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Article

Initial Problem for Two-Dimensional Hyperbolic Equation with a Nonlocal Term

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Abstract: In this paper, we study the Cauchy problem in a strip for a two-dimensional hyperbolic equation containing the sum of a differential operator and a shift operator acting on a spatial variable that varies over the real axis. An operating scheme is used to construct the solutions of the equation. The solution of the problem is obtained in the form of a convolution of the function found using the operating scheme and the function from the initial conditions of the problem. It is proved that classical solutions of the considered initial problem exist if the real part of the symbol of the differential-difference operator in the equation is positive.

Keywords: hyperbolic equation; differential-difference equation; initial problem; Fourier transform; operational scheme

MSC: 5L15; 42A38

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1. Introduction

In recent years, functional-differential equations, or, their special case, differential equations with a deviating argument, have become widespread in applications of mathematics. The systematic study of equations with a deviating argument began in the 1940s in connection with applications to automatic control theory and it was associated with the research by Pinney [1], Bellman and Cooke [2], Hale [3], and other authors.

Interest in problems for differential-difference equations is due to their numerous applications: in the mechanics of a deformable solid body; in relativistic electrodynamics; when studying the processes of vortex formation and the formation of complex coherent spots; when solving some problems related to plasma; in simulation of vibrations of the crystal lattice; in problems of nonlinear optics; in the study of neural networks; when studying models of population dynamics in mathematical biology; in the study of environmental and economic processes; in a wide range of tasks in the theory of automatic control; when solving problems of optimizing the treatment of oncological diseases (see, for example, works [4–6]); etc.

Differential-difference equations form a special class of functional-differential equations for which the theory of boundary value problems is currently developed. Problems for elliptic differential-difference equations in bounded domains have been studied quite comprehensively by now; the theory for such equations was created and developed by Skubachevskii [7,8].

Problems for parabolic and hyperbolic differential-difference equations have been studied to a much lesser extent [9–11].

As far as the authors know, at present, there are few papers dealing with hyperbolic differential-difference equations containing shifts with respect to the spatial variable. In [12–14], the families of classical solutions are constructed for two-dimensional hyperbolic equations with shifts in the only space variable x ranging over the real line; the shifts occur either in the potentials or in the highest derivative. Some similar problems for elliptic equations were studied in [15,16].

In this paper, we study the solvability of the Cauchy problem in a strip for a two-dimensional hyperbolic equation with a nonlocal potential.

Let $D = \{(x,t) : x \in \mathbf{R}, 0 < t < T\}$ be the coordinate plane area Oxt, where $T > 0$ is the given real number, $\overline{D} = \{(x,t) : x \in \mathbf{R}, 0 \le t \le T\}$. Let us consider in the domain D the hyperbolic differential-difference equation, which contains the sum of the differential operator and the shift operator with respect of the spatial variable x:

$$Lu \stackrel{\text{def}}{=} \frac{\partial^2 u(x,t)}{\partial t^2} - a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + b\, u(x-h,t) = 0, \tag{1}$$

where $a, b > 0$, $h \neq 0$ are given real numbers.

Suppose that for all $\xi \in \mathbf{R}$ the inequality

$$a^2\xi^2 + b\cos(h\xi) > 0, \tag{2}$$

holds.

Inequality (2) means that the real part of the symbol of the differential-difference operator in Equation (1) is positive.

Consider the function $a^2\xi^2 + b\cos(h\xi)$, $\xi \in [0, +\infty)$. The derivative of this function is

$$2a^2\xi - bh\sin(h\xi) = 2a^2\xi\left(1 - \frac{bh^2}{2a^2}\frac{\sin(h\xi)}{h\xi}\right).$$

Since $\sin(h\xi)/h\xi \longrightarrow 1$ at $\xi \longrightarrow 0$, and $\sin(h\xi)/h\xi \longrightarrow 0$ at $\xi \to +\infty$, then the derivative is non-negative on the interval $\xi \in [0, +\infty)$ if

$$0 < b \le \frac{2a^2}{h^2}. \tag{3}$$

In this case, the function $a^2\xi^2 + b\cos(h\xi)$ at $\xi \in [0, +\infty)$ is non-decreasing and its smallest value is equal to $b > 0$; hence,

$$a^2\xi^2 + b\cos(h\xi) \ge b > 0, \tag{4}$$

for all $\xi \in [0, +\infty)$.

Since the function $a^2\xi^2 + b\cos(h\xi)$ is even, this value b is the smallest for all real $\xi \in (-\infty, +\infty)$. Thus, the condition (2) is satisfied if the coefficients a, b and the shift h of the equality (1) satisfy the inequalities (3).

Formulation of the problem. Find a function $u(x,t)$ that satisfies the conditions

$$u(x,t) \in C^1(\overline{D}) \cap C^2(D); \tag{5}$$

$$Lu(x,t) \equiv 0, \quad (x,t) \in D; \tag{6}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = 0, \quad x \in \mathbf{R}, \tag{7}$$

where the initial function satisfies the conditions $u_0(x) \in L_1(\mathbf{R})$, and $u_0(x) \in C^1(\mathbf{R})$.

Definition 1. *A function $u(x,t)$ is called a classical solution of the problem (5)–(7), if*

- *It is continuous and continuously differentiable with respect to the variables x and t in the set $\overline{D}$;*
- *It has continuous derivatives u_{xx} and u_{tt} in the domain D;*
- *It satisfies at each point $(x,t) \in D$ Equation (1);*
- *For each point $x_0 \in (-\infty, +\infty)$, the limits of the functions $u(x_0,t) - u_0(x_0)$ and $u_t(x_0,t)$ at $t \longrightarrow 0+$ exists and are equal to zero.*

This paper studies the initial problem (5)–(7) for two-dimensional hyperbolic equation with a nonlocal term. The solution of the problem is obtained in the form of a convolution of the function found by using the operating scheme and the initial conditions (7).

2. Construction of Solutions of the Equation

The fundamental solution of a linear differential operator L with constant coefficients is a generalized function $\mathcal{E}(x,t)$, that satisfies the equation

$$L\mathcal{E}(x,t) \equiv \frac{\partial^2 \mathcal{E}(x,t)}{\partial t^2} - a^2 \frac{\partial^2 \mathcal{E}(x,t)}{\partial x^2} + b\,\mathcal{E}(x-h,t) = \delta(x,t), \tag{8}$$

where $\delta(x,t)$ is the Dirac δ-function.

We formally apply the Fourier transform with respect to the variable x to Equation (1), and passes to the dual variable ξ. For the function $\widehat{\mathcal{E}}(\xi,t) := F_x[\mathcal{E}](\xi,t)$, we obtain the equation

$$\frac{\partial^2 \widehat{\mathcal{E}}(x,t)}{\partial t^2} + \left(a^2\xi^2 + b\,e^{ih\xi}\right)\widehat{\mathcal{E}}(x,t) = 1(\xi)\delta(t). \tag{9}$$

The solution of Equation (9) has the form

$$\widehat{\mathcal{E}}(\xi,t) = \theta(t)Z(t), \tag{10}$$

where $\theta(t)$ is the Heaviside step function and the function $Z(t)$ satisfies the equation

$$Z''(t) + \left(a^2\xi^2 + b\,e^{ih\xi}\right)Z(t) = 0, \tag{11}$$

with the initial conditions

$$Z(0) = 0, \quad Z'(0) = 1. \tag{12}$$

The characteristic equation for Equation (11) has the roots

$$k_{1,2} = \pm\sqrt{-\left(a^2\xi^2 + b\,e^{ih\xi}\right)} = \pm\, i\sqrt{a^2\xi^2 + b\,e^{ih\xi}} = \pm\, i\rho(\xi)e^{i\,\varphi(\xi)},$$

where the functions $\rho(\xi)$ and $\varphi(\xi)$ are denoted by

$$\rho(\xi) := \left[\left(a^2\xi^2 + b\cos(h\xi)\right)^2 + b^2\sin^2(h\xi)\right]^{1/4}, \tag{13}$$

$$\varphi(\xi) := \frac{1}{2}\mathrm{arctg}\frac{b\sin(h\xi)}{a^2\xi^2 + b\cos(h\xi)}. \tag{14}$$

Note that, whenever the condition (2) is satisfied, Functions (13) and (14) are defined correctly.

The general solution of Equation (11) is determined by the formula

$$Z(t) = C_1(\xi)\cos\left(\rho(\xi)[\cos\varphi(\xi) + i\sin\varphi(\xi)]t\right) + C_2(\xi)\sin\left(\rho(\xi)[\cos\varphi(\xi) + i\sin\varphi(\xi)]t\right),$$

where $C_1(\xi)$ and $C_2(\xi)$ are arbitrary constants depending on the parameter ξ. To find these constants, substitute we substitute the last expression to the conditions (12):

$$C_1(\xi) = 0, \quad C_2(\xi) = \frac{1}{\rho(\xi)[\cos\varphi(\xi) + i\sin\varphi(\xi)]}.$$

As a result, the solution to the problem (11), (12) has the form

$$Z(t) = \frac{\sin\left(\rho(\xi)[\cos\varphi(\xi) + i\sin\varphi(\xi)]t\right)}{\rho(\xi)[\cos\varphi(\xi) + i\sin\varphi(\xi)]}.$$

Taking into account Equation (10), the solution of Equation (9) is determined by the formula

$$\widehat{\mathcal{E}}(\xi,t)=\theta(t)\frac{\sin\left(\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]t\right)}{\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]}.$$

Applying the inverse Fourier transform F_{ξ}^{-1} to the last expression, we obtain

$$\begin{aligned}\mathcal{E}(x,t)&=\theta(t)F_{\xi}^{-1}\left[\frac{\sin\left(\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]t\right)}{\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]}\right]\\&=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\sin\left(\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]t\right)}{\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]}e^{-ix\xi}d\xi\\&=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\sin\left(\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]t\right)}{\rho(\xi)}e^{-i(\varphi(\xi)+x\xi)}d\xi.\end{aligned}$$

Transform this expression using the equalities $\rho(-\xi)=\rho(\xi)$ and $\varphi(-\xi)=\varphi(\xi)$:

$$\begin{aligned}\mathcal{E}(x,t)=&\frac{1}{2\pi}\int\limits_{-\infty}^{0}\frac{\sin\left(\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]t\right)}{\rho(\xi)}e^{-i(\varphi(\xi)+x\xi)}d\xi\\&+\frac{1}{2\pi}\int\limits_{0}^{+\infty}\frac{\sin\left(\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]t\right)}{\rho(\xi)}e^{-i(\varphi(\xi)+x\xi)}d\xi\\&=\frac{1}{2\pi}\int\limits_{0}^{+\infty}\frac{\sin\left(\rho(\xi)[\cos\varphi(\xi)-i\sin\varphi(\xi)]t\right)}{\rho(\xi)}e^{i(\varphi(\xi)+x\xi)}d\xi\\&+\frac{1}{2\pi}\int\limits_{0}^{+\infty}\frac{\sin\left(\rho(\xi)[\cos\varphi(\xi)+i\sin\varphi(\xi)]t\right)}{\rho(\xi)}e^{-i(\varphi(\xi)+x\xi)}d\xi\\=&\frac{1}{\pi}\int\limits_{0}^{+\infty}\frac{1}{\rho(\xi)}[\sin\left(t\,\rho(\xi)\cos\varphi(\xi)\right)\cos\left(\varphi(\xi)+x\xi\right)\cos\left(i\,t\,\rho(\xi)\sin\varphi(\xi)\right)\\&-i\cos\left(t\,\rho(\xi)\cos\varphi(\xi)\right)\sin\left(\varphi(\xi)+x\xi\right)\sin\left(i\,t\,\rho(\xi)\sin\varphi(\xi)\right)]d\xi.\end{aligned}$$

Define the functions

$$G_1(\xi):=\rho(\xi)\sin\varphi(\xi),\quad G_2(\xi):=\rho(\xi)\cos\varphi(\xi). \tag{15}$$

Since $\cos(ix)=\mathrm{ch}x$ and $-i\sin(ix)=\mathrm{sh}x$, we can write $\mathcal{E}(x,t)$ in the form

$$\begin{aligned}\mathcal{E}(x,t)=&\frac{1}{\pi}\int\limits_{0}^{+\infty}\frac{1}{\rho(\xi)}[\sin\left(t\,G_2(\xi)\right)\cos\left(\varphi(\xi)+x\xi\right)\mathrm{ch}(t\,G_1(\xi))\\&+\cos\left(t\,G_2(\xi)\right)\sin\left(\varphi(\xi)+x\xi\right)\mathrm{sh}(t\,G_1(\xi))]d\xi\\=&\frac{1}{2\pi}\int\limits_{0}^{+\infty}\frac{1}{\rho(\xi)}\Big[[\sin\left(t\,G_2(\xi)\right)\cos\left(\varphi(\xi)+x\xi\right)+\cos\left(t\,G_2(\xi)\right)\sin\left(\varphi(\xi)+x\xi\right)]e^{t\,G_1(\xi)}\\&+[\sin\left(t\,G_2(\xi)\right)\cos\left(\varphi(\xi)+x\xi\right)-\cos\left(t\,G_2(\xi)\right)\sin\left(\varphi(\xi)+x\xi\right)]e^{-t\,G_1(\xi)}\Big]d\xi\\=&\frac{1}{2\pi}\int\limits_{0}^{+\infty}\frac{1}{\rho(\xi)}\Big[\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)e^{t\,G_1(\xi)}+\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\xi\right)e^{-t\,G_1(\xi)}\Big]d\xi.\end{aligned}$$

We will use the resulting integral to construct a solution to the system (5)–(7).

We introduce a weight function $A(\xi)$ (according to [15]), that is continuous, non-negative for each $\xi \in [0, +\infty)$, and satisfies the conditions:

(1) For any arbitrarily small number $\alpha > 0$:

$$\lim_{\xi \to +\infty} A(\xi) e^{t\,G_1(\xi)} \xi^{1+\alpha} = 0, \quad \lim_{\xi \to +\infty} A(\xi) e^{-t\,G_1(\xi)} \xi^{1+\alpha} = 0; \tag{16}$$

(2) Improper integrals

$$\int_0^{+\infty} \frac{A(\xi)\,\xi}{e^{-t\,G_1(\xi)}} d\xi, \quad \int_0^{+\infty} \frac{A(\xi)\,\xi}{e^{t\,G_1(\xi)}} d\xi, \quad \int_0^{+\infty} \frac{A(\xi)\,\xi^2}{e^{-t\,G_1(\xi)}} d\xi, \quad \int_0^{+\infty} \frac{A(\xi)\,\xi^2}{e^{t\,G_1(\xi)}} d\xi, \tag{17}$$

converge for each $t \in (0, T]$;

(3) Improper integrals

$$\int_0^{+\infty} \frac{A(\xi)}{e^{-t\,G_1(\xi)}} d\xi, \quad \int_0^{+\infty} \frac{A(\xi)}{e^{t\,G_1(\xi)}} d\xi \tag{18}$$

converge for each $t \in [0, T]$.

As an example of such weight function $A(\xi)$, which is continuous and non-negative for any value of $\xi \in [0, +\infty)$, and satisfies conditions (16)–(18), one can take any function $\xi^\beta e^{-TC\xi}$ where $\beta \geq 0$ and $C > a > 0$ are any real constants.

Indeed, Function (13) is represented in the following form:

$$\rho(\xi) = \left[\left(a^2\xi^2 + b\cos(h\xi) \right)^2 + b^2 \sin^2(h\xi) \right]^{1/4} =$$
$$= \left[a^4\xi^4 + 2a^2 b \xi^2 \cos(h\xi) + b^2 \right]^{1/4} = a|\xi| \left[1 + \frac{2b\cos(h\xi)}{a^2\xi^2} + \frac{b^2}{a^4\xi^4} \right]^{1/4},$$

that is, for $\xi \longrightarrow +\infty$, the function $\rho(\xi)$ is equivalent to the function $a\xi(1+\varepsilon)$, where $\varepsilon > 0$ is any arbitrarily small number.

From Formula (14), it follows that $|\varphi(\xi)| < \pi/4$, which means that $|\sin\varphi(\xi)| < \sqrt{2}/2$. Thus, for Function (15) for $\xi \longrightarrow +\infty$ we have the estimate

$$|G_1(\xi)| = |\rho(\xi)\sin\varphi(\xi)| < \frac{\sqrt{2}}{2} a(1+\varepsilon)\xi.$$

Since the inequalities

$$-a\xi < -\frac{\sqrt{2}}{2} a(1+\varepsilon)\xi < G_1(\xi) < \frac{\sqrt{2}}{2} a(1+\varepsilon)\xi < a\xi,$$

hold for any arbitrarily small number $\varepsilon > 0$ and $\xi \to +\infty$, we obtain the conditions

$$TC\xi - tG_1(\xi) > (TC - ta)\xi > 0, \quad TC\xi + tG_1(\xi) > (TC - ta)\xi > 0. \tag{19}$$

Using the inequalities

$$0 \leq \frac{\xi^\beta}{e^{TC\xi - t\,G_1(\xi)}} \xi^{1+\alpha} < \frac{\xi^{1+\alpha+\beta}}{e^{(TC-ta)\xi}}$$

and Lopital's rule, one can show that

$$\lim_{\xi \to +\infty} \frac{\xi^{1+\alpha+\beta}}{e^{(TC-ta)\xi}} = 0,$$

therefore,

$$\lim_{\xi\to+\infty}\frac{\xi^{1+\alpha+\beta}}{e^{TC\xi-t\,G_1(\xi)}}=0.$$

Thus, for the function $A(\xi)=\xi^{\beta}e^{-TC\xi}$, the first condition from (16) is satisfied.

Similarly, taking into account the second inequality from (19), we can check that the second condition from (16) for the function $A(\xi)=\xi^{\beta}e^{-TC\xi}$ is also satisfied.

To prove the convergence of the integrals (17) and (18), we use the criterion for the convergence of improper integrals: if there is a finite limit $\lim\limits_{x\to+\infty}|f(x)|\cdot x^{p}$ at $p>1$, then the integral $\int\limits_{a}^{+\infty}f(x)dx$ converges. The existence of finite limits

$$\lim_{\xi\to+\infty}\frac{\xi^{1+\alpha+\beta}}{e^{TC\xi-t\,G_1(\xi)}}=0,\quad \lim_{\xi\to+\infty}\frac{\xi^{1+\alpha+\beta}}{e^{TC\xi+t\,G_1(\xi)}}=0$$

implies the convergence of the integrals (18) for the function $A(\xi)=\xi^{\beta}e^{-TC\xi}$ for any fixed values $t\in[0,T]$, $\beta\geq 0$ and $C>a>0$.

Using the inequalities (19) and Lopital's rule, one can check that all four limits

$$\lim_{\xi\to+\infty}\frac{\xi^{1+\beta}}{e^{TC\xi-t\,G_1(\xi)}}\xi^{1+\alpha},\quad \lim_{\xi\to+\infty}\frac{\xi^{1+\beta}}{e^{TC\xi+t\,G_1(\xi)}}\xi^{1+\alpha};$$

$$\lim_{\xi\to+\infty}\frac{\xi^{2+\beta}}{e^{TC\xi-t\,G_1(\xi)}}\xi^{1+\alpha},\quad \lim_{\xi\to+\infty}\frac{\xi^{2+\beta}}{e^{TC\xi+t\,G_1(\xi)}}\xi^{1+\alpha}$$

are equal to zero for any arbitrarily small number $\alpha>0$. This means that the integrals (17) converge for the function $A(\xi)=\xi^{\beta}e^{-TC\xi}$ ($\beta\geq 0$, $C>a>0$) and each $t\in[0,T]$.

Let us prove the following assertion.

Lemma 1. *Under condition (2), the function*

$$G(x,t):=\int\limits_{0}^{+\infty}\frac{A(\xi)}{\rho(\xi)}\Big[\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)e^{t\,G_1(\xi)}+\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\xi\right)e^{-t\,G_1(\xi)}\Big]d\xi \tag{20}$$

satisfies the equality (1) in the classical sense. Here, $A(\xi)$ is a definite and non-negative for any value $\xi\in[0,+\infty)$ function, which satisfies the conditions (16)–(18); functions $G_1(\xi)$ and $G_2(\xi)$ are determined by Equation (15).

Proof. As noted above, if the condition (2) is satisfied, the functions $\rho(\xi)$ and $\varphi(\xi)$ are defined correctly for all values of the parameters a, b, h, and ξ. Moreover, the function $\rho(\xi)$ is non-zero. That is, the integrand in (20) is continuous at every point $\xi\in[0,+\infty)$ as a composition of continuous functions.

Let us first investigate the convergence of the integral:

$$\int\limits_{0}^{+\infty}F(x,t;\xi)d\xi:=\int\limits_{0}^{+\infty}\frac{A(\xi)\,\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)}{\rho(\xi)e^{-t\,G_1(\xi)}}d\xi. \tag{21}$$

It follows from Formulas (4) and (13) that for the function $\rho(\xi)$ satisfies the estimate

$$\rho(\xi)=\left[\left(a^2\xi^2+b\cos\left(h\xi\right)\right)^2+b^2\sin^2\left(h\xi\right)\right]^{1/4}\geq\left[b^2+b^2\sin^2\left(h\xi\right)\right]^{1/4}\geq\sqrt{b} \tag{22}$$

Consider the expression

$$\left|\frac{A(\xi)\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)}{\rho(\xi)e^{-t\,G_1(\xi)}}\xi^{1+\alpha}\right|,$$

where $\alpha > 0$ is any arbitrarily small number. Taking into account (22), we have

$$0 \le \left|\frac{A(\xi)\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)}{\rho(\xi)e^{-t\,G_1(\xi)}}\xi^{1+\alpha}\right| \\ \le \frac{A(\xi)\,|\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)|}{\sqrt{b}\,e^{-t\,G_1(\xi)}}\xi^{1+\alpha} \le \frac{1}{\sqrt{b}}\frac{A(\xi)}{e^{-t\,G_1(\xi)}}\xi^{1+\alpha}.$$

Using the first condition from (16) we obtain

$$\lim_{\xi\to+\infty}\left|\frac{A(\xi)\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)}{\rho(\xi)e^{-t\,G_1(\xi)}}\xi^{1+\alpha}\right| = 0,$$

that is, the integral $\int\limits_0^{+\infty} F(x,t;\xi)d\xi$ converges.

Similarly, it can be shown that when the second condition in (16) is satisfied, the integral

$$\int\limits_0^{+\infty} H(x,t;\xi)d\xi := \int\limits_0^{+\infty} \frac{A(\xi)\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\xi\right)}{\rho(\xi)e^{t\,G_1(\xi)}}d\xi \tag{23}$$

converges.

Let us now check that Function (21) satisfies Equation (1). To do this, we formally differentiate Function (21) with respect to the variables x and t up to the second order under the integral sign.

$$\int\limits_0^{+\infty} F_x(x,t;\xi)d\xi = \int\limits_0^{+\infty} \frac{A(\xi)\,\xi\,\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)}{\rho(\xi)e^{-t\,G_1(\xi)}}d\xi; \tag{24}$$

$$\int\limits_0^{+\infty} F_{xx}(x,t;\xi)d\xi = -\int\limits_0^{+\infty} \frac{A(\xi)\,\xi^2\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)}{\rho(\xi)e^{-t\,G_1(\xi)}}d\xi; \tag{25}$$

$$\int\limits_0^{+\infty} F_t(x,t;\xi)d\xi = \int\limits_0^{+\infty} \frac{A(\xi)}{\rho(\xi)e^{-t\,G_1(\xi)}}[G_2(\xi)\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right) \\ + G_1(\xi)\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)]d\xi; \tag{26}$$

$$\int\limits_0^{+\infty} F_{tt}(x,t;\xi)d\xi = \int\limits_0^{+\infty} \frac{A(\xi)}{\rho(\xi)e^{-t\,G_1(\xi)}}\left[\left(G_1^2(\xi)-G_2^2(\xi)\right)\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)\right. \\ \left. +2G_1(\xi)G_2(\xi)\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)\right]d\xi. \tag{27}$$

Taking into account Equation (15), we obtain $2G_1(\xi)G_2(\xi) = \rho^2(\xi)\sin 2\varphi(\xi)$. Since $\varphi(\xi)$ is determined by Equation (14), the inequality $|2\varphi(\xi)| < \pi/2$ is satisfied, and therefore, $\cos 2\varphi(\xi) > 0$. Therefore,

$$\sin 2\varphi(\xi) = \frac{\operatorname{tg} 2\varphi(\xi)}{\sqrt{1+\operatorname{tg}^2 2\varphi(\xi)}}$$
$$= \operatorname{tg}\left(\operatorname{arctg}\frac{b\sin(h\xi)}{a^2\xi^2+b\cos(h\xi)}\right)\left[1+\operatorname{tg}^2\left(\operatorname{arctg}\frac{b\sin(h\xi)}{a^2\xi^2+b\cos(h\xi)}\right)\right]^{-1/2}$$
$$= \frac{b\sin(h\xi)}{a^2\xi^2+b\cos(h\xi)}\left[1+\frac{b^2\sin^2(h\xi)}{(a^2\xi^2+b\cos(h\xi))^2}\right]^{-1/2}$$
$$= \frac{b\sin(h\xi)}{a^2\xi^2+b\cos(h\xi)}\left[\frac{(a^2\xi^2+b\cos(h\xi))^2}{(a^2\xi^2+b\cos(h\xi))^2+b^2\sin^2(h\xi)}\right]^{1/2}$$

is true.

Taking into account the inequality (2) and Equation (13), we deduce that $\sin 2\varphi(\xi) = b\sin(h\xi)/\rho^2(\xi)$, whence, $2G_1(\xi)G_2(\xi) = b\sin(h\xi)$.

If the inequality $\cos 2\varphi(\xi) > 0$ and the condition (2) are satisfied, we can calculate

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi)\left[\sin^2\varphi(\xi) - \cos^2\varphi(\xi)\right]$$
$$= -\rho^2(\xi)\cos 2\varphi(\xi) = -\frac{\rho^2(\xi)}{\sqrt{1+\operatorname{tg}^2 2\varphi(\xi)}}$$
$$= -\rho^2(\xi)\left[\frac{(a^2\xi^2+b\cos(h\xi))^2}{(a^2\xi^2+b\cos(h\xi))^2+b^2\sin^2(h\xi)}\right]^{1/2} = -a^2\xi^2 - b\cos(h\xi).$$

Using the obtained expressions for $G_1^2(\xi) - G_2^2(\xi)$ and $2G_1(\xi)G_2(\xi)$, from equality (27) we obtain

$$\int_0^{+\infty} F_{tt}(x,t;\xi)d\xi = \int_0^{+\infty} \frac{A(\xi)}{\rho(\xi)e^{-t\,G_1(\xi)}}\Big[-\left(a^2\xi^2+b\cos(h\xi)\right)\sin(t\,G_2(\xi)+\varphi(\xi)+x\xi)$$
$$+b\sin(h\xi)\cos(t\,G_2(\xi)+\varphi(\xi)+x\xi)\Big]d\xi. \quad (28)$$

Substitute the obtained expressions of the derivatives $\int_0^{+\infty} F_{tt}(x,t;\xi)d\xi$ and $\int_0^{+\infty} F_{xx}(x,t;\xi)d\xi$ to Equation (1):

$$\int_0^{+\infty} F_{tt}(x,t;\xi)d\xi - a^2\int_0^{+\infty} F_{xx}(x,t;\xi)d\xi$$
$$= -b\int_0^{+\infty} A(\xi)\frac{\sin(t\,G_2(\xi)+\varphi(\xi)+x\xi)\cdot\cos(h\xi) - \cos(t\,G_2(\xi)+\varphi(\xi)+x\xi)\cdot\sin(h\xi)}{\rho(\xi)e^{-t\,G_1(\xi)}}d\xi$$
$$= -b\int_0^{+\infty}\frac{A(\xi)\sin(t\,G_2(\xi)+\varphi(\xi)+(x-h)\xi)}{\rho(\xi)e^{-t\,G_1(\xi)}}d\xi = -b\int_0^{+\infty} F(x-h,t;\xi)d\xi.$$

Thus, Function (21) satisfies Equation (1) in the classical sense.

Now let us prove the uniform convergence of the integrals (24) and (25) with respect to the variable x on any segment $[x_1, x_2] \subset (-\infty, +\infty)$, and the uniform convergence of the integrals (26) and (28) with respect to the variable t on any segment $[t_1, t_2] \subset (0, T]$. Note that the integrands of all these integrals have no singularities at the point $\xi = 0$.

Let us investigate the integral (24) for the uniform convergence, taking into account the estimate (22) and using the Weierstrass criterion:

$$\int_0^{+\infty} |F_x(x,t;\xi)|d\xi = \int_0^{+\infty} \left| \frac{A(\xi)\,\xi\,\cos\left(t\,G_2(\xi) + \varphi(\xi) + x\xi\right)}{\rho(\xi)e^{-t\,G_1(\xi)}} \right| d\xi \le \frac{1}{\sqrt{b}} \int_0^{+\infty} \frac{A(\xi)\,\xi}{e^{-t\,G_1(\xi)}} d\xi.$$

According to the convergence of the integrals (17), the integral on the right-hand side of the latter inequality converges and the integrand does not depend on the variable x; hence, the integral (24) converges uniformly with respect to the variable x on any finite segment $[x_1, x_2] \subset (-\infty, +\infty)$.

Let us investigate the uniform convergence of the integral (25) with respect to the variable x on any finite segment $[x_1, x_2] \subset (-\infty, +\infty)$. Using the inequality (22), we calculate

$$\int_0^{+\infty} |F_{xx}(x,t;\xi)|d\xi = \int_0^{+\infty} \left| \frac{A(\xi)\,\xi^2\sin\left(t\,G_2(\xi) + \varphi(\xi) + x\xi\right)}{\rho(\xi)e^{-t\,G_1(\xi)}} \right| d\xi \le \frac{1}{\sqrt{b}} \int_0^{+\infty} \frac{A(\xi)\,\xi^2}{e^{-t\,G_1(\xi)}} d\xi.$$

By virtue of the convergence of the integrals (17), the integral on the right-hand side of the latter inequality converges, and the integrand does not depend on the variable x. Thus, the integral (25) converges uniformly, which means that differentiation under the integral sign of Function (21) with respect to the variable x up to the second order including was legal.

It remains to check the uniform convergence of the integral (26) with respect to the variable t on any finite segment $[t_1, t_2] \subset [0, T]$, and the integral (28) with respect to the variable t on any finite segment $[t_1, t_2] \subset (0, T]$. Using the definition (15), we can write Function (26) as

$$\int_0^{+\infty} F_t(x,t;\xi)d\xi = \int_0^{+\infty} \frac{A(\xi)}{e^{-t\,G_1(\xi)}}[\cos\varphi(\xi)\cos\left(t\,G_2(\xi) + \varphi(\xi) + x\xi\right)$$

$$+\sin\varphi(\xi)\sin\left(t\,G_2(\xi) + \varphi(\xi) + x\xi\right)]d\xi = \int_0^{+\infty} \frac{A(\xi)}{e^{-t\,G_1(\xi)}}\cos\left(t\,G_2(\xi) + x\xi\right)d\xi.$$

Hence,

$$\int_0^{+\infty} |F_t(x,t;\xi)|d\xi \le \int_0^{+\infty} \frac{A(\xi)}{e^{-t\,G_1(\xi)}}d\xi \le \begin{cases} \int_0^{+\infty} \frac{A(\xi)}{e^{-t_2\,G_1(\xi)}}d\xi, & G_1(\xi) \ge 0, \\ \int_0^{+\infty} \frac{A(\xi)}{e^{-t_1\,G_1(\xi)}}d\xi, & G_1(\xi) < 0. \end{cases}$$

Since the integral (18) converges, the integrals on the right side of the latter inequality converge and not depend on the variable t. Therefore, the integral (26) converges uniformly for any value of $t \in [0, T]$.

Let us now estimate the integral (28):

$$\int_0^{+\infty} |F_{tt}(x,t;\xi)|d\xi = \int_0^{+\infty} \left| \frac{A(\xi)}{\rho(\xi)e^{-t\,G_1(\xi)}} \right| \Big| -\left(a^2\xi^2 + b\cos(h\xi)\right)\sin\left(t\,G_2(\xi) + \varphi(\xi) + x\xi\right)$$

$$+ b\sin(h\xi)\cos(t\,G_2(\xi)+\varphi(\xi)+x\xi)|d\xi$$

$$\leq \frac{1}{\sqrt{b}}\int\limits_0^{+\infty}\frac{A(\xi)\,(a^2\xi^2+2b)}{e^{-t\,G_1(\xi)}}d\xi \leq \begin{cases} \frac{1}{\sqrt{b}}\int\limits_0^{+\infty}\frac{A(\xi)\,(a^2\xi^2+2b)}{e^{-t_2\,G_1(\xi)}}d\xi, & G_1(\xi)\geq 0, \\ \frac{1}{\sqrt{b}}\int\limits_0^{+\infty}\frac{A(\xi)\,(a^2\xi^2+2b)}{e^{-t_1\,G_1(\xi)}}d\xi, & G_1(\xi)<0. \end{cases}$$

Since the integrand in the latter integral does not depend on t, then, according to the Weierstrass test, the integral (28) converges uniformly in the variable t on any finite segment $[t_1, t_2] \subset (0, T]$. This means that differentiation under the integral sign in (21) with respect to the variable t up to the second order including was legal if the integral

$$\int\limits_0^{+\infty}\frac{A(\xi)\,(a^2\xi^2+2b)}{e^{-t\,G_1(\xi)}}d\xi = a^2\int\limits_0^{+\infty}\frac{A(\xi)\,\xi^2}{e^{-t\,G_1(\xi)}}d\xi + 2b\int\limits_0^{+\infty}\frac{A(\xi)}{e^{-t\,G_1(\xi)}}d\xi$$

converges for any value $t \in (0, T]$. And this is true, since the integrals (17) and (18) converge.

It can be shown similarly that the improper integrals obtained after the formal differentiation under the sign of the integral over the variables x and t up to the second order including of the integrand in (23) converge uniformly if the integrals

$$\int\limits_0^{+\infty}\frac{A(\xi)}{e^{t\,G_1(\xi)}}d\xi, \quad \int\limits_0^{+\infty}\frac{A(\xi)\,\xi}{e^{t\,G_1(\xi)}}d\xi, \quad \int\limits_0^{+\infty}\frac{A(\xi)\,\xi^2}{e^{t\,G_1(\xi)}}d\xi$$

converge. Their convergence was considered above, see (17) and (18).

Thus, we have proved that the function $G(x,t)$, defined by Equation (20), exists at every point of the area D and satisfies the equality (1) in the classical sense. Hence, the lemma is proved.

□

Theorem 1. *Under conditions* (2) *and* $u_0(x) \in L_1(\mathbf{R})$, *the function*

$$u(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} G(x-\eta,t)u_0(\eta)d\eta, \tag{29}$$

where $G(x,t)$ *is defined by the equality* (20), *satisfies the equality* (1) *in the classical sense.*

Proof. Function (29) has the following form

$$u(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} G(x-\eta,t)u_0(\eta)d\eta$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} u_0(\eta)\int\limits_0^{+\infty}\Bigg[\frac{A(\xi)\,\sin(t\,G_2(\xi)+\varphi(\xi)+x\xi-\eta\xi)}{\rho(\xi)e^{-t\,G_1(\xi)}} + \frac{A(\xi)\,\sin(t\,G_2(\xi)-\varphi(\xi)-x\xi+\eta\xi)}{\rho(\xi)e^{t\,G_1(\xi)}}\Bigg]d\xi d\eta.$$

Function (29) exists in the region D if the condition

$$\int\limits_{-\infty}^{+\infty}|u_0(\eta)|d\eta\int\limits_{-\infty}^{+\infty}|G(\eta,t)|d\eta < +\infty$$

holds.

Since $u_0(x)$ is an integrable function for all $x \in \mathbf{R}$, we are to check the fulfillment of condition $G(x,t) \in L_1(\mathbf{R})$ for any $t \in (0,T]$.

Let us find the majorant of Function (20). From the equality (13), we obtain the estimate

$$\rho(\xi) = \left[a^4\xi^4 + 2a^2\xi^2 b\cos(h\xi) + b^2\right]^{1/4} \le \left[a^4\xi^4 + 2a^2\xi^2 b + b^2\right]^{1/4} = \sqrt{a^2\xi^2 + b}. \quad (30)$$

The equality (14) implies that $|\varphi(\xi)| < \pi/4$, hence; $\sqrt{2}/2 < \cos\varphi(\xi) \le 1$. Then, from (15) and (30), we have

$$|G_2(\xi)| = \rho(\xi)|\cos\varphi(\xi)| \le \frac{\sqrt{2}}{2}\sqrt{a^2\xi^2 + b}. \quad (31)$$

Taking into account the inequalities (22), (31) and $|\sin\alpha/\alpha| < 1$, we can estimate the function

$$|G(x,t)| \le \int_0^{+\infty} \left| \frac{A(\xi)\,\sin(t\,G_2(\xi) + \varphi(\xi) + x\xi)}{\rho(\xi)e^{-tG_1(\xi)}} + \frac{A(\xi)\,\sin(t\,G_2(\xi) - \varphi(\xi) - x\xi)}{\rho(\xi)e^{tG_1(\xi)}} \right| d\xi$$

$$\le \int_0^{+\infty} \left(\frac{A(\xi)\,|\sin(t\,G_2(\xi) + \varphi(\xi) + x\xi)|}{\rho(\xi)e^{-tG_1(\xi)}} + \frac{A(\xi)\,|\sin(t\,G_2(\xi) - \varphi(\xi) - x\xi)|}{\rho(\xi)e^{tG_1(\xi)}} \right) d\xi$$

$$\le \frac{1}{\sqrt{b}} \int_0^{+\infty} \left(\frac{A(\xi)\,|\sin(t\,G_2(\xi) + \varphi(\xi) + x\xi)|}{e^{-tG_1(\xi)}} + \frac{A(\xi)\,|\sin(t\,G_2(\xi) - \varphi(\xi) - x\xi)|}{e^{tG_1(\xi)}} \right) d\xi$$

$$< \frac{1}{\sqrt{b}} \int_0^{+\infty} \left(\frac{A(\xi)\,|t\,G_2(\xi) + \varphi(\xi) + x\xi|}{e^{-tG_1(\xi)}} + \frac{A(\xi)\,|t\,G_2(\xi) - \varphi(\xi) - x\xi|}{e^{tG_1(\xi)}} \right) d\xi$$

$$\le \frac{1}{\sqrt{b}} \int_0^{+\infty} \left(\frac{t|G_2(\xi)| + |\varphi(\xi)| + |x|\xi}{e^{-tG_1(\xi)}} + \frac{t|G_2(\xi)| + |\varphi(\xi)| + |x|\xi}{e^{tG_1(\xi)}} \right) A(\xi)d\xi$$

$$\le \frac{1}{\sqrt{b}} \int_0^{+\infty} \left(\frac{\frac{\sqrt{2}}{2}\sqrt{a^2\xi^2 + b}\,t + \pi/4 + |x|\xi}{e^{-tG_1(\xi)}} + \frac{\frac{\sqrt{2}}{2}\sqrt{a^2\xi^2 + b}\,t + \pi/4 + |x|\xi}{e^{tG_1(\xi)}} \right) A(\xi)d\xi.$$

Replace the integration variable in the latter integral according to the formula $|x|\xi = \tau$ for $x \ne 0$:

$$|G(x,t)| <$$

$$< \frac{1}{\sqrt{b}\,|x|} \int_0^{+\infty} \left(\frac{\frac{\sqrt{2}}{2}\sqrt{a^2\tau^2/x^2 + b}\,t + \pi/4 + \tau}{e^{-tG_1(\tau/|x|)}} + \frac{\frac{\sqrt{2}}{2}\sqrt{a^2\tau^2/x^2 + b}\,t + \pi/4 + \tau}{e^{tG_1(\tau/|x|)}} \right) A(\tau/|x|)d\tau$$

$$= \frac{t}{\sqrt{b}\,x^2} \int_0^{+\infty} \left(\frac{\frac{\sqrt{2}}{2}\sqrt{a^2\tau^2 + bx^2} + \frac{\pi|x|}{4t} + \frac{|x|}{t}\tau}{e^{-tG_1(\tau/|x|)}} + \frac{\frac{\sqrt{2}}{2}\sqrt{a^2\tau^2 + bx^2} + \frac{\pi|x|}{4t} + \frac{|x|}{t}\tau}{e^{tG_1(\tau/|x|)}} \right) A(\tau/|x|)d\tau. \quad (32)$$

The integrals on the right-hand side of (32), due to conditions (17) and (3), converge for any $t \in (0,T]$ and any $x \in \mathbf{R}\backslash\{0\}$. Thus, we have shown that the function $|G(x,t)|$ is majorized by the function $\widetilde{C}t/x^2$, where $t \in (0,T]$, $x \ne 0$, where $\widetilde{C}$ is an absolute constant. This implies the convergence of the integral $\int_{-\infty}^{+\infty} G(\eta,t)d\eta$, that is $G(x,t) \in L_1(\mathbf{R})$ for any $t \in (0,T]$. This means that Function (20) exists in the domain D and, by virtue of the proved Lemma, it is a classical solution of Equation (1).

Note also that, by virtue of the same Lemma, Function (29) belongs to the class $C^1(\overline{D}) \cap C^2(D)$ (the integrand in (29) is continuous, the integrals $u_x(x,t)$ and $u_{xx}(x,t)$ converge uniformly in the variable x on any finite segment $[x_1, x_2] \subset (-\infty, +\infty)$, the integrals $u_t(x,t)$ and $u_{tt}(x,t)$ converge uniformly in the variable t on any finite segment $[t_1, t_2] \subset (0, T]$, and the integral $u_t(x,t)$ converges at the border $t = 0$). Thus, the theorem is proved.

□

3. Fulfillment of the Initial Conditions of the Problem

Theorem 2. *Under conditions* (2), $u_0(x) \in L_1(\mathbf{R})$ *and* $u_0(x) \in C^1(\mathbf{R})$ *the limit relations*

$$\lim_{t\to 0+} u(x_0, t) = u_0(x_0), \quad \lim_{t\to 0+} u_t(x_0, t) = 0 \tag{33}$$

are valid for any $x_0 \in (-\infty, +\infty)$.

Proof. 1. Let $x_0 \in (-\infty, +\infty)$. In the equality (29), make the change of variable $(x_0 - \eta)/t = \tau$, and consider the difference

$$\begin{aligned} u(x_0,t) - u_0(x_0) &= \frac{t}{2\pi} \int_{-\infty}^{+\infty} G(t\tau, t) u_0(x_0 - t\tau) d\tau - \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u_0(x_0)}{1+\tau^2} d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[t\, G(t\tau, t) u_0(x_0 - t\tau) - \frac{2u_0(x_0)}{1+\tau^2} \right] d\tau \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{+\infty} \right] \stackrel{\text{def}}{=} I_{1,A} + I_{2,A} + I_{3,A}. \end{aligned} \tag{34}$$

Using the inequality (2) and the equality $\operatorname{arctg} x = \arccos(1/\sqrt{1+x^2})$, we write the functions (15) as follows:

$$\begin{aligned} G_1(\xi) &= \rho(\xi) \sin \varphi(\xi) = \rho(\xi) \sqrt{\frac{1 - \cos 2\varphi(\xi)}{2}} \\ &= \frac{\rho(\xi)}{\sqrt{2}} \left[1 - \cos\left(\operatorname{arctg} \frac{b \sin(h\xi)}{a^2\xi^2 + b\cos(h\xi)} \right) \right]^{1/2} \\ &= \frac{\rho(\xi)}{\sqrt{2}} \left[1 - \frac{|a^2\xi^2 + b\cos(h\xi)|}{\sqrt{(a^2\xi^2 + b\cos(h\xi))^2 + b^2\sin^2(h\xi)}} \right]^{1/2} \\ &= \frac{\rho(\xi)}{\sqrt{2}} \left[\frac{\rho^2(\xi) - a^2\xi^2 - b\cos(h\xi)}{\rho^2(\xi)} \right]^{1/2} = \frac{1}{\sqrt{2}} \left[\rho^2(\xi) - a^2\xi^2 - b\cos(h\xi) \right]^{1/2} \\ &= \frac{1}{\sqrt{2}} \left[\sqrt{a^4\xi^4 + 2a^2b\xi^2\cos(h\xi) + b^2} - a^2\xi^2 - b\cos(h\xi) \right]^{1/2}, \end{aligned} \tag{35}$$

and

$$G_2(\xi) = \rho(\xi)\cos\varphi(\xi) = \rho(\xi)\sqrt{\frac{1+\cos 2\varphi(\xi)}{2}}$$

$$= \frac{\rho(\xi)}{\sqrt{2}}\left[1 + \frac{|a^2\xi^2 + b\cos(h\xi)|}{\sqrt{(a^2\xi^2 + b\cos(h\xi))^2 + b^2\sin^2(h\xi)}}\right]^{1/2} = \frac{\rho(\xi)}{\sqrt{2}}\left[\frac{\rho^2(\xi) + a^2\xi^2 + b\cos(h\xi)}{\rho^2(\xi)}\right]^{1/2}$$

$$= \frac{1}{\sqrt{2}}\left[\sqrt{a^4\xi^4 + 2a^2b\xi^2\cos(h\xi) + b^2} + a^2\xi^2 + b\cos(h\xi)\right]^{1/2}. \quad (36)$$

Obviously, the resulting radical expressions in (35) and (36) are always non-negative. Then,

$$t\,G_1(\xi) = \frac{1}{\sqrt{2}}\left[\sqrt{a^4t^4\xi^4 + 2a^2bt^4\xi^2\cos(h\xi) + b^2t^4} - a^2t^2\xi^2 - bt^2\cos(h\xi)\right]^{1/2},$$

$$t\,G_2(\xi) = \frac{1}{\sqrt{2}}\left[\sqrt{a^4t^4\xi^4 + 2a^2bt^4\xi^2\cos(h\xi) + b^2t^4} + a^2t^2\xi^2 + bt^2\cos(h\xi)\right]^{1/2},$$

and the function $t\,G(t\tau, t)$ can be written as

$$t\,G(t\tau, t) = t\int_0^{+\infty}\left[\frac{A(\xi)\,\sin(t\,G_2(\xi) + \varphi(\xi) + t\tau\xi)}{\rho(\xi)e^{-t\,G_1(\xi)}} + \frac{A(\xi)\,\sin(t\,G_2(\xi) - \varphi(\xi) - t\tau\xi)}{\rho(\xi)e^{t\,G_1(\xi)}}\right]d\xi$$

$$= t\int_0^{+\infty}\left[\frac{A(\xi)\,\sin\left(\frac{1}{\sqrt{2}}\left[\sqrt{a^4t^4\xi^4 + 2a^2bt^4\xi^2\cos(h\xi) + b^2t^4} + a^2t^2\xi^2 + bt^2\cos(h\xi)\right]^{1/2} + \varphi(\xi) + t\tau\xi\right)}{\rho(\xi)e^{-\frac{1}{\sqrt{2}}\left[\sqrt{a^4t^4\xi^4 + 2a^2bt^4\xi^2\cos(h\xi) + b^2t^4} - a^2t^2\xi^2 - bt^2\cos(h\xi)\right]^{1/2}}}\right.$$

$$\left. + \frac{A(\xi)\,\sin\left(\frac{1}{\sqrt{2}}\left[\sqrt{a^4t^4\xi^4 + 2a^2bt^4\xi^2\cos(h\xi) + b^2t^4} + a^2t^2\xi^2 + bt^2\cos(h\xi)\right]^{1/2} - \varphi(\xi) - t\tau\xi\right)}{\rho(\xi)e^{\frac{1}{\sqrt{2}}\left[\sqrt{a^4t^4\xi^4 + 2a^2bt^4\xi^2\cos(h\xi) + b^2t^4} - a^2t^2\xi^2 - bt^2\cos(h\xi)\right]^{1/2}}}\right]d\xi.$$

After the substitution $t\xi = z$, from the latter equality we obtain

$$t\,G(t\tau, t) = \int_0^{+\infty}\left[\frac{A(z/t)}{\rho(z/t)e^{-\frac{1}{\sqrt{2}}\left[\sqrt{a^4z^4 + 2a^2bt^2z^2\cos(hz/t) + b^2t^4} - a^2z^2 - bt^2\cos(hz/t)\right]^{1/2}}}\right.$$

$$\times\sin\left(\frac{1}{\sqrt{2}}\left[\sqrt{a^4z^4 + 2a^2bt^2z^2\cos(hz/t) + b^2t^4} + a^2z^2 + bt^2\cos(hz/t)\right]^{1/2} + \varphi(z/t) + \tau z\right)$$

$$+ \frac{A(z/t)}{\rho(z/t)e^{\frac{1}{\sqrt{2}}\left[\sqrt{a^4z^4 + 2a^2bt^2z^2\cos(hz/t) + b^2t^4} - a^2z^2 - bt^2\cos(hz/t)\right]^{1/2}}}$$

$$\left.\sin\left(\frac{1}{\sqrt{2}}\left[\sqrt{a^4z^4 + 2a^2bt^2z^2\cos(hz/t) + b^2t^4} + a^2z^2 + bt^2\cos(hz/t)\right]^{1/2} - \varphi(z/t) - \tau z\right)\right]dz.$$

Denote the functions

$$g_1(z, t) := \frac{1}{\sqrt{2}}\left[\sqrt{a^4z^4 + 2a^2bt^2z^2\cos(hz/t) + b^2t^4} - a^2z^2 - bt^2\cos(hz/t)\right]^{1/2}, \quad (37)$$

and

$$g_2(z,t) := \frac{1}{\sqrt{2}}\left[\sqrt{a^4z^4 + 2a^2bt^2z^2\cos(hz/t) + b^2t^4} + a^2z^2 + bt^2\cos(hz/t)\right]^{1/2}. \tag{38}$$

Using this notation, we can write

$$t\,G(t\tau,t) = \int_0^{+\infty}\left[\frac{A(z/t)\sin(g_2(z,t)+\varphi(z/t)+\tau z)}{\rho(z/t)e^{-g_1(z,t)}} + \frac{A(z/t)\sin(g_2(z,t)-\varphi(z/t)-\tau z)}{\rho(z/t)e^{g_1(z,t)}}\right]dz. \tag{39}$$

2. Let us now prove that the limit relation

$$\lim_{t\to 0+} t\,G(t\tau,t) = \frac{2}{1+\tau^2} \tag{40}$$

is satisfied uniformly with respect to $\tau \in (-\infty,+\infty)$. To do this, it suffices to show that for any arbitrarily small number $\varepsilon > 0$ there is a number $0 < \delta \le T$ such that for any $t \in (0,\delta)$ and $\tau \in (-\infty,+\infty)$ holds the inequality

$$\left|t\,G(t\tau,t) - \frac{2}{1+\tau^2}\right| < \varepsilon. \tag{41}$$

Represent the function in the form $2/(1+\tau^2) = 2\int_0^{+\infty} e^{-z}\cos(\tau z)dz$ and consider the difference

$$\begin{aligned}
t\,G(t\tau,t) - \frac{2}{1+\tau^2} &= \int_0^{+\infty}\left[\frac{A(z/t)\sin(g_2(z,t)+\varphi(z/t)+\tau z)}{\rho(z/t)e^{-g_1(z,t)}} + \frac{A(z/t)\sin(g_2(z,t)-\varphi(z/t)-\tau z)}{\rho(z/t)e^{g_1(z,t)}}\right]dz \\
&\quad - 2\int_0^{+\infty} e^{-z}\cos(\tau z)dz \\
&= \int_0^{+\infty}\left[\frac{A(z/t)e^{g_1(z,t)}}{\rho(z/t)}\sin(g_2(z,t)+\varphi(z/t)+\tau z) - e^{-z}\cos(\tau z)\right]dz \\
&\quad + \int_0^{+\infty}\left[\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\sin(g_2(z,t)-\varphi(z/t)-\tau z) - e^{-z}\cos(\tau z)\right]dz \\
&= \int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)}\sin(g_2(z,t)+\varphi(z/t)) - 1\right]\cos(\tau z)dz \\
&\quad + \int_0^{+\infty}\frac{A(z/t)e^{g_1(z,t)}}{\rho(z/t)}\cos(g_2(z,t)+\varphi(z/t))\sin(\tau z)dz
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)}\sin\left(g_2(z,t)-\varphi(z/t)\right)-1\right]\cos\left(\tau z\right)dz \\
&\qquad -\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\cos\left(g_2(z,t)-\varphi(z/t)\right)\sin\left(\tau z\right)dz \\
&= \int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)}\sin\left(g_2(z,t)+\varphi(z/t)\right)-1\right]\cos\left(\tau z\right)dz \\
&+ \int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)}\sin\left(g_2(z,t)-\varphi(z/t)\right)-1\right]\cos\left(\tau z\right)dz \\
&+ \int_0^{+\infty}\frac{A(z/t)e^{g_1(z,t)}}{\rho(z/t)}\left(1-2\sin^2\frac{g_2(z,t)+\varphi(z/t)}{2}\right)\sin\left(\tau z\right)dz \\
&- \int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(1-2\sin^2\frac{g_2(z,t)-\varphi(z/t)}{2}\right)\sin\left(\tau z\right)dz \\
&= \int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)}\sin\left(g_2(z,t)+\varphi(z/t)\right)-1\right]\cos\left(\tau z\right)dz \\
&+ \int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)}\sin\left(g_2(z,t)-\varphi(z/t)\right)-1\right]\cos\left(\tau z\right)dz \\
&\qquad -2\int_0^{+\infty}\frac{A(z/t)e^{g_1(z,t)}}{\rho(z/t)}\sin^2\frac{g_2(z,t)+\varphi(z/t)}{2}\sin\left(\tau z\right)dz \\
&\qquad +2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\sin^2\frac{g_2(z,t)-\varphi(z/t)}{2}\sin\left(\tau z\right)dz \\
&\qquad\qquad +\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(e^{2g_1(z,t)}-1\right)\sin\left(\tau z\right)dz.
\end{aligned}
$$

We use the formulas for the sine of the sum and difference in the third and fourth integrals in the last expression. Taking into account their squaring and the formula for the sine of a double angle, we obtain

$$
\begin{aligned}
t\,G(t\tau,t)-\frac{2}{1+\tau^2} &= \int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)}\sin\left(g_2(z,t)+\varphi(z/t)\right)-1\right]\cos\left(\tau z\right)dz \\
&+ \int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)}\sin\left(g_2(z,t)-\varphi(z/t)\right)-1\right]\cos\left(\tau z\right)dz
\end{aligned}
$$

$$
\begin{aligned}
&-2\int_0^{+\infty}\frac{A(z/t)e^{g_1(z,t)}}{\rho(z/t)}\left(\sin^2\frac{g_2(z,t)}{2}\cos^2\frac{\varphi(z/t)}{2}+\cos^2\frac{g_2(z,t)}{2}\sin^2\frac{\varphi(z/t)}{2}\right.\\
&\qquad\left.+\frac{1}{2}\sin g_2(z,t)\sin\varphi(z/t)\right)\sin(\tau z)dz\\
&+2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(\sin^2\frac{g_2(z,t)}{2}\cos^2\frac{\varphi(z/t)}{2}+\cos^2\frac{g_2(z,t)}{2}\sin^2\frac{\varphi(z/t)}{2}\right.\\
&\qquad\left.-\frac{1}{2}\sin g_2(z,t)\sin\varphi(z/t)\right)\sin(\tau z)dz\\
&+\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(e^{2g_1(z,t)}-1\right)\sin(\tau z)dz\\
&=\int_0^{+\infty}e^{-z}\left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)}\sin(g_2(z,t)+\varphi(z/t))-1\right]\cos(\tau z)dz\\
&+\int_0^{+\infty}e^{-z}\left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)}\sin(g_2(z,t)-\varphi(z/t))-1\right]\cos(\tau z)dz\\
&-\int_0^{+\infty}\frac{A(z/t)e^{g_1(z,t)}}{\rho(z/t)}\sin g_2(z,t)\sin\varphi(z/t)\sin(\tau z)dz\\
&-\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\sin g_2(z,t)\sin\varphi(z/t)\sin(\tau z)dz\\
&-2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\sin^2\frac{g_2(z,t)}{2}\cos^2\frac{\varphi(z/t)}{2}\sin(\tau z)dz\\
&-2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\cos^2\frac{g_2(z,t)}{2}\sin^2\frac{\varphi(z/t)}{2}\sin(\tau z)dz\\
&+\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(e^{2g_1(z,t)}-1\right)\sin(\tau z)dz.
\end{aligned}
$$

In the resulting expression, we expand the sine of the sum in the integrand in the first integral, which we then group with the third integral, and write the sine of the difference in the second integral and group it with the fourth integral as follows:

$$
\int_0^{+\infty}e^{-z}\left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)}\sin g_2(z,t)\cos\varphi(z/t)-1\right]\cos(\tau z)dz
$$

$$+\int_0^{+\infty}\frac{A(z/t)e^{g_1(z,t)}}{\rho(z/t)}[\cos g_2(z,t)\cos(\tau z)-\sin g_2(z,t)\sin(\tau z)]\sin\varphi(z/t)dz$$

$$+\int_0^{+\infty}e^{-z}\left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)}\sin g_2(z,t)\cos\varphi(z/t)-1\right]\cos(\tau z)dz$$

$$-\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}[\cos g_2(z,t)\cos(\tau z)+\sin g_2(z,t)\sin(\tau z)]\sin\varphi(z/t)dz$$

$$-2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\sin^2\frac{g_2(z,t)}{2}\cos^2\frac{\varphi(z/t)}{2}\sin(\tau z)dz$$

$$-2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\cos^2\frac{g_2(z,t)}{2}\sin^2\frac{\varphi(z/t)}{2}\sin(\tau z)dz$$

$$+\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(e^{2g_1(z,t)}-1\right)\sin(\tau z)dz$$

$$=\int_0^{+\infty}e^{-z}\left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)}\sin g_2(z,t)\cos\varphi(z/t)-1\right]\cos(\tau z)dz$$

$$+\int_0^{+\infty}e^{-z}\left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)}\sin g_2(z,t)\cos\varphi(z/t)-1\right]\cos(\tau z)dz$$

$$+\int_0^{+\infty}\frac{A(z/t)e^{g_1(z,t)}}{\rho(z/t)}\cos(g_2(z,t)+\tau z)\sin\varphi(z/t)dz$$

$$-\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\cos(g_2(z,t)-\tau z)\sin\varphi(z/t)dz$$

$$-2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\sin^2\frac{g_2(z,t)}{2}\cos^2\frac{\varphi(z/t)}{2}\sin(\tau z)dz$$

$$-2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\cos^2\frac{g_2(z,t)}{2}\sin^2\frac{\varphi(z/t)}{2}\sin(\tau z)dz$$

$$+\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(e^{2g_1(z,t)}-1\right)\sin(\tau z)dz$$

$$=\int_0^{+\infty}e^{-z}\left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)}\sin g_2(z,t)\cos\varphi(z/t)-1\right]\cos(\tau z)dz$$

$$+\int_0^{+\infty} e^{-z}\left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)}\sin g_2(z,t)\cos\varphi(z/t)-1\right]\cos(\tau z)dz$$
$$+\int_0^{+\infty}\frac{A(z/t)e^{g_1(z,t)}\cos(g_2(z,t)+\tau z)-A(z/t)e^{-g_1(z,t)}\cos(g_2(z,t)-\tau z)}{\rho(z/t)}\sin\varphi(z/t)dz$$
$$-2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\sin^2\frac{g_2(z,t)}{2}\cos^2\frac{\varphi(z/t)}{2}\sin(\tau z)dz$$
$$-2\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\cos^2\frac{g_2(z,t)}{2}\sin^2\frac{\varphi(z/t)}{2}\sin(\tau z)dz$$
$$+\int_0^{+\infty}\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(e^{2g_1(z,t)}-1\right)\sin(\tau z)dz$$
$$\stackrel{\text{def}}{=} I_1(\tau,t)+I_2(\tau,t)+I_3(\tau,t)-2I_4(\tau,t)-2I_5(\tau,t)+I_6(\tau,t).$$

3. Consider first the integral $I_6(\tau,t)$. Using the definition (37) and inequality (4), transform the expression

$$2g_1^2(z,t)=\sqrt{a^4z^4+2a^2b\,t^2z^2\cos(hz/t)+b^2t^4}-a^2z^2-bt^2\cos(hz/t)$$
$$=\frac{b^2t^4-b^2t^4\cos^2(hz/t)}{\sqrt{a^4z^4+2a^2b\,t^2z^2\cos(hz/t)+b^2t^4}+a^2z^2+bt^2\cos(hz/t)}$$
$$=\frac{b^2t^4\sin^2(hz/t)}{\sqrt{a^4z^4+2a^2b\,t^2z^2\cos(hz/t)+b^2t^4}+t^2\left(a^2\frac{z^2}{t^2}+b\,\cos(hz/t)\right)}.$$

Taking into account that $\sqrt{a^4z^4+2a^2b\,t^2z^2\cos(hz/t)+b^2t^4}\geq 0$ and the inequality (4), from the last expression, we obtain

$$\frac{b^2t^4\sin^2(hz/t)}{\sqrt{a^4z^4+2a^2b\,t^2z^2\cos(hz/t)+b^2t^4}+t^2\left(a^2\frac{z^2}{t^2}+b\,\cos(hz/t)\right)}\leq\frac{b^2t^4\sin^2(hz/t)}{bt^2}\leq\frac{b^2t^4}{bt^2}=bt^2,$$

whence,

$$0\leq g_1(z,t)\leq\sqrt{\frac{b}{2}}t. \tag{42}$$

Using the inequalities (22) and (42), we estimate the integral $I_6(\tau,t)$:

$$|I_6(\tau,t)|\leq\int_0^{+\infty}\left|\frac{A(z/t)e^{-g_1(z,t)}}{\rho(z/t)}\left(e^{2g_1(z,t)}-1\right)\sin(\tau z)\right|dz$$
$$=\int_0^{+\infty}\frac{A(z/t)\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)e^{g_1(z,t)}}|\sin(\tau z)|dz\leq\frac{1}{\sqrt{b}}\int_0^{+\infty}\frac{A(z/t)\left(e^{\sqrt{2b}\,t}-1\right)}{e^{g_1(z,t)}}dz.$$

Since the asymptotic representation of the function $e^{\sqrt{2bt}}=1+\sqrt{2b}\,t+o(t)$ is valid for $t\longrightarrow 0+$, the latter integral for $t\longrightarrow 0+$ can be bounded as

$$\frac{1}{\sqrt{b}}\int\limits_0^{+\infty}\frac{A(z/t)(\sqrt{2b}t+o(t))}{e^{g_1(z,t)}}dz=\sqrt{2}\,t\int\limits_0^{+\infty}\frac{A(z/t)(1+o(1))}{e^{g_1(z,t)}}dz\le \mathrm{const}\cdot\sqrt{2}\,t\int\limits_0^{+\infty}\frac{A(z/t)}{e^{g_1(z,t)}}dz.$$

After the reverse change of the variable according to the formula $t\xi=z$, we finally obtain

$$|I_6(\tau,t)|\le \mathrm{const}\cdot\sqrt{2}\,t\int\limits_0^{+\infty}\frac{A(z/t)}{e^{g_1(z,t)}}dz=\mathrm{const}\cdot\sqrt{2}\,t^2\int\limits_0^{+\infty}\frac{A(\xi)}{e^{t\,G_1(\xi)}}d\xi.$$

Since the resulting integral on the right-hand side is a convergent integral from the series (18), the last inequality for $t\longrightarrow 0+$ implies the estimate

$$|I_6(\tau,t)|<\frac{\varepsilon}{6}. \tag{43}$$

4. Similarly, we estimate the integrals $I_4(\tau,t)$ and $I_5(\tau,t)$ for $t\longrightarrow 0+$.

$$|I_4(\tau,t)|\le\int\limits_0^{+\infty}\left|\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\sin^2\frac{g_2(z,t)}{2}\cos^2\frac{\varphi(z/t)}{2}\sin(\tau z)\right|dz$$
$$\le\frac{1}{\sqrt{b}}\int\limits_0^{+\infty}\frac{A(z/t)\left(e^{\sqrt{2b}t}-1\right)}{e^{g_1(z,t)}}dz=\sqrt{2}\,t\int\limits_0^{+\infty}\frac{A(z/t)(1+o(1))}{e^{g_1(z,t)}}dz\le\mathrm{const}\cdot\sqrt{2}\,t^2\int\limits_0^{+\infty}\frac{A(\xi)}{e^{t\,G_1(\xi)}}d\xi;$$

and

$$|I_5(\tau,t)|\le\int\limits_0^{+\infty}\left|\frac{A(z/t)e^{-g_1(z,t)}\left(e^{2g_1(z,t)}-1\right)}{\rho(z/t)}\cos^2\frac{g_2(z,t)}{2}\sin^2\frac{\varphi(z/t)}{2}\sin(\tau z)\right|dz$$
$$\le\frac{1}{\sqrt{b}}\int\limits_0^{+\infty}\frac{A(z/t)\left(e^{\sqrt{2b}t}-1\right)}{e^{g_1(z,t)}}dz=\sqrt{2}\,t\int\limits_0^{+\infty}\frac{A(z/t)(1+o(1))}{e^{g_1(z,t)}}dz\le\mathrm{const}\cdot\sqrt{2}\,t^2\int\limits_0^{+\infty}\frac{A(\xi)}{e^{t\,G_1(\xi)}}d\xi.$$

From the last two expressions for $t\longrightarrow 0+$, we can deduce

$$|I_4(\tau,t)|<\frac{\varepsilon}{12},\quad |I_5(\tau,t)|<\frac{\varepsilon}{12}, \tag{44}$$

for any arbitrarily small fixed number $\varepsilon>0$.

5. Let us now estimate the integral $I_3(\tau,t)$. Taking into account the inequalities (22) and (42), we obtain

$$|I_3(\tau,t)|\le\int\limits_0^{+\infty}\left|\frac{A(z/t)e^{g_1(z,t)}\cos(g_2(z,t)+\tau z)-A(z/t)e^{-g_1(z,t)}\cos(g_2(z,t)-\tau z)}{\rho(z/t)}\right||\sin\varphi(z/t)|dz$$

$$\le\int\limits_0^{+\infty}A(z/t)\frac{e^{g_1(z,t)}+e^{-g_1(z,t)}}{\rho(z/t)}dz=\int\limits_0^{+\infty}A(z/t)\frac{e^{-g_1(z,t)}\left(e^{2g_1(z,t)}+1\right)}{\rho(z/t)}dz$$
$$\le\int\limits_0^{+\infty}A(z/t)\frac{e^{-g_1(z,t)}\left(e^{\sqrt{2b}\,t}+1\right)}{\rho(z/t)}dz\le\frac{e^{\sqrt{2b}\,T}+1}{\sqrt{b}}\int\limits_0^{+\infty}\frac{A(z/t)}{e^{g_1(z,t)}}dz=\frac{e^{\sqrt{2b}\,T}+1}{\sqrt{b}}\,t\int\limits_0^{+\infty}\frac{A(\xi)}{e^{t\,G_1(\xi)}}d\xi.$$

Since the resulting integral converges as the partial case of (18), the latter chain of inequalities for $t \longrightarrow 0+$ implies the estimate

$$|I_3(\tau,t)| < \frac{\varepsilon}{6} \tag{45}$$

for any arbitrarily small number $\varepsilon > 0$.

6. Consider now the integral $I_2(\tau,t)$. Let the function $A(z/t)$ be such that for $z \in [1,+\infty)$ the integrand in the integral $I_2(\tau,t)$ is majorized (in absolute value) by the function $|\mathrm{Const}+1|e^{-z}$, where $\mathrm{Const} > 0$ is a real constant. Therefore, you can choose such a large enough number $B > 0$ such that the inequality

$$\left| \int_B^{+\infty} e^{-z} \left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) - 1 \right] \cos(\tau z) dz \right| < \frac{\varepsilon}{12}$$

will be satisfied.

Fix the number $B > 0$ and evaluate for $0 \le z \le B$ the expression

$$\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t).$$

Let at $t = 0$ the value of the function

$$\widetilde{g}_1(z,t) := \frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) \tag{46}$$

be equal to one for any z. Let us show that the function defined in this way tends to unity at $t \longrightarrow 0+$ uniformly with respect to $z \in [0,B]$.

Suppose the contrary, that is, there is such a number $\varepsilon_0 > 0$ that for any positive number δ there are $t(\delta) \in (0,\delta)$ and $z(\delta) \in [0,B]$ such that the inequality

$$\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) > 1 + \varepsilon_0$$

is satisfied.

Consider the sequence $\delta_n = 1/n$, $n = 1,2,\ldots$. There exist the sequences $\{t_n\} \subset (0,1/n)$ and $\{z_n\} \subset [0,B]$ for which the inequality

$$|\widetilde{g}_1(z_n,t_n)| > 1 + \varepsilon_0$$

holds for any $n \in \mathbf{N}$. Without loss of generality, we can assume that the sequence $\{(z_n,t_n)\}$ of elements $(z_n,t_n) \in \mathbf{R}^2$ converges, and we denote its limit as (z_0,t_0). Obviously, $z_0 \in [0,B]$ and $t_0 = 0$; hence, $|\widetilde{g}_1(z_0,0)| > 1 + \varepsilon_0$, which contradicts the definition of the function $\widetilde{g}_1(z,t)$ on the axis $t = 0$. Thus, we have proved the inequality

$$\left| \int_0^{B} e^{-z} \left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) - 1 \right] \cos(\tau z) dz \right| < \frac{\varepsilon}{12}.$$

So, we have proved the estimate

$$|I_2(\tau,t)| \leq \left| \int_0^B e^{-z} \left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) - 1 \right] \cos(\tau z) dz \right|$$
$$+ \left| \int_B^{+} e^{-z} \left[\frac{A(z/t)e^{z-g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) - 1 \right] \cos(\tau z) dz \right| < \frac{\varepsilon}{24} + \frac{\varepsilon}{24} = \frac{\varepsilon}{6}. \quad (47)$$

7. Similarly, it can be shown that if, for $t = 0$, the value of the function

$$\widetilde{g}_2(z,t) := \frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) \quad (48)$$

is equal to one for any z, then the following inequality holds:

$$|I_1(\tau,t)| \leq \left| \int_0^B e^{-z} \left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) - 1 \right] \cos(\tau z) dz \right|$$
$$+ \left| \int_B^{+\infty} e^{-z} \left[\frac{A(z/t)e^{z+g_1(z,t)}}{\rho(z/t)} \sin g_2(z,t) \cos \varphi(z/t) - 1 \right] \cos(\tau z) dz \right| < \frac{\varepsilon}{12} + \frac{\varepsilon}{12} = \frac{\varepsilon}{6}. \quad (49)$$

Estimates (43)–(45), (47), and (49) prove the fulfillment of inequality (41), which implies the validity of relation (40).

8. Let us now estimate the expression (34). From the inequality (32), it follows that for $x \neq 0$ and $t \in (0,T]$ the inequality

$$|G(x,t)| \leq \frac{\widetilde{C}t}{x^2}$$

is satisfied, where $\widetilde{C} > 0$ is a constant. Hence, for $\tau \neq 0$ and $t \in (0,T]$, the estimate

$$|t\,G(t\tau,t)| \leq \frac{\widetilde{C}}{\tau^2}$$

holds. This means that the integrand in the integral $I_{3,A}$ is majorized by the function $2\widetilde{C} \sup |u_0|/\tau^2$, that is, for any $A > 0$, the inequality

$$|I_{3,A}| \leq \frac{\widetilde{C} \sup |u_0|}{\pi A}$$

is satisfied. Choose the number $A > 0$ sufficiently large so that we obtain the estimate

$$|I_{3,A}| < \varepsilon/3,$$

for any arbitrarily small number $\varepsilon > 0$.

The integral $I_{1,A}$ is estimated in a similar way:

$$|I_{1,A}| < \varepsilon/3.$$

Write the integrand in the integral $I_{2,A}$ in the following form:

$$t\,G(t\tau,t)u_0(x_0-t\tau)-t\,G(t\tau,t)u_0(x_0)+t\,G(t\tau,t)u_0(x_0)-\frac{2u_0(x_0)}{1+\tau^2}$$
$$=t\,G(t\tau,t)[u_0(x_0-t\tau)-u_0(x_0)]+u_0(x_0)\left[t\,G(t\tau,t)-\frac{2}{1+\tau^2}\right].$$

Let us write down $|I_{2,A}|=\left|\int\limits_{-A}^{0}+\int\limits_{0}^{A}\right|\leq 2\left|\int\limits_{0}^{A}\right|$. Given Equation (40), there is such $t_1\in(0,T]$ that the inequality

$$\left|t\,G(t\tau,t)-\frac{2}{1+\tau^2}\right|<\frac{\varepsilon}{12A\sup|u_0|}$$

is satisfied for any $t\in(0,t_1)$ and $\tau\in\mathbf{R}$, and for a chosen sufficiently large number $A>0$. Let $\varepsilon<12A\sup|u_0|$, then for any $t\in(0,t_1)$ we obtain the inequalities

$$\frac{2}{1+\tau^2}-1<t\,G(t\tau,t)<\frac{2}{1+\tau^2}+1,$$

that is, $|t\,G(t\tau,t)|\leq 3$.

Since the function $u_0(x)$ is continuous over the real line (together with its first derivative), there exists $t_0\in(0,t_1)$, such that the inequality

$$|u_0(x_0-t\tau)-u_0(x_0)|<\frac{\varepsilon}{36A\sup|u_0|}$$

is satisfied for any $t\in(0,t_0)$ and $\tau\in[-A,A]$. Thus, it is proved that for $0<t<t_0\leq T$ the following estimate holds:

$$|I_{2,A}|<\varepsilon/3.$$

The resulting estimates $|I_{1,A}|<\varepsilon/3$, $|I_{2,A}|<\varepsilon/3$ and $|I_{3,A}|<\varepsilon/3$, by virtue of an arbitrary choice of $\varepsilon>0$ and $x_0\in(-\infty,+\infty)$, prove that the first relation in (33) holds.

9. Let us now prove the second relation in (33). Using the definition (15), we first calculate

$$G_t(x,t)=\int\limits_0^{+\infty}\frac{A(\xi)}{\rho(\xi)}\Big[G_2(\xi)\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)e^{t\,G_1(\xi)}$$
$$+\,G_1(\xi)\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)e^{t\,G_1(\xi)}+G_2(\xi)\cos\left(t\,G_2(\xi)-\varphi(\xi)-x\xi\right)e^{-t\,G_1(\xi)}$$
$$-G_1(\xi)\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\xi\right)e^{-t\,G_1(\xi)}\Big]d\xi$$
$$=\int\limits_0^{+\infty}A(\xi)\Big[[\cos\varphi(\xi)\cos\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)+\sin\varphi(\xi)\sin\left(t\,G_2(\xi)+\varphi(\xi)+x\xi\right)]e^{t\,G_1(\xi)}$$
$$+[\cos\varphi(\xi)\cos\left(t\,G_2(\xi)-\varphi(\xi)-x\xi\right)-\sin\varphi(\xi)\sin\left(t\,G_2(\xi)-\varphi(\xi)-x\xi\right)]e^{-t\,G_1(\xi)}\Big]d\xi$$
$$=\int\limits_0^{+\infty}A(\xi)\Big[\cos\left(t\,G_2(\xi)+x\xi\right)e^{t\,G_1(\xi)}+\cos\left(t\,G_2(\xi)-x\xi\right)e^{-t\,G_1(\xi)}\Big]d\xi.$$

Let $x_0\in(-\infty,+\infty)$ be an arbitrary value. Consider the function

$$u_t(x_0,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} G_t(x_0-\eta,t)u_0(\eta)d\eta$$
$$= \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} u_0(\eta)\int\limits_{0}^{+\infty}\left(\frac{A(\xi)\,\cos\left(t\,G_2(\xi)+x_0\xi-\eta\xi\right)}{e^{-t\,G_1(\xi)}} + \frac{A(\xi)\,\cos\left(t\,G_2(\xi)-x_0\xi+\eta\xi\right)}{e^{t\,G_1(\xi)}}\right)d\xi\,d\eta.$$

In the last expression, make the change of variable $(x_0-\eta)/t=\tau$ and obtain

$$u_t(\eta+t\tau,t) = \frac{t}{2\pi}\int\limits_{-\infty}^{+\infty} G_t(t\tau,t)u_0(x_0-t\tau)d\tau.$$

Since the condition

$$\int\limits_{-\infty}^{+\infty}|u_0(\eta)|d\eta\int\limits_{-\infty}^{+\infty}|G_t(\eta,t)|d\eta < +\infty$$

is satisfied, the function $\int\limits_{-\infty}^{+\infty} G_t(t\tau,t)u_0(x_0-t\tau)d\tau$ exists in the region $\overline{D}$.

Since, according to the condition of the theorem, the function $u_0(x)$ is integrable over the real line, it is enough to check the condition $G_t(x,t)\in L_1(\mathbf{R})$. To do this, taking into account inequalities (30), we estimate the function

$$|G_t(x,t)| \le \int\limits_{0}^{+\infty}\left(\left|\frac{A(\xi)\,\cos\left(t\,G_2(\xi)+x\xi\right)}{e^{-t\,G_1(\xi)}}\right| + \left|\frac{A(\xi)\,\cos\left(t\,G_2(\xi)-x\xi\right)}{e^{t\,G_1(\xi)}}\right|\right)d\xi$$
$$\le \int\limits_{0}^{+\infty}\frac{A(\xi)}{e^{-t\,G_1(\xi)}}d\xi + \int\limits_{0}^{+\infty}\frac{A(\xi)}{e^{t\,G_1(\xi)}}d\xi.$$

The integrals on the right-hand side in the latter expression converge as the partial case of the integrals (18) for any $t\in[0,T]$. Thus, we have shown that $G_t(x,t)\in L_1(\mathbf{R})$ for any $t\in[0,T]$, which means that the function $\int\limits_{-\infty}^{+\infty} G_t(t\tau,t)u_0(x_0-t\tau)d\tau$ exists in the area $\overline{D}$.

Thus, for $t\longrightarrow 0+$, the estimate

$$|u_t(x_0,t)| = \left|\frac{t}{2\pi}\int\limits_{-\infty}^{+\infty} G_t(t\tau,t)u_0(x_0-t\tau)d\tau\right| < \varepsilon$$

is satisfied for any arbitrarily small number $\varepsilon>0$. This implies the fulfillment of the second relation in (33). Thus, the theorem is proved.

□

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