Reservoir Permeability Identification under Three-Phase Filtration Using a Priori Information on Wells

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Abstract: A method is proposed to solve the identification problem of the permeability field of a three-dimensional reservoir under conditions of three-phase fluid filtration. The peculiarity of the method lies in the fact that the wells retain the proportionality of the values of the permeability coefficients of the layers obtained a priori from the results of geophysical studies. The approximation of the permeability field was carried out in layers using surface splines. The problem of identifying the permeability field belongs to the class of inverse coefficient problems for a system of nonlinear partial differential equations describing the process of fluid flow in a porous medium. The solution of the problem was reduced to minimizing the residual function constructed on the total liquid production rate at the wells. Minimization of the residual function was carried out by the Levenberg–Marquardt method. The black oil model was used to describe the fluid filtration process. The filtration equations were solved numerically by the method of the simultaneous solution, and the control volume method was used to approximate the equations by spatial variables. The proposed approach was used to solve a model problem. The influence of various types of errors on the results of identification was investigated.

Keywords: multiphase flow in porous media; reservoir simulation; permeability identification; surface spline

MSC: 76S05; 35R30

1. Introduction

Currently, numerical modeling of the oil field development process is the main tool used for solving various field management and forecasting problems [1–4]. In numerical modeling, the filtration-capacitance characteristics of the reservoir must be known. There are various approaches to determining unknown parameters: core analysis, geophysical methods, local hydrodynamic methods, and non-local hydrodynamic methods. Geophysical methods make it possible to estimate the permeability values in near-well areas. For this, theoretical and empirical correlations between the measured geophysical parameters and the reservoir properties of the rock are used. The data obtained during the core analysis are the initial data to construct petrophysical dependencies and correct the results of geophysical studies. The permeability values are determined either by the porosity coefficient, by formulas similar to the Kozeny–Carman formula, or by nomograms and palettes [5,6]. Local and non-local hydrodynamic methods belong to the class of inverse coefficient problems. Various methods for solving inverse coefficient problems were considered by [7–19], etc. Local hydrodynamic methods [7–9] as well as geophysical methods are used to determine the parameters in near-well areas, while non-local hydrodynamic methods are used to determine the parameter field of reservoirs of complex geometry, opened by a large number of wells. The solution methods of inverse coefficient problems can be divided into two groups: explicit and implicit [10]. The values of the identified
parameters in explicit methods are calculated from the solution of a nonlinear system of
equations, while the fields characterizing the state of the reservoir (pressure and saturation
fields) must be known. In practice, the pressure and saturation fields are known very
approximately, which introduces a significant error in the solution of the inverse problem.
In implicit methods, various iterative procedures are used to construct the parameter field.
In the present work, the second approach was used to solve the problem of determining the
permeability field of a three-dimensional reservoir from known values of bottom hole pres-
sure and flow rate at wells under three-phase filtration conditions. Bottom hole pressure
values were used to solve the forward problem. Methods for solving forward problems of
fluid flow in a porous medium have been considered in many works [20–24]. The flow rate
values were used to construct the residual function in the process of minimizing which the
permeability field was determined.

The information to solve the inverse problem is always limited. This leads to the need
to limit the number of unknown parameters. To achieve this, various parameterization
methods are used, which can be divided into two groups: stochastic [13,14] and determin-
istic. In stochastic methods, permeability is treated as a random field. With a deterministic
approach, various approximations of the permeability field are constructed. For exam-
ple, in the zonation method, the solution domain is divided into zones, each of which
is characterized by a constant parameter value [25,26]. Thus, the unknown parameter is
approximated by a piecewise constant function with the number of unknown values equal
to the number of zones. Various methods can be used to represent the parameter field as a
continuous function: finite element representation of parameters [27,28], approximation
of parameters using a spline [27,29,30], polynomial method [31], or kriging [32]. In the
pilot point method, the unknown parameter value field is approximated by the kriging
method with unknown values at the control points. One of the main tasks of this method is
to determine the number and optimal locations of control points [33].

The reservoir permeability field is approximated in layers. The permeability of each
layer is represented as a surface spline [34] with interpolation nodes located in the wells.
Using the surface spline allows recalculation of the permeability field to any grid to obtain
smooth permeability fields. The permeability values in the near-well areas can be estimated
using geophysical methods of well investigation [6]. These values can be used as a priori
information. As a rule, a priori information is used to construct a regularizing functional
according to Tikhonov’s method [35]. Other approaches for using a priori information can
be found in [10,13,16]. In the present work, the solution process was constructed so that
for each near-well area, the relative change in the permeability coefficient in the layers
was the same. This was a consequence of the assumption that the relative error of the
geophysical methods to determine permeability in the well is the same for different layers
of the reservoir.

The proposed approach was used to solve a model problem. In model problems,
the exact solution (the true permeability field) is known, therefore, when solving inverse
filtration problems, they are used to test the proposed algorithms. It also allows for studying
the influence of various types of errors on the calculation results. The true field in the
model problem was the permeability field constructed from the given permeability values
in the wells. The flow rate measurements obtained by solving the forward problem with
this permeability field were used to construct the residual function. Furthermore, the
permeability field was considered unknown and was restored according to the proposed
algorithm.

The main element of novelty of the presented work is the preservation of the structure
of a priori data on permeability of the wells, obtained by geophysical methods. This
information, along with well operation data, makes it possible to determine the permeability
field of a three-dimensional multilayer reservoir.
2. Methods

An isothermal three-phase flow of fluids (oil, gas, water) in a three-dimensional isotropic reservoir was considered. Capillary and gravitational forces were not taken into account. The black oil model was used to describe the filtration process. Water and oil did not mix and did not exchange masses; gas was soluble in oil and insoluble in water. The viscosities of all phases were assumed to be constant. Under these assumptions, the filtration equations were written as follows. For the water component:

$$\frac{\partial}{\partial t} \left( \phi S_w \frac{B_w}{B_w} \right) = \nabla \cdot \left( \frac{u_w}{B_w} \right) + \frac{q_w}{B_w}$$

(1)

For the oil component:

$$\frac{\partial}{\partial t} \left( \phi S_o \frac{B_o}{B_o} \right) = \nabla \cdot \left( \frac{u_o}{B_o} \right) + \frac{q_o}{B_o}$$

(2)

For the gas component:

$$\frac{\partial}{\partial t} \left( \phi \left( S_g + R_s S_o \right) \frac{B_g}{B_o} \right) = \nabla \cdot \left( \frac{u_g}{B_g} + R_s \frac{u_o}{B_o} \right) + \frac{q_g}{B_g} + R_s \frac{q_o}{B_o}$$

(3)

where $t$ is the time; $\phi$ is the porosity; $S_\alpha$, $B_\alpha$, $u_\alpha$ are, respectively, the saturation, formation volume factor, and volumetric velocity of the $\alpha$-phase, $\alpha = w, o, g$, (water, oil and gas phases, respectively); $R_s$ is the gas solubility; and $q_\alpha$ represents the external sources and sinks of the $\alpha$ component, $\alpha = W, O, G$, (water, oil and gas component, respectively). The system of Equations (1)–(3) expresses the mass conservation law for each component. In addition to (1)–(3), the law of momentum conservation in the form of Darcy’s law was used. Darcy’s law for fluid flow through a porous medium defined, for each of the three phases, a linear relationship between the fluid velocity and the pressure gradient:

$$u_\alpha = \frac{k_{\alpha}}{\mu_\alpha} \nabla p, \ \alpha = w, o, g$$

where $k_\alpha$ and $\mu_\alpha$ are the relative permeability and viscosity, respectively, of the $\alpha$-phase; $k$ is the absolute permeability; and $p$ is the pressure. The saturation constraint was:

$$S_w + S_o + S_g = 1$$

(4)

For systems (1)–(4) the following initial and boundary conditions were assigned:

$$p|_{t=0} = p_0, \ S_w|_{t=0} = S_w^0, \ S_o|_{t=0} = S_o^0$$

(5)

$$p|_{\Gamma_1} = p_{\Gamma_2}, \ u_\alpha \cdot n|_{\Gamma_2} = 0, \ S_w|_{\Gamma_{in}} = S_w^0, \ S_o|_{\Gamma_{in}} = S_o^0$$

(6)

where $\Gamma_1 + \Gamma_2 = \Gamma$ is the boundary of the reservoir, $\Gamma_{in}$ is the inflow boundary ($\Gamma_{in} \subset \Gamma_1$), and $n$ is the outward unit normal to the boundary.

The results of experiments on measuring the functions of relative permeabilities for a three-phase flow indicated that it was permissible to use the following assumptions: $k_{rw}$ depends only on water saturation $S_w$, $k_{rg}$ depends only on gas saturation, and $S_g$ and $k_{ro}$ depend on $S_w$ and $S_g$. In the present work, the relative permeabilities were determined using Stone’s second model [36]:

$$k_{rw} = \tilde{k}_{rw}, \ k_{rg} = \tilde{k}_{rg},$$

$$k_{ro} = k_{rc} \left[ \left( \frac{\tilde{k}_{ro}}{k_{rc}} + k_{rw} \right) + \left( \frac{\tilde{k}_{ro}}{k_{rc}} + k_{rg} \right) - (k_{rw} + k_{rg}) \right]$$
where \( \tilde{k}_{\text{row}}, \tilde{k}_{\text{rw}} \) and \( \tilde{k}_{\text{rog}} \) are the relative permeabilities in two-phase systems oil–water and oil–gas, \( k_{\text{rc}} = \tilde{k}_{\text{row}}(S_{\text{wc}}) = \tilde{k}_{\text{rog}}(S_{g} = 0) \); and \( S_{\text{wc}} \) is the critical water saturation in the oil–water system. Two approaches can be used to model wells. In the first approach, the solution domain is multiply connected, and the operation of wells is modeled by setting boundary conditions on the surface well. In the second approach, the well is modeled by a point source and is included in the mass conservation equation. In this case, the calculated pressure in the grid block containing the well will differ from the bottom hole pressure at the point source and is included in the mass conservation equation. In this case, the calculated pressure in the grid block containing the well will differ from the bottom hole pressure at the well. In this paper, the second approach was used, and Peaceman’s formula was used to calculate the bottom hole pressure [23,37]:

\[
q_{W} = \sum_{i=1}^{N} W_{i} \frac{k_{\text{row}}}{\mu_{w}} (p_{\text{bh}i} - p) \delta(x - x^{i})
\]

(7)

\[
q_{O} = \sum_{i=1}^{N} W_{i} \frac{k_{\text{ro}}}{\mu_{o}} (p_{\text{bh}i} - p) \delta(x - x^{i})
\]

(8)

\[
q_{G} = \sum_{i=1}^{N} W_{i} \frac{k_{\text{rg}}}{\mu_{g}} (p_{\text{bh}i} - p) \delta(x - x^{i})
\]

(9)

where \( N \) is the number of wells, each of which has one perforated zone; \( p_{\text{bh}i} \) is the bottom hole pressure of the \( i \)-th well; \( \delta(x) \) is the Dirac delta function, \( W_{i} = 2\pi k^{i} l^{i} / \ln(r_{e}^{i} / r^{i}) \); \( k^{i} = k(x^{i}) \); \( x^{i} \) is the central location of the perforated zone of the \( i \)-th well; \( l^{i} \) is the length of the perforated zone of the \( i \)-th well; \( r^{i} \) is the radius of the \( i \)-th well; and \( r_{e}^{i} \) is the drainage radius of the \( i \)-th well.

System (1)–(6) were solved numerically by the method of simultaneous solution using implicit approximation of time and an iterative procedure according to the Newton–Raphson method [23]. When solving the filtration equations, three main schemes can be distinguished [20]: the IMPES (implicit in pressure and explicit in saturation), the fully implicit (all equations solved simultaneously and implicitly), and the sequential (all equations decoupled and each solved implicitly and sequentially). The authors of [38] compared the solution schemes for the black oil model and concluded that in the presence of free gas, the fully implicit scheme was the most stable and robust. The use of an implicit scheme makes it possible to reduce the time required to solve the direct problem by choosing a larger time step. In turn, this significantly reduces computational costs in solving the inverse problem. The solution method is presented in Appendix A.

On solving the identification problem, the permeability field was searched in the following form:

- The reservoir was layered, the values of permeability remained constant across the thickness for each layer \( k(x, y, z) = k_{j}(x, y) \) in the \( j \)-th layer;
- The logarithm of the permeability of each layer was represented as a surface spline. A spline surface is a model of a thin plate bent under the action of external forces applied at some points (interpolation nodes). To construct a spline surface, it was necessary to solve the variational problem of finding the minimum free energy of the plate, which led to the following formula [34]:

\[
\ln k_{j}(x, y) = \sum_{i=1}^{N} c_{i} r_{i}^{2} \ln r_{i}^{2} + c_{N+1} + c_{N+2} x + c_{N+3} y
\]

where \( r_{i}^{2} = (x - x_{i})^{2} + (y - y_{i})^{2} \), \( (x_{i}, y_{i}) \) are the coordinates of the wells. The use of the logarithm of absolute permeability made it possible to obtain positive permeability values at any point of the reservoir. The coefficient \( c_{i} \) was determined by solution of the system of equations:

\[
\ln k_{j}(x_{i}, y_{i}) = \ln k_{ji}, \ i = 1, N, \ \sum_{i=1}^{N} c_{i} = 0, \ \sum_{i=1}^{N} x_{i} c_{i} = 0, \ \sum_{i=1}^{N} y_{i} c_{i} = 0
\]
where $k_{ji}$ represents the value of permeability at $i$-th wells in $j$-th layer;

- The value of permeability in the $i$-th well in the $j$-th layer was calculated by the formula $k_{ji} = \lambda_i k_{ji}^0$, where $\lambda_i$ is the unknown coefficient, and $k_{ji}^0$ is the a priori value of $k_{ji}$.

When using such an approximation of the permeability in the identification process, it is required to find $N$ and the coefficient $\lambda_i$. The coefficient $\lambda_i$ was determined in the process of minimizing the residual function, in the form:

$$J(\mathbf{X}) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{M_i} \left( Q_{ij} - Q_{ij}^* \right)^2$$

(10)

where $\mathbf{X} = \{ \lambda_i, \ i = 1, N \}$; and $Q_{ij}$, $Q_{ij}^*$ are the calculated and measured values of the total liquid production rate at different instants of time, $M = \sum_{i=1}^{N} M_i$. From (7)–(9) it follows that:

$$Q_{ij} = \frac{\sum_{i=1}^{N} W_i \left( \frac{K_{rw}}{\mu_w} + \frac{K_{ro}}{\mu_o} \right) \left( p_{bh}^i - p_{ij} \right) \delta \left( \mathbf{x} - \mathbf{x}^i \right)}{\lambda_i}$$

where $p_{bh}^i$, $p_{ij}$ are the known values of bottom hole pressure at the well and those resulting from the solution of (1)–(4).

Minimization of the residual function (10) was carried out by the iterative Levenberg–Marquardt method [39]. At each iteration, moving forward to the new parameter values was carried out according to the formula:

$$\mathbf{X}^n = \mathbf{X}^{n-1} - (H + \eta^n I)^{-1} \mathbf{g}$$

where $I$ is the unit matrix, $H = A^T A$ is the approximate matrix of second derivatives, $A$ is the sensitivity matrix, $\mathbf{g}$ is the gradient of the residual function, $\eta^n$ is the Marquardt parameter, and $n$ is the number of iterations. The initial value of the Marquardt parameter was chosen to be an order of magnitude larger than the maximum singular number of the matrix $H$ [40]. If $J(\mathbf{X}^n) < J(\mathbf{X}^{n-1})$ then this iteration ended and $\eta^{n+1} = \eta^n / 2$. Otherwise, $\eta^n = 2\eta^n$ until the condition for reducing the residual function was performed. To stop the minimization process, three criteria were used: $\max_{i,j} \left| Q_{ij} - Q_{ij}^* \right| < \delta$, where $\delta$ was the specified accuracy; $J^n - J^{n+1} < 0.01 J^n$ in the course of three iterations (slow convergence); and $\eta^n > \eta^3$ (there was no transition to new values).

In general, the proposed method for solving the problem of the permeability field determination is shown in Figure 1.

**Figure 1.** General scheme of the solution method.
3. Results

A three-phase fluid flow was considered for a reservoir 2000 m × 2000 m, consisting of 10 layers, where the thickness of layers 1 to 9 were each 5 m, and the thickness of layer 10 was 15 m. The porosity of the reservoir was constant, \( \varphi = 0.2 \). The reservoir was opened by 16 evenly spaced vertical producing wells; the distance between the wells was 500 m, the distance from the wells to the lateral boundary of the reservoir was 250 m. The perforation zones of all wells were located in the sixth layer and their length was 5 m. A bottom hole pressure of 15 MPa was set for all wells. The roof and bottom of the reservoir were impermeable. The lateral surfaces from the 1st to the 9th layer were impermeable, on the lateral surface of the 10th layer the pressure was 20 MPa and the saturation condition was \( S_{bw} = 1 \) (water enters). The initial pressure in the reservoir was 20 MPa. The initial saturations were: for the 1st layer, \( S_{0w} = 0.2, S_{0o} = 0 \); for 2nd to 9th layers, \( S_{0w} = 0.2, S_{0o} = 0.8 \); and for the 10th layer, \( S_{0w} = 1 \). Viscosities of oil, water and gas were: \( \mu_o = 10 \text{ mPa} \cdot \text{s}, \mu_w = 1 \text{ mPa} \cdot \text{s}, \) and \( \mu_g = 0.01 \text{ mPa} \cdot \text{s} \), respectively. The volume coefficients of oil, gas, and the gas solubility coefficient are given in Table 1. The dependence of the water volume coefficient on the pressure was in the form \( B_w = \frac{B_{wb}}{1 + c_w(p - p_b)} \), where \( B_{wb} = 1.0142, p_b = 0.101 \text{ MPa}, c_w = 0.000435 \text{ MPa}^{-1} \). The relative permeabilities \( \tilde{k}_{row}, \tilde{k}_{rw}, \) and \( \tilde{k}_{rog} \) in two-phase systems oil–water and oil–gas, are defined in Table 2.

Table 1. PVT property data.

<table>
<thead>
<tr>
<th>( p ), MPa</th>
<th>( B_o )</th>
<th>( B_g )</th>
<th>( R_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.025</td>
<td>0.02</td>
<td>300</td>
</tr>
<tr>
<td>10</td>
<td>1.045</td>
<td>0.01</td>
<td>600</td>
</tr>
<tr>
<td>15</td>
<td>1.07</td>
<td>0.006</td>
<td>900</td>
</tr>
<tr>
<td>20</td>
<td>1.09</td>
<td>0.0045</td>
<td>1200</td>
</tr>
<tr>
<td>25</td>
<td>1.11</td>
<td>0.0035</td>
<td>1400</td>
</tr>
</tbody>
</table>

Table 2. Saturation functions.

<table>
<thead>
<tr>
<th>( S_w )</th>
<th>( \tilde{k}_{rw} )</th>
<th>( \tilde{k}_{row} )</th>
<th>( S_g )</th>
<th>( \tilde{k}_{rg} )</th>
<th>( \tilde{k}_{rog} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.00</td>
<td>1.00</td>
<td>0.0</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.3</td>
<td>0.05</td>
<td>0.40</td>
<td>0.05</td>
<td>0.00</td>
<td>0.60</td>
</tr>
<tr>
<td>0.4</td>
<td>0.10</td>
<td>0.10</td>
<td>0.1</td>
<td>0.01</td>
<td>0.30</td>
</tr>
<tr>
<td>0.6</td>
<td>0.30</td>
<td>0.005</td>
<td>0.2</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td>0.8</td>
<td>0.60</td>
<td>0.00</td>
<td>0.3</td>
<td>0.10</td>
<td>0.01</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
<td>0.00</td>
<td>0.4</td>
<td>0.20</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.6</td>
<td>0.50</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.7</td>
<td>0.70</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.8</td>
<td>1.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The rectangular parallelepipeds of 100 m × 100 m × 5 \( h_j \) (\( h_j \) being the thickness of the \( j \)-th layer) were used as control volumes for the approximation of the system (A3)–(A6) in the spatial variables. The time step was taken equal to 1 day. When choosing a time step, it was necessary to take into account the following: the convergence of the Newton–Raphson method, when solving a nonlinear system of partial differential equations, could be violated at large values of the time step; small values of the time step would significantly increase the time to solve the problem; the selected time step should provide a solution to the problem at the time of the flow rate measurements. The chosen step, equal to 1 day, was not the only possible one, but was enough to demonstrate the operation of the proposed algorithm.

The model problem of absolute permeability identification was formulated as follows. The true permeability field, \( k^{tr} \), was built based on the given permeability values in the wells. The permeability values [\( \mu m^2 \)] in the wells were chosen randomly: for layers 1 and 5, from the interval (0.01; 0.09); and for the remaining layers, from the interval (0.1; 0.9). Then
from the solution of system (1)–(6) for the field \( k^f \), the values of the fluid flow rate in the wells \( Q_{ij}^f \) [m³/day] were calculated at various points in time. The start time of the wells was different: the first row of four wells was turned on at the initial time, the second row of wells was turned on at the time of 30 days, the third row of wells was turned on at the time of 60 days, and the fourth row of wells was turned on at the time of 90 days. The first flow rate measurements at the wells were performed a day after they were launched; the remaining measurements were performed every 30 days. The time of the last measurement at all wells was at 121 days.

Further, the permeability field was considered unknown and was determined in the process of minimizing the residual function (10), where \( Q_{ij}^f = Q_{ij}^f + \varepsilon_i Q_{ij}^f \), \( \varepsilon_i \) were the given errors. Various variants of errors were considered: (1) the same values for all wells \( \varepsilon_i = 0.1, \varepsilon_i = -0.1, \varepsilon_i = 0.05, \varepsilon_i = -0.05, \varepsilon_i = 0.01, \varepsilon_i = -0.01; \) and, (2) uniformly distributed random variables on given intervals \( |\varepsilon_i| \in (0.0075; 0.01), |\varepsilon_i| \in (0.025; 0.05), |\varepsilon_i| \in (0.075; 0.1) \). At the same time, a priori values of permeability at wells \( k_{ji}^t \) were known, which were defined as \( k_{ji}^t = k_{ji}^f / k_{ji}^r \), where \( j \) is the layer number, \( i \) is the well number, and \( k_{ji}^r \) is the permeability value at the perforated zones of wells. To compare the calculated permeability fields with the true field, the values of maximum and standard deviations were given: \( \Delta K_{\text{max}} = \max \sum_{i=1}^{N_i} |\lg k_{ji}^r - \lg k_{ji}^t| \), \( \delta_r = \max_i Q_{ij}^t - Q_{ij}^f \), \( \Delta K_{\text{av}} = \sum_{i=1}^{N_i} \frac{(\lg k_{ji}^r - \lg k_{ji}^t)^2}{N_i} \), where \( k_{ji}^t \) represents the calculated permeability values at the perforated zones of wells, also given the number of iterations, \( N_i \). The results obtained when solving problems without errors in the flow rates and with various errors in the flow rates are shown in Table 3. The values \( \Delta K_i = |\lg k_{ji}^r - \lg k_{ji}^t| \), sorted in ascending order \( k_{ji}^r \), for random error values, are shown in Figure 2. The true and calculated with errors \( |\varepsilon_i| \in (0.0075; 0.01) \) permeability fields by layers, are shown in Figures 3–5.

**Table 3.** The obtained results.

<table>
<thead>
<tr>
<th>( \varepsilon_i )</th>
<th>( \Delta K_{\text{max}} )</th>
<th>( \Delta K_{\text{av}} )</th>
<th>( \delta_r )</th>
<th>( N_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 2.08 \times 10^{-4} )</td>
<td>( 5.42 \times 10^{-5} )</td>
<td>( 0.014 )</td>
<td>6</td>
</tr>
<tr>
<td>( \varepsilon_i = 0.01 )</td>
<td>0.00450</td>
<td>0.00428</td>
<td>1.593</td>
<td>11</td>
</tr>
<tr>
<td>( \varepsilon_i = -0.01 )</td>
<td>0.00454</td>
<td>0.00435</td>
<td>1.708</td>
<td>11</td>
</tr>
<tr>
<td>(</td>
<td>\varepsilon_i</td>
<td>\in (0.0075; 0.01) )</td>
<td>0.00437</td>
<td>0.00388</td>
</tr>
<tr>
<td>( \varepsilon_i = 0.05 )</td>
<td>0.02179</td>
<td>0.02055</td>
<td>7.389</td>
<td>10</td>
</tr>
<tr>
<td>( \varepsilon_i = -0.05 )</td>
<td>0.02823</td>
<td>0.02273</td>
<td>4.905</td>
<td>13</td>
</tr>
<tr>
<td>(</td>
<td>\varepsilon_i</td>
<td>\in (0.025; 0.05) )</td>
<td>0.02144</td>
<td>0.01535</td>
</tr>
<tr>
<td>( \varepsilon_i = 0.1 )</td>
<td>0.04355</td>
<td>0.03958</td>
<td>15.127</td>
<td>11</td>
</tr>
<tr>
<td>( \varepsilon_i = -0.1 )</td>
<td>0.08357</td>
<td>0.04937</td>
<td>23.377</td>
<td>9</td>
</tr>
<tr>
<td>(</td>
<td>\varepsilon_i</td>
<td>\in (0.075; 0.1) )</td>
<td>0.04497</td>
<td>0.03683</td>
</tr>
</tbody>
</table>

**Figure 2.** The values \( \Delta K_i \) sorted in ascending order \( k_{ji}^r \), for random error values: \( |\varepsilon_i| \in (0.075; 0.1) \) ( ), \( |\varepsilon_i| \in (0.025; 0.05) \) ( ■ ), \( |\varepsilon_i| \in (0.0075; 0.01) \) ( ■ ).
When solving the problem without error in the flow rate measurements ($\epsilon_i = 0$), the specified accuracy $\delta = 0.1$ m$^3$/day was achieved. In case of presence of errors $\epsilon_i$, the minimization process was interrupted by the slow convergence criterion. From the results given in Table 1 and Figure 2, it can be seen that in the absence of errors, the calculated permeability field coincided, in practice, with the true one; and with a decrease in the absolute values of errors, the calculated absolute permeability fields approached the true field. The type of error (fixed or random values from a given interval) did not affect the results obtained. The execution time of one iteration of the minimization of the residual function in all of the above tasks was approximately the same. Therefore, by the number of iterations performed (Table 3), it was possible to judge the total time to solve the problem. The results showed that in the presence of errors in the flow rate measurements, the convergence rate slowed down and the time to solve the problem increased.
Figure 4. True (on the left), and calculated with errors $|\varepsilon_i| \in (0.0075; 0.01)$ (on the right), permeability fields of layers 4 to 7.
The problem of identifying the permeability field of a three-layer reservoir without taking into account a priori information was considered in [41]. The reservoir was opened by 81 wells, and 48 parameter values were determined. It was shown that in the case of wells located in one layer of the reservoir, the accuracy of determining the permeability field (ΔK_{max}, ΔK_{av}) reduced by an order of magnitude, and the task also became less stable to errors in flow rate measurements. The use of a priori information about the values of permeability of wells allows for a smaller number of wells and measurements to obtain more accurate solutions, and also increases the stability of the problem to errors in flow measurements.

4. Discussion and Conclusions

An algorithm was constructed to solve the problem of identifying the permeability field in the form of a spline function, which preserved a priori information about the distribution of permeability in wells. In this work, a linear dependence of the calculated permeability values on a priori values was used in the wells. The proposed algorithm can be used for other dependencies.
The algorithm testing was carried out on the model problem of identifying the permeability field of a three-dimensional reservoir under three-phase fluid filtration. The model reservoir, consisting of 10 layers, was opened by 16 wells, which required the determination of 160 unknown permeability values. Due to the use of a priori information, the number of required parameters was reduced to 16. The effect of errors in the flow rate measurements on the solution of the problem was studied. The obtained results showed that as the errors decreased, the calculated permeability fields approached the true field. When solving the problem without errors in flow rate measurements in the residual function minimization process, the calculated permeability field gradually approached the true field, and when small residual values were reached (less than 0.1 m³/day at all wells), the calculated field practically coincided with the true field ($\Delta K_{\text{max}} = 2.08 \times 10^{-4}$). Obtaining an exact solution (true permeability field) in a problem without error can be used as a proof of the correct operation of the proposed algorithm.

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**Nomenclature**

- $\Gamma$ boundary of the reservoir
- $\Gamma_1$ part of the boundary on which the boundary condition of the first kind is specified
- $\Gamma_2$ part of the boundary on which the boundary condition of the second kind is specified
- $\Gamma_{\text{in}}$ inflow boundary ($\Gamma_{\text{in}} \subset \Gamma_1$)
- $A$ sensitivity matrix
- $B_\alpha$ formation volume factor of phase $\alpha$
- $g$ gradient of the residual function
- $H$ approximate matrix of second derivatives
- $I$ unit matrix
- $J(X)$ residual function
- $k$ absolute permeability
- $k_{\alpha}$ relative permeability of phase $\alpha$
- $k_{\text{rg}}$ gas phase relative permeability of oil–gas system
- $k_{\text{rog}}$ oil phase relative permeability of oil–gas system
- $k_{\text{row}}$ oil phase relative permeability of oil–water system
- $k_{\text{rw}}$ water phase relative permeability of oil–water system
- $l^i$ length of the perforated zone of the $i$-th well
- $M$ total number of measurements for all wells
- $M_i$ number of measurements on $i$-th well
- $N$ number of wells
- $n$ outward unit normal to the boundary
- $p$ pressure
- $p_{bh}^i$ bottom hole pressure of the $i$-th well
- $t$ time
- $q_\alpha$ external sources and sinks of component $\alpha$
- $Q_{ij}$ calculated values of the total liquid production rate
- $Q_{ij}^*$ measured values of the total liquid production rate
- $r^i$ radius of the $i$-th well
- $r_{\text{ie}}^i$ drainage radius of the $i$-th well
After substituting (A2) into (A1), a system of linear equations is obtained for the unknowns \( \boldsymbol{y}^l \):

\[
L_m \left[ F_m \left( \boldsymbol{y}^l \right) + \nabla F_m \left( \boldsymbol{y}^l \right) \delta \boldsymbol{y}^l \right] = f_m(\boldsymbol{x}), \quad m = 1, 2, \ldots, M
\]

or

\[
L_m \left[ \nabla F_m \left( \boldsymbol{y}^l \right) \delta \boldsymbol{y}^l \right] = f_m(\boldsymbol{x}) - L_m \left[ F_m \left( \boldsymbol{y}^l \right) \right], \quad m = 1, 2, \ldots, M
\]

The transition to a new vector \( \boldsymbol{y}^{l+1} \) is performed according to the formula \( \boldsymbol{y}^{l+1} = \boldsymbol{y}^l + \delta \boldsymbol{y}^l \).

Equations (1)–(4) relative to \( \delta S_{\alpha}, \delta S_{\beta}, \delta S_{\phi}, \delta p \) at the \( l \)-th iteration with the introduced notation for the transmissibility

\[
T_{\alpha} = \frac{k_{\alpha}}{\mu_{\alpha} \varphi}
\]

of the \( \alpha \)-phase, can be written in the following form. For the water component:

\[
\frac{1}{\Delta t} \left[ \left( \frac{\varphi S_w}{B_w} \right)^l - \left( \frac{\varphi S_w}{B_w} \right)^n + c_w \delta p + c_{ww} \delta S_w \right] = \\
\nabla \cdot \left( \left( T_{\alpha}^l + E_{ww} \delta p + E_{ww} \delta S_w \right) \nabla p^l + \nabla \cdot \left( T_{\alpha}^l \nabla (\delta p) \right) \right) + \\
\frac{1}{B_w^l} \sum_{i=1}^N W_i \left( \epsilon_{ww}^l \delta p + \epsilon_{ww}^l \delta S_w \delta \left( \boldsymbol{x} - \boldsymbol{x}^i \right) \right) + q_{ww}^l \left( \frac{d}{dp} \left( \frac{1}{B_w} \right) \right)^l \delta p
\]

where

\[
\epsilon_w = \left( \frac{\varphi S_w}{B_w} \right)^l, \quad c_w = \left( \frac{\varphi}{B_w} \right)^l
\]

\[
E_{ww} = \left( \frac{k_{ww}}{\mu_w} \frac{d}{dp} \left( \frac{1}{B_w} \right) \right)^l, \quad E_{ww} = \left( \frac{1}{\mu_w B_w} \frac{dS_w}{dp} \right)^l k
\]

\[
\epsilon_{ww}^l = - \frac{k_{ww}}{\mu_w}, \quad \epsilon_{ww}^l = \frac{1}{\mu_w} \left( \frac{dS_w}{dp} \left( \frac{1}{B_w} \right) \right)^l
\]
For the oil component:

\[
\frac{1}{\Delta t} \left[ \left( \frac{\varphi_S o}{B_o} \right)^{i} - \left( \frac{\varphi_S o}{B_o} \right)^{i+1} \right] + c_{op} \delta p + c_{ow} \delta S_o = \nabla \cdot \left( \left( T_o^i + E_{op} \delta p + E_{ow} \delta S_w + E_{oo} \delta S_o \right) \nabla p^i \right) + \nabla \cdot \left( T_o^i \nabla (\delta p) \right) + \Delta \left( R_o^i T_o^i \nabla (\delta p) \right)
\]

(A4)

where

\[
c_{op} = \left( \frac{\varphi S_o}{B_o} \frac{d}{dp} \left( \frac{1}{B_o} \right) \right)^{i}, \quad c_{ow} = \left( \frac{\varphi S_o}{B_o} \frac{d}{dp} \left( \frac{1}{B_o} \right) \right)^{i} \mu_o
\]

\[
E_{oo} = - \left( \frac{1}{\mu_o B_o} \frac{dk_{ro}}{dS_g} \right)^{i} k, \quad E_{ow} = \left( \frac{1}{\mu_o B_o} \frac{dk_{ro}}{dS_g} \right)^{i} k, \quad c_{ow} = - \frac{k_{ro}}{\mu_o}
\]

\[
e_{ow} = \frac{1}{\mu_o} \left( \left( \frac{d k_{ro}}{dS_w} - \frac{d k_{ro}}{dS_g} \right) \left( p_{sh} - p \right) \right)^{i}, \quad e_{oo} = - \frac{1}{\mu_o} \left( \frac{dk_{ro}}{dS_g} \left( p_{sh} - p \right) \right)^{i}
\]

For the gas component:

\[
\frac{1}{\Delta t} \left[ \left( \frac{\varphi_S g}{B_g} + R_S S_o \right)^{i} - \left( \frac{\varphi_S g}{B_g} + R_S S_o \right)^{i+1} \right] + c_{gp} \delta p + c_{gw} \delta S_w + c_{go} \delta S_o = \nabla \cdot \left( \left( T_g^i + E_{gp} \delta p + E_{gs} (\delta S_w + \delta S_o) \right) \nabla p^i \right) + \nabla \cdot \left( \left( R_g^i T_g^i \nabla (\delta p) \right) \nabla p^i \right) + \nabla \cdot \left( \left( R_g^i T_g^i \nabla (\delta p) \right) \nabla p^i \right)
\]

(A5)

where

\[
c_{gp} = \left( \frac{\varphi S_g}{B_g} \frac{d}{dp} \left( \frac{1}{B_g} \right) \right)^{i} \mu_g
\]

\[
c_{gw} = - \left( \frac{\varphi S_g}{B_g} \frac{d}{dp} \left( \frac{1}{B_g} \right) \right)^{i} \mu_g
\]

\[
E_{gp} = \left( \frac{k_{rg} \frac{d}{dp} \left( \frac{1}{B_g} \right) \right)^{i} k, \quad E_{gs} = - \left( \frac{1}{\mu_g B_g} \frac{dk_{rg}}{dS_g} \right)^{i} k
\]

\[
r_{sp} = \left( \frac{d R_s}{dp} \right)^{i}, \quad e_{gp} = - \frac{k_{rg}}{\mu_g}, \quad e_{gs} = - \frac{1}{\mu_g} \left( \frac{dk_{rg}}{dS_g} \left( p_{sh} - p \right) \right)^{i}
\]

For the saturations of the phases:

\[
\delta S_w + \delta S_o + \delta S_g = 0
\]

(A6)
The approximation of system (A3)–(A6) in spatial variables is carried out using the method of control volumes. The resulting system of linear algebraic equations is solved by the biconjugate gradient stabilized method (BiCGStab) with SSOR preconditioner.

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