Natural Convection of Heat-Generating Liquid of Variable Viscosity under Wall Cooling Impact

Alexander I. Kudrov and Mikhail A. Sheremet

Abstract: This research presents a computational investigation of the thermal convection of a heat-generating liquid having variable viscosity in a semi-cylindrical cavity. The analysis is carried out to obtain the time patterns of the average Nusselt number at the lower border of the chamber and understand the impact of the variable viscosity, the Prandtl number, and the Rayleigh number on this parameter. The natural convection in the cavity is defined by the set of non-dimensional equations based on the Boussinesq approach employing the non-primitive parameters such as vorticity and stream function. These governing equations are worked out numerically based on the finite difference technique. The time dependencies have been obtained at the Rayleigh number equal to $10^4$, $10^5$, and $10^6$ and the Prandtl number taking values of 7.0, 70, and 700. The results obtained for variable and constant viscosity have been compared. Additionally, the paper represents maps of isotherms and streamlines for the mentioned values of the Rayleigh number. The influence of variable viscosity on the parameters of natural convection is poorly studied in closed systems; therefore, this research gives necessary data to understand the general time nature of the average Nusselt number at cooling surface of various parameters. Additionally in this research, the model for simulating the natural convection in non-primitive variables is presented in polar coordinates when the dynamic viscosity varies with temperature. The computational model designed could be used to simulate the free convection in systems with inner heat production such as chemical reactors, inductive metal melting facilities, or corium in-vessel retention to analyze the impact of various factors on the parameters of the natural convection in such systems.

Keywords: natural convection; internal heat generation; variable viscosity; semi-cylindrical cavity; numerical technique

MSC: 35Q30; 76R10; 80A19; 80M20

1. Introduction

The free convection in units with inner thermal production became a relevant topic of scientific interest over the last few decades, since such a process is observed in various fields of engineering such as chemical reactors, nuclear and metallurgical industries, etc. The free convection in the engineering systems is often influenced by various complicating elements, one of which is variable viscosity of the fluid. However, a majority of the research analyzing natural convection under the inner thermal production have been conducted at constant properties of the fluid [1–5]. Yet, it is well known that variable viscosity can affect the properties of the natural convection. Yamasaki and Irvine [6] have numerically simulated a laminar free convection of a fluid in a vertical tube to study the effect of variable viscosity on the energy transfer and the average velocity of the fluid in the domain. According to their results, the variable viscosity effect significantly increases the average velocity and the total energy transport strength in the system. Hyun and Lee [7] have reported the results of a numerical analysis on a similar problem; however, the domain under study has had a
square shape, where the left border is hot, and the right one is cold. This study has shown that accounting for variable viscosity also leads to overall enhancement of the convective flows and energy transport in the system. Another example is the research of Jin and Chen [8], where the authors have conducted a computational analysis of free convection in a vertical slot to study transition processes there. They also have supposed that the fluid viscosity varies exponentially according to four parameters of exponential law. Given these conditions, it has been discovered that the critical value of the Grashof number providing a multicellular flow structure is smaller compared to that at constant viscosity.

The studies devoted to the influence of variable viscosity on the parameters of natural convection are not exhausted just by the case of closed domains. As an example, we can use the work by Hossain et al. [9] that presents a numerical simulation of thermal convection above a vertical wavy sheet. The authors have reported that accounting for variable viscosity leads to identical results obtained by Yamasaki and Irvine [6] and Hyun and Lee [7]. Apart from numerical analyses, we can also highlight a research study by Cordoba et al. [10] where, besides a numerical simulation, they have conducted experiments on natural convection in a cube and shown the changes in the velocity field that the variable viscosity causes.

The works represented above have been conducted when the heat source is differential. However, some endeavors have been made to study the influence of temperature-dependent viscosity in systems where the fluid generates heat. Here, we can mention the works of Bagai [11], Siddiqa et al. [12], and Alim et al. [13] where the effects of variable viscosity in case of the natural convection boundary layer problem is studied at various complicating factors. Thus, Bagai [11] has represented a numerical analysis on thermal convection along bodies of various geometries, whereas Siddiqa et al. [12] have performed computations of this process along a plane-inclined surface at various angles of inclination. Finally, Alim et al. [13] have carried out the analysis for a surface that has been placed vertically and has had a wavy structure. The results of these works have shown a reduction of the energy transfer intensity and an increment of the skin friction when the fluid generates heat, and its viscosity varies with temperature.

Although the foregoing review shows attempts to study the thermal convection in units having inner thermal production if the viscosity of the liquid varies with temperature, the analyses mentioned are limited to only a boundary layer problem. However, the influence of variable viscosity under the effect of internal thermal generation needs to be analyzed in closed domains since this phenomenon is observed in the fields of engineering application mentioned earlier. Therefore, in the present research, the objective is to perform a computational analysis on a time-dependent thermogravitational convection of a thermally producing liquid within a semi-cylindrical chamber when the dynamic viscosity of the liquid is reduced with temperature exponentially.

2. Mathematical Simulation

As shown in Figure 1, the domain under study has a form of a semi-cylindrical chamber where the top border is thermally insulated, and the lower one is isothermal and has temperature $T_0$. The cavity encloses a heat-generating fluid with the volumetric heat rate that is constant in both space and time. Initially, the liquid does not move, and the initial temperature is the same with the lower border temperature. Additionally, we suppose that dynamic viscosity of the liquid varies with temperature. According to [6,7], this dependence is given by:

$$\mu = \mu(T_0) \exp\left(-c\frac{T - T_0}{\Delta T}\right)$$

(1)

Here, $\Delta T$ is the reference temperature difference that will be defined later in this paper.

It should be noted that here we have used the exponential form for the viscosity proposed by Torrance and Turcotte [14]. This exponential approach has a very good description of the available experimental outcomes [6].
Here, $\Delta T$ is the reference temperature difference that will be defined later in this paper.

The other properties of the liquid are fixed, and they have been determined at initial temperature $T_0$.

In the cavity, a combined effect of the inner thermal production and the gravitational field leads to a formation of convective flows, the behavior of which is defined by the set of governing equations based on the Boussinesq approach. The set includes the equations of momentum, continuity, and energy. These governing equations are represented below.

The equation of momentum (the Navier–Stokes equation) is [15,16]:

$$
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \varphi} - \frac{u}{r} \right) = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \varphi} \left( \mu \left( \frac{\partial u}{\partial \varphi} \right) + \frac{\partial v}{\partial \varphi} \right)$$

$$
2\mu \left( \frac{\partial u}{\partial r} - \frac{1}{r} \frac{\partial v}{\partial \varphi} - \frac{v}{r} \right) - \rho g \beta (T - T_0) \sin(\varphi)
$$

The continuity equation is:

$$\frac{\partial (ur)}{\partial r} + \frac{\partial v}{\partial \varphi} = 0$$

The energy equation is:

$$
\rho c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \varphi} \right) = k \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} \right) + q_v
$$

Given the formulation of the problem, the additional restrictions for Equations (2)–(5) are:

$$
t = 0, r \in [0, R_c], \varphi \in [0, \pi]; u = v = 0, T = T_0;
$$

$$
r = 0, \varphi \in [0, \pi]; u = v = 0, \frac{\partial T}{\partial r} = 0;
$$

$$
r = R_c, \varphi \in [0, \pi]; u = v = 0, T = T_0;
$$

$$
\varphi = 0 \text{ and } \varphi = \pi, r \in (0, R_c); u = v = 0, \frac{\partial T}{\partial \varphi} = 0
$$

The mathematical formulation represented above can be simplified by eliminating the pressure terms from the system. It can be performed if we introduce the non-primitive parameters including vorticity, $\omega$, and stream function, $\psi$, which are defined as [15,16]:

$$
\omega = \frac{1}{r} \frac{\partial}{\partial \varphi} \left( r v - \frac{\partial u}{\partial r} \right) - \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( -\frac{\partial \psi}{\partial \varphi} \right) ; u = -\frac{\partial \psi}{\partial r}, v = \frac{\partial \psi}{\partial r}
$$
As a result of the substitution, System (2)–(5) can be transformed into:

\[ \rho \left( \frac{\partial \omega}{\partial t} + \frac{\partial (\omega u)}{\partial r} + \frac{\partial (\omega v)}{\partial \phi} + \frac{\omega u}{r} \right) = \frac{\partial^2 (\mu \omega)}{\partial r^2} + \frac{1}{r} \frac{\partial (\mu \omega)}{\partial r} + \frac{1}{r} \frac{\partial^2 (\mu \omega)}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2 (\mu \omega)}{\partial \phi^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{1}{r} \frac{\partial^2 \omega}{\partial \phi^2} \]

\[ 2 \left[ \frac{1}{r} \frac{\partial^2 \mu}{\partial r^2} \left( v - \frac{\partial u}{\partial r} + \frac{\partial v}{\partial \phi} \right) + \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 u}{\partial \phi^2} \right] + 2 \frac{\partial u}{\partial r} \left( \frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} \right) \]

\[ \rho \Omega^2 \left( \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} \cos(\phi) - \frac{1}{r} \frac{\partial \phi}{\partial r} \sin(\phi) \right) \]

\[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} = \Omega \]

\[ \rho \rho_r \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + v \frac{\partial T}{\partial \phi} \right) = k \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} \right) + q_v \]

In the system above, Equations (9) and (10) are referred to as the equation of vorticity and the Poisson equation for stream function, respectively.

Furthermore, we non-dimensionalize the formulation in non-primitive variables using the following scales of quantities:
- The scale of length is the radius of the cavity, \( R_c \);
- The scale of temperature is \( \Delta T = \frac{q_v R_c^2}{\kappa} \) (see [17,18]);
- The scale of velocity is the convection velocity \( w_0 = \sqrt{g \beta \Delta T R_c} \) (see [19]);
- The scale of stream function is \( \psi_0 = R_c w_0 \);
- The scale of vorticity is \( \omega_0 = \omega_0 / R_c \);
- The scale of dynamic viscosity is \( \mu_0 = \mu(T_0) \); thus, the dependence of the dimensionless viscosity on the dimensionless temperature is:

\[ M = \frac{\mu(T)}{\mu(T_0)} = \exp(-C \theta) \]

As a result, the non-dimensional variables have been defined as follows:

\[ R = \frac{r}{R_c}, \quad U = \frac{u}{w_0}, \quad V = \frac{v}{w_0}, \quad \Psi = \psi / \psi_0, \quad \Omega = \omega / \omega_0, \quad \theta = \frac{T - T_0}{\Delta T} \]

The scales introduced enable us to obtain the system of non-dimensional equations in Boussinesq approximation in terms of non-primitive variables. These equations are as follows:

\[ \frac{\partial \Omega}{\partial t} + \frac{\partial (U \Omega)}{\partial r} + \frac{\partial (V \Omega)}{\partial \phi} + \frac{\Omega U}{R} = \sqrt{\frac{Pr}{Ra}} \left( \frac{\partial^2 (M \Omega)}{\partial r^2} + \frac{1}{R} \frac{\partial (M \Omega)}{\partial r} + \frac{1}{R^2} \frac{\partial^2 (M \Omega)}{\partial \phi^2} \right) + 2 \sqrt{\frac{Pr}{Ra}} \frac{1}{R} \frac{\partial^2 M}{\partial r^2} \left( V - \frac{\partial U}{\partial \phi} \right) + \frac{1}{R} \frac{\partial V}{\partial r} \left( \frac{\partial M}{\partial r} + \frac{1}{R} \frac{\partial^2 M}{\partial \phi^2} \right) + 2 \frac{\partial U}{\partial r} \left( \frac{\partial^2 M}{\partial r^2} + \frac{1}{R} \frac{\partial M}{\partial \phi^2} \right) \]

\[ \frac{\partial \phi}{\partial R} \cos(\phi) - \frac{\partial \phi}{\partial \phi} \sin(\phi) \]

\[ \frac{\partial^2 \Psi}{\partial R^2} + \frac{1}{R} \frac{\partial \Psi}{\partial r} + \frac{1}{R^2} \frac{\partial^2 \Psi}{\partial \phi^2} = \Omega \]

\[ \frac{\partial \theta}{\partial t} + U \frac{\partial \theta}{\partial r} + V \frac{\partial \theta}{\partial \phi} = \frac{1}{\sqrt{PrRa}} \left( \frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R} \frac{\partial \theta}{\partial R} + \frac{1}{R^2} \frac{\partial^2 \theta}{\partial \phi^2} \right) + 1 \]

The initial and boundary restrictions for the dimensionless mentioned equations are given as:

\[ \tau = 0, \ R \in [0, 1], \ \phi \in [0, \pi]; \ \Psi = \theta = \Omega = 0; \]

\[ R = 0, \ \phi \in [0, \pi]; \ \Psi = \Omega = 0, \ \frac{\partial \theta}{\partial R} = 0; \]

\[ R = 1, \ \phi \in [0, \pi]; \ \Psi = \theta = 0, \ \Omega = \frac{\partial^2 \psi}{\partial R^2}; \]

\[ \phi = 0 \text{ and } \phi = \pi, \ R \in (0, 1); \ \Psi = 0, \ \frac{\partial \theta}{\partial \phi} = 0, \ \Omega = \frac{1}{R^2} \frac{\partial^2 \Psi}{\partial \phi^2}; \]
3. Solution Technique

In the developed code, Equations (14)–(16) have been worked out employing the finite difference technique in a uniform grid with the number of divisions along radius and polar angle equal to \( N \) and \( M \), respectively. For each time level, since the equations are time-dependent, we firstly worked out the Poisson equation for the stream function (15). This equation has been discretized using the five-point difference pattern where the second order derivatives have been approximated by the symmetric differences, and the first order derivatives have been discretized by the central differences. The finite difference approximation of Equation (15) for the nodes within the domain, where \( i = 1, N - 1 \) and \( j = 1, M - 1 \), is represented below.

\[
\frac{\Psi^n_{i+1,j} - 2\Psi^n_{i,j} + \Psi^n_{i-1,j}}{\Delta R^2} + \frac{1}{R_i} \left( \frac{\Psi^n_{i+1,j} - \Psi^n_{i,j}}{2\Delta R} + \frac{1}{R_i^2 \Delta \phi^2} \right) = \Omega_i^n
\]

Here, we have used the magnitudes of vorticity using the previous time level \( \eta \), and as a result, in the numerical code, the values of the dimensionless vorticity at the current time level \( n + 1 \) are still unknown. After approximation, we have obtained a set of linear algebraic equations, which has been solved then by the successive over-relaxation algorithm. As the magnitudes of the non-dimensional \( \Psi \) have just been found at the time level \( n + 1 \), next, we have computed the projections of non-dimensional velocity employing Equation (8) the central differences as follows:

\[
U^n_{i,j+1} = \frac{\Psi^n_{i+1,j+1} - \Psi^n_{i-1,j+1}}{2R_i \Delta \phi} + O \left( (\Delta \phi)^2 \right) ; \quad V^n_{i,j+1} = -\frac{\Psi^n_{i,j+1} - \Psi^n_{i,j-1}}{2\Delta R} + O \left( (\Delta R)^2 \right) ; \quad j = 1, N - 1
\]

Furthermore, we have worked out Equation (14), which has been approximated through the locally one-dimensional scheme of Samarkis [20], where the equation is jointly discretized at two time levels, namely \( n + 1/2 \) and \( n + 1 \). At both time levels, the convective terms have been discretized via the special difference schemes using the central differences. Thus for \( i = 1, N - 1 \) and \( j = 1, M - 1 \), the approximation of Equation (14) is as follows:

\[
\begin{align*}
\left( \frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{\Delta R^2} + \frac{1}{R_i} \left( \frac{U^n_{i+1,j} - U^n_{i,j}}{2\Delta R} + \frac{1}{R_i^2 \Delta \phi^2} \right) \right) & = \frac{\Delta \phi}{2 \Delta \phi} \left( \frac{U^n_{i+1,j} - 2U^n_{i,j} + U^n_{i-1,j}}{\Delta R^2} \right) \\
& + \frac{1}{R_i} \left( \frac{U^n_{i+1,j} - U^n_{i,j}}{2\Delta R} \right) \\
& + \frac{1}{R_i^2 \Delta \phi^2} \left( \frac{U^n_{i+1,j} - U^n_{i,j}}{2\Delta R} \right) \\
& + \frac{1}{R_i} \left( \frac{U^n_{i+1,j} - U^n_{i,j}}{2\Delta R} \right) \\
& + \frac{1}{R_i^2 \Delta \phi^2} \left( \frac{U^n_{i+1,j} - U^n_{i,j}}{2\Delta R} \right) \\
\end{align*}
\]

Finally, we have discretized the convection-diffusion term for the stream function (15) within the domain at both time levels, namely \( n + 1/2 \) and \( n + 1 \). At both time levels, the convective terms have been discretized via the special difference schemes using the central differences. Thus for \( i = 1, N - 1 \) and \( j = 1, M - 1 \), the approximation of Equation (14) is as follows:

\[
\begin{align*}
\frac{\Omega_i^{n+0.5} - \Omega_i^n}{\Delta \phi} & + \frac{1}{R_i} \left( \frac{\Omega_i^{n+0.5} - \Omega_i^n}{2\Delta \phi} + \frac{1}{R_i^2 \Delta \phi^2} \right) \\
& + \frac{1}{R_i^2 \Delta \phi^2} \left( \frac{\Omega_i^{n+0.5} - \Omega_i^n}{2\Delta \phi} \right) \\
& + \frac{1}{R_i} \left( \frac{\Omega_i^{n+0.5} - \Omega_i^n}{2\Delta \phi} \right) \\
\end{align*}
\]
Here, for instance, the notation $M_{ij}^n$ means $M_{ij}^n = M\left(\theta_{ij}^n\right)$. The other notations of this kind have similar meaning to the above approximations. The obtained difference Equations (21) and (22) include a tridiagonal shape, which enabled us to employ the Thomas technique to work them out.

The values of the dimensionless vorticity at the boundaries of the cavity have been obtained using the Pearson relation [21] as follows:

$$
\Omega_{Nj}^{n+1} = \frac{8\Psi_{N-1,j}^{n+1} - \Psi_{N,j}^{n+1}}{2\Delta R^2} + O(\Delta R^2), j = 1, M - 1;
$$

$$
\Omega_{ij,0}^{n+1} = \frac{1}{R_f^2} \left( \frac{8\Psi_{i-1,j}^{n+1} - \Psi_{ij}^{n+1}}{2\Delta \phi^2} + O(\Delta \phi^2) \right), i = 1, N - 1;
$$

$$
\Omega_{ij,M}^{n+1} = \frac{1}{R_f^2} \left( \frac{8\Psi_{i,j-1}^{n+1} - \Psi_{ij}^{n+1}}{2\Delta \phi^2} + O(\Delta \phi^2) \right), i = 1, N - 1.
$$

Finally, we have solved the energy equation that has been discretized like Equation (14). The only difference is that we have approximated the convective terms with the Samarksii monotonic scheme [20]. The obtained finite difference equations at the time levels $n + 0.5$ and $n + 1$ are as follows:

$$
\frac{\theta_{ij}^{n+0.5} - \theta_{ij}^{n}}{\Delta \tau} + U_{ij}^{n+1} \frac{\theta_{i+1,j}^{n+0.5} - \theta_{ij}^{n+0.5}}{2\Delta \tau} - \left| U_{ij}^{n+1} \right| \frac{\theta_{i+1,j}^{n+0.5} - 2\theta_{ij}^{n+0.5} + \theta_{i-1,j}^{n+0.5}}{2\Delta \tau} =
$$

$$
\frac{1}{\sqrt{\text{Fr} \text{Ra}}} \left( 1 + \left| U_{ij}^{n+1} \right| \frac{\Delta \phi \sqrt{\text{Fr} \text{Ra}}}{2} \right) \frac{1}{\Delta R} \left( \frac{\theta_{i+1,j}^{n+0.5} - \theta_{i-1,j}^{n+0.5}}{\Delta \phi} \right) + \frac{1}{\sqrt{\text{Fr} \text{Ra}}} \left( 1 + \left| U_{ij}^{n+1} \right| \frac{\Delta \phi \sqrt{\text{Fr} \text{Ra}}}{2} \right) \frac{1}{\Delta R} \left( \frac{\theta_{i+1,j}^{n+0.5} - \theta_{i-1,j}^{n+0.5}}{\Delta \phi} \right)
$$

$$
= \frac{1}{\sqrt{\text{Fr} \text{Ra}}} \left( 1 + \left| U_{ij}^{n+1} \right| \frac{\Delta \phi \sqrt{\text{Fr} \text{Ra}}}{2} \right) \frac{1}{\Delta R} \left( \frac{\theta_{i+1,j}^{n+0.5} - \theta_{i-1,j}^{n+0.5}}{\Delta \phi} \right) - \frac{1}{\sqrt{\text{Fr} \text{Ra}}} \left( 1 + \left| U_{ij}^{n+1} \right| \frac{\Delta \phi \sqrt{\text{Fr} \text{Ra}}}{2} \right) \frac{1}{\Delta R} \left( \frac{\theta_{i+1,j}^{n+0.5} - \theta_{i-1,j}^{n+0.5}}{\Delta \phi} \right)
$$

The finite difference equations above also have a tridiagonal structure; therefore, we have used the TDMA to solve them.

The total approximation has an accuracy of the second order in space, $\{O(\Delta R^2 + \Delta \phi^2)\}$ yet the accuracy in time is of the first order $\{O(\Delta \tau)\}$, as in the time-dependent equations the transient term has been discretized using the forward difference.

To prove the validity of the designed computational algorithm, we have solved the problem considered by Chudanov and Strizhov [1]. In that paper, the authors have conducted a numerical analysis on the thermal convection of a thermally producing liquid within a square chamber having isothermal borders.

Figure 2 shows a good agreement with Figure 1 from Chudanov and Strizhov [1] that reflects an operability and quality of the present computational algorithm. Thus, it enables us to employ the designed code to work out the problem formulated earlier.

Prior to performing a numerical simulation, it is important to select an optimal grid, which provides both sufficient accuracy and small computational time. In this work, the optimal grid has been determined based on the time dependence of the average Nusselt number at the lower border, built for grids of $17 \times 17$, $26 \times 26$, $51 \times 51$, and $101 \times 101$ nodes. As Figure 3 illustrates, the difference between the solutions for the grids of $51 \times 51$ and $101 \times 101$ nodes is insignificant; thus, the grid of $51 \times 51$ nodes has been selected as optimal.
mely, it takes more time for the fluid, as in the time-valuation, the difference between the solutions for the grids of 51 and 21 nodes at $Pr = 7.0$, $Ra = 10^5$, and $C = 2$.

The values of the Prandtl number are $7.0$, $70$, and $700$, and $Ra$ varies between $10^4$ and $10^6$. Additionally, we have studied how variable viscosity and the $C$-factor in Equation (12) impact the integral parameter of heat transfer if the $C$-factor takes values 1, 2, and 3.

Firstly, in the discussion, we consider the effect of the Rayleigh number on $Nu_{av}$, which is represented by Figure 4 illustrating time dependencies of $Nu_{av}$ at constant Prandtl number and various values of $Ra$ from the mentioned range. As Figure 4 shows, the average $Nu$ initially raises due to warming up, and when the net balance between heat supply and heat removal from the system settles, the process becomes steady. At both parts of the

Figure 2. Streamlines (the left one) and isotherms (the right one) from the developed code at $Ra = 6.4 \times 10^5$ and $Pr = 7.0$, $\Psi^{max} = 8.73 \times 10^{-1}$, $\Psi^{min} = -8.73 \times 10^{-1}$, $\theta^{max} = 4.70 \times 10^{-2}$, while from [1] $\Psi^{max} = 8.78 \times 10^{-1}$, $\Psi^{min} = -8.78 \times 10^{-1}$, $\theta^{max} = 4.77 \times 10^{-2}$.

Figure 3. Average Nusselt number on the lower border for grids of $17 \times 17$, $27 \times 27$, $51 \times 51$, and $101 \times 101$ nodes at $Pr = 7.0$, $Ra = 10^5$, and $C = 2$.

4. Results and Discussion

In the present paper, the influence of the Rayleigh and Prandtl numbers on the mean $Nu$ at the lower surface of the chamber is studied when the dynamic viscosity of the heat-generating fluid decreases with temperature. The average $Nu$ has been defined as:

$$ Nu_{av} = \frac{1}{\pi} \int_0^\pi \left. \frac{\partial \theta}{\partial R} \right|_{R=1} d\varphi $$

(26)

The values of $Nu_{av}$ are 1.01, 0.5, and 0.2 for grids of 51 and 101, and $Ra$ varies between $10^4$ and $10^6$. Additionally, we have studied how variable viscosity and the $C$-factor in Equation (12) impact the integral parameter of heat transfer if the $C$-factor takes values 1, 2, and 3.

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process, an increment of $Ra$ causes a diminution of the mean Nusselt number, which may result from weakening of the convective circulations inside the chamber due to uniform heating. This is indicated by Figure 5 where we can observe a reduction of the maximal $Ψ$ with increasing $Ra$. Figure 5 also illustrates that at higher $Ra$, the transient part of the process lasts longer since the increment of the Rayleigh number is associated with a growth of the volumetric heat rate of the liquid. Namely, it takes more time for the fluid to attain a certain temperature level at high rates of heat supply.

Figure 4. Time profiles of average $Nu$ on the lower border at $Pr = \text{const}$ and $Ra = \text{var}$ for $C = 2$. 

$Pr = 7.0$

$Pr = 70$

$Pr = 700$
Figure 5. Time profiles of the maximum $\Psi$ at $Pr = \text{const}$ and $Ra = \text{var}$ for $C = 2$.

The impact of $Pr$ on the mean $Nu$ is represented in Figure 6, which shows that the steady-state magnitudes of the integral heat transfer parameters are insignificantly influenced by $Pr$. However, the unsteady values of $Nu_{\text{av}}$ are lower at higher $Pr$ since the growth of $Pr$ results in less intense movement of the fluid. The effect of slow fluid motion at high $Pr$ also leads to prolonging the transient stage of the process.
Additionally, we have researched how variable viscosity influences the mean Nusselt number. The computations have shown no impact of $M = \text{var}$ on $\text{Nu}_{av}$; therefore, such results are not represented in the paper. This effect could be caused by the fact that greater changes in the viscosity are found in the center of the chamber where the temperatures are maximum, and in the layers neighboring upon the bottom wall, the properties of the fluid remain virtually unchanged. However, as Figure 7 shows, taking into account the
variable viscosity, one can find a strengthening of the fluid motion since the maximum $\Psi$ is increased when the viscosity varies with temperature. Further growth in $C$ only enhances the mentioned effect. The increased intensity of the fluid motion eventually results in a reduction of the temperature within the chamber as enhanced mixing of the cold and hot layer of the liquid makes the temperature field more uniform. This effect is also more pronounced at high values of $C$.

![Figure 7. Time profiles of the maximum dimensionless stream function and temperature at various values of factor $C$ for $Ra = 10^4$, $10^5$ and $Pr = 70$.](image)

To understand the nature of the convective structure and the temperature field within the cavity, we have built a series of maps of isotherms and streamlines for a steady-state part of the process at $Pr = 700$ and $C = 1$. They are represented in Figure 8. Firstly, we should highlight that two convective cells appear within the domain. The fluid rotates clockwise inside the right cell, whereas the direction of the rotation is opposite for the left cell. As we can see in Figure 8, at low $Ra$, fluid motion has a minute impact on the temperature field. However, as $Ra$ increases, the isotherms arc upwards, but the intensity of the fluid motion is still insufficient to develop a thermal jet. Moreover, we can observe that with rising $Ra$, the cores of the convective cells move upwards near the corners of the chamber due to combined influence of increased $q_v$, that the Rayleigh number depends on in this problem, and adiabatic wall. Moving upwards, the cores shift temperature maximum closer to the top boundary, and this reduces the density of the isotherms in the bottom part of the domain causing the decrease in the temperature gradient near the lower border and
eventually contributing to a reduction of the average $Nu$. The structure and behavior of these distributions are similar to those obtained for the other considered $Pr$ values. The variable viscosity and the factor $C$ do not influence the distributions significantly; therefore, such results are also omitted from the discussion.

Figure 8. Steady-state distributions of streamlines and isotherms at $Pr = 700$ and $C = 2$ for different $Ra$.

5. Conclusions

In the present research, the thermogravitational convection of a thermally producing liquid with variable viscosity has been studied numerically. The domain of interest has geometry of a horizontal semi-cylinder with thermally insulated upper border and isothermal lower border. According to the results of the simulation, it has been discovered that the average Nusselt number decreases with a growth of $Ra$ due to weakening of the fluid motion. Additionally, it has been found that a growth of the Rayleigh number prolongs the transient part of the process due to increased heat supply to the system. The Prandtl number causes no influence on the steady-state average Nusselt number; however, at high $Pr$, the unsteady values of $Nu_{av}$ are lower, and the transient part of the process lasts longer due to less intense fluid motion. Finally, we have found that the variable viscosity, as well the values of $C$-factor, impacts the dimensionless heat transfer rate insignificantly. Yet, it enhances the convective circulations and decreases the temperatures within the chamber. Both effects are more pronounced when the value of $C$ increases.

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Nomenclature

- $c_p$: heat capacity
- $C$: viscosity variation parameter
- $g$: acceleration due to gravity
- $K$: thermal conductivity
- $M$: non-dimensional dynamic viscosity
- $n$: time index
- $Nu_{av}$: average Nusselt number
- $p$: Pressure
- $Pr$: Prandtl number
- $q_v$: inner volumetric heat generation density
- $R$: non-dimensional polar radial coordinate
- $R_c$: cylinder radius
- $Ra$: Rayleigh number
- $T$: Time
- $T_0$: initial temperature
- $u, v$: velocity components
- $U, V$: non-dimensional velocity components
- $w_0$: reference velocity
- $x, y$: Cartesian coordinates

Greek symbols

- $\beta$: thermal expansion parameter
- $\Delta R$: step for the polar coordinate
- $\Delta T$: reference temperature difference
- $\Delta \tau$: time step
- $\Delta \phi$: step for the polar angle
- $\theta$: non-dimensional temperature
- $\mu$: dynamic viscosity
- $\mu_0$: reference dynamic viscosity value
- $\rho$: density
- $\tau$: non-dimensional time
- $\phi$: polar angle
- $\psi$: stream function
- $\psi_0$: reference stream function value
- $\Psi$: non-dimensional stream function
- $\omega$: vorticity
- $\omega_0$: reference vorticity value
- $\Omega$: non-dimensional vorticity

Subscripts

- $0$: reference
- $av$: average
- $c$: cylinder
- $i, j$: indices for the spatial nodes
- $M$: maximum index for polar angle coordinate
- $\text{max}$: maximum value
- $\text{min}$: minimum value
- $N$: maximum index for polar radial coordinate
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