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Degenerate Multi-Term Equations with Gerasimov–Caputo Derivatives in the Sectorial Case

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Abstract: The unique solvability for the Cauchy problem in a class of degenerate multi-term linear equations with Gerasimov–Caputo derivatives in a Banach space is investigated. To this aim, we use the condition of sectoriality for the pair of operators at the oldest derivatives from the equation and the general conditions of the other operators' coordination with invariant subspaces, which exist due to the sectoriality. An abstract result is applied to the research of unique solvability issues for the systems of the dynamics and of the thermoconvection for some viscoelastic media.

Keywords: Gerasimov–Caputo derivative; fractional differential equation; analytic resolving family of operators; degenerate evolution equation; multi-term fractional equation; initial value problem; initial boundary value problem

MSC: 34G10; 34K37; 35R11

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1. Introduction

Fractional integro-differential calculus provides effective tools for the study of applied mathematical problems in various fields of science, such as physics, mathematical biology, theory of financial markets and many others. A large number of mathematical models of various real processes have appeared in the scientific literature, described in terms of equations with fractional derivatives and integrals [1–9]. At the same time, such equations are also of theoretical interest for the theory of differential equations and, therefore, have been the objects of research in a multitude of papers over the past few decades (see monographs [10–15] and the bibliographies therein).

In the theory of differential equations, a separate class consists of degenerate evolution equations, the special properties of which are entailed by the presence of a degenerate operator at the highest-order derivative. Various classes of degenerate evolution equations of an integer order have been studied by many authors [16–22]. Degenerate evolution equations with Gerasimov–Caputo, Riemann–Liouville and Dzhrbashyan–Nersesyan fractional derivatives were studied in [23–29].

In the present work, we study the unique solvability of a special initial value problem in the degenerate multi-term linear equation

$$D^{\alpha}Lx(t) = \sum_{l=1}^{n} D^{\alpha_l}M_l x(t) + g(t), \tag{1}$$

with the Gerasimov–Caputo derivatives D^{β}, $\beta \geq 0$, the Riemann–Liouville integrals D^{β}, $\beta < 0$ and the linear operators $L, M_1, M_2, \ldots, M_n$, which act from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$, $\ker L \neq \{0\}$. Here, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, where some of α_l may be negative, $m-1 < \alpha \leq m$, $m_n - 1 < \alpha_n \leq m_n$, $T > 0$ and $g : [0, T] \to \mathcal{Z}$. The unique solvability of the Cauchy problem in such an equation with bounded operators $M_1, M_2, \ldots, M_n$ in the nondegenerate case ($\mathcal{X} = \mathcal{Y}$, $L = I$) was proven in [30]. In [31], the Cauchy problem was researched for nondegenerate Equation (1) under the more general condition $(M_1, M_2, \ldots, M_n) \in \mathcal{A}^n_{\alpha,G}$ on linear, closed, densely defined operators $M_1, M_2, \ldots, M_n$.

In the study of degenerate equations of the form $D^\alpha Lx(t) = Mx(t)$ and $\ker L \neq \{0\}$, the conditions for the pair of operators (L, M) are often used, entailing the existence of the so-called pairs of invariant subspaces. We are talking about the representation of two Banach spaces in the form of the direct sums of the subspaces $\mathcal{X} = \mathcal{X}^0 \oplus \mathcal{X}^1$ and $\mathcal{Y} = \mathcal{Y}^0 \oplus \mathcal{Y}^1$, for which $L, M : \mathcal{X}^r \to \mathcal{Y}^r$ and there exist operators M_0^{-1} and L_1^{-1}, where $L_r = L|_{D_L \cap \mathcal{X}^r}$, $M_r = M|_{D_M \cap \mathcal{X}^r}$ and $r = 0, 1$. The direct sums correspond to the projectors P along $\mathcal{X}^0$ on $\mathcal{X}^1$ and Q along $\mathcal{Y}^0$ on $\mathcal{Y}^1$. Such an approach was used in [21] with the condition of an (L, p)-bounded operator and $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and in [25] with the condition $(L, M) \in \mathcal{H}_\alpha(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$. This makes it possible to reduce the degenerate equation to a system of two simpler equations on two subspaces. A generalization of this approach to the case of three or more operators $L, M_1, M_2, \ldots, M_n$ for degenerate equations is not evident, since in this case, we need to work with a pencil of operators $\mu^\alpha L - \mu^{\alpha_1} N_1 - \mu^{\alpha_2} N_2 - \ldots$, and the standard technique does not look applicable due to the presence of several fractional powers of the parameter μ. However, the same conditions can be used for a pair of operators (L, M_n) if the action of the remaining operators $M_1, M_2, \ldots, M_{n-1}$ is coordinated with the subspaces $\mathcal{X}^0$, $\mathcal{X}^1$, $\mathcal{Y}^0$ and $\mathcal{Y}^1$. The simplest variant of such a coordination is the equality $M_l P = QM_l$, implying that $M_l : \mathcal{X}^r \to \mathcal{Y}^r$, $r = 0, 1$ and $l = 1, 2, \ldots, n-1$. This is how multi-term degenerate Equation (1) with bounded operators $L, M_1, M_2, \ldots, M_n$ was investigated in [30], namely by reducing to the system of two simpler equations on two subspaces under the condition of $(L, 0)$-boundedness of the operator M_n. In this paper, when studying Equation (1) with unbounded operators L, M_n, a condition $(L, M_n) \in \mathcal{H}_\alpha(\theta_0, a_0)$ [25] is used that allows us to obtain pairs of invariant subspaces. At the same time, the coordination of the other operators $M_1, M_2, \ldots, M_{n-1}$ has a general form $M_l P = QM_l + (I - Q)N_l P$ with some bounded operators N_l, where $l = 1, 2, \ldots, n-1$.

In the second section, the preliminaries are given, including theorems on unique solvability of the Cauchy problem for two classes of nondegenerate ($\mathcal{X} = \mathcal{Y}$, $L = I$) multi-term equations (Equation (1)) with the Gerasimov–Caputo derivatives, where one of them has bounded operators $M_1, M_2, \ldots, M_n$ [30], and for the other one, the condition $(M_1, M_2, \ldots, M_n) \in \mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ is satisfied, which implies the existence of analytic resolving families of the operators [31]. In the third section, the theorem on the existence of a unique solution to the problem

$$D^k x(0) = x_k, \quad k = 0, 1, \ldots, m_n - 1, \quad D^k Px(0) = x_k, \quad k = m_n, m_n + 1, \ldots, m - 1,$$

for the degenerate multi-term Equation (1) is proven under conditions $(L, M_n) \in \mathcal{H}_\alpha(\theta_0, a_0)$ and $M_l P = QM_l + (I - Q)N_l P$ with some bounded operators N_l, where $l = 1, 2, \ldots, n-1$. To this aim, Equation (1) is reduced to a system of two nondegenerate multi-term equations on the subspaces of two classes, which are described in the second section. Abstract results are applied to the study of unique solvability issues for the initial boundary value problems of some systems of the dynamics of viscoelastic fluids in the framework of the abstract, non-degenerate multi-term equation and for the system of the thermoconvection for the Kelvin–Voigt fluid as a degenerate, multi-term equation in a Banach space.

2. Preliminaries

We define the Riemann–Liouville fractional integral of the order $\beta > 0$ [12,14] as follows:

$$J^\beta h(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} h(s) ds, \ t > 0.$$

Let $m-1<\alpha\leq m\in\mathbb{N}$, D^m be the derivative of the order $m\in\mathbb{N}$ and D^α be the fractional Gerasimov–Caputo derivative of the order α [14,32]:

$$D^\alpha h(t):=D^m J^{m-\alpha}\left(h(t)-\sum_{k=0}^{m-1}D^k h(0)\frac{t^k}{k!}\right).$$

For $\beta<0$, by defintion, we will mean $D^\beta h(t):=J^{-\beta}h(t)$. Hereafter, with $D^\beta h(0)$ for $\beta\in\mathbb{R}$, we denote the limit $\lim\limits_{t\to 0+}D^\beta h(t)$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, denoting with $\mathcal{L}(\mathcal{X};\mathcal{Y})$ the Banach space of all linear bounded operators acting from $\mathcal{X}$ into $\mathcal{Y}$ and with $\mathcal{C}l(\mathcal{X};\mathcal{Y})$ the set of all linear closed operators acting on $\mathcal{Y}$ with a dense domain in $\mathcal{X}$. We also denote $\mathcal{L}(\mathcal{X};\mathcal{X}):=\mathcal{L}(\mathcal{X})$ and $\mathcal{C}l(\mathcal{X};\mathcal{X}):=\mathcal{C}l(\mathcal{X})$, for $A\in\mathcal{C}l(\mathcal{X})$ $R_\mu(A):=(\mu I-A)^{-1}$ and for $L,M\in\mathcal{C}l(\mathcal{X};\mathcal{Y})$ $R^L_\mu(M):=(\mu L-M)^{-1}L$, while $L^L_\mu(M):=L(\mu L-M)^{-1}$, $\rho^L(M)$ is the set of $\mu\in\mathbb{C}$ such that $\mu L-M:D_L\cap D_M\to\mathcal{Y}$ is injective mapping and $R^L_\mu(M)\in\mathcal{L}(\mathcal{X})$, $L^L_\mu(M)\in\mathcal{L}(\mathcal{Y})$. We will assume that $\ker L\neq\{0\}$.

2.1. Theorem on Pairs of Invariant Subspaces

Definition 1. [32]. *An operator $A\in\mathcal{C}l(\mathcal{X})$ belongs to the class $\mathcal{A}_\alpha(\theta_0,a_0)$ if*

(1) *there exist $\theta_0\in(\pi/2,\pi)$ and $a_0\geq 0$ such that for all $\lambda\in S_{\theta_0,a_0}:=\{\mu\in\mathbb{C}:|\arg(\mu-a_0)|<\theta_0,\ \mu\neq a_0\}$, we have $\lambda^\alpha\in\rho(A):=\{\mu\in\mathbb{C}:(\mu I-A)^{-1}\in\mathcal{L}(\mathcal{X})\}$ and*

(2) *for every $\theta\in(\pi/2,\theta_0)$, $a>a_0$, there exists a constant $K=K(\theta,a)>0$ such that, for all $\lambda\in S_{\theta,a}$, we have*

$$\|R_{\lambda^\alpha}(A)\|_{\mathcal{L}(\mathcal{X})}\leq\frac{K(\theta,a)}{|\lambda^{\alpha-1}(\lambda-a)|}.$$

Definition 2. [25]. *Let $L,M\in\mathcal{C}l(\mathcal{X};\mathcal{Y})$. A pair (L,M) belongs to the class $\mathcal{H}_\alpha(\theta_0,a_0)$ if*

(1) *there exist $\theta_0\in(\pi/2,\pi)$ and $a_0\geq 0$ such that, for all $\lambda\in S_{\theta_0,a_0}$, we have $\lambda^\alpha\in\rho^L(M)$, and*

(2) *for every $\theta\in(\pi/2,\theta_0)$, $a>a_0$, there exists a constant $K=K(\theta,a)>0$ such that, for all $\lambda\in S_{\theta,a}$, we have*

$$\max\{\|R^L_{\lambda^\alpha}(M)\|_{\mathcal{L}(\mathcal{X})},\|L^L_{\lambda^\alpha}(M)\|_{\mathcal{L}(\mathcal{Y})}\}\leq\frac{K(\theta,a)}{|\lambda^{\alpha-1}(\lambda-a)|}.$$

Remark 1. *In the case of the inverse operator $L^{-1}\in\mathcal{L}(\mathcal{X})$ existing, we have $(L,M)\in\mathcal{H}_\alpha(\theta_0,a_0)$ if and only if $L^{-1}M\in\mathcal{A}_\alpha(\theta_0,a_0)$ and $ML^{-1}\in\mathcal{A}_\alpha(\theta_0,a_0)$.*

From the pseudo-resolvent identity, which is valid for $R^L_\mu(M)$ and for $L^L_\mu(M)$ separately, it follows that the subspaces $\ker R^L_\mu(M)=\ker L$, $\mathrm{im}R^L_\mu(M)$ and $\ker L^L_\mu(M)$, $\mathrm{im}L^L_\mu(M)$ do not depend on $\mu\in\rho^L(M)$. We introduce the denotations $\ker R^L_\mu(M):=\mathcal{X}^0$ and $\ker L^L_\mu(M):=\mathcal{Y}^0$. With $\mathcal{X}^1$ ($\mathcal{Y}^1$), we denote the closure of the image $\mathrm{im}R^L_\mu(M)$ ($\mathrm{im}L^L_\mu(M)$) in the norm of the space $\mathcal{X}$ ($\mathcal{Y}$). With L_r (M_r), the restriction of the operator L (M) on $D_{L_r}:=D_L\cap\mathcal{X}^r$ ($D_{M_r}:=D_M\cap\mathcal{X}^r$) will be denoted, where $r=0,1$.

Theorem 1. [25]. *Let the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ be reflexive, where $(L,M)\in\mathcal{H}_\alpha(\theta_0,a_0)$. Then, the following are true:*

(1) $\mathcal{X}=\mathcal{X}^0\oplus\mathcal{X}^1$ *and* $\mathcal{Y}=\mathcal{Y}^0\oplus\mathcal{Y}^1$.
(2) *The projection P (Q) on the subspace $\mathcal{X}^1$ ($\mathcal{Y}^1$) along $\mathcal{X}^0$ ($\mathcal{Y}^0$) has the form $P:=s\text{-}\lim\limits_{n\to\infty}nR^L_n(M)$ ($Q:=s\text{-}\lim\limits_{n\to\infty}nL^L_n(M)$).*
(3) $L_0=0$, $M_0\in\mathcal{C}l(\mathcal{X}^0;\mathcal{Y}^0)$ *and* $L_1,M_1\in\mathcal{C}l(\mathcal{X}^1;\mathcal{Y}^1)$.
(4) *There exist inverse operators $L_1^{-1}\in\mathcal{C}l(\mathcal{Y}^1;\mathcal{X}^1)$ and $M_0^{-1}\in\mathcal{L}(\mathcal{Y}^0;\mathcal{X}^0)$.*
(5) $\forall x\in D_L$ $Px\in D_L$ *and* $LPx=QLx$.

(6) *$\forall x \in D_M$ $Px \in D_M$ and $MPx = QMx$.*
(7) *Let $S := L_1^{-1}M_1 : D_S \to \mathcal{X}^1$. Then, $D_S := \{x \in D_{M_1} : M_1x \in \mathrm{im}L_1\}$ is dense in $\mathcal{X}$.*
(8) *Let $T := M_1L_1^{-1} : D_T \to \mathcal{Y}^1$. Then, $D_T := \{y \in \mathrm{im}L_1 : L_1^{-1}y \in D_{M_1}\}$ is dense in $\mathcal{Y}$.*
(9) *If $L_1 \in \mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$ or $M_1 \in \mathcal{L}(\mathcal{X}^1;\mathcal{Y}^1)$, then $S \in \mathcal{C}l(\mathcal{X}^1)$, and moreover, $S \in \mathcal{A}_\alpha(\theta_0, a_0)$.*
(10) *If $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$ or $M_1^{-1} \in \mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$, then $T \in \mathcal{C}l(\mathcal{Y}^1)$, and aside from that, $T \in \mathcal{A}_\alpha(\theta_0, a_0)$.*

2.2. Nondegenerate Multi-Term Equation

Let $m - 1 < \alpha \leq m \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \leq m_l \in \mathbb{N}$, $l = 1, 2, \ldots, n$. Some of α_l may be negative. Consider the Cauchy problem

$$D^k z(0) = z_k, \quad k = 0, 1, \ldots, m-1, \tag{2}$$

for a linear multi-term fractional differential equation

$$D^\alpha z(t) = \sum_{l=1}^{n} D^{\alpha_l} A_l z(t) + f(t), \quad t \in (0, T], \tag{3}$$

where the operators $A_l \in \mathcal{C}l(\mathcal{X})$ have domains D_{A_l}, $l = 1, 2, \ldots, n$ and $f \in C([0, T]; \mathcal{X})$. A solution to problem (2), (3) is a function $z \in C^{m-1}([0, T]; \mathcal{X})$, for which $D^\alpha z, D^{\alpha_l} A_l z_l \in C((0, T]; \mathcal{X})$, $l = 1, 2, \ldots, n$, and conditions (2) and equality (3) for all $t \in (0, T]$ hold.

We denote $D := \bigcap_{l=1}^{n} D_{A_l}$, $R_\lambda := \left(\lambda^\alpha I - \sum\limits_{l=1}^{n} \lambda^{\alpha_l} A_l\right)^{-1} : \mathcal{X} \to D$ and endow the set D with the norm $\|\cdot\|_D = \|\cdot\|_\mathcal{X} + \sum\limits_{l=1}^{n} \|A_l \cdot\|_\mathcal{X}$, with respect to which D is a Banach space, since it is the intersection of the Banach spaces $D_{A_1}, D_{A_2}, \ldots, D_{A_n}$ with the corresponding graph norms.

We also denote $n_k := \min\{l \in \{1, 2, \ldots, n\} : k \leq m_l - 1\}$ for $k = 0, 1, \ldots, m-1$. If the set $\{l \in \{1, 2, \ldots, n\} : k \leq m_l - 1\}$ is empty for some $k \in \{0, 1, \ldots, m-1\}$ (it is valid if and only if $\alpha_n \leq k$), then we apply $n_k := n + 1$:

Definition 3. *A tuple of operators $(A_1, A_2, \ldots, A_n)$ belongs to the class $\mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ at some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$ if the following are true:*

(1) *D is dense in $\mathcal{X}$.*

(2) *For all $\lambda \in S_{\theta_0, a_0}$, $k = 0, 1, \ldots, m-1$, there exist operators $R_\lambda \cdot \left(I - \sum\limits_{l=n_k}^{n} \lambda^{\alpha_l - \alpha} A_l\right) \in \mathcal{L}(\mathcal{X})$.*

(3) *For any $\theta \in (\pi/2, \theta_0)$, $a > a_0$, there exists such a $K(\theta, a) > 0$ that for all $\lambda \in S_{\theta,a}$, $k = 0, 1, \ldots, m-1$,*

$$\|R_\lambda\|_{\mathcal{L}(\mathcal{X})} \leq \frac{K(\theta, a)}{|\lambda - a||\lambda|^{\alpha-1}}, \quad \left\| R_\lambda \left(I - \sum_{l=n_k}^{n} \lambda^{\alpha_l - \alpha} A_l \right) \right\|_{\mathcal{L}(\mathcal{X})} \leq \frac{K(\theta, a)}{|\lambda - a||\lambda|^{\alpha-1}}.$$

Remark 2. *If $n_k = n + 1$, then by the definition, $\sum\limits_{l=n_k}^{n} \lambda^{\alpha_l - \alpha} A_l := 0$.*

Remark 3. *In [31], the same class $\mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ of tuples of operators is denoted by $\mathcal{A}^{n,r}_{\alpha,G}(\theta_0, a_0)$, since in that case, r operators at a negative α_l value were grouped separately.*

Remark 4. *It is easy to show that in the case $\alpha_l = 0$ for some $l \in \{0, 1, \ldots, n\}$ the condition $(0, \ldots, 0, A_l, 0, \ldots, 0) \in \mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$ is satisfied if and only if $A_l \in \mathcal{A}_\alpha(\theta_0, a_0)$.*

We denote at $t > 0$ that

$$Z_k(t) = \frac{1}{2\pi i}\int_\Gamma R_\lambda \left(\lambda^{\alpha-k-1}I - \sum_{l=n_k}^{n} \lambda^{\alpha_l-k-1}A_l\right) e^{\lambda t} d\lambda,\ k = 1,2,\ldots,m-1,$$

$$Z(t) := \frac{1}{2\pi i}\int_\Gamma R_\lambda e^{\lambda t} d\lambda,$$

where $\Gamma := \Gamma^+ \cup \Gamma^- \cup \Gamma^0$, $\Gamma^0 := \{\lambda \in \mathbb{C} : |\lambda - a| = r_0 > 0, \arg\lambda \in (-\theta,\theta)\}$, $\Gamma^\pm := \{\lambda \in \mathbb{C} : \arg(\lambda - a) = \pm\theta, |\lambda - a| \in [r_0,\infty)\}$, $\theta \in (\pi/2,\theta_0)$, $a > a_0$ and $r_0 > 0$.

In [31], it is shown that there exist resolving families of operators $\{Z_k \in \mathcal{L}(\mathcal{X}) : t \geq 0\}$, $k = 0,1,\ldots,m-1$ of the homogeneous Equation (3) ($f \equiv 0$) if and only if $(A_1, A_2, \ldots, A_n) \in \mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$. Therein, the following unique solvability theorem was proved for the Cauchy problem in the inhomogeneous equation:

Theorem 2. [31]. *Let $m-1 < \alpha \leq m \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \leq m_l \in \mathbb{N}$, $l = 1,2,\ldots,n$, $(A_1, A_2, \ldots, A_n) \in \mathcal{A}^n_{\alpha,G}(\theta_0, a_0)$, $z_k \in D$, $k = 0,1,\ldots,m-1$ and $f \in C([0,T]; D)$. Then, there exists a unique solution to problem (2), (3), and it has the form*

$$z(t) = \sum_{k=0}^{m-1} Z_k(t) z_k + \int_0^t Z(t-s) f(s) ds. \tag{4}$$

In the case of bounded operators $A_1, A_2, \ldots, A_n$, an analogous result was obtained in [30]:

Theorem 3. [30]. *Let $m-1 < \alpha \leq m \in \mathbb{N}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \leq m_l \in \mathbb{N}$, $l = 1,2,\ldots,n$, $A_1, A_2, \ldots, A_n \in \mathcal{L}(\mathcal{X})$, $z_k \in \mathcal{X}$, $k = 0,1,\ldots,m-1$ and $f \in C([0,T]; \mathcal{X})$. Then, there exists a unique solution to problem (2), (3), and it has form (4).*

3. An Initial Value Problem for a Degenerate Equation

Suppose that $n \in \mathbb{N}$, $M_1, M_2, \ldots, M_{n-1} \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ and $M_n, L \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$ and that D_{M_n} and D_L are domains of the operators M_n, L, respectively, with the respective graph norms $\ker L \neq \{0\}$.

Let Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ be reflexive, $(L, M_n) \in \mathcal{H}_\alpha(\theta_0, a_0)$, $\alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha$, $m-1 < \alpha \leq m$, $m_l - 1 < \alpha_l \leq m_l$, $l = 1,2,\ldots,n$ and $g \in C([0,T]; \mathcal{Y})$. Some of α_l may be negative. Consider the initial value problem

$$D^k x(0) = x_k,\ k = 0,1,\ldots,m_n-1, \quad D^k P x(0) = x_k,\ k = m_n, m_n+1, \ldots, m-1, \tag{5}$$

for a multi-term fractional linear inhomogeneous equation

$$D^\alpha L x(t) = \sum_{l=1}^{n} D^{\alpha_l} M_l x(t) + g(t), \tag{6}$$

which is called degenerate in the case where $\ker L \neq \{0\}$. The projector P is defined in Theorem 1.

A solution to problem (5), (6) is a function $x : [0,T] \to D_L \cap D_{M_n}$ such that $x \in C^{m_n-1}([0,T]; \mathcal{X})$, $Px \in C^{m-1}([0,T]; \mathcal{X})$, $D^\alpha Lx, D^{\alpha_l} M_l x \in C((0,T]; \mathcal{Y})$, $l = 1,2,\ldots,n$, equality (6) for all $t \in (0,T]$ and conditions (5) are valid.

Lemma 1. *Let $(L, M_n) \in \mathcal{H}_\alpha(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, where $a_0 \geq 0$ and $\alpha > \alpha_n \geq 0$. Then, for every $\theta \in (\pi/2, \theta_0)$, $a > \max\{1, a_0^{\alpha/(\alpha-\alpha_n)}\}$, there exists $K_1(\theta, a) > 0$ such that*

$$\max\{\|(\mu^\alpha L - \mu^{\alpha_n} M_n)^{-1} L\|_{\mathcal{L}(\mathcal{X})}, \|L(\mu^\alpha L - \mu^{\alpha_n} M_n)^{-1}\|_{\mathcal{L}(\mathcal{Y})}\} \leq \frac{K_1(\theta, a)}{|\mu - a||\mu|^{\alpha-1}}.$$

Proof. Take $\theta \in (\pi/2, \theta_0)$, $a > \max\{1, a_0^{\alpha/(\alpha-\alpha_n)}\}$, $\mu \in S_{\theta,a}$ and $\lambda = \mu^{1-\alpha_n/\alpha}$ in the sense of the principal branch of the power function. Then, $\lambda \in S_{\theta_0,a_0}$, since $1 - \alpha_n/\alpha \in (0, 1)$. Hence, we have

$$\|(\mu^\alpha L - \mu^{\alpha_n} M_n)^{-1} L\|_{\mathcal{L}(\mathcal{X})} = |\mu|^{-\alpha_n} \|R^L_{\mu^{\alpha-\alpha_n}}(M_n)\|_{\mathcal{L}(\mathcal{X})} = |\mu|^{-\alpha_n} \|R^L_{\lambda^\alpha}(M_n)\|_{\mathcal{L}(\mathcal{X})} \leq$$

$$\leq \frac{K(\theta, a)}{|\lambda - a||\lambda|^{\alpha-1}|\mu|^{\alpha_n}} = \frac{K(\theta, a)}{|\mu^{1-\alpha_n/\alpha} - a||\mu|^{(1-\alpha_n/\alpha)(\alpha-1)}|\mu|^{\alpha_n}} \leq \frac{K_1(\theta, a)}{|\mu - a||\mu|^{\alpha-1}}.$$

Analogously, we can obtain a similar inequality for $\|L(\mu^\alpha L - \mu^{\alpha_n} M_n)^{-1}\|_{\mathcal{L}(\mathcal{Y})}$. □

For a negative α_n, we can obtain a similar result:

Lemma 2. *Let $(L, M_n) \in \mathcal{H}_\alpha(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, where $a_0 \geq 0$ and $\alpha > 0 > \alpha_n > \alpha(1 - 2\theta_0/\pi)$. Then, for every $\theta \in (\pi/2, \alpha\theta_0/(\alpha - \alpha_n))$, where $a > \max\{1, a_0\}$, there exists $K_1(\theta, a) > 0$ such that*

$$\max\{\|(\mu^\alpha L - \mu^{\alpha_n} M_n)^{-1} L\|_{\mathcal{L}(\mathcal{X})}, \|L(\mu^\alpha L - \mu^{\alpha_n} M_n)^{-1}\|_{\mathcal{L}(\mathcal{Y})}\} \leq \frac{K_1(\theta, a)}{|\mu - a||\mu|^{\alpha-1}}.$$

Proof. Since $1 - \alpha_n/\alpha > 1$, for $\theta \in (\pi/2, \alpha\theta_0/(\alpha - \alpha_n))$, $a > \max\{1, a_0\}$ and $\mu \in S_{\theta,a}$, we have $\lambda = \mu^{1-\alpha_n/\alpha} \in S_{\theta_0,a_0}$. The remaining part of the proof is the same as for the previous lemma. □

We denote for brevity that $P_0 := I - P$, $Q_0 := I - Q$, L_r ($M_{l,r}$) is the restriction of L (M_l) on $D_{L_r} := D_L \cap \mathcal{X}^r$ (on $D_{M_{l,r}} := D_{M_l} \cap \mathcal{X}^r$ for $l = 1, 2, \ldots, n$), where $r = 0, 1$. Due to Theorem 1 $LP = QL$ for $x \in D_L$, $M_n P x = Q M_n x$ for $x \in D_{M_n}$, and hence $M_{n,r} \in \mathcal{C}l(\mathcal{X}^r; \mathcal{Y}^r)$ and $L_r \in \mathcal{L}(\mathcal{X}^r; \mathcal{Y}^r)$, where $r = 0, 1$. That aside, there exist $M_{n,0}^{-1} \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$ and $L_1^{-1} \in \mathcal{C}l(\mathcal{Y}^1; \mathcal{X}^1)$.

Theorem 4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be reflexive Banach spaces, $(L, M_n) \in \mathcal{H}_\alpha(\theta_0, a_0)$, $M_l \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $l = 1, \ldots, n-1$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$ and $\alpha_n > \alpha(1 - 2\theta_0/\pi)$. Then, $(M_{1,1}L_1^{-1}, M_{2,1}L_1^{-1}, \ldots, M_{n,1}L_1^{-1}) \in \mathcal{A}^n_{\alpha,G}(\theta_1, a_1)$ for some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$.*

Proof. Since $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, by Theorem 1 (10), we have $M_{l,1}L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1)$, where $l = 1, 2, \ldots, n-1$. Due to Lemma 1 for $\alpha_n \geq 0$ or Lemma 2 in the case where $\alpha_n \in (\alpha(1 - 2\theta_0/\pi), 0)$ for some $\theta_1 \in (\pi/2, \theta_0]$, $a_1 \geq a_0$ and $\mu \in S_{\theta,a}$, we have

$$\left(\mu^\alpha I - \sum_{l=1}^{n} \mu^{\alpha_l} M_{l,1} L_1^{-1}\right)^{-1} =$$

$$= (\mu^\alpha I - \mu^{\alpha_n} M_{n,1} L_1^{-1})^{-1} \left(I - \sum_{l=1}^{n-1} \mu^{\alpha_l} M_{l,1} L_1^{-1} (\mu^\alpha I - \mu^{\alpha_n} M_{n,1} L_1^{-1})^{-1}\right)^{-1},$$

$$\left\|\sum_{l=1}^{n-1} \mu^{\alpha_l} M_{l,1} L_1^{-1} (\mu^\alpha I - \mu^{\alpha_n} M_{n,1} L_1^{-1})^{-1}\right\|_{\mathcal{L}(\mathcal{Y}^1)} \leq$$

$$\leq \sum_{l=1}^{n-1} |\mu|^{\alpha_l} \|M_{l,1} L_1^{-1}\|_{\mathcal{L}(\mathcal{Y}^1)} \left\|L(\mu^\alpha L - \mu^{\alpha_n} M_n)^{-1}\right\|_{\mathcal{L}(\mathcal{Y})} \leq$$

$$\leq \frac{\sum\limits_{l=1}^{n-1} |\mu|^{\alpha_l - \alpha + 1} \|M_{l,1}L_1^{-1}\|_{\mathcal{L}(\mathcal{Y})} K_1(\theta, a)}{|\mu - a|} < q < 1$$

for some $q \in (0,1)$, and hence

$$\left\| \left(\mu^\alpha I - \sum_{l=1}^{n} \mu^{\alpha_l} M_{l,1} L_1^{-1} \right)^{-1} \right\|_{\mathcal{L}(\mathcal{Y}^1)} \leq \frac{K_2(\theta, a)}{(1-q)|\mu - a||\mu|^{\alpha-1}}.$$

Finally, we have

$$\left(\mu^\alpha I - \sum_{l=1}^{n} \mu^{\alpha_l} M_{l,1} L_1^{-1} \right)^{-1} \left(I - \sum_{l=n_k}^{n} \mu^{\alpha_l - \alpha} M_{l,1} L_1^{-1} \right) =$$

$$= \mu^{-\alpha} \left(I + \left(\mu^\alpha I - \sum_{l=1}^{n} \mu^{\alpha_l} M_{l,1} L_1^{-1} \right)^{-1} \sum_{l=1}^{n_k - 1} \mu^{\alpha_l} M_{l,1} L_1^{-1} \right),$$

$$\left\| \left(\mu^\alpha I - \sum_{l=1}^{n} \mu^{\alpha_l} M_{l,1} L_1^{-1} \right)^{-1} \left(I - \sum_{l=n_k}^{n} \mu^{\alpha_l - \alpha} M_{l,1} L_1^{-1} \right) \right\|_{\mathcal{L}(\mathcal{Y}^1)} \leq$$

$$\leq |\mu|^{-\alpha} \left(1 + \frac{K_2(\theta, a)}{(1-q)|\mu - a||\mu|^{\alpha-1}} \cdot \sum_{l=1}^{n_k - 1} |\mu|^{\alpha_l} \|M_{l,1}L_1^{-1}\|_{\mathcal{L}(\mathcal{Y}^1)} \right) \leq \frac{K_3(\theta, a)}{|\mu - a||\mu|^{\alpha-1}}.$$

□

Theorem 5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be reflexive Banach spaces, $(L, M_n) \in \mathcal{H}_\alpha(\theta_0, a_0)$, $M_l \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $M_l P = QM_l + Q_0 N_l P$ for some $N_l \in \mathcal{L}(\mathcal{X}^1; \mathcal{Y})$, $l = 1, 2, \ldots, n-1$, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $\alpha_n > \alpha(1 - 2\theta_0/\pi)$, $g \in C([0,T]; \mathcal{Y})$, $Qg \in C([0,T]; D_{M_{n,1}L_1^{-1}})$, $x_k \in D_{M_{n,1}} \dot{+} \mathcal{X}^0$ for $k = 0, 1, \ldots, m_n - 1$ and $x_k \in D_{M_{n,1}}$ for $k = m_n, m_n + 1, \ldots, m-1$. Then, there exists a unique solution to problem (5), (6).*

Proof. Note that $M_l P_0 = M_l(I - P) = M_l - QM_l - Q_0 N_l P = Q_0(M_l - N_l P)$ for $l = 1, 2, \ldots, n-1$. Establish that $P_0 x(t) := w(t)$, $y(t) := Lx(t) = L_1 Px(t) + L_0 w(t) = L_1 Px(t)$, and then $x(t) = Px(t) + w(t) = L_1^{-1} y(t) + w(t)$. Thus, for $l = 1, 2, \ldots, n-1$, we have $M_l x = M_l(L_1^{-1} y(t) + w(t)) = (QM_l + Q_0 N_l P) L_1^{-1} y(t) + Q_0(M_l - N_l P) w(t) = (QM_l + Q_0 N_l) L_1^{-1} y(t) + Q_0 M_l w(t)$.

Using the operator $M_{n,0}^{-1} Q_0 \in \mathcal{L}(\mathcal{Y}^0; \mathcal{X}^0)$, problem (5), (6) can be written as the system

$$D^\alpha y(t) = \sum_{l=1}^{n} D^{\alpha_l} QM_{l,1} L_1^{-1} y(t) + Qg(t), \tag{7}$$

$$D^{\alpha_n} w(t) = -\sum_{l=1}^{n-1} D^{\alpha_l} M_{n,0}^{-1} Q_0 M_{l,0} w(t) - \sum_{l=1}^{n-1} D^{\alpha_l} M_{n,0}^{-1} Q_0 N_l L_1^{-1} y(t) - M_{n,0}^{-1} Q_0 g(t), \tag{8}$$

with the initial conditions

$$D^k y(0) = L_1 P x_k, \; k = 0, 1, \ldots, m-1, \tag{9}$$

$$D^k w(0) = P_0 x_k, \; k = 0, 1, \ldots, m_n - 1. \tag{10}$$

In the considered case $D := \bigcap\limits_{l=1}^{n} D_{M_{l,1}L_1^{-1}} = D_{M_{n,1}L_1^{-1}}$ with the graph norm of the operator $D_{M_{n,1}L_1^{-1}}$, since $x_k \in D_{M_{n,1}} \dot{+} \mathcal{X}^0$, then $L_1 P x_k \in D_{M_{n,1}L_1^{-1}}$ for $k = 0, 1, \ldots, m_n - 1$. Hence, through Theorem 2, there exists a unique solution to problem (7), (9). Problem (8), (10) have a unique solution due to Theorem 3, since the operators $M_{n,0}^{-1}Q_0M_{l,0}$, $l = 1, 2, \dots, n-1$ are bounded and $\sum\limits_{l=1}^{n-1} D^{\alpha_l} M_{n,0}^{-1} Q_0 N_l L_1^{-1} y + M_{n,0}^{-1} Q_0 g \in C([0,T]; \mathcal{X}^0)$ is a known function. □

Remark 5. *The proof of Theorem 5 implies that the Cauchy problem $x^{(l)}(0) = x_l$, $l = 0, \dots, m-1$ for Equation (6) has a unique solution under the additional conditions $P_0 x_l = D^l w(0)$, $l = m_n, m_n + 1, \dots, m-1$ only. Here, w is a unique solution to problem (8), (10).*

4. Some Initial Value Problems for Viscoelastic Media Systems

Consider the initial boundary value problem

$$D_t^k v(s,0) = v_k(s), \quad s \in \Omega, \quad k = 0, 1, \dots, m-1, \tag{11}$$

$$v(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0,T], \tag{12}$$

$$D_t^\alpha v(s,t) = \chi D_t^\beta \Delta v(s,t) + \nu D_t^\gamma \Delta v(s,t) + \kappa D_t^\delta \Delta v(s,t) - r(s,t) + h(s,t),\ (s,t) \in \Omega \times (0,T], \tag{13}$$

$$\nabla \cdot v(s,t) = 0, \quad (s,t) \in \Omega \times (0,T], \tag{14}$$

in a bounded region $\Omega \subset \mathbb{R}^d$ with a smooth boundary $\partial\Omega$, $\chi, \nu, \kappa \in \mathbb{R}$, $m - 1 < \alpha \le m \in \mathbb{N}$, $\alpha > \beta > \gamma > \delta$, where some of numbers $\alpha, \beta, \gamma, \delta$ may be negative. Here, D_t^ε is a fractional Gerasimov–Caputo derivative of the order $\varepsilon \ge 0$ (or fractional Riemann–Liouville integral of the order $-\varepsilon > 0$ in the case where $\varepsilon < 0$) with respect to t, the velocity $v = (v_1, v_2, \dots, v_d)$ and the pressure gradient $r = (r_1, r_2, \dots, r_d) = \nabla p$ are unknown, and $h : \Omega \times [0,T] \to \mathbb{R}^d$ is a given function.

If $\alpha = \beta = 1$, $\gamma = 0$ and $\delta < 0$, then the system of Equations (13) and (14) is the linearization for the generalized Oskolkov system of the viscoelastic fluid dynamics with the kernel $h(s,t) = \kappa(t-s)^{-\delta-1}/\Gamma(-\delta)$ in the integral operator (see system (2.1.1), (2.1.2) in [33]). With $\alpha = 1$, $\beta > 0$, $\gamma = 0$ and $\kappa = 0$, it will be the linearized Kelvin–Voigt fluid system [34,35]. If, moreover, $\nu = 0$, then (13), (14) is the linearized system of the Scott-Blair fluid dynamics.

With $\mathbb{L}_2 := (L_2(\Omega))^n$, $\mathbb{H}^1 := (H^1(\Omega))^n$, $\mathbb{H}^2 := (H^2(\Omega))^n$, the closure of the subspace $\mathcal{L} := \{z \in (C_0^\infty(\Omega))^n : \nabla \cdot z = 0\}$ in the norm of the space $\mathbb{L}_2$ will be denoted by $\mathbb{H}_\sigma$, and in the norm of $\mathbb{H}^1$, it will be denoted by $\mathbb{H}_\sigma^1$. We denote $\mathbb{H}_\sigma^2 := \mathbb{H}_\sigma^1 \cap \mathbb{H}^2$, where $\mathbb{H}_\pi$ is the orthogonal complement for $\mathbb{H}_\sigma$ in $\mathbb{L}_2$ and $\Sigma : \mathbb{L}_2 \to \mathbb{H}_\sigma$, $\Pi = I - \Sigma$ are the corresponding orthoprojectors.

The operator $B := \Sigma\Delta$, extended to a closed operator in $\mathbb{H}_\sigma$ with the domain $\mathbb{H}_\sigma^2$, has a real negative discrete spectrum with finite multiplicities, which is condensed only at $-\infty$ [36].

The system of Equations (13) and (14) is equivalent to the equation

$$D_t^\alpha v(s,t) = \chi D_t^\beta B v(s,t) + \nu D_t^\gamma B v(s,t) + \kappa D_t^\delta B v(s,t) + \Sigma h(s,t),\ (s,t) \in \Omega \times (0,T], \tag{15}$$

since

$$r(s,t) = \chi D_t^\beta \Pi\Delta v(s,t) + \nu D_t^\gamma \Pi\Delta v(s,t) + \kappa D_t^\delta \Pi\Delta v(s,t) + \Pi h(s,t), \quad (s,t) \in \Omega \times (0,T].$$

Therefore, we need to study problem (11), (12), (15). If $\alpha > \beta > \gamma > \delta$, $m - 1 < \alpha \le m \in \{1,2\}$, $\beta > -\alpha$, $\chi > 0$ and $\nu, \kappa \in \mathbb{R}$. Due to incompressibility Equation (14) take $\mathcal{X} = \mathbb{H}_\sigma$, $A_1 = \kappa B$, $A_2 = \nu B$ and $A_3 = \chi B$ are closed, densely defined operators. Then, by Lemma 3 from [31], $(A_1, A_2, A_3) \in \mathcal{A}_{\alpha,G}^3$, and by Theorem 2, for any $v_0, v_1 \in D = \mathbb{H}_\sigma^2$, $\Sigma h \in C([0,T]; \mathbb{H}_\sigma^2)$, there exist a unique solution to problem (11), (12), (15). Therefore, problem (11)–(14) also have a unique solution.

If $\beta > \alpha > \gamma > \delta$, $m-1 < \alpha \leq m \in \mathbb{N}$ and $\chi, \nu, \kappa \in \mathbb{R}$, we rewrite Equation (15) into the form

$$D_t^{\beta} v(s,t) = \chi^{-1} D_t^{\alpha} B^{-1} v(s,t) - \chi^{-1} \nu D_t^{\gamma} v(s,t) - \chi^{-1} \kappa D_t^{\delta} v(s,t) - B^{-1} \Sigma h(s,t)$$

for $(s,t) \in \Omega \times (0,T]$. By setting $\mathcal{X} = \mathbb{H}_{\sigma}$, $A_1 = \chi^{-1} D_t^{\alpha} B^{-1}$, $A_2 = -\chi^{-1} \nu I$ and $A_3 = -\chi^{-1} \kappa I$, and by Theorem 3, since A_1, A_2, A_3 are bounded operators, for any $v_0, v_1 \in \mathbb{H}_{\sigma}$, $B^{-1}\Sigma h \in C([0,T]; \mathbb{H}_{\sigma})$, there exist a unique solution to problem (11)–(14).

Now, consider the initial boundary value problem

$$v(s,0) = v_0(s), \quad (m-1) D_t^1 v(s,0) = (m-1) v_1(s), \quad s \in \Omega, \tag{16}$$

$$\tau(s,0) = \tau_0(s), \quad (m-1) D_t^1 \tau(s,0) = (m-1) \tau_1(s), \quad s \in \Omega, \tag{17}$$

$$v(s,t) = 0, \quad \tau(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0,T], \tag{18}$$

for the linearized system of the thermoconvection in the same medium

$$D_t^{\alpha} v(s,t) = \chi D_t^{\alpha} \Delta v(s,t) + \nu \Delta v(s,t) + \kappa D_t^{\delta} \Delta v(s,t) - r(s,t) + h(s,t), \quad (s,t) \in \Omega \times (0,T], \tag{19}$$

$$\nabla \cdot v(s,t) = 0, \quad (s,t) \in \Omega \times (0,T], \tag{20}$$

$$D_t^{\alpha} \tau(s,t) = \varrho \triangle \tau(s,t) + \varsigma v_n(s,t) + f(s,t), \quad (s,t) \in \Omega \times (0,T]. \tag{21}$$

where $m-1 < \alpha \leq m \in \{1,2\}$, $\delta < 0$, $\chi, \nu, \kappa, \varrho, \varsigma \in \mathbb{R}$ and $\triangle$ is the Laplace operator with the domain $H_0^2(\Omega) := \{w \in H^2(\Omega) : w(x) = 0,\ x \in \partial\Omega\}$, which is dense in $L_2(\Omega)$.

Remark 6. *If $\chi = 0$, then system of Equations (19)–(21) is the linear approximation of the thermoconvection in viscous media and not in viscoelastic media. In part, for $\chi = 0$, $\alpha = 1$ and $\kappa = 0$, we have the linearization of the Boussinesq system, which models the thermoconvection in viscous media. Operator methods close to the methods of this work are used for studying an initial boundary value problem and some control problems of the linearized Boussinesq system in* [37].

Set

$$\mathcal{X} = \mathbb{H}_{\sigma}^2 \times \mathbb{H}_{\pi} \times L_2(\Omega), \quad \mathcal{Y} = \mathbb{L}_2 \times L_2(\Omega) = \mathbb{H}_{\sigma} \times \mathbb{H}_{\pi} \times L_2(\Omega), \tag{22}$$

$$L = \begin{pmatrix} I - \chi B & \mathbb{O} & \mathbb{O} \\ -\chi \Pi \Delta & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & I \end{pmatrix}, \quad M_1 = \begin{pmatrix} \kappa B & \mathbb{O} & \mathbb{O} \\ \kappa \Pi \Delta & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \nu B & \mathbb{O} & \mathbb{O} \\ \nu \Pi \Delta & -I & \mathbb{O} \\ \varsigma P_n & \mathbb{O} & \varrho \triangle \end{pmatrix}, \tag{23}$$

$$g(t) = \begin{pmatrix} \Sigma h(\cdot,t) \\ \Pi h(\cdot,t) \\ f(\cdot,t) \end{pmatrix}, \quad t \in [0,T].$$

Here, P_n is the projector $(v_1, v_2, \ldots, v_n) \to v_n$. Then, $L, M_1 \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $M_2 \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$ and $D_{M_2} = \mathbb{H}_{\sigma}^2 \times \mathbb{H}_{\pi} \times H_0^2(\Omega)$. We have $x(t) \in \mathcal{X}$, where $x(t) = (v(\cdot,t), r(\cdot,t), \tau(\cdot,t))$.

Lemma 3. *Let $\alpha \in (0,2)$, $\chi, \nu, \varsigma \in \mathbb{R}$, $\chi \neq 0$, $\chi^{-1} \notin \sigma(B)$, $\varrho > 0$, spaces $\mathcal{X}$ and $\mathcal{Y}$ have form (22), and operators L and M_2 be defined by (23). Then, $(L, M_2) \in \mathcal{H}_{\alpha}(\theta_0, a_0)$ for some $a_0 \geq 0$, $\theta_0 \in (\pi/2, \pi)$, and in this case, we have*

$$P = \begin{pmatrix} I & \mathbb{O} & \mathbb{O} \\ \nu \Pi \Delta (I - \chi B)^{-1} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & I \end{pmatrix}, \quad Q = \begin{pmatrix} I & \mathbb{O} & \mathbb{O} \\ -\chi \Pi \Delta (I - \chi B)^{-1} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & I \end{pmatrix},$$

where $\mathcal{X}^0 = \{0\} \times \mathbb{H}_{\pi} \times \{0\}$, $\mathcal{X}^1 = \{(z, \nu \Pi \Delta (I - \chi B)^{-1} z, w) : z \in \mathbb{H}_{\sigma}^2,\ w \in L_2(\Omega)\}$, $\mathcal{Y}^0 = \{0\} \times \mathbb{H}_{\pi} \times \{0\}$ and $\mathcal{Y}^1 = \{(z, -\chi \Pi \Delta (I - \chi B)^{-1} z, w) : z \in \mathbb{H}_{\sigma},\ w \in L_2(\Omega)\}$.

Proof. The Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ are reflexive since they are Hilbert spaces. The operators $(I-\chi B)^{-1}:\mathbb{H}_\sigma\to\mathbb{H}^2_\sigma$, $(I-\chi B)^{-1}B = B(I-\chi B)^{-1}:\mathbb{H}_\sigma\to\mathbb{H}_\sigma$ and $(I-\chi B)^{-1}B = B(I-\chi B)^{-1}:\mathbb{H}^2_\sigma\to\mathbb{H}^2_\sigma$ are bounded. Therefore, we can choose $\theta_1\in(\pi/2,\pi)$, $a_0>0$ such that the disc $\{\mu\in\mathbb{C}: |\mu|\le 2^{-1/\alpha}|\nu|^{1/\alpha}\max\{\|(I-\chi B)^{-1}B\|^{1/\alpha}_{\mathbb{H}_\sigma}, \|(I-\chi B)^{-1}B\|^{1/\alpha}_{\mathbb{H}^2_\sigma}\}\}$ is situated outside the sector S_{θ_1,a_0}. Then, for $\mu\in S_{\theta_1,a_0}$, using the Neumann series, we obtain

$$\|(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1}\|_{\mathbb{H}_\sigma}\le\frac{1}{|\mu|^\alpha-|\nu|\|(I-\chi B)^{-1}B\|_{\mathbb{H}_\sigma}}\le\frac{2}{|\mu|^\alpha}. \tag{24}$$

$$\|(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1}\|_{\mathbb{H}^2_\sigma}\le\frac{1}{|\mu|^\alpha-|\nu|\|(I-\chi B)^{-1}B\|_{\mathbb{H}^2_\sigma}}\le\frac{2}{|\mu|^\alpha}. \tag{25}$$

Now, we take $\alpha\in[1,2)$, $\delta\in(0,\pi(1/\alpha-1/2))$ and $\theta_0=\min\{\theta_1,\pi/2+\delta\}$. Then, $(\mu^\alpha I-\varrho\triangle)^{-1}\in\mathcal{L}(\mathbb{H}_\sigma)$ for all $\mu\in S_{\theta_0,a_0}$, since $|\arg\mu^\alpha|\in(\pi/2,\pi)$ and the spectrum of the operator $\varrho\triangle$ is real and negative. Moreover, for $w\in L_2(\Omega)$, we have

$$\|(\mu^\alpha I-\varrho\triangle)^{-1}w\|^2_{L_2(\Omega)}=\sum_{k=0}^\infty\frac{|\langle w,\varphi_k\rangle|^2}{|\mu^\alpha-\varrho\lambda_k|^2}\le\frac{\|w\|^2_{L_2(\Omega)}}{\sin^2\theta_0|\mu|^{2\alpha}}, \tag{26}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(\Omega)$, $\{\lambda_k\}$ is the eigenvalues of $\triangle$ and $\{\varphi_k\}$ is the orthonormal system of the corresponding eigenfunctions.

Thus, for $\mu\in S_{\theta_0,a_0}$, we have

$$\mu^\alpha L-M_2=\begin{pmatrix}\mu^\alpha(I-\chi B)-\nu B & \mathbb{O} & \mathbb{O}\\ -\mu^\alpha\chi\Pi\Delta-\nu\Pi\Delta & I & \mathbb{O}\\ -\varsigma P_n & \mathbb{O} & \mu^\alpha I-\varrho\triangle\end{pmatrix},$$

$$(\mu^\alpha L-M)^{-1}=$$

$$=\begin{pmatrix}(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1}(I-\chi B)^{-1} & \mathbb{O} & \mathbb{O}\\ (\mu^\alpha\chi\Pi\Delta+\nu\Pi\Delta)(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1}(I-\chi B)^{-1} & I & \mathbb{O}\\ \varsigma(\mu^\alpha I-\varrho\triangle)^{-1}P_n(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1}(I-\chi B)^{-1} & \mathbb{O} & (\mu^\alpha I-\varrho\triangle)^{-1}\end{pmatrix},$$

$$R^L_{\mu^\alpha}(M)=\begin{pmatrix}(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1} & \mathbb{O} & \mathbb{O}\\ (\mu^\alpha\chi\Pi\Delta+\nu\Pi\Delta)(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1}-\chi\Pi\Delta & \mathbb{O} & \mathbb{O}\\ \varsigma(\mu^\alpha I-\varrho\triangle)^{-1}P_n(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1} & \mathbb{O} & (\mu^\alpha I-\varrho\triangle)^{-1}\end{pmatrix}=$$

$$=\begin{pmatrix}(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1} & \mathbb{O} & \mathbb{O}\\ \nu\Pi\Delta(I-\chi B)^{-1}(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1} & \mathbb{O} & \mathbb{O}\\ \varsigma(\mu^\alpha I-\varrho\triangle)^{-1}P_n(\mu^\alpha I-\nu(I-\chi B)^{-1}B)^{-1} & \mathbb{O} & (\mu^\alpha I-\varrho\triangle)^{-1}\end{pmatrix},$$

$$L^L_{\mu^\alpha}(M)=\begin{pmatrix}(\mu^\alpha I-\nu B(I-\chi B)^{-1})^{-1} & \mathbb{O} & \mathbb{O}\\ -\chi\Pi\Delta(I-\chi B)^{-1}(\mu^\alpha I-\nu B(I-\chi B)^{-1})^{-1} & \mathbb{O} & \mathbb{O}\\ \varsigma(\mu^\alpha I-\varrho\triangle)^{-1}P_n(I-\chi B)^{-1}(\mu^\alpha I-\nu B(I-\chi B)^{-1})^{-1} & \mathbb{O} & (\mu^\alpha I-\varrho\triangle)^{-1}\end{pmatrix}.$$

Thus, $R^L_{\mu^\alpha}(M)\in\mathcal{L}(\mathcal{X})$ and $L^L_{\mu^\alpha}(M)\in\mathcal{L}(\mathcal{Y})$. Using inequalities (24)–(26), we obtain that $(L,M)\in\mathcal{H}_\alpha(a_0,\theta_0)$.

For $\alpha\in(0,1)$, the proof is similar..

The projectors P and Q and subspaces $\mathcal{X}^0=\ker P$, $\mathcal{X}^1=\operatorname{im}P$, $\mathcal{Y}^0=\ker Q$ and $\mathcal{Y}^1=\operatorname{im}Q$ can be calculated using Theorem 1 (2): □

Remark 7. *It is evident that in this case,* $L_1^{-1}\in\mathcal{L}(\mathcal{Y}^1;\mathcal{X}^1)$.

Theorem 6. *Let* $\alpha\in(0,2)$, $\delta<0$, $\chi,\nu,\kappa,\varsigma\in\mathbb{R}$, $\chi\ne0$, $\chi^{-1}\notin\sigma(B)$, $\varrho>0$, $v_0\in\mathbb{H}_\sigma$ *and* $\tau_0\in H^2(\Omega)$ *for* $\alpha\in(0,1]$, *and* $v_0,v_1\in\mathbb{H}_\sigma$, $\tau_0,\tau_1\in H^2(\Omega)$ *for* $\alpha\in(1,2)$; $h\in C([0,T];\mathbb{L}_2)$, $\Sigma h\in C([0,T];\mathbb{H}^2_\sigma)$ *and* $f\in C([0,T];H^2(\Omega))$. *Then, there exist a unique solution to problem* (16)–(21).

Proof. We reduce problem (16)–(21) to problem (5), (6) with $n = 2$, using operators (23) in spaces (22). Note that in this case, $\alpha_1 = \delta < 0$, $\alpha_2 = 0$ and $m_2 = 0$. Hence, conditions (5) have the form $Px(0) = x_0$ for $\alpha \in (0,1]$, $Px(0) = x_0$ and $D^1 Px(0) = x_1$ for $\alpha \in (1,2)$, which are equivalent to conditions (16) and (17) due to the form of the projector P (see Lemma 3). Here, $m = 1$ for $\alpha \in (0,1]$ and $m = 2$ for $\alpha \in (1,2)$. Therefore, for $m = 1$, the second condition in (16) and in (17) is absent.

According to Remark 7, $L_1^{-1} \in \mathcal{L}(\mathcal{Y}^1; \mathcal{X}^1)$, and moreover, $D_{M_{n,1}L_1^{-1}} = L[D_{M_n}] = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times H_0^2(\Omega)$. Hence, $(v_0, \nu\Pi\Delta(I-\chi B)^{-1}v_0, \tau_0), (v_1, \nu\Pi\Delta(I-\chi B)^{-1}v_1, \tau_1) \in D_{M_{n,1}L_1^{-1}}$ under the conditions of the present theorem. We also have $Qg(t) = (\Sigma h(\cdot,t), -\chi\Pi\Delta(I - \chi B)^{-1}\Sigma h(\cdot,t), f(\cdot,t)) \in C([0,T]; D_{M_{n,1}L_1^{-1}})$. Finally, we have

$$M_1 P - QM_1 = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \kappa\Pi\Delta(I-\chi B)^{-1} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix} := N_1 \in \mathcal{L}(\mathcal{X}; \mathcal{Y}).$$

It is obvious that $N_1 = Q_0 N_1 P$. Under Theorem 5, we obtain the required statement. □

5. Conclusions

An initial value problem for a class of degenerate multi-term linear equations in Banach spaces with Gerasimov–Caputo derivatives was studied by the methods of pairs of invariant subspaces. Under the conditions of the operators at the two oldest derivatives, by implying the existence of pairs of invariant subspaces and analytic resolving families of operators for the linear homogeneous equation with these two operators, we reduced the degenerate equation to a system of two nondegenerate equations in the subspaces. This allowed us to prove the existence of a unique solution. The obtained abstract unique solvability theorem was used for the research of the initial boundary value problems for the systems of the dynamics and of the thermoconvection of the Kelvin–Voigt-type media.

As for the development of the results obtained and their significance, we note that the results for the solvability of initial problem (5), (6) will further allow us to consider other problems for Equation (6) (boundary value problems on a segment, nonlocal problems, etc.). Aside from that, the proof of the solvability theorem (Theorem 5), coupled with solution formula (4) for the nondegenerate equation, gives the form of a solution to the degenerate equation, which can become a starting point for finding new methods for the numerical solutions of initial boundary value problem (16)–(21).

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References

1. Oldham, K.B.; Spanier, J. *The Fractional Calculus*; Academic Press: Boston, MA, USA, 1974.
2. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach Science Publishers: Philadelphia, PA, USA, 1993.
3. Hilfer, R. *Applications of Fractional Calculus in Physics*; WSPC: Singapore, 2000.
4. Nakhushev, A.M. *Fractional Calculus ant Its Applications*; Fizmatlit: Moscow, Russia, 2003. (In Russian)

5. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*; Springer: New York, NY, USA, 2011.
6. Uchaikin, V.V. *Fractional Derivatives for Physicists and Engineers, Volume I Background and Theory, Volume II Applications*; Springer: Berlin/Heidelberg, Germany, 2013.
7. Sibatov, R.T.; Uchaikin, V.V. Fractional kinetics of charge carriers in supercapacitors. In *Handbook of Fractional Calculus with Applications*; Tenreiro Machado, J.A., Ed.; De Gruyter: Berlin, Germany; Boston, MA, USA, 2019.
8. Thach, T.N.; Tuan, N.H. Stochastic pseudo-parabolic equations with fractional derivative and fractional Brownian motion. *Stoch. Anal. Appl.* **2021**, *40*, 1906274. [CrossRef]
9. Tuan, N.H.; Phuong, N.D.; Thach, T.N. New well-posedness results for stochastic delay Rayleigh—Stokes equations. *Discret. Contin. Dyn. Syst. Ser. B* **2022**, *28*, 347–358. [CrossRef]
10. Nishimoto, K. *Fractional Calculus and Its Applications*; Nihon University: Koriyama, Japan, 1990.
11. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley & Sons: Hoboken, NJ, USA, 1993.
12. Podlubny, I. *Fractional Differential Equations*; Academic Press: Boston, MA, USA, 1999.
13. Pskhu, A.V. *Partial Differential Equations of Fractional Order*; Nauka: Moscow, Russia, 2005. (In Russian)
14. Kilbas, A.A.; Srivastava H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
15. Diethelm, K. *The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer: Berlin/Heidelberg, Germany, 2010.
16. Caroll, R.W.; Showalter, R.E. *Singular and Degenerate Cauchy Problems*; Academic Press: New York, NY, USA; San Francisco, CA, USA, 1976.
17. Favini, A.; Yagi, A. *Degenerate Differential Equations in Banach Spaces*; Marcel Dekker Inc.: New York, NY, USA, 1999.
18. Pyatkov, S.G. *Operator Theory: Nonclassical Problems*; VSP: Utrecht, The Netherlands, 2002.
19. Sidorov, N.; Loginov, B.; Sinitsyn, A.; Falaleev, M. *Lyapunov—Schmidt Method in Nonlinear Analysis and Applications*; Kluwer Academic Publishers: Dordrecht, Germany, 2002.
20. Demidenko, G.V.; Uspenskii, S.V. *Partial Differential Equations and Systems Not Solvable with Respect to the Highest-Order Derivative*; CRC Press: Boca Raton, FL, USA, 2003.
21. Sviridyuk, G.A.; Fedorov, V.E. *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*; VSP: Utrecht, The Netherlands, 2003.
22. Kostić, M. *Abstract Volterra Integro-Differential Equations*; CRC Press: Boca Raton, FL, USA, 2015.
23. Plekhanova, M.V. Strong solutions of quasilinear equations in Banach spaces not solvable with respect to the highest-order derivative. *Discret. Contin. Dyn. Syst. Ser.* **2016**, *9*, 833–847. [CrossRef]
24. Plekhanova, M.V. Distributed control problems for a class of degenerate semilinear evolution equations. *J. Comput. Appl. Math.* **2017**, *312*, 39–46. [CrossRef]
25. Fedorov, V.E.; Romanova, E.A.; Debbouche, A. Analytic in a sector resolving families of operators for degenerate evolution fractional equations. *J. Math. Sci.* **2018**, *228*, 380–394. [CrossRef]
26. Fedorov, V.E.; Avilovich, A.S. A Cauchy type problem for a degenerate equation with the Riemann—Liouville derivative in the sectorial case. *Sib. Math. J.* **2019**, *60*, 359–372. [CrossRef]
27. Debbouche, A.; Fedorov, V.E. A class of fractional degenerate evolution equations with delay. *Mathematics* **2020**, *8*, 1700. [CrossRef]
28. Fedorov, V.E.; Plekhanova, M.V.; Izhberdeeva, E.M. Initial value problems of linear equations with the Dzhrbashyan—Nersesyan derivative in Banach spaces. *Symmetry* **2021**, *13*, 1058. [CrossRef]
29. Fedorov, V.E.; Turov, M.M. The defect of a Cauchy type problem for linear equations with several Riemann—Liouville derivatives. *Sib. Math. J.* **2021**, *62*, 925–942. [CrossRef]
30. Fedorov, V.E.; Boyko, K.V.; Phuong, T.D. Initial value problems for some classes of linear evolution equations with several fractional derivatives. *Math. Notes Nefu* **2021**, *28*, 85–104.
31. Boyko, K.V.; Fedorov, V.E. The Cauchy problem for a class of multi-term equations with Gerasimov—Caputo derivatives. *Lobachevskii J. Math.* **2022**, *43*, 1293–1302. [CrossRef]
32. Bajlekova, E.G. Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 2001.
33. Zvyagin V.G., Turbin M.V. The study of initial-boundary value problems for mathematical models of the motion of Kelvin — Voigt fluids. *J. Math. Sci.* **2010**, *168*, 157–308. [CrossRef]
34. Oskolkov, A.P. Initial-boundary value problems for equations of motion of Kelvin—Voight fluids and Oldroyd fluids. *Proc. Steklov Inst. Math.* **1989**, *179*, 137–182.
35. Mainardi, F.; Spada, G. Creep, relaxation and viscosity properties for basic fractional models in rheology. *Eur. Phys. J. Spec. Top.* **2011**, *193*, 133–160. [CrossRef]
36. Ladyzhenskaya, O.A. *The Mathematical Theory of Viscous Incompressible Flow*; Gordon and Breach Science Publishers: New York, NY, USA, 1963.
37. Plekhanova, M.V.; Islamova, A.F. Problems with a robust mixed control for the linearized Boussinesq equation. *Differ. Equ.* **2012**, *48*, 574–585. [CrossRef]