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Numerical Solutions of Inverse Nodal Problems for a Boundary Value Problem

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Abstract: In this paper, we study inverse nodal problems for a boundary value problem. A uniqueness result for the potential function and a reconstruction method are obtained. By using the nodal points as input data, we compute the approximation solution of the potential function for the boundary value problem by the first kind Chebyshev wavelet method. Two numerical examples show that the first kind Chebyshev wavelet method for solving the inverse nodal problems for the boundary value problem is valid.

Keywords: inverse nodal problem; boundary value problem; potential function; Chebyshev wavelet

MSC: 34A55; 47E05

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1. Introduction

We are concerned with the inverse nodal problem for the boundary value problem (BVP) $L := L(q, h, a)$ defined by

$$ly := -y'' + q(x)y = \rho^2 y, \quad 0 < x < 1, \tag{1}$$

associated with boundary conditions:

$$y'(0, \rho) - hy(0, \rho) = 0, \tag{2}$$
$$ay'(1, \rho) + \rho y(1, \rho) = 0, \tag{3}$$

where $a \neq 0, a, h \in \mathbb{R}$, ρ is the spectral parameter, $q(x)$ is a real-valued function and $q \in L^2[0, 1]$.

Differential operators with boundary conditions having the spectral parameter frequently arise in nuclear physics, mathematics, quantum mechanics (see [1–9] and the references therein). In 2010, using the method of spectral mapping, Freiling and Yurko [1] studied three inverse problems for the Sturm–Liouville equation with boundary conditions polynomially dependent on the spectral parameter, and provided procedures to reconstruct this operator. In 1965, Li [2] showed that only one spectrum is sufficient to determine the potential function $q(x)$ of BVP $L(q, \infty, a)$ on $[0, 1]$ by the quantum theory of scattering and presented an example to show that Li's theorem does not hold for $a = 0$.

The classical Sturm–Liouville operator $L_0 := L(q, h, H)$ is of the form (see [10]):

$$\begin{cases} ly := -y'' + q(x)y = \lambda y, \quad 0 < x < 1, \\ y'(0, \lambda) - hy(0, \lambda) = 0, \\ y'(1, \lambda) + Hy(1, \lambda) = 0, \end{cases}$$

where h and $q(x)$ are defined the above, λ is the spectral parameter, $H \in \mathbb{R}$. Borg [3] showed that two spectra with one common boundary condition and another differential boundary conditions are sufficient to determine the potential on $[0,1]$ together with the coefficients of the boundary conditions. Although there is only a little difference between operators L and L_0, many features are unlike, for example, Li's theorem for L and Borg's theorem for L_0. However, Hochstadt [4] studied the relationship between Li's theorem and Borg's theorem and improved the Li's theorem. He proved that one spectrum of operator L is equivalent to two spectra of operators $L_0(q,\infty,0)$ and $L_0(q,\infty,\infty)$.

The inverse nodal problem for differential operators is to recover the potential function and coefficients of boundary conditions by using its nodal data (see [11–13]), which was firstly studied by McLaughlin [11], Shen [12], Hald and McLaughlin [13]. Later on, there have been a lot of study of recovering the potential function by less nodal data. The uniqueness theorems and the reconstruction formulae were given by partial nodal data, for example, X.F. Yang [14]; Cheng, Law, and Tsay [15]; Guo and Wei [16]; C.-F. Yang [17]; Buterin and Shieh [18,19]; Wang and Yurko [20]; Wang, Shieh, and Wei [21]; and Wei, Miao, Ge, and Zhao [22], and the references therein). In particular, Chen, Cheng, and Law studied the stability of the inverse nodal problem for the Sturm–Liouville operator L_0 [23]. Since BVP $L(q,h,a)$ is not a special case of the operator in [1], Theorems 1 and 2 are also new results (see [9,20–22]).

In recent years, some numerical methods have been studied to determine the approximation solution of 1st type of Fredholm integral equation by Rashed [24,25]; Maleknejad, Saeedipoor, and Dehbozorgi [26]; Zhou and Xu [27], or other works. The approximation solutions of the inverse nodal problem for differential operators were studied by Akbarpoor, Koyunbakan, and Dabbaghian [28]; Gulsen, Yilmaz, and Akbarpoor [29]; Neamaty, Akbarpoor, and Yilmaz [30], respectively. In this study, we compute the approximation solution of the inverse nodal problem of BVP L by the first kind Chebyshev wavelet method and apply the first kind Chebyshev wavelet method for solving this problem.

In Section 2, we establish the uniqueness theorem for BVP L and give the reconstruction procedure. In Section 3, we find an approximation solution of the potential function $q(x)$ of BVP L from the first kind Chebyshev wavelet method. In Section 4, we present two numerical examples to show that the numerical method is valid.

2. Inverse Nodal Problem

In this section, we study the asymptotic formula of nodal points of the boundary value problem (1)–(3) and establish a uniqueness theorem for the inverse node problem with given nodal data.

Let $S(x,\rho), C(x,\rho), \varphi(x,\rho)$, and $\psi(x,\rho)$ be solutions of (1) with the initial conditions

$$S(0,\rho)=0, S'(0,\rho)=1, C(0,\rho)=1, C'(0,\rho)=0,$$

$$\varphi(0,\rho)=1, \varphi'(0,\rho)=h, \psi(1,\rho)=a, \psi'(1,\rho)=-\rho.$$

Denote $\tau=|\operatorname{Im}\rho|$, $\varphi_0=\arctan\frac{1}{a}$ and

$$q_1(x):=h+\frac{1}{2}\int_0^x q(t)\mathrm{d}t.$$

Then, the asymptotic formulae of $\varphi(x,\rho)$ and $\psi(x,\rho)$ are as follows:

$$\varphi(x,\rho) = \cos\rho x+q_1(x)\frac{\sin\rho x}{\rho}+\mathrm{o}\left(\frac{\mathrm{e}^{\tau x}}{|\rho|}\right), \quad 0\le x\le 1, \tag{4}$$

$$\begin{aligned}
\varphi'(x,\rho) &= -\rho\sin\rho x+q_1(x)\cos\rho x+\mathrm{o}(\mathrm{e}^{\tau x}), \quad 0\le x\le 1,\\
\psi(x,\rho) &= \sqrt{1+a^2}\cos(\rho(1-x)-\varphi_0)+\mathrm{O}\left(\frac{\mathrm{e}^{\tau(1-x)}}{|\rho|}\right), \quad 0\le x\le 1, \qquad (5)\\
\psi'(x,\rho) &= \sqrt{1+a^2}\rho\sin(\rho(1-x)-\varphi_0)+\mathrm{O}\left(\mathrm{e}^{\tau(1-x)}\right), \quad 0\le x\le 1.
\end{aligned}$$

The characteristic function $\Delta(\rho)$ of L is defined by

$$\Delta(\rho) := \langle \psi, \varphi \rangle (x, \rho),$$

where $\langle \psi, \varphi \rangle (\rho, x) := \psi(x,\rho)\varphi'(x,\rho) - \psi'(x,\rho)\varphi(x,\rho)$, which is called Wronskian of ψ and φ. Clearly $\Delta(\rho)$ is independent of x (see [10]), and zeros of $\Delta(\rho)$ are called the eigenvalues of L. Denote the index set $\mathbf{A} := \{\pm 0, \pm 1, \pm 2, \cdots\}$ (For details, see [31]) and $\sigma(L) := \{\rho_n : \rho_n \in \mathbf{A}\}$ be the set of eigenvalues. Therefore, the asymptotic formula of $\Delta(\rho)$ is

$$\begin{aligned}\Delta(\rho) =& a\varphi'(x,\rho) + \rho\varphi(x,\rho) \\ =& \sqrt{1+a^2}[-\rho \sin(\rho - \varphi_0) + \omega \cos(\rho - \varphi_0) + \mathrm{o}(\mathrm{e}^{\tau})],\end{aligned}$$

where

$$\omega = h + \frac{1}{2}\int_0^1 q(t)\mathrm{d}t.$$

We have the asymptotic formulae of eigenvalue ρ_n:

$$\rho_n = n\pi + \varphi_0 + \frac{\omega}{n\pi + \varphi_0} + \frac{\kappa_n}{n}, \quad n \in \mathbf{A}, \quad |n| \gg 1, \tag{6}$$

where $\{\kappa_n\} \in l^2$. It follows from (6), all eigenvalues are real and simple for sufficiently large $|n|$. By direct calculation, we see that the n-th eigenfunction $\varphi(x, \rho_n)$ has exactly $|n|$ zeros $x_n^j \in (0,1)$, which satisfies the following formula:

$$\begin{aligned}&0 < x_n^1 < x_n^2 < \cdots < x_n^n < 1, \quad \text{if} \quad n > 0, \\ &0 < x_n^0 < x_n^{-1} < \cdots < x_n^{n+1} < 1, \text{ if} \quad n < 0,\end{aligned}$$

$$x_n^j = \frac{(j-1/2)\pi}{n\pi + \varphi_0} + \frac{2h + \int_0^{x_n^j} q(t)\mathrm{d}t}{2(n\pi + \varphi_0)^2} - \frac{(j-1/2)\pi\omega}{(n\pi+\varphi_0)^3} + \mathrm{o}\left(\frac{1}{n^2}\right). \tag{7}$$

for $n \gg 1$ uniformly with respect to j. Denote $l_n^j := x_n^{j+1} - x_n^j$,

$$X := X_+ \bigcup X_-, \quad X_+ := \bigcup_{n=1}^{\infty} \left\{x_n^j\right\}_{j=1}^{n-1} \quad \text{and} \quad X_- := \bigcup_{n=-\infty}^{-1} \left\{x_n^j\right\}_{j=n+1}^{0}.$$

It follows from (7)

$$l_n^j = \frac{\pi}{n\pi + \varphi_0} + \mathrm{o}\left(\frac{1}{n^2}\right) \tag{8}$$

and X is dense on $(0,1)$. We have

Theorem 1. *Given $X_0 \subseteq X$, where X_0 be dense on $(0,1)$. For each fixed $x \in [0,1]$, select a nodal sequence $\left\{x_{n_k}^{j_{n_k}}\right\} \subseteq X_0$ such that $\lim_{|n_k|\to\infty} x_{n_k}^{j_{n_k}} = x$, then*

$$\begin{aligned}\varphi_0 =& -\pi \lim_{|n_k|\to\infty} n_k \left(n_k l_{n_k}^{j_{n_k}} - 1\right), \\ f(x) :=& \lim_{|n_k|\to\infty} 2(n_k\pi + \varphi_0)^2 \left(x_{n_k}^{j_{n_k}} - \frac{(j_{n_k} - \frac{1}{2})\pi}{n_k\pi + \varphi_0}\right) \\ =& \int_0^x q(t)\mathrm{d}t + 2h - 2\omega x.\end{aligned}$$

Proof. It follows from (8)

$$\varphi_0 = -\pi \lim_{|n_k|\to\infty} \Big(n_k (n_k l_{n_k-1}^{j_{n_k}} - 1) + \mathrm{o}(1) \Big) = -\pi \lim_{|n_k|\to\infty} n_k \Big(n_k l_{n_k-1}^{j_{n_k}} - 1 \Big).$$

We reconstruct the coefficient a by

$$\frac{1}{a} = \tan \varphi_0. \tag{9}$$

It follows from (7)

$$\begin{aligned} f(x) &:= \lim_{|n_k|\to\infty} 2(n_k\pi + \varphi_0)^2 \left(x_{n_k}^{j_{n_k}} - \frac{(j_{n_k} - \frac{1}{2})\pi}{n_k\pi + \varphi_0} \right) \\ &= \lim_{|n_k|\to\infty} \left(\int_0^{\frac{(j_{n_k} - \frac{1}{2})\pi}{n_k\pi + \varphi_0}} q(t)\mathrm{d}t + 2h - 2\omega \frac{(j_{n_k} - \frac{1}{2})\pi}{n_k\pi + \varphi_0} + \mathrm{o}(1) \right) \\ &= \int_0^x q(t)\mathrm{d}t + 2h - 2\omega x. \end{aligned} \tag{10}$$

In (10), let $x = 0$, we obtain

$$h = \frac{f(0)}{2}. \tag{11}$$

Taking the derivative with respect to x in (10), we get

$$f'(x) \overset{\text{a.e.}}{=} q(x) - 2\omega.$$

Then, we reconstruct the coefficient $q(x) - \int_0^1 q(t)\mathrm{d}t$ as follows

$$q(x) - \int_0^1 q(t)\mathrm{d}t \overset{\text{a.e.}}{=} f'(x) + 2h. \tag{12}$$

□

According to the Theorem 1, the coefficient of L is reconstructed from the nodal subset X_0 on $(0,1)$, and the reconstruction procedure is as follows:

Algorithm 1.

(1) *For each fixed* $x \in [0,1]$, *choose a sequence* $\left\{ x_{n_k}^{j_{n_k}} \right\} \subseteq X_0$ *such that* $\lim_{|n_k|\to\infty} x_{n_k}^{j_{n_k}} = x$;
(2) *The coefficient* a *is reconstructed by* (9);
(3) *Calculate the function* $f(x)$ *from* (10);
(4) *Reconstruct the coefficient* h *of the boundary conditions from* (11);
(5) *Recover the function* $q(x) - \int_0^1 q(t)\mathrm{d}t$ *by* (12).

From Algorithm 1, we obtain the uniqueness theorem

Theorem 2. $(q(x) - \int_0^1 q(t)\mathrm{d}t, h, a)$ *can be uniquely determined by the dense nodal subset* X_0 *on* $(0,1)$.

3. Numerical Solution of Inverse Nodal Problems

In this section, we study the following numerical solution of inverse node problems of L.

Numerical solution: for sufficiently large n, given the nodal points $x_n^j, j = \overline{1,n}$ and constants h, a, and $\int_0^1 q(t)\mathrm{d}t$, reconstruct the potential function $q(x)$. The solution $\varphi(x,\rho)$

of (1) can be written as follows (see [10]):

$$\varphi\left(x_n^j,\lambda_n\right)=\cos(\rho_n x_n^j)+q_1\left(x_n^j\right)\frac{\sin(\rho_n x_n^j)}{\rho_n}+\frac{1}{2\rho_n}\int_0^{x_n^j}q(t)\sin(\rho_n x_n^j)\cos(2\rho_n t)\,\mathrm{d}t$$
$$-\frac{1}{2\rho_n}\int_0^{x_n^j}q(t)\cos(\rho_n x_n^j)\sin(2\rho_n t)\,\mathrm{d}t+\mathrm{O}\left(\frac{1}{\rho_n^2}\right)=0. \tag{13}$$

This implies

$$\cos(\rho_n x_n^j)=\mathrm{O}\left(\frac{1}{\rho_n}\right). \tag{14}$$

By virtue of (13) and (14), we obtain

$$\int_0^{x_n^j}q(t)\cos^2(\rho_n t)\,\mathrm{d}t=-\rho_n\cot(\rho_n x_n^j)-h+\mathrm{O}\left(\frac{1}{\rho_n}\right)$$
$$=-\rho_n^0\cot(\rho_n^0 x_n^j)-h+\mathrm{O}\left(\frac{1}{n}\right), \tag{15}$$

where $\rho_n^0:=n\pi+\varphi_0+\frac{\omega}{n\pi+\varphi_0}$. Therefore, (15) implies

$$\int_0^{x_n^j}q(t)\cos^2(\rho_n^0 t)\,\mathrm{d}t=-\rho_n^0\cot(\rho_n^0 x_n^j)-h+\mathrm{O}\left(\frac{1}{n}\right). \tag{16}$$

It follows from (16)

$$\int_0^{x_n^j}q(t)\cos^2(\rho_n^0 t)\mathrm{d}t\cong-\rho_n^0\cot(\rho_n^0 x_n^j)-h. \tag{17}$$

It is well known that Equation (17) is the first kind Fredholm integral equation. We convert the integral Equation (17) to a system of linear Equation (18) (see below). Then, the solution of the system of linear Equation (18) is an approximation solution of the potential function $q(t)$. We approximate the potential function $q(t)$ with the first kind Chebyshev polynomials as the basis functions.

Consider the Chebyshev wavelets on the interval $[0,1)$ (see [26])

$$\psi_{l,m}(t)=\begin{cases}2^{k/2}\widetilde{T}_m\left(2^k t-2l+1\right), & \dfrac{l-1}{2^{k-1}}\le t<\frac{l}{2^{k-1}},\\ 0, & \text{otherwise},\end{cases}$$

where

$$\widetilde{T}_m(t)=\begin{cases}\frac{1}{\sqrt{\pi}}, & m=0,\\ \sqrt{\frac{2}{\pi}}T_m(t), & m>0,\end{cases}$$

where $l=1,2,\ldots,2^{k-1}, m=0,1,\ldots,M-1$, k can be any positive integer, and $M\gg 1$. The functions $T_m(t)$ are the Chebyshev polynomials of degree m of the first kind on the interval $[-1,1]$, given by the following recursive formula:

$$T_0(t)=1,\quad T_1(t)=t,$$
$$T_{m+1}(t)=2tT_m(t)-T_{m-1}(t),\quad m=1,2,\ldots.$$

The function $f(t)$ on the interval $[0,1)$ is expressed as

$$f(t)=\sum_{l=1}^{2^{k-1}}\sum_{m=0}^{\infty}c_{l,m}\psi_{l,m}(t),$$

where

$$c_{l,m}=<q(t),\psi_{l,m}(t)>_{L_w^2[0,1)}=\int_0^1 q(t)\psi_{l,m}(t)w(2^k t-2l+1)\mathrm{d}t,$$

$< \cdot, \cdot >_{L^2_w[0,1)}$ is the inner product on $L^2_w[0,1)$, $\|q\|_{2,w}$ is the norm of $q(x)$ on $L^2_w[0,1)$, and the weight function is $w(x) = \frac{1}{\sqrt{1-x^2}}$.

Using the Chebyshev wavelets to approximate the potential function $q(t)$, we have

$$q(t) \cong \sum_{l=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{l,m}\psi_{l,m}(t) = \mathbf{C}^{\mathrm{T}}\mathbf{\Psi}(\mathbf{t}), \tag{18}$$

where

$$\mathbf{C} = \left[c_{1,0}, \ldots, c_{1,M-1}, c_{2,0}, \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},M-1}\right]^{\mathrm{T}},$$
$$\mathbf{\Psi}(\mathbf{t}) = \left[\psi_{1,0}(t), \ldots, \psi_{1,M-1}(t), \psi_{2,0}(t), \ldots, \psi_{2,M-1}(t), \ldots, \psi_{2^{k-1},0}(t), \ldots, \psi_{2^{k-1},M-1}(t)\right]^{\mathrm{T}}.$$

Substituting (18) into (17), we get

$$\sum_{l=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{l,m}\left(\int_0^{x_n^j} \psi_{l,m}(t)\cos^2(\rho_n^0 t)\mathrm{d}t\right) \cong -\rho_n^0 \cot(\rho_n^0 x_n^j) - h, \quad j = \overline{1,n}.$$

Next, we present the following theorem for the convergence of the given method. The readers refer to the references [25,26] for the convergence of the first kind Chebyshev wavelet method.

Theorem 3. *For each fixed $n = 2^{k-1}M$, $M \gg 1$, given n nodal points $\{x_n^j\}_{j=1}^n$ satisfying (7) together with the coefficients $(h, a, Q(1))$, then the potential function $q(x)$ can be written as an infinite sum of the first kind Chebyshev wavelets and this series converges to $q(x)$, that it is*

$$q(x) = \sum_{l=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{l,m}\psi_{l,m}(x)$$

and the approximation of potential function $q(x)$ is as follows:

$$q_{k,M}(x) = \sum_{l=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{l,m}\psi_{l,m}(x).$$

The solution of the inverse node problem can be computed by the following steps:

(1) Choose k, M. Set $n = 2^{k-1}M$.

(2) Calculate the unknown vector $\mathbf{C}$ by the following linear equation:

$$\mathbf{AC} = \mathbf{B},$$

where

$$\mathbf{A} = \begin{pmatrix} a_{1,0}^1 & a_{1,1}^1 & \cdots & a_{1,M-1}^1 & a_{2,0}^1 & \cdots & a_{2,M-1}^1 & \cdots & a_{2^{k-1},0}^1 & \cdots & a_{2^{k-1},M-1}^1 \\ a_{1,0}^2 & a_{1,1}^2 & \cdots & a_{1,M-1}^1 & a_{20}^2 & \cdots & a_{2,M-1}^2 & \cdots & a_{2^{k-1},0}^2 & \cdots & a_{2^{k-1},M-1}^2 \\ \vdots & & & & \vdots & & & & \vdots & & \\ a_{1,0}^{2^{k-1}M} & a_{1,1}^{2^{k-1}M} & \cdots & a_{1,M-1}^{2^{k-1}M} & a_{2,0}^{2^{k-1}M} & \cdots & a_{2,M-1}^{2^{k-1}M} & \cdots & a_{2^{k-1},0}^{2^{k-1}M} & \cdots & a_{2^{k-1},M-1}^{2^{k-1}M} \end{pmatrix},$$

where

$$a_{l,m}^j = \int_0^{x_n^j} \psi_{l,m}(t)\cos^2(\rho_n^0)t\mathrm{d}t,$$
$$l = \overline{1, 2^{k-1}}, \quad m = \overline{0, M-1}, \quad n = 2^{k-1}M, \quad j = \overline{1,n},$$

$$\mathbf{B} = \begin{pmatrix} -\rho_n^0 \cot(\rho_n^0 x_n^1) - h \\ -\rho_n^0 \cot(\rho_n^0 x_n^2) - h \\ \cdot \\ \cdot \\ \cdot \\ -\rho_n^0 \cot(\rho_n^0 x_n^n) - h \end{pmatrix},$$

(3) To approximate $q(t_i), i = \overline{1, 2^{k-1}M}$, we use the following formula:

$$[q(t_i)] = \mathbf{C}^{\mathrm{T}}\mathbf{\Phi},$$

where

$$t_i = \frac{2i-1}{2^k M}, \quad i = 1, 2, \ldots, 2^{k-1}M,$$

and

$$\mathbf{\Phi} = \left[\mathbf{\Psi}\left(\frac{1}{2^k M}\right), \mathbf{\Psi}\left(\frac{3}{2^k M}\right), \ldots, \mathbf{\Psi}\left(\frac{2^k M - 1}{2^k M}\right)\right].$$

4. Numerical Examples

In this section, we use the first kind Chebyshev wavelet method to solve the inverse nodal problem for (1)–(3) and provide two numerical examples to demonstrate the accuracy of the numerical method by the Matlab software program.

Example 1. *Let the potential function* $q(x) = \cos(4\pi x)$ *and* $h = 1, a = 1$. *If* $k = 4$ *and* $M = 5, 7, 9$ *and* x_n^j *satisfy the Formula (7), respectively, we find three approximation solutions of the potential function* $q(x)$.

We use the first kind Chebyshev wavelet method to obtain an approximation of the potential function $q(x)$ as a solution of the inverse nodal problem for L. Numerical values of $q(x)$ and exact solutions of $q(x)$ with $k = 4$ and $M = 5, 7, 9$, respectively, we obtain three approximation solutions of the potential function $q(x)$ by the first kind Chebyshev wavelet method (see Figure 1).

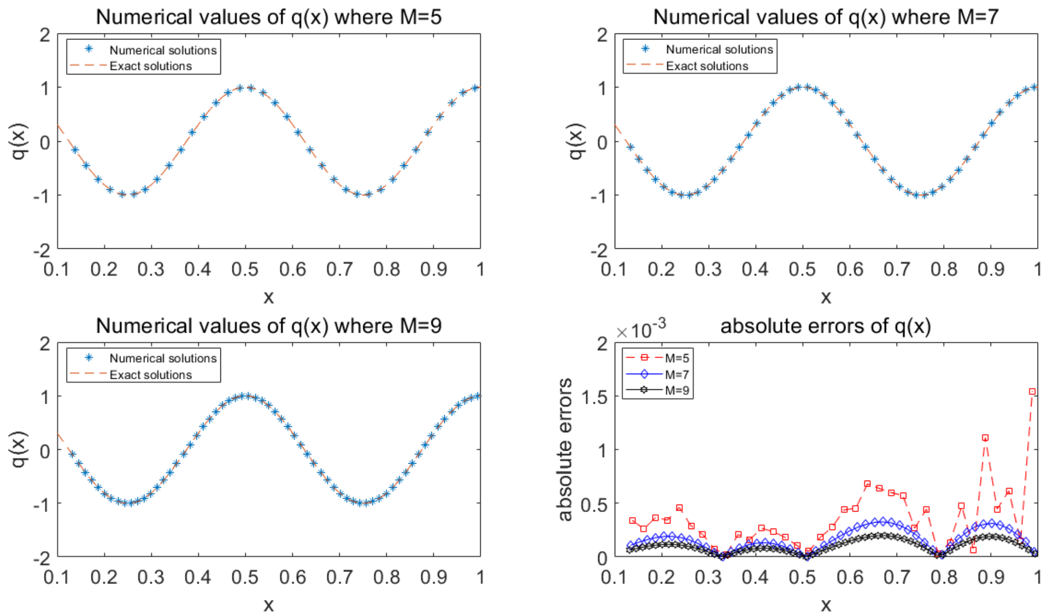

Figure 1. Numerical values of $q(x)$ and absolute errors between the approximation and exact solutions of $q(x)$ with $k = 4$ and $M = 5, 7, 9$ in Example 1.

Now, we give the second example for numerical values of $q(x)$ and exact solutions of $q(x)$ with $k = 4$ and $M = 5, 7, 9$ and $k = 5, 7$, $M = 5$ and $k = 6$, $M = 10$, respectively.

Example 2. *Let the potential function* $q(x) = 3x^2 - 1$ *and* $h = 1, a = 1$.

(1) *If* $k = 4$ *and* $M = 5,7,9$ *and* x_n^j *satisfy the Formula* (7)*, we find three approximation solutions of the potential function* $q(x)$*;*

(2) *If* $k = 5,7$*,* $M = 5$ *and* $k = 6$*,* $M = 10$ *and* x_n^j *satisfy the Formula* (7)*, we find three approximation solutions of the potential function* $q(x)$*.*

We obtain approximation solutions of the potential function $q(x)$ by the first kind Chebyshev wavelet method (see Figures 2 and 3).

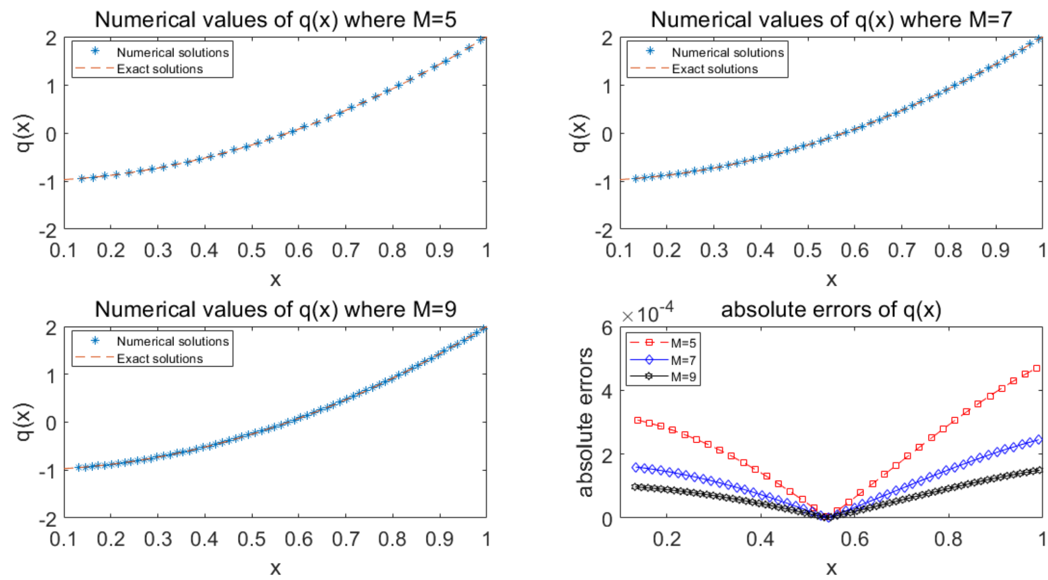

Figure 2. Numerical values of $q(x)$ and absolute errors between the approximation and exact solutions of $q(x)$ with $k = 4$ and $M = 5, 7, 9$ in Example 2.

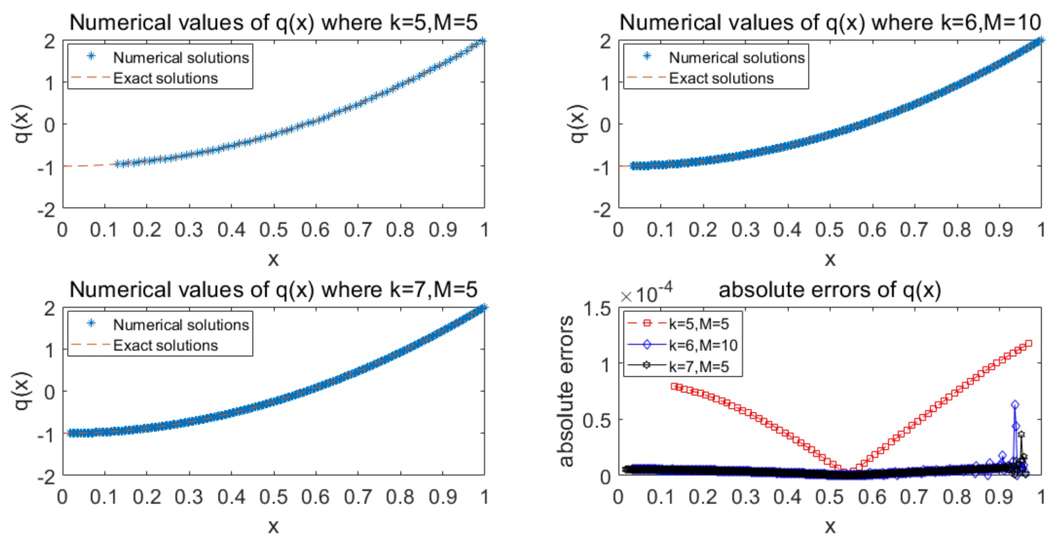

Figure 3. Numerical values of $q(x)$ and absolute errors between the approximation and exact solutions of $q(x)$ with $k = 5,7$, $M = 5$, and $k = 6$, $M = 10$ in Example 2.

In the Figures 1–3, it can be seen that by increasing the values of n, the approximation solution of BVP L for the inverse nodal problem by the first kind Chebyshev wavelet method becomes more accurate and the error decreases. However, if n is not large, the errors near the boundary points are larger than others. If $n = 2^{7-1} \times 5$, the numerical solution is more effective.

5. Conclusions

In this work, we study the inverse nodal problem for Sturm–Liouville equation with one boundary condition having spectral parameter. The uniqueness theorem for BVP L and the reconstruction procedure are presented from the dense nodal set on the whole interval. By applying the first kind Chebyshev wavelet method, we compute three approximation solutions of BVP L for $k = 4$ and $M = 5, 7, 9$, respectively, in two examples. We still compute three approximation solutions of BVP L for $k = 5, 7$, $M = 5$ and $k = 6$, $M = 10$, respectively, in Example 2. With increasing the values of n, the approximation solution of BVP L for the inverse nodal problem by the first kind Chebyshev wavelet method becomes more accurate and the error decreases. It is also proved that the first kind Chebyshev wavelet method for the approximation solution of BVP L for the inverse nodal problem is an effective method.

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