gamma(\cdot)$ as the Gamma function given by

$$\Gamma(q) = \int_0^{+\infty} e^{-r}r^{q-1}\mathrm{d}r, \quad q > 0.$$

Definition 2 ([34]). *Let $n \in \mathbb{N}$ and $\psi, v \in C^n([a_1, a_2], \mathbb{R})$, where ψ is introduced above. The ψ-Caputo fractional derivative of order q for the function v is defined by*

$${}^{c}D^{\psi;q}_{a_1^+}v(s) = I^{\psi;n-q}_{a_1^+}\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n v(s),$$

with $n = [q] + 1$ for $q \notin \mathbb{N}$ and $n = q$ for $q \in \mathbb{N}$, where $[q]$ denotes the largest integer less than or equal to q.

To simplify in writing, the abbreviated symbol

$$v^{[n]}_{\psi}(s) = \left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n v(s), \tag{6}$$

can be used. By definition,

$$ {}^{c}D_{a_1^+}^{\psi;q} v(s) = \begin{cases} \displaystyle\int_{a_1}^{s} \frac{\psi'(r)(\psi(s)-\psi(r))^{n-q-1}}{\Gamma(n-q)} v_{\psi}^{[n]}(r)\mathrm{d}r &, \quad q \notin \mathbb{N}, \\ v_{\psi}^{[n]}(s) &, \quad q = n \in \mathbb{N}. \end{cases} \tag{7} $$

This extension (7) gives the Caputo fractional derivative if $\psi(s) = s$. For $\psi(s) = \ln s$, the Caputo–Hadamard fractional derivative is obtained.

Next, we provide a property in relation to the composition of the generalzied fractional ψ-derivatives with ψ-integrals.

Lemma 1 ([35])**.** *Let $n \in \mathbb{N}$, $n-1 < q < n$, and $v \in C^n([a_1, a_2], \mathbb{R})$. Then, the following relation holds:*

$$ I_{a_1^+}^{\psi;q} \, {}^{c}D_{a_1^+}^{\psi;q} v(s) = v(s) - \sum_{j=0}^{n-1} \frac{v_{\psi}^{[j]}(a_1)}{j!} [\psi(s) - \psi(a_1)]^j, $$

for all $s \in [a_1, a_2]$. Furthermore, for $m \in \mathbb{N}$ and $v \in C^{n+m}([a_1, a_2], \mathbb{R})$, we have

$$ \left(\frac{1}{\psi'(s)} \frac{\mathrm{d}}{\mathrm{d}s} \right)^m {}^{c}D_{a_1^+}^{\psi;q} v(s) = {}^{c}D_{a_1^+}^{\psi;q+m} v(s) + \sum_{j=0}^{m-1} \frac{[\psi(s) - \psi(a_1)]^{j+n-q-m}}{\Gamma(j+n-q-m+1)} v_{\psi}^{[j+n]}(a_1). $$

Lemma 2 ([33,35])**.** *Let $q, p > 0$, and $v \in C([a_1, a_2], \mathbb{R})$. Then, for each $s \in [a_1, a_2]$, we have*

1. $I_{a_1^+}^{\psi;q} I_{a_1^+}^{\psi;p} v(s) = I_{a_1^+}^{\psi;q+p} v(s)$,
2. ${}^{c}D_{a_1^+}^{\psi;q} I_{a_1^+}^{\psi;q} v(s) = v(s)$,
3. $I_{a_1^+}^{\psi;q} (\psi(s) - \psi(a_1))^{p-1} = \frac{\Gamma(p)}{\Gamma(p+q)} (\psi(s) - \psi(a_1))^{p+q-1}$,
4. ${}^{c}D_{a_1^+}^{\psi;q} (\psi(s) - \psi(a_1))^{p-1} = \frac{\Gamma(p)}{\Gamma(p-q)} (\psi(s) - \psi(a_1))^{p-q-1}$,
5. ${}^{c}D_{a_1^+}^{\psi;q} (\psi(s) - \psi(a_1))^{j} = 0$, $j \in \{0, \dots, n-1\}, n \in \mathbb{N}, n-1 < q < n$.

Lemma 3 ([36])**.** *(Jensen's inequality) Assume that μ is a positive measure and $\mathbb{B}$ is a measurable set such that $\mu(\mathbb{B}) = 1$. If $K \in L^1(\mu)$ is a real-valued function and for each $x \in \mathbb{B}$, $a < K(x) < b$, and ϕ is a real-valued convex function on (a, b), then*

$$ \phi\left(\int_{\mathbb{B}} K \mathrm{d}\mu \right) \le \int_{\mathbb{B}} (\phi \circ K) \mathrm{d}\mu. \tag{8} $$

For K with the concavity property on (a, b), (8) is satisfied with $\ge$ instead of $\le$.

3. Green's Function

Green's function plays a fundamental role in the theory of integral equations [37–39]. Here, we discuss some properties of the relevant Green's function in the thermostat ψ-model.

Proposition 1. *Let $q \in (1, 2]$, $p \in (a_1, a_2)$, $\mu > 0$ and $\mathbb{A} \in C_{\mathbb{R}}(\mathbb{I})$, where $C_{\mathbb{R}}(\mathbb{I})$ denotes the family of all continuous real-valued functions on the interval $\mathbb{I}$. A function $v \in C_{\mathbb{R}}(\mathbb{I})$ is a solution for the linear thermostat ψ-model*

$$ \begin{cases} -{}^{c}D^{\psi;q} v(s) = \mathbb{A}(s), \quad (s \in \mathbb{I} := [a_1, a_2]), \\ {}^{c}D^{\psi;1} v(a_1) = 0, \quad v(p) + \mu \, {}^{c}D^{\psi;q-1} v(a_2) = 0, \end{cases} \tag{9} $$

which is given by the integral equation

$$v(s) = -\int_{a_1}^{s} \frac{\psi'(r)(\psi(s)-\psi(r))^{q-1}}{\Gamma(q)}\mathbb{A}(r)\mathrm{d}r + \int_{a_1}^{p} \frac{\psi'(r)(\psi(p)-\psi(r))^{q-1}}{\Gamma(q)}\mathbb{A}(r)\mathrm{d}r + \mu\int_{a_1}^{a_2}\psi'(r)\mathbb{A}(r)\mathrm{d}r. \tag{10}$$

Proof. If v satisfies the linear ψ-thermostat Equation (9), then ${}^cD^{\psi;q}v(s) = -\mathbb{A}(s)$. As $1 < q \leq 2$, by integrating, it becomes

$$v(s) = -\frac{1}{\Gamma(q)}\int_{a_1}^{s}\psi'(r)(\psi(s)-\psi(r))^{q-1}\mathbb{A}(r)\mathrm{d}r + c_0 + c_1(\psi(s)-\psi(a_1)), \tag{11}$$

where we need to find values of the coefficients $c_0, c_1 \in \mathbb{R}$. Moreover, the properties of the ψ-Caputo fractional derivative give

$${}^cD^{\psi;1}v(s) = -\frac{1}{\Gamma(q-1)}\int_{a_1}^{s}\psi'(r)(\psi(s)-\psi(r))^{q-2}\mathbb{A}(r)\mathrm{d}r + c_1, \tag{12}$$

and for $0 < q-1 \leq 1$, we obtain

$${}^cD^{\psi;q-1}v(s) = -\int_{a_1}^{s}\psi'(r)\mathbb{A}(r)\mathrm{d}r + c_1\frac{(\psi(s)-\psi(a_1))^{2-q}}{\Gamma(3-q)}. \tag{13}$$

By the condition ${}^cD^{\psi;1}v(a_1) = 0$ and (12), we obtain $c_1 = 0$. Moreover, the Equations (11) and (13) and the condition $v(p) + \mu^cD^{\psi;q-1}v(a_2) = 0$ imply that

$$-\frac{1}{\Gamma(q)}\int_{a_1}^{p}\psi'(r)(\psi(p)-\psi(r))^{q-1}\mathbb{A}(r)\mathrm{d}r + c_0 - \mu\int_{a_1}^{a_2}\psi'(r)\mathbb{A}(r)\mathrm{d}r = 0,$$

and thus, we have

$$c_0 = \frac{1}{\Gamma(q)}\int_{a_1}^{p}\psi'(r)(\psi(p)-\psi(r))^{q-1}\mathbb{A}(r)\mathrm{d}r + \mu\int_{a_1}^{a_2}\psi'(r)\mathbb{A}(r)\mathrm{d}r.$$

Finally, if we substitute the obtained coefficients c_0 and c_1 in (11), then the proof is completed. □

Remark 1. *Note that one can rewrite* (10) *by means of Green's function as*

$$v(s) = \int_{a_1}^{a_2} G_\psi(s,r)\psi'(r)\mathbb{A}(r)\mathrm{d}r, \tag{14}$$

where

$$G_\psi(s,r) = \begin{cases} -\frac{(\psi(s)-\psi(r))^{q-1}}{\Gamma(q)} + \frac{(\psi(p)-\psi(r))^{q-1}}{\Gamma(q)} + \mu, & a_1 \leq r \leq \min\{p,s\} \\ \frac{(\psi(p)-\psi(r))^{q-1}}{\Gamma(q)} + \mu, & a_1 \leq s \leq r \leq p, \\ -\frac{(\psi(s)-\psi(r))^{q-1}}{\Gamma(q)} + \mu, & p \leq r \leq s \leq a_2, \\ \mu, & \max\{p,s\} \leq r \leq a_2. \end{cases} \tag{15}$$

Proposition 2. *For Green's function given by* (15)*, we have*

(i) $\min_{a_1 \leq r,s \leq a_2} G_\psi(s,r) = -\frac{(\psi(a_2)-\psi(p))^{q-1}}{\Gamma(q)} + \mu.$

(ii) $\max_{a_1 \leq r,s \leq a_2} G_\psi(s,r) = \mu + \frac{(\psi(p)-\psi(a_1))^{q-1}}{\Gamma(q)}.$

Proof. We have

$$\frac{\partial G_\psi(s,r)}{\partial s}\begin{cases} -\frac{(q-1)\psi'(s)(\psi(s)-\psi(r))^{q-2}}{\Gamma(q)}, & a_1 \le r \le s \le a_2, \\ 0, & a_1 \le s \le r \le a_2. \end{cases}$$

This shows that $G_\psi(s,r)$ is a non-increasing function with respect to the first variable s.
(i). From the above result, we deduce that

$$\min_{a_1 \le s \le a_2} G_\psi(s,r) =$$
$$G_\psi(a_2,r) = \begin{cases} -\frac{(\psi(a_2)-\psi(r))^{q-1}}{\Gamma(q)} + \frac{(\psi(p)-\psi(r))^{q-1}}{\Gamma(q)} + \mu, & \text{for } a_1 \le r \le p \\ -\frac{(\psi(a_2)-\psi(r))^{q-1}}{\Gamma(q)} + \mu, & \text{for } p \le r \le a_2. \end{cases} \tag{16}$$

Using the fact that $a^x - b^x \le (a-b)^x$ for any $a \ge b \ge 0$ and $x \in (0,1]$, we have

$$0 \le (\psi(a_2) - \psi(r))^{q-1} - (\psi(p) - \psi(r))^{q-1} \le (\psi(a_2) - \psi(p))^{q-1},$$

for each $a_1 \le r \le p$. This leads to

$$\begin{aligned} G_\psi(a_2,r) &= -\frac{(\psi(a_2) - \psi(r))^{q-1}}{\Gamma(q)} + \frac{(\psi(p) - \psi(r))^{q-1}}{\Gamma(q)} + \mu \\ &\ge -\frac{(\psi(a_2) - \psi(p))^{q-1}}{\Gamma(q)} + \mu, \end{aligned}$$

for each $a_1 \le r \le p$. On the other hand, we can easily see that

$$\min_{p \le r \le a_2} G_\psi(a_2,r) = G_\psi(a_2,p) = -\frac{(\psi(a_2) - \psi(p))^{q-1}}{\Gamma(q)} + \mu.$$

Combining two last inequalities yield that

$$\min_{a_1 \le r \le a_2} G_\psi(a_2,r) = -\frac{(\psi(a_2) - \psi(p))^{q-1}}{\Gamma(q)} + \mu.$$

Using (16) and the latter inequality, we obtain

$$\min_{a_1 \le r,s \le a_2} G_\psi(s,r) = -\frac{(\psi(a_2) - \psi(p))^{q-1}}{\Gamma(q)} + \mu.$$

(ii). We have

$$\max_{a_1 \le s \le a_2} G_\psi(s,r) = G_\psi(a_1,r) = \begin{cases} \frac{(\psi(p)-\psi(r))^{q-1}}{\Gamma(q)} + \mu, & \text{for } a_1 \le r \le p, \\ \mu, & \text{for } p \le r \le a_2. \end{cases} \tag{17}$$

It is obvious that

$$\max_{a_1 \le r \le p} G_\psi(a_1,r) = G_\psi(a_1,a_1) = \frac{(\psi(p) - \psi(a_1))^{q-1}}{\Gamma(q)} + \mu > \mu.$$

Combining the latter inequality and (17), we obtain

$$\max_{a_1 \le r,s \le a_2} G_\psi(s,r) = \frac{(\psi(p) - \psi(a_1))^{q-1}}{\Gamma(q)} + \mu.$$

The proof of Proposition 2 is completed. □

Remark 2. *Since ψ is a non-decreasing function, from Proposition 2, we have*

$$\max_{a_1\le r,s\le a_2}|G_\psi(s,r)| = \max\left\{\frac{\mu\Gamma(q)+(\psi(p)-\psi(a_1))^{q-1}}{\Gamma(q)}, \frac{\left|(\psi(a_2)-\psi(p))^{q-1}-\mu\Gamma(q)\right|}{\Gamma(q)}\right\}.$$

Moreover, if $\mu \ge \dfrac{(\psi(a_2)-\psi(p))^{q-1}}{\Gamma(q)}$, *then*

$$\max_{a_1\le r,s\le a_2}|G_\psi(s,r)| = \frac{\mu\Gamma(q)+(\psi(p)-\psi(a_1))^{q-1}}{\Gamma(q)}.$$

Proposition 3. *For Green's function given by (15), the following inequality holds:*

$$\begin{aligned}&\int_{a_1}^{a_2}|G_\psi(s,r)|\psi'(r)\mathrm{d}r\\ &\le \max\left\{\frac{(\psi(p)-\psi(a_1))^q}{\Gamma(q+1)}+\mu(\psi(a_2)-\psi(a_1)), \frac{(\psi(a_2)-\psi(p))^q}{\Gamma(q+1)}-\mu(\psi(a_2)-\psi(a_1))\right\},\end{aligned}$$

for each $s\in[a_1,a_2]$.

Proof. Using and by direct computations, we reach the desired result of the Proposition. □

4. Lyapunov-Type Inequality

In this section, we obtain a Lyapunov-type inequality for the proposed thermostat control ψ-model. We consider the following assumption:

- **Assumption** $(\mathcal{A}1)$**:** There exist $\kappa : [a_1,a_2] \to \mathbb{R}$ and a positive, concave and non-decreasing function $g:\mathbb{R}\to\mathbb{R}$ such that

$$|K(s,v)| \le |\kappa(s)||g(v)|,$$

for each $s\in[a_1,a_2]$ and $v\in\mathbb{R}$.

Using the proposed assumption and above notations, we state the main result of this section as follows. Here, we define $\|v\| = \max_{a_1\le s\le a_2}|v(s)|$.

Theorem 1. *Assume that Assumption $(\mathcal{A}1)$ holds. If $\psi'(\cdot)\kappa(\cdot)\in L^1[a,b]$, and the fractional ψ-model of thermostat control (4) has a non-trivial solution, $v\in C[a_1,a_2]$, then*

$$\int_{a_1}^{a_2}\psi'(r)|\kappa(r)|\mathrm{d}r \ge \min\left\{\frac{\Gamma(q)}{\mu\Gamma(q)+(\psi(p)-\psi(a_1))^{q-1}}, \frac{\Gamma(q)}{\left|(\psi(a_2)-\psi(p))^{q-1}-\mu\Gamma(q)\right|}\right\}\frac{\|v\|}{g(\|v\|)}.$$

If $\mu\ge\frac{(\psi(a_2)-\psi(p))^{q-1}}{\Gamma(q)}$, *then*

$$\int_{a_1}^{a_2}\psi'(r)|\kappa(r)|\mathrm{d}r \ge \frac{\Gamma(q)\|v\|}{(\mu\Gamma(q)+(\psi(p)-\psi(a_1))^{q-1})g(\|v\|)}.$$

Proof. If $v\in C[a_1,a_2]$ is a non-trivial solution of the fractional ψ-model of thermostat control (4), we find from (14) that

$$v(s) = \int_{a_1}^{a_2}G_\psi(s,r)\psi'(r)K(r,v(r))\mathrm{d}r. \tag{18}$$

For each $s\in[a_1,a_2]$, by using Jensen's inequality, and from (18), we have

$$|v(s)| \le \int_{a_1}^{a_2}|G_\psi(s,r)\psi'(r)K(r,v(r))|\mathrm{d}r$$

$$\leq \max_{a_1\leq s,r\leq a_2}|G_\psi(s,r)|\int_{a_1}^{a_2}\psi'(r)|\kappa(r)||g(v(r))|\mathrm{d}r$$
$$\leq \max_{a_1\leq s,r\leq a_2}|G_\psi(s,r)|\|\psi'(\cdot)\kappa(\cdot)\|_{L^1[a,b]}\int_{a_1}^{a_2}\frac{\psi'(r)|\kappa(r)|}{\|\psi'\|\|\kappa\|_{L^1[a,b]}}|g(v(r))|\mathrm{d}r$$
$$\leq \max_{a_1\leq s,r\leq a_2}|G_\psi(s,r)|\|\psi'(\cdot)\kappa(\cdot)\|_{L^1[a,b]}g\left(\int_{a_1}^{a_2}\frac{\psi'(r)|\kappa(r)|}{\|\psi'\|\|\kappa\|_{L^1[a,b]}}|v(r)|\mathrm{d}r\right)$$
$$\leq \max_{a_1\leq s,r\leq a_2}|G_\psi(s,r)|\|\psi'(\cdot)\kappa(\cdot)\|_{L^1[a,b]}g(\|u\|).$$

It follows from Proposition 2 that

$$\|\psi'(\cdot)\kappa(\cdot)\|_{L^1[a,b]}\geq\frac{1}{\max_{a_1\leq s,r\leq a_2}|G_\psi(s,r)|}\frac{\|u\|}{g(\|u\|)}$$
$$=\min\left\{\frac{\Gamma(q)}{\mu\Gamma(q)+(\psi(p)-\psi(a_1))^{q-1}},\ \frac{\Gamma(q)}{|(\psi(a_2)-\psi(p))^{q-1}-\mu\Gamma(q)|}\right\}\frac{\|v\|}{g(\|v\|)}.$$

This completes the proof. □

Corollary 1. *For $K(s,v(s))=\kappa(s)v(s)$, $\psi(s)=s$ and $\mu\geq\frac{(a_2-p)^{q-1}}{\Gamma(q)}$, we have*

$$\int_{a_1}^{a_2}|\kappa(r)|\mathrm{d}r\geq\frac{\Gamma(q)}{\mu\Gamma(q)+(p-a_1)^{q-1}}.$$

The result coincides with the one in [32].

Proof. Apply Theorem 1 for $\psi(s)=s$ and $g(v)=v$. □

5. Some Existence and Non-Existence Results

In this section, we investigate the existence and non-existence of a mild solution for the thermostat control ψ-model (4). We begin with the definition of mild solutions.

Definition 3. *The function $v\in C[a_1,a_2]$ is called a mild solution of the thermostat control model* (4) *if it satisfies the following integral equation:*

$$v(s)=\int_{a_1}^{a_2}G_\psi(s,r)\psi'(r)K(r,v(r))\mathrm{d}r. \quad (19)$$

To study the existence of a mild solution for our problem, the following assumption will be considered:

- **Assumption** $(\mathcal{A}2)$**:** There exists $\kappa:[a_1,a_2]\to[0,+\infty)$ such that

$$|K(s,v)-K(s,w)|\leq\kappa(s)|v-w|,$$

for each $(s,v),(s,w)\in[a_1,a_2]\times\mathbb{R}$.

Continuously, from now on, for $\varphi\in C[a_1,a_2]$, we denote $\|\varphi\|=\max_{a_1\leq s\leq a_2}|\varphi(s)|$. Based on the above assumption and definition, we can state and prove the existence and uniqueness result for our ψ-model.

Theorem 2. *Suppose that K is a continuous function which satisfies Assumption $(\mathcal{A}2)$. If $\psi\in C^1[a_1,a_2]$, $\kappa\in L^1[a_1,a_2]$, and*

$$\|\kappa\|_{L^1[a_1,a_2]}<\frac{1}{\|\psi'\|}\min\left\{\frac{\Gamma(q)}{\mu\Gamma(q)+(\psi(p)-\psi(a_1))^{q-1}},\ \frac{\Gamma(q)}{|(\psi(a_2)-\psi(p))^{q-1}-\mu\Gamma(q)|}\right\},$$

then the ψ-model (4) of the thermostat has a unique mild solution.

Proof. Let us consider the operator $Q : C[a_1, a_2] \to C[a_1, a_2]$ defined by

$$Qv(s) = \int_{a_1}^{a_2} G_\psi(s,r)\psi'(r)K(r,v(r))\mathrm{d}r.$$

Note that it is well-defined in virtue of the continuity of the functions ψ', G_ψ and K. Then

$$\begin{aligned}|Qv(s) - Qw(s)| &\leq \|\psi'\| \max_{a_1 \leq s,r \leq a_2} |G_\psi(s,r)| \int_{a_1}^{a_2} |K(r,v(r)) - K(r,w(r))|\mathrm{d}r \\ &\leq \|\psi'\| \max_{a_1 \leq s,r \leq a_2} |G_\psi(s,r)| \int_{a_1}^{a_2} \kappa(s)|v(r) - w(r)|\mathrm{d}r \\ &\leq \|\psi'\| \max_{a_1 \leq s,r \leq a_2} |G_\psi(s,r)| \|\kappa\|_{L^1[a_1,a_2]} \|v - w\|.\end{aligned}$$

It follows

$$\|Qv - Qw\| \leq \|\psi'\| \max_{a_1 \leq s,r \leq a_2} |G_\psi(s,r)| \|\kappa\|_{L^1[a_1,a_2]} \|v - w\|. \tag{20}$$

Note that

$$\begin{aligned}\|\kappa\|_{L^1[a_1,a_2]} &< \frac{1}{\|\psi'\|} \min\left\{ \frac{\Gamma(q)}{\mu\Gamma(q) + (\psi(p) - \psi(a_1))^{q-1}}, \frac{\Gamma(q)}{|(\psi(a_2) - \psi(p))^{q-1} - \mu\Gamma(q)|} \right\} \\ &= \left(\|\psi'\| \max_{a_1 \leq s,r \leq a_2} |G_\psi(s,r)| \right)^{-1}.\end{aligned}$$

Thus, we conclude from (20) that Q is contraction in $C[a_1, a_2]$. Hence, Q has a unique fixed point in $C[a_1, a_2]$, which is a mild solution of the thermostat control ψ-model (4). The proof of Theorem 2 is completed. □

To complete this section, we give a non-existence result for our problem. Herein, we use the following assumption:

- **Assumption** $(\mathcal{A}3)$**:** There exists a constant $L_K > 0$ such that

$$|K(s,v)| \leq L_K|v|,$$

for each $(s,v) \in [a_1, a_2] \times \mathbb{R}$.

Theorem 3. *Suppose that Assumption $(\mathcal{A}3)$ holds. If $L_K C_\psi < 1$ with*

$$C_\psi = \max\left\{ \frac{(\psi(p) - \psi(a_1))^q}{\Gamma(q+1)} + \mu(\psi(a_2) - \psi(a_1)), \frac{(\psi(a_2) - \psi(p))^q}{\Gamma(q+1)} - \mu(\psi(a_2) - \psi(a_1)) \right\},$$

then the thermostat control ψ-model (4) has no non-trival mild solution.

Proof. We prove by contradiction that the thermostat control ψ-model (4) has a mild solution. Then, from (19), we have

$$\begin{aligned}\|v\| &= \max_{a_1 \leq s \leq a_2} \left| \int_{a_1}^{a_2} G_\psi(s,r)\psi'(r)K(r,v(r))\mathrm{d}r \right| \\ &\leq \max_{a_1 \leq s \leq a_2} \int_{a_1}^{a_2} |G_\psi(s,r)|\psi'(r)|K(r,v(r))|\mathrm{d}r \\ &\leq L_K\|v\| \max_{a_1 \leq s \leq a_2} \int_{a_1}^{a_2} |G_\psi(s,r)|\psi'(r)\mathrm{d}r\end{aligned}$$

$$\leq L_K C_\psi \|v\|$$
$$< \|v\|,$$

due to Proposition 3. This is a contradiction. The proof of Theorem 3 is completed. □

6. Conclusions

In the present reaserch study, we considered a new applied model of thermostat control by defining the relevant differential equation and boundary value conditions with the help of ψ-operators in which ψ is a non-decreasing function. This model covers all previous fractional BVPs of thermostat control. To follow the study, a mathematical structure of Green's function was obtained, and then maximum and minimum values of it over the given interval were calculated. By using some estimates and the Jensen's inequality, the Lyapunov-type inequality was proved under the supposed conditions for the generalized ψ-model of the thermostat. Moreover, based on functional analysis techniques, the non-existence and existence of mild solutions for such a generalized ψ-system were established. Due to the importance of real models of processes, we can continue such studies for other forms of applied mathematical models via the newly defined mathematical operators.

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