Article

On Solvability of Fractional (p,q)-Difference Equations with (p,q)-Difference Anti-Periodic Boundary Conditions

Ravi P. Agarwal [1] (ORCID), **Hana Al-Hutami** [2] (ORCID) and **Bashir Ahmad** [2,*] (ORCID)

1   Department of Mathematics, Texas A&M University, Kingsville, TX 78363-8202, USA
2   Nonlinear Analysis and Applied Mathematics (NAAM)—Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
*   Correspondence: bahmad@kau.edu.sa or bashirahmad_qau@yahoo.com

Abstract: We discuss the solvability of a (p,q)-difference equation of fractional order $\alpha \in (1,2]$, equipped with anti-periodic boundary conditions involving the first-order (p,q)-difference operator. The desired results are accomplished with the aid of standard fixed point theorems. Examples are presented for illustrating the obtained results.

Keywords: fractional Caputo fractional (p,q)-derivative; (p,q)-difference operator; anti-periodic boundary conditions; existence; fixed point

MSC: 26A33; 39A13; 34B15

1. Introduction

The subject of (p,q)-calculus is known as the extension of q-calculus to its two-parameter (p,q) variant, and it has efficient applications in many fields. One can find some useful information about the (p,q)-calculus in articles [1–3]. On the other hand, details about (p,q)-analogues of the Laplace transform, convolution formula, two-dimensional conformal field theory, Kantorovich type Bernstein–Stancu–Schurer operator, and hypergeometric functions related to quantum groups can, respectively, be found in [4–8]. For a (p,q)-oscillator realization of two-parameter quantum algebras and application of the (p,q)-gamma function to Szász Durrmeyer operators, we refer the reader to articles [9,10], respectively.

Let us now address some recent developments on boundary value problems of (p,q)-difference equations. Kamsrisuk et al. [11] proved the existence and uniqueness of solutions for a first-order quantum (p,q)-difference equation with a nonlocal integral condition. In [12], the authors studied a second-order (p,q)-difference equation with separated boundary conditions. The authors in [13] obtained some existence results for impulsive quantum (p,q)-difference equations. In [14], some existence results for a boundary value problem of (p,q)-integrodifference equations with nonlocal fractional (p,q)-integral boundary conditions were presented. The existence of multiple positive solutions for a fractional (p,q)-difference equation under (p,q)-integral boundary conditions was discussed in [15]. More recently, the authors investigated the existence of solutions for a nonlinear fractional (p,q)-difference equation subject to separated nonlocal boundary conditions in [16], whereas some existence results for a sequential fractional Caputo (p,q)-integrodifference equation with three-point fractional Riemann–Liouville (p,q)-difference boundary conditions were obtained in [17]. However, one can notice that the study on the boundary value problems of fractional (p,q)-difference equations is at its initial phase and needs further attention for the enrichment of the topic.

In this paper, we introduce and study a new class of boundary value problems of fractional (p,q)-difference equations supplemented with anti-periodic boundary conditions involving (p,q)-difference operator given by

$$^{c}D_{p,q}^{\alpha}x(t) = f(t, x(p^{\alpha}t)), \ t \in [0, T/p^{\alpha}], \ 1 < \alpha \leq 2, \ T > 0, \tag{1}$$

$$x(0) + x(T/p^{\alpha}) = 0, \ D_{p,q}x(0) + D_{p,q}x(T/p^{\alpha}) = 0, \tag{2}$$

where $^{c}D_{p,q}^{\alpha}$ and $D_{p,q}$, respectively, denote the Caputo type fractional (p,q)-derivative operator of order $\alpha \in (1,2]$ and the first-order (p,q)-difference operator, $0 < q < p \leq 1$, $f \in C([0, T/p^{\alpha}] \times \mathbb{R}, \mathbb{R})$.

The objective of the present work is to develop sufficient criteria for the existence of solutions for an anti-periodic boundary value problem involving fractional (p,q)-difference and (p,q)-difference operators. The standard tools of the fixed point theory are applied to accomplish the desired results.

In the rest of the paper, we recall some fundamental concepts of (p,q)-calculus and prove an auxiliary lemma related to the linear version of the given problem in Section 2. The main results are presented in Section 3, and the illustrative examples for these results are discussed in Section 4.

2. Preliminaries

Let us first describe some fundamental concepts of q-calculus and (p,q)-calculus [16]. We also establish an auxiliary lemma that will be used in obtaining the main results of the paper.

Throughout this paper, let $[a, b] \subset \mathbb{R}$ be an interval with $a < b$, and $0 < q < p \leq 1$ be constants. We define

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}, \quad k \in \mathbb{N},$$

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q} = \displaystyle\prod_{i=1}^{k} \frac{p^i - q^i}{p - q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

The q-analogue of the power function $(a - b)_q^{(n)}$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ can be expressed as

$$(a - b)_q^{(0)} = 1, \quad (a - b)_q^{(n)} := \prod_{k=0}^{n-1}(a - bq^k), \quad a, b \in \mathbb{R}.$$

The (p,q)-analogue of the power function $(a - b)_{p,q}^{(n)}$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ is defined by

$$(a - b)_{p,q}^{(0)} = 1, \quad (a - b)_{p,q}^{(n)} := \prod_{k=0}^{n-1}(ap^k - bq^k), \quad a, b \in \mathbb{R}.$$

For $t \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$, (p,q)-gamma function is defined by

$$\Gamma_{p,q}(t) = \frac{(p - q)_{p,q}^{(t-1)}}{(p - q)^{t-1}},$$

and an equivalent definition of (p,q)-gamma function is

$$\Gamma_{p,q}(t) = p^{\frac{t(t-1)}{2}} \int_0^\infty x^{t-1} E_{p,q}^{-qx} d_{p,q}x,$$

where

$$E_{p,q}^{-qx} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!}(-qx)^n, \quad \binom{n}{2} = \frac{\Gamma(n+1)}{\Gamma(2+1)\Gamma(n-2+1)}.$$

Obviously, $\Gamma_{p,q}(t+1) = [t]_{p,q}\Gamma_{p,q}(t)$.

Definition 1. *Let $0 < q < p \leq 1$. Then, the (p,q)-derivative of f is defined by*

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \ t \neq 0,$$

with $D_{p,q}f(0) = \lim_{t \to 0} D_{p,q}f(t)$, provided that f is differentiable at 0.

Definition 2. *Let f be an arbitrary function, and t be a real number. The (p,q)-integral of f is defined as*

$$\int_0^t f(s)d_{p,q}s = (p-q)t \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}t\right),$$

provided that the series on the right-hand side converges.

Definition 3. *Let $f : [0,T] \to \mathbb{R}$ be a continuous function and $\alpha \geq 0$. The fractional (p,q)-integral of Riemann–Liouville type is given by $\left(I_{p,q}^0 f\right)(t) = f(t)$ and*

$$\left(I_{p,q}^\alpha f\right)(t) = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$= \frac{(p-q)t}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(t - \frac{q^{n+1}}{p^{n+1}}t\right)_{p,q}^{(\alpha-1)} f\left(\frac{q^n}{p^{\alpha+n}}t\right).$$

Definition 4. *The fractional (p,q)-derivative of Riemann–Liouville type of order $\alpha \geq 0$ of a continuous function f is defined by $(D_{p,q}^0 f)(t) = f(t)$ and*

$$(D_{p,q}^\alpha f)(t) = \left(D_{p,q}^{[\alpha]} I_{p,q}^{[\alpha]-\alpha}\right)(t),$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Definition 5. *The fractional (p,q)-derivative of Caputo type of order $\alpha \geq 0$ of a continuous function f is defined by $\left({}^c D_{p,q}^0 f\right)(t) = f(t)$ and*

$$({}^c D_{p,q}^\alpha f)(t) = \left(I_{p,q}^{[\alpha]-\alpha} D_{p,q}^{[\alpha]} f\right)(t),$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Lemma 1. *Let f be a continuous function and $\alpha, \beta \geq 0$. Then the following formulas hold:*

(i) $\left(I_{p,q}^\beta I_{p,q}^\alpha f\right)(t) = \left(I_{p,q}^{\alpha+\beta} f\right)(t);$

(ii) $\left(D_{p,q}^\alpha I_{p,q}^\alpha f\right)(t) = f(t);$

(iii) $\left(I_{p,q}^\alpha D_{p,q}^n\right)(t) = \left(D_{p,q}^n I_{p,q}^\alpha f\right)(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^{\alpha-n+k}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha-n+k+1)} \left(D_{p,q}^k\right)(0), n \in \mathbb{N};$

(iv) $\left(I_{p,q}^\alpha {}^c D_{p,q}^\alpha\right)(t) = f(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(k+1)} \left(D_{p,q}^k\right)(0).$

For details of the above concepts, see [16] and the references cited therein. In order to define the solution for the problem in (1) and (2), we need the following lemma.

Lemma 2. *For a given $g \in C([0, T/p^\alpha], \mathbb{R})$, the unique solution of the following boundary value problem:*

$$\begin{cases} {}^c D_{p,q}^\alpha x(t) = g(t), \ t \in [0, T/p^\alpha], \\ x(0) + x(T/p^\alpha) = 0, \\ D_{p,q} x(0) + D_{p,q} x(T/p^\alpha) = 0, \end{cases} \tag{3}$$

is given by

$$x(t) = \int_0^t \frac{(t - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s - \frac{t}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha-1)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha-1)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s$$

$$- \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s. \tag{4}$$

Proof. Applying fractional (p, q)-integral operator on both sides of the fractional (p, q)-difference equation in (3), we get

$$x(t) = \int_0^t \frac{(t - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s + b_0 t + b_1, \tag{5}$$

where b_0, b_1 are constants and $t \in [0, T/p^\alpha]$. The (p, q)-derivative of (5) is

$$D_{p,q} x(t) = \int_0^t \frac{(t - qs)^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha-1)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s + b_0. \tag{6}$$

Using the boundary conditions of (3) in (5) and (6) and solving the resulting system of equations for b_0 and b_1, we get

$$b_0 = -\frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha-1)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s,$$

and

$$b_1 = \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha-1)} g\left(\frac{s}{p^{\alpha-2}}\right) d_{p,q}s - \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha)} g\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s.$$

Substituting the above values of b_0 and b_1 in (5) and (6), we obtain (4). The converse of the lemma follows by direct computation. Therefore, the proof is complete. □

3. Main Results

Let $\mathcal{C} := \mathbf{C}([0, T/p^\alpha], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, T/p^\alpha]$ to $\mathbb{R}$ endowed with a norm defined by $\|x\| = \sup\{|x(t)| : t \in [0, T/p^\alpha]\}$. In view of Lemma 2, we define an operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ associated with the problem in (1) and (2) as

$$(\mathcal{G}x)(t) = \int_0^t \frac{(t-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, x(p^{\alpha-1}s))\, d_{p,q}s - \frac{t}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} f(s, x(p^\alpha s))\, d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} f(s, x(p^\alpha s))\, d_{p,q}s$$

$$- \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, x(p^{\alpha-1}s))\, d_{p,q}s. \tag{7}$$

Observe that the problem in (1) and (2) has a solution if the operator equation $\mathcal{G}x = x$ has a fixed point, where the operator $\mathcal{G}$ is given by (7).

Theorem 1. *Let $f : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and that there exists a (p,q)-integrable function $L : [0, T/p^\alpha] \to \mathbb{R}$ such that*

(H_1) $|f(t,x) - f(t,y)| \le L(t)|x - y|$, for each $t \in [0, T/p^\alpha]$ and $x, y \in \mathbb{R}$. Then the boundary value problem (1) and (2) has a unique solution on $[0, T/p^\alpha]$ provided that

$$\Phi < 1, \tag{8}$$

where

$$\Phi = \left(I_{p,q}^\alpha L\right)(T/p^\alpha) + \frac{T}{2p^\alpha}\left(I_{p,q}^{\alpha-1}L\right)(T/p^\alpha) + \frac{T}{4p^\alpha}\left(I_{p,q}^{\alpha-1}L\right)(T/p^\alpha) + \frac{1}{2}\left(I_{p,q}^\alpha L\right)(T/p^\alpha). \tag{9}$$

Proof. We transform the problem in (1) and (2) into a fixed point problem $\mathcal{G}x = x$, where the operator $\mathcal{G}$ is given by (7). Applying Banach's contraction mapping principle, we will show that $\mathcal{G}$ has a unique fixed point. Define a ball $B_\sigma = \{x \in \mathcal{C} : \|x\| \le \sigma\}$ with $\sigma \ge (M\Omega)(1-\Phi)^{-1}$, where $M = \sup\limits_{t \in [0, T/p^\alpha]} |f(t,0)|$ and

$$\Omega = \frac{T^\alpha}{p^{\alpha^2} p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha+1)} + \frac{T^\alpha}{2p^{\alpha^2} p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha)} + \frac{T^\alpha}{4p^{\alpha^2} p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha)} + \frac{T^\alpha}{2p^{\alpha^2} p^{\binom{\alpha}{2}} \Gamma_{p,q}(\alpha+1)}. \tag{10}$$

Then, by the condition (H_1), we have $|f(t, x(t))| \le |f(t, x(t)) - f(t,0)| + |f(t,0)| \le L(t)\sigma + M$. Now, we shall show that $\mathcal{G}B_\sigma \subset B_\sigma$. For any $x \in B_\sigma$, we find that

$$|(\mathcal{G}x)(t)|$$

$$\le \int_0^t \frac{(t-qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ \frac{|t|}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))|\, d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))|\, d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$\le \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))| \, d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))| \, d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$\leq \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left(L\left(\frac{s}{p^{\alpha-1}}\right)\sigma + M \right) d_{p,q}s$$

$$+ \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \left(L\left(\frac{s}{p^\alpha}\right)\sigma + M \right) d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \left(L\left(\frac{s}{p^\alpha}\right)\sigma + M \right) d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left(L\left(\frac{s}{p^{\alpha-1}}\right)\sigma + M \right) d_{p,q}s$$

$$\leq M \left\{ \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} d_{p,q}s + \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} d_{p,q}s \right.$$

$$\left. + \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} d_{p,q}s + \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} d_{p,q}s \right\}$$

$$+ \sigma \left\{ \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s + \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} L\left(\frac{s}{p^\alpha}\right) d_{p,q}s \right.$$

$$\left. + \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} L\left(\frac{s}{p^\alpha}\right) d_{p,q}s + \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right\}$$

$$\leq M \left\{ \frac{T^\alpha}{p^{\alpha^2} p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)} + \frac{T^\alpha}{2p^{\alpha^2} p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)} + \frac{T^\alpha}{4p^{\alpha^2} p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)} + \frac{T^\alpha}{2p^{\alpha^2} p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)} \right\}$$

$$+ \sigma \left\{ \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s + \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} L\left(\frac{s}{p^\alpha}\right) d_{p,q}s \right.$$

$$\left. + \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} L\left(\frac{s}{p^\alpha}\right) d_{p,q}s + \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right\},$$

which, in the view of (9) and (10), implies that

$$\|\mathcal{G}x\| \leq M\Omega + \sigma\Phi \leq \sigma.$$

This shows that $\mathcal{G}\mathcal{B}_\sigma \subset \mathcal{B}_\sigma$. Now, for $x, y \in \mathcal{C}$, we obtain

$$\|\mathcal{G}x - \mathcal{G}y\|$$

$$\leq \sup_{t \in [0, T/p^\alpha]} \left\{ \int_0^t \frac{(t - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |f(s, x(p^{\alpha-1}s)) - f(s, y(p^{\alpha-1}s))| \, d_{p,q}s \right.$$

$$\left. + \frac{T}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s)) - f(s, y(p^\alpha s))| \, d_{p,q}s \right.$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |f(s, x(p^\alpha s)) - f(s, y(p^\alpha s))| \, d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |f(s, x(p^{\alpha-1}s)) - f(s, y(p^{\alpha-1}s))| \, d_{p,q}s \Bigg\}$$

$$\leq \|x - y\| \Bigg\{ \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} L\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \Bigg\}$$

$$= \Bigg\{ \left(I_{p,q}^\alpha L\right)(T/p^\alpha) + \frac{T}{2p^\alpha}\left(I_{p,q}^{\alpha-1} L\right)(T/p^\alpha) + \frac{T}{4p^\alpha}\left(I_{p,q}^{\alpha-1} L\right)(T/p^\alpha) + \frac{1}{2}\left(I_{p,q}^\alpha L\right)(T/p^\alpha) \Bigg\} \|x - y\|,$$

which, by (9), implies that

$$\|\mathcal{G}x - \mathcal{G}y\| \leq \Phi\|x - y\|.$$

Since $\Phi \in (0,1)$ by assumption (8), therefore $\mathcal{G}$ is a contraction. Hence, it follows by Bannach's contraction mapping principle that the problem in (1) and (2) has a unique solution on $[0, T/p^\alpha]$. □

If we take $L(t) = L$ (L is a positive constant), condition (8) becomes $L < \frac{1}{\Omega}$, where Ω is defined by (10) and Theorem 1 can be phrased as follows.

Corollary 1. *If $f : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and that there exists $L \in (0, \frac{1}{\Omega})$ with Ω is given by (10), and let $|f(t, x) - f(t, y)| \leq L|x - y|$, for each $t \in [0, T/p^\alpha]$ and $x, y \in \mathbb{R}$, then the boundary value problem (1) and (2) has a unique solution.*

Our next existence result is based on Krasnoselskii's fixed point theorem.

Lemma 3 ([18]). (Krasnoselskii's fixed point theorem) *Let M be a closed, bounded, convex, and non-empty subset of a Banach space X. Let A, B be two operators such that:*

(i) $Ax + By \in M$, whenever $x, y \in M$;
(ii) A is compact and continuous;
(iii) B is contraction mapping.
    Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2. *Let $f : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ be a continuous functions satisfying (H_1). In addition, we assume that*

(H_2) There exists a function $\psi \in \mathbf{C}([0, T/p^\alpha], \mathbb{R}^+)$ and a non-decreasing function $\vartheta \in \mathbf{C}([0, T/p^\alpha], \mathbb{R}^+)$ such that $|f(t, x)| \leq \psi(t)\vartheta(|x|)$, where $(t, x) \in [0, T/p^\alpha] \times \mathbb{R}$;

(H_3) There exists a constant r with

$$r \geq \Omega \, \vartheta(r) \, \|\psi\|, \quad \|\psi\| = \sup_{t \in [0, T/p^\alpha]} |\psi(t)|. \tag{11}$$

Then the boundary value problem (1)–(2) *has at least one solution on* $[0, T/p^\alpha]$, *provided that*

$$\int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s < 1. \tag{12}$$

Proof. Let us define $\mathcal{B}_r := \{x \in \mathcal{C} : \|x\| \le r\}$, where r is given (11) and introduce the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on $\mathcal{B}_r$ to $\mathbb{R}$ as

$$
\begin{aligned}
(\mathcal{G}_1 x)(t) = & -\frac{t}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} f(s, x(p^\alpha s)) \, d_{p,q}s \\
& + \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} f(s, x(p^\alpha s)) \, d_{p,q}s \\
& - \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, x(p^{\alpha-1} s)) \, d_{p,q}s,
\end{aligned}
$$

and

$$(\mathcal{G}_2 x)(t) = \int_0^t \frac{(t - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(s, x(p^{\alpha-1} s)) \, d_{p,q}s.$$

Observe that $\mathcal{G}_1 x + \mathcal{G}_2 x = \mathcal{G}x$. For $x, y \in \mathcal{B}_r$, we have

$$
\begin{aligned}
|(\mathcal{G}_1 x + \mathcal{G}_2 y)(t)| \le & \int_0^t \frac{(t - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, y(p^{\alpha-1} s)) \right| d_{p,q}s \\
& + \frac{|t|}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))| \, d_{p,q}s \\
& + \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))| \, d_{p,q}s \\
& + \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1} s)) \right| d_{p,q}s \\
\le & \, \vartheta(r)\|\psi\| \left[\frac{T^\alpha}{p^{\alpha^2} p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)} + \frac{T^\alpha}{2p^{\alpha^2} p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)} + \frac{T^\alpha}{4p^{\alpha^2} p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)} \right. \\
& \left. + \frac{T^\alpha}{2p^{\alpha^2} p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)} \right] \\
\le & \, r.
\end{aligned}
$$

Thus, $\mathcal{G}_1 x + \mathcal{G}_2 y \in \mathcal{B}_r$. From (H_1) and (12), it follows that $\mathcal{G}_2$ is a contraction mapping. It follows from the continuity of f that the operator $\mathcal{G}_1$ is continuous. Moreover, it can easily be verified that

$$\|\mathcal{G}_1\| \le \vartheta(r)\|\psi\| \left[\frac{T^\alpha}{2p^{\alpha^2} p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)} + \frac{T^\alpha}{4p^{\alpha^2} p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha)} + \frac{T^\alpha}{2p^{\alpha^2} p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)} \right].$$

Therefore, the set $\mathcal{G}_1(\mathcal{B}_r)$ is uniformly bounded. Next we show the compactness of the operator $\mathcal{G}_1$. Let us fix

$$\overline{f} = \sup_{(t,x) \in [0, T/p^\alpha] \times \mathcal{B}_r} |f(t,x)| < \infty,$$

and take $t_1, t_2 \in [0, T/p^\alpha]$ with $t_1 < t_2$. Then we get

$$|(\mathcal{G}_1 x)(t_2) - (\mathcal{G}_1 x)(t_1)|$$

$$\leq \frac{(t_2 - t_1)}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha - 1)} |f(s, x(p^\alpha s))| \, d_{p,q}s$$

$$\leq \overline{f} \left(\frac{T^{\alpha-1}}{2 p^{\alpha(\alpha-1)} p^{\binom{\alpha-1}{2}} \Gamma_{p,q}(\alpha)} \right) (t_2 - t_1),$$

which is independent of x and tends to zero as $t_1 \to t_2$. Therefore, the set $\mathcal{G}_1(B_r)$ is equicontinuous. Thus, the conclusion of the Arzelá–Ascoli theorem applies and hence $\mathcal{G}_1$ is compact on B_r. In the foregoing steps, it has been shown that the hypothesis of Lemma 3 is satisfied. Hence, by the conclusion of Lemma 3, the boundary value problem (1) and (2) has at least one solution on $[0, T/p^\alpha]$. $\quad\square$

As special case, for $\vartheta(|x|) \equiv 1$, there always exists a positive real number r such that (11) holds true. In consequence, we have the following corollary.

Corollary 2. *Let $f : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1) with $L(t) = L$. In addition, we assume that*

$$|f(t, x)| \leq \psi(t), \ \forall (t, x) \in [0, T/p^\alpha] \times \mathbb{R} \text{ and } \mu \in \mathbf{C}([0, T/p^\alpha], \mathbb{R}^+).$$

Then the boundary value problem (1) and (2) has at least one solution on $[0, T/p^\alpha]$ provided that (12) holds.

In the next existence result, we apply the Leray–Schauder nonlinear alternative [19].

Lemma 4 ([19]). *(Nonlinear alternative for single value maps) Let $\mathcal{C}$ be a closed and convex subset of the Banach space E and $\mathcal{U}$ be an open subset of $\mathcal{C}$ with $u \in \mathcal{U}$. Suppose that $\mathcal{F} : \overline{\mathcal{U}} \to \mathcal{C}$ is continuous and compact map, that is, $\mathcal{F}(\overline{\mathcal{U}})$ is a relatively compact subset of $\mathcal{C}$. Then either*

(i) $\mathcal{F}$ has a fixed point in $\overline{\mathcal{U}}$, or

(ii)   there is a $u \in \partial\mathcal{U}$ (the boundary of $\mathcal{U}$ in $\mathcal{C}$) and $\lambda \in (0, 1)$ with $u = \lambda \mathcal{F} u$.

Theorem 3. *Let $f : [0, T/p^\alpha] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and the following assumptions hold:*

(H_4) *There exist functions $\varphi_1, \varphi_2 \in \mathbf{C}([0, T/p^\alpha], \mathbb{R}^+)$, and a nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$|f(t, x)| \leq \varphi_1(t)\Psi(|x|) + \varphi_2(t), \ \ (t, x) \in [0, T/p^\alpha] \times \mathbb{R};$$

(H_5) *There exists a number $\varrho > 0$ such that*

$$\frac{\varrho}{\Psi(\varrho)\varpi_1 + \varpi_2} > 1, \tag{13}$$

where

$$\varpi_i = \left(I_{p,q}^\alpha \varphi_i \right)(T/p^\alpha) + \frac{T}{2p^\alpha} \left(I_{p,q}^{\alpha-1} \varphi_i \right)(T/p^\alpha) + \frac{T}{4p^\alpha} \left(I_{p,q}^{\alpha-1} \varphi_i \right)(T/p^\alpha)$$

$$+ \frac{1}{2} \left(I_{p,q}^\alpha \varphi_i \right)(T/p^\alpha), \ i = 1, 2. \tag{14}$$

Then the boundary value problem (1) and (2) has at least one solution on $[0, T/p^\alpha]$.

Proof. Consider the operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ defined by (7). We first show that $\mathcal{G}$ is continuous. Let $\{x_n\}$ be a sequence of functions such that $x_n \to x$ on $[0, T/p^\alpha]$. Given that f is a continuous function on $[0, T/p^\alpha]$, we have $f(t, x_n(p^\alpha t)) \to f(t, x(p^\alpha t))$. Therefore, we obtain

$$
\begin{aligned}
&|(\mathcal{G}x_n)(t) - (\mathcal{G}x)(t)| \\
&\leq \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |f(s, x_n(p^{\alpha-1}s)) - f(s, x(p^{\alpha-1}s))|\, d_{p,q}s \\
&\quad + \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x_n(p^\alpha s)) - f(s, x(p^\alpha s))|\, d_{p,q}s \\
&\quad + \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |f(s, x_n(p^\alpha s)) - f(s, x(p^\alpha s))|\, d_{p,q}s \\
&\quad + \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |f(s, x_n(p^{\alpha-1}s)) - f(s, x(p^{\alpha-1}s))|\, d_{p,q}s,
\end{aligned}
$$

which implies that

$$
\|\mathcal{G}x_n - \mathcal{G}x\| \to 0, \quad \text{as } n \to \infty.
$$

Thus, the operator $\mathcal{G}$ is continuous. Next, we show that $\mathcal{G}$ maps bounded set into bounded set in $\mathbf{C}([0, T], \mathbb{R})$. For a positive number $\rho > 0$, let $B_\rho = \{x \in \mathbf{C}([0, T]) : \|x\| \leq \rho\}$. Then, for any $x \in B_\rho$, we have

$$
\begin{aligned}
|(\mathcal{G}x)(t)| &\leq \int_0^t \frac{(t - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1}s)) \right|\, d_{p,q}s \\
&\quad + \frac{|t|}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))|\, d_{p,q}s \\
&\quad + \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))|\, d_{p,q}s \\
&\quad + \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1}s)) \right|\, d_{p,q}s \\
&\leq \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[\varphi_1\left(\frac{s}{p^{\alpha-1}}\right) \Psi(\|x\|) + \varphi_2\left(\frac{s}{p^{\alpha-1}}\right) \right]\, d_{p,q}s \\
&\quad + \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \left[\varphi_1\left(\frac{s}{p^\alpha}\right) \Psi(\|x\|) + \varphi_2\left(\frac{s}{p^\alpha}\right) \right]\, d_{p,q}s \\
&\quad + \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \left[\varphi_1\left(\frac{s}{p^\alpha}\right) \Psi(\|x\|) + \varphi_2\left(\frac{s}{p^\alpha}\right) \right]\, d_{p,q}s \\
&\quad + \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[\varphi_1\left(\frac{s}{p^{\alpha-1}}\right) \Psi(\|x\|) + \varphi_2\left(\frac{s}{p^{\alpha-1}}\right) \right]\, d_{p,q}s \\
&\leq \Psi(\rho) \Bigg\{ \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \varphi_1\left(\frac{s}{p^{\alpha-1}}\right)\, d_{p,q}s \\
&\quad + \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \varphi_1\left(\frac{s}{p^\alpha}\right)\, d_{p,q}s
\end{aligned}
$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \varphi_1\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \varphi_1\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \Bigg\}$$

$$+ \Bigg\{ \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \varphi_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \varphi_2\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \varphi_2\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \varphi_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \Bigg\}.$$

In consequence, we deduce that

$$\|\mathcal{G}x\| \leq \Psi(\rho)\varpi_1 + \varpi_2,$$

where $\varpi_i, i = 1, 2$, are given in (14).

Therefore, the set $\mathcal{G}B_\rho$ is uniformly bounded. Now we show that $\mathcal{G}$ maps bounded sets into equicontinuous sets of $\mathbf{C}([0, T/p^\alpha], \mathbb{R})$. Let $t_1, t_2 \in [0, T/p^\alpha]$ with $t_1 < t_2$ be two points and B_ρ be a bounded ball in $\mathcal{G}$. Then, for any $x \in B_\rho$, we get

$$|(\mathcal{G}x)(t_2) - (\mathcal{G}x)(t_1)|$$

$$\leq \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$- \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| f(s, x(p^{\alpha-1}s)) \right| d_{p,q}s$$

$$+ \frac{(t_2 - t_1)}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} |f(s, x(p^\alpha s))| \, d_{p,q}s$$

$$\leq \int_0^{t_2} \frac{[(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}]}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[\varphi_1\left(\frac{s}{p^{\alpha-1}}\right)\Psi(\rho) + \varphi_2\left(\frac{s}{p^{\alpha-1}}\right) \right] d_{p,q}s$$

$$+ \int_{t_1}^{t_2} \frac{(t_1 - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left[\varphi_1\left(\frac{s}{p^{\alpha-1}}\right)\Psi(\rho) + \varphi_2\left(\frac{s}{p^{\alpha-1}}\right) \right] d_{p,q}s$$

$$+ \frac{(t_2 - t_1)}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \left[\varphi_1\left(\frac{s}{p^\alpha}\right)\Psi(\rho) + \varphi_2\left(\frac{s}{p^\alpha}\right) \right] d_{p,q}s.$$

Obviously, the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t_2 \to t_1$. Thus, it follows by the Arzelá–Ascoli theorem that $\mathcal{G} : \mathbf{C}([0, T/p^\alpha], \mathbb{R}) \to \mathbf{C}([0, T/p^\alpha], \mathbb{R})$ is completely continuous. Hence the operator $\mathcal{G}$ satisfies all the conditions of Lemma 4, and therefore, by its conclusion, either condition (*i*) or condition (*ii*) holds. Now we show that the conclusion (*ii*) is not possible. Let $\mathcal{Z}_\varrho = \{ x \in \mathbf{C}([0, T/p^\alpha], \mathbb{R}) : \|x\| \leq \varrho \}$ with $\Psi(\varrho)\varpi_1 + \varpi_2 < \varrho$. Then, we obtain

$$|\mathcal{G}x| \leq \Psi(\|x\|) \Bigg\{ \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \varphi_1\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \varphi_1\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \varphi_1\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \varphi_1\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \Bigg\}$$

$$+ \Bigg\{ \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \varphi_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

$$+ \frac{T}{2p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \varphi_2\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{T}{4p^\alpha} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-2)}}{p^{\binom{\alpha-1}{2}}\Gamma_{p,q}(\alpha-1)} \varphi_2\left(\frac{s}{p^\alpha}\right) d_{p,q}s$$

$$+ \frac{1}{2} \int_0^{T/p^\alpha} \frac{(T/p^\alpha - qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \varphi_2\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \Bigg\}$$

$$\leq \Psi(\varrho)\varpi_1 + \varpi_2 \leq \varrho.$$

Suppose there exists $x \in \partial\mathcal{Z}_\varrho$ and $\kappa \in (0,1)$ such that $x = \kappa \mathcal{G}x$. Then, for such a choice of x and κ, we have

$$\varrho = \|x\| = \kappa\|\mathcal{G}x\| < \Psi(\|x\|)\varpi_1 + \varpi_2 < \varrho.$$

This leads to a contradiction. Accordingly, by Lemma 4, we have that $\mathcal{G}$ has a fixed point $x \in \overline{\mathcal{Z}}_\varrho$, which is a solution of the problem in (1) and (2). Therefore, the proof is complete. □

Remark 1. *If φ_1, φ_2 in (H_4) are continuous, then $\varpi_i = \Omega\|\varphi_i\|$, $i = 1, 2$, where Ω is defined by* (10).

4. Examples

Here, we construct examples for the illustration of the results obtained in the last section.

Example 1. *Consider the following fractional (p,q)-difference equation with (p,q)-difference anti-periodic boundary conditions:*

$$^cD_{1/2,1/3}^{3/2}x(t) = f\left(t, x\left(\frac{t}{2\sqrt{2}}\right)\right), \; t \in [0, 2\sqrt{2}], \tag{15}$$

$$x(0) + x(2\sqrt{2}) = 0, \; D_{p,q}x(0) + D_{p,q}x(2\sqrt{2}) = 0. \tag{16}$$

Here, $p = 1/2, q = 1/3, \alpha = 3/2, T = 1$.

I. Illustration of Corollary 1. *We take*

$$f\left(t, x\left(\frac{t}{2\sqrt{2}}\right)\right) = L\left(\sin\left(\frac{t}{2\sqrt{2}}\right) + \arctan x\right),$$

where L is a constant. It is not difficult to show that

$$|f(t,x) - f(t,y)| \leq L|x - y|, \; \text{for each } t \in [0, 2\sqrt{2}],$$

and

$$\Phi = \frac{L}{\Gamma_{1/2,1/4}(3/2)}\left[\frac{3\sqrt[8]{2}(4-\sqrt{2})+3\sqrt[8]{2^{25}}}{4-\sqrt{2}}\right],$$

where

$$L < \left[\frac{1}{\Gamma_{1/2,1/4}(3/2)}\left(\frac{3\sqrt[8]{2}(4-\sqrt{2})+3\sqrt[8]{2^{25}}}{4-\sqrt{2}}\right)\right]^{-1}.$$

Clearly, all the conditions of Corollary 1 are satisfied. Therefore, the conclusion of Corollary 1 applies to the problem in (15) and (16).

II. Illustration of Theorem 2. *Let us consider*

$$f\left(t, x\left(\frac{t}{2\sqrt{2}}\right)\right) = \frac{1}{512}\left(\frac{t^2\left|x\left(\frac{t}{2\sqrt{2}}\right)\right|}{1+\left|x\left(\frac{t}{2\sqrt{2}}\right)\right|}+1\right),$$

and observe that

$$\left|f\left(t, x\left(\frac{t}{2\sqrt{2}}\right)\right)\right| \leq \frac{1}{512}(t^2+1) = \psi(t), \quad \text{for each } t \in [0, 2\sqrt{2}].$$

With the given data, it is found that $L = 1/8$ as

$$\left|f\left(t, x\left(\frac{t}{2\sqrt{2}}\right)\right) - f\left(t, y\left(\frac{t}{2\sqrt{2}}\right)\right)\right| \leq \frac{1}{8}|x-y|, \quad \text{for each } t \in [0, 2\sqrt{2}]$$

and

$$\int_0^{\frac{t}{2\sqrt{2}}} \frac{\left(\frac{t}{2\sqrt{2}} - qs\right)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} L\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \simeq 0.00357786 < 1,$$

that is, condition (12) is satisfied. Thus, all the conditions of Theorem 2 hold true and hence its conclusion implies that the problem in (15) and (16) has at least one solution on $[0, 2\sqrt{2}]$.

III. Illustration of Theorem 3. *Consider*

$$f\left(t, x\left(\frac{t}{2\sqrt{2}}\right)\right) = \frac{1}{20}x\left(\frac{t}{2\sqrt{2}}\right) + \frac{1}{15}\sin\left(x\left(\frac{t}{2\sqrt{2}}\right)\right),$$

and note that

$$|f(t, x)| \leq \frac{1}{20}\left|x\left(\frac{t}{2\sqrt{2}}\right)\right| + \frac{1}{15}.$$

Obviously, $\varphi_1 = 1/20$, $\varphi_2 = 1/15$, $\Psi(\varrho) = \varrho$. In consequence $\varpi_1 \simeq 0.00284058$, $\varpi_2 \simeq 0.00378745$. By condition (13), it follows that $\varrho > 0.00379823$. Thus, all the assumptions of Theorem 3 hold true. Therefore, the problem in (15) and (16) has at least one solution on $[0, 2\sqrt{2}]$.

5. Conclusions

By applying the standard tools of fixed point theory, we established the existence criteria for solutions of a (p, q)-difference equation of fractional order $\alpha \in (1, 2]$, supplemented with anti-periodic boundary conditions involving the first-order (p, q)-difference operator. Our results are new in the given configuration and contribute significantly to the existing literature on the topic addressed in this paper.

References

1. Hounkonnou, M.N.; Désiré, J.; Kyemba, B.R. (p,q)-calculus: Differentiation and integration. *SUT J. Math.* **2013**, *49*, 145–167. [CrossRef]
2. Sadjang, P.N. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. *Results Math.* **2018**, *73*, 21.
3. Soontharanon, J.; Sitthiwirattham, T. On fractional (p,q)-calculus. *Adv. Differ. Equ.* **2020**, *1*, 1–18. [CrossRef]
4. Sadjang, P.N. On two (p,q)-analogues of the Laplace transform. *J. Differ. Equ. Appl.* **2017**, *23*, 1562–1583.
5. Médics, A.D.; Leroux, P. Generalized Stirling numbers, convolution formulae and p,q-analogues. *Canad. J. Math.* **1995**, *47*, 474–499. [CrossRef]
6. Burban, I. Two-parameter deformation of the oscillator algebra and (p,q)-analog of two-dimensional conformal field theory. Symmetry in nonlinear mathematical physics, Vol. 1 (Kiev, 1995). *J. Nonlinear Math. Phys.* **1995**, *2*, 384–391. [CrossRef]
7. Cai, Q.-B.; Zhou, G. On (p,q)-analogue of Kantorovich type Bernstein-Stancu-Schurer operators. *Appl. Math. Comput.* **2016**, *275*, 12–20. [CrossRef]
8. Burban, I.M.; Klimyk, A.U. P,Q-differentiation, P,Q-integration, and P,Q-hypergeometric functions related to quantum groups. *Integral Transform. Spec. Funct.* **1994**, *2*, 15–36. [CrossRef]
9. Chakrabarti, R.; Jagannathan, R. A (p,q)-oscillator realization of two-parameter quantum algebras. *J. Phys. A* **1991**, *24*, L711–L718. [CrossRef]
10. Aral, A.; Gupta, V. Applications of (p,q)-gamma function to Szász durrmeyer operators. *Publ. Inst. Math. (Beograd) (N.S.)* **2017**, *102*, 211–220. [CrossRef]
11. Kamsrisuk, N.; Promsakon, C.; Ntouyas, S.K.; Tariboon, J. Nonlocal boundary value problems for (p,q)-difference equations. *Differ. Equ. Appl.* **2018**, *10*, 183–195. [CrossRef]
12. Promsakon, C.; Kamsrisuk, N.; Ntouyas, S.K.; Tariboon, J. On the second-order (p,q)-difference equation with separated boundary conditions. *Adv. Math. Phys.* **2018**, *2018*, 1–9. [CrossRef]
13. Nuntigrangjana, T.; Putjuso, S.; Ntouyas, S.K.; Tariboon, J. Impulsive quantum (p,q)-difference equations. *Adv. Differ. Equ.* **2020**, *2020*, 1–20. [CrossRef]
14. Dumrongpokaphan, T.; Ntouyas, S.K.; Sitthiwirattham, T. Separate fractional (p,q)-integrodifference equations via nonlocal fractional (p,q)-integral boundary conditions. *Symmetry* **2021**, *13*, 1–15. [CrossRef]
15. Liu, Y.; Liu, Y. Multiple positive solutions for a class of boundary value problem of fractional (p,q)-difference equations under (p,q)-integral boundary conditions. *J. Math.* **2021**, *2021*, 1–13. [CrossRef]
16. Neang, P.; Nonlaopon, K.; Tariboon, J.; Ntouyas, S.K.; Ahmad, B. Nonlocal boundary value problems of nonlinear fractional (p,q)- difference equations. *Fractal Fract.* **2021**, *5*, 270. [CrossRef]
17. Soontharanon, J.; Sitthiwirattham, T. On sequential fractional Caputo (p,q)-integrodifference equations via three-point fractional Riemann-Liouville (p,q)-difference boundary condition. *AIMS Math.* **2022**, *7*, 704–722. [CrossRef]
18. Krasnoselskii, M.A. Two remarks on the method of successive approximations. *Uspekhi. Mat. Nauk.* **1995**, *10*, 123–127.
19. Granas, A.; Dugundji, J. *Fixed Point Theory and Applications*; Springer: New York, NY, USA, 2003.