mathematics

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Operators in Rigged Hilbert Spaces, Gel'fand Bases and Generalized Eigenvalues

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Abstract: Given a self-adjoint operator A in a Hilbert space $\mathcal{H}$, we analyze its spectral behavior when it is expressed in terms of generalized eigenvectors. Using the formalism of Gel'fand distribution bases, we explore the conditions for the generalized eigenspaces to be one-dimensional, i.e., for A to have a simple spectrum.

Keywords: rigged Hilbert space; generalized eigenvectors; simple spectrum

MSC: 47A70; 42C15; 42C30

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Citation: Antoine, J.-P.; Trapani, C. Operators in Rigged Hilbert Spaces, Gel'fand Bases and Generalized Eigenvalues. *Mathematics* **2023**, *11*, 195. https://doi.org/10.3390/math11010195

Academic Editor: Manuel Gadella

Received: 29 November 2022
Revised: 21 December 2022
Accepted: 23 December 2022
Published: 30 December 2022

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{D}$ a dense domain in $\mathcal{H}$, equipped with a locally convex topology τ, finer than the norm topology. Let S be an essentially self-adjoint operator in $\mathcal{D}$ which maps $\mathcal{D}[\tau]$ into $\mathcal{D}[\tau]$ continuously. S has a continuous extension $\widehat{S}$ given by the conjugate duality (the adjoint, in other words; i.e., $\widehat{S} = S^\dagger$) from the conjugate dual space $\mathcal{D}^\times$ into itself. A *generalized eigenvector* of S, with eigenvalue $\lambda \in \mathbb{C}$, is then an eigenvector of $\hat{S}$; that is, a conjugate linear functional $\Phi_\lambda \in \mathcal{D}^\times$ such that:

$$\langle \Phi_\lambda | S f \rangle = \lambda \langle \Phi_\lambda | f \rangle, \quad \forall f \in \mathcal{D},$$

which we rewrite as $\widehat{S}\Phi_\lambda = S^\dagger \Phi_\lambda = \lambda \Phi_\lambda$. The completeness of the set $\{\Phi_\lambda; \lambda \in \sigma(\widehat{S})\}$ is expressed through the Parseval identity:

$$\|f\| = \left(\int_{\sigma(\widehat{S})} |\langle f | \Phi_\lambda \rangle|^2 d\mu(\lambda) \right)^{1/2}, \quad \forall f \in \mathcal{D},$$

where the positive measure μ may be, in general, both discrete and continuous.

In the present context, the Gel'fand–Maurin theorem [1] states that if $\mathcal{D}[\tau]$ is a nuclear domain in the Hilbert space, $\mathcal{H}$ and S is an essentially self-adjoint operator in $\mathcal{D}$ which maps $\mathcal{D}[\tau]$ into $\mathcal{D}[\tau]$ continuously, then S admits a *complete set of generalized eigenvectors*.

Let A now be a self-adjoint operator with dense domain $D(A)$ in a *separable* Hilbert space $\mathcal{H}$, which can be represented as

$$Af = \sum_{n=1}^{\infty} \lambda_n \langle f | e_n \rangle e_n, \quad f \in D(A), \tag{1}$$

with $\{e_n\}$ an orthonormal basis of $\mathcal{H}$. The notion of *simple spectrum* for the operator A corresponds to the fact that each eigenvalue has multiplicity 1. Of course this definition is of little use when the operator A has (also or only) a continuous spectrum. To cover this case a different definition has been proposed which relies on the notion of *generating* vector [2] (Definition 5.1) or [3] (§69). Let us denote by $E(\cdot)$ the spectral measure associated with the self-adjoint operator A.

Definition 1. *A vector $f \in \mathcal{H}$ is called a* generating *vector or* cyclic *vector for A if the linear span of vectors $E(\Delta)f$, $\Delta \in \mathcal{B}(\mathbb{R})$ (the σ-algebra of Borel sets of the real line), is dense in $\mathcal{H}$. We say that A has a simple spectrum if A has a generating vector.*

This definition is motivated by the fact that for operators of the form (1) one can find such a generating vector quite easily; in fact, the vector $h_0 = \sum_{k=1}^{\infty} \frac{e_k}{2^k}$ achieves this. This can also be expressed by stating that the von Neumann algebra $\mathfrak{A}$ generated by the spectral resolution $E(\Delta)$, $\Delta \in \mathcal{B}(\mathbb{R})$ of A is maximal abelian; i.e., $\mathfrak{A}' = \mathfrak{A}$. [Recall : An abelian von Neumann algebra in separable Hilbert space is maximal abelian if and only if it has a generating vector: [4] (Cor. 5.5.17; Cor. 7.2.16)].

In this paper, we will study the spectral behavior of self-adjoint operators A which are represented in the form

$$Af = \int_{\mathbb{R}} \lambda \langle f|\zeta_\lambda\rangle \zeta_\lambda d\lambda,$$

where the ζ_λ's are generalized eigenvectors. In particular, we determine the relationship between the spectral family of A and the sesquilinear forms defined by the ζ_λ's. Moreover, we discuss the notion of *simple spectrum* in this situation; in other words, we expect to find conditions for every subspace $\mathcal{D}_\lambda^\times$ of generalized eigenvectors, corresponding to $\lambda \in \mathbb{R}$, to be one-dimensional. An interesting tool in this context is that of *Gel'fand distribution basis*, introduced recently in [5] and defined in Section 3. In Section 4, we will see how this notion may be utilized for characterizing the class of operators we consider.

Before going further, we fix some notations and give some preliminary results on self-adjoint operators.

Let A be a self-adjoint operator with dense domain $D(A)$. As usual we denote by $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$ the point spectrum (i.e., the set of *true* eigenvalues), the continuous spectrum and the residual spectrum of an operator A, respectively. If A is self-adjoint then $\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$ and $\sigma_r(A) = \emptyset$.

If the spectrum of A contains some *true* eigenvalue λ_0 , then the spectral measure of the singleton $\{\lambda_0\}$ is given by the jump of the spectral function $E(\lambda_0) - E(\lambda_0^-)$ (remember that the spectral family is strongly right continuous, i.e., $\lim_{\lambda\to\lambda_0^+} E(\lambda)f = E(\lambda_0)f$, for every $\lambda_0 \in \mathbb{R}$ and every $f \in \mathcal{H}$. This can be deduced from the following theorem which describes the spectrum of a self-adjoint operator A in terms of its spectral family [3] (Ch.6, §68).

Proposition 1. *Let $\{E(\lambda)\}$ be the spectral family of the self-adjoint operator A and λ_0 a real number. Then,*

(i) $\lambda_0 \notin \sigma(A)$ *if, and only if,* $E(\lambda)$ *is constant in a neighborhood of* λ_0.
(ii) λ_0 *is an eigenvalue if, and only if,* $E(\lambda)$ *is discontinuous in* λ_0.
(iii) $\lambda_0 \in \sigma_c(A)$ *if, and only if,* $E(\lambda)$ *is continuous in* λ_0*, but non constant in every neighborhood of* λ_0.

As a consequence, a real number λ_0 belongs to the spectrum $\sigma(A)$ of A if, and only if, $E(\lambda_0 + \epsilon) - E(\lambda_0 - \epsilon) \neq 0$, for every $\epsilon > 0$.

Remark 1. *Since the Hilbert space $\mathcal{H}$ is separable, the point spectrum $\sigma_p(A)$ of A consists at most of a countable set of* true *eigenvalues. Therefore, the continuous spectrum $\sigma_p(A)$ is a Borel set and, for every $f \in \mathcal{H}$, the restriction of the spectral measure $\langle E(\lambda)f|f\rangle$ to $\sigma_c(A)$ may be, for certain f, absolutely continuous with respect to the Lebesgue measure (restricted to the same set). Indeed, according to [6] (Ch.VII.2) one could consider a different decomposition of the spectrum into pure point $\sigma_{pp}(A)$, continuous $\sigma_{cont}(A)$, absolutely continuous $\sigma_{ac}(A)$ and singular spectrum $\sigma_s(A)$ (these sets need not be disjoint). Corresponding to this, the Hilbert space $\mathcal{H}$ decomposes as $\mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_s$ where $\mathcal{H}_{pp}$, $\mathcal{H}_{ac}$, $\mathcal{H}_s$ are, respectively, the pure point, absolutely continuous* *and singular parts of $\mathcal{H}$ obtained via the corresponding Lebesgue decomposition of the measures generated by the functions $\langle E(\lambda)f|f\rangle$, $f \in \mathcal{H}$.*

We will not go into further details on the spectral analysis of self-adjoint operators, for which we refer to [2,3,6]; for a discussion of this matter in the framework of RHS's, see also [7].

2. Rigged Hilbert Spaces and Quantum Mechanics

2.1. Rigged Hilbert Spaces

Let, as before, $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. A locally convex topology τ on $\mathcal{D}$ finer than the topology induced by the Hilbert norm defines, in standard fashion, a *rigged Hilbert space* (RHS, for short)

$$\mathcal{D}[\tau] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[\tau^\times], \tag{2}$$

where $\mathcal{D}^\times$ is the vector space of all continuous conjugate linear functionals on $\mathcal{D}[\tau]$, i.e., the conjugate dual of $\mathcal{D}[\tau]$, endowed with the *strong dual topology* $\tau^\times = \beta(\mathcal{D}^\times, \mathcal{D})$, which can be defined by the seminorms

$$q_{\mathcal{M}}(F) = \sup_{g \in \mathcal{M}} |\langle F|g\rangle|, \quad F \in \mathcal{D}^\times, \tag{3}$$

where $\mathcal{M}$ is a bounded subset of $\mathcal{D}[\tau]$.

Since the Hilbert space $\mathcal{H}$ can be identified with a subspace of $\mathcal{D}^\times[\tau^\times]$, we will systematically read (2) as a chain of topological inclusions: $\mathcal{D}[\tau] \subset \mathcal{H} \subset \mathcal{D}^\times[\tau^\times]$. These identifications imply that the sesquilinear form $B(\cdot,\cdot)$ which puts $\mathcal{D}$ and $\mathcal{D}^\times$ in duality is an extension of the inner product of $\mathcal{H}$; i.e., $B(\xi,\eta) = \langle \xi|\eta\rangle$, for every $\xi,\eta \in \mathcal{D}$ (to simplify notations we adopt the symbol $\langle\cdot|\cdot\rangle$ for both of them) and also that the embedding map $I_{\mathcal{D},\mathcal{D}^\times} : \mathcal{D} \to \mathcal{D}^\times$ can be taken to act on $\mathcal{D}$ as $I_{\mathcal{D},\mathcal{D}^\times} f = f$ for every $f \in \mathcal{D}$.

Let now $\mathcal{D}[\tau] \subset \mathcal{H} \subset \mathcal{D}^\times[\tau^\times]$ be a rigged Hilbert space, and let $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ denote the vector space of all continuous linear maps from $\mathcal{D}[\tau]$ into $\mathcal{D}^\times[\tau^\times]$. If $\mathcal{D}[\tau]$ is barreled (e.g., reflexive), an involution $X \mapsto X^\dagger$ can be introduced in $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ by the equality

$$\left\langle X^\dagger \eta|\xi\right\rangle = \overline{\langle X\xi|\eta\rangle}, \quad \forall \xi,\eta \in \mathcal{D}.$$

Hence, in this case, $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ is a †-invariant vector space.

If $\mathcal{D}[\tau]$ is a smooth space (e.g., Fréchet and reflexive), then $\mathcal{L}(\mathcal{D},\mathcal{D}^\times)$ is a quasi *-algebra over $\mathcal{L}^\dagger(\mathcal{D})$ [8] (Definition 2.1.9).

We also denote by $\mathcal{L}(\mathcal{D})$ the algebra of all continuous linear operators $Y : \mathcal{D}[\tau] \to \mathcal{D}[\tau]$ and by $\mathcal{L}(\mathcal{D}^\times)$ the algebra of all continuous linear operators $Z : \mathcal{D}^\times[\tau^\times] \to \mathcal{D}^\times[\tau^\times]$. If $\mathcal{D}[\tau]$ is reflexive, for every $Y \in \mathcal{L}(\mathcal{D})$ there exists a unique operator $Y^\times \in \mathcal{L}(\mathcal{D}^\times)$, the adjoint of Y, such that

$$\langle \Phi|Yg\rangle = \langle Y^\times \Phi|g\rangle, \quad \forall \Phi \in \mathcal{D}^\times, g \in \mathcal{D}.$$

In a similar way, an operator $Z \in \mathcal{L}(\mathcal{D}^\times)$ has an adjoint $Z^\times \in \mathcal{L}(\mathcal{D})$ such that $(Z^\times)^\times = Z$.

A typical example of RHS can be constructed starting from a self-adjoint operator A with domain $D(A)$ in the Hilbert space $\mathcal{H}$. Let $\mathcal{D} := \bigcap_{n\in\mathbb{N}} D(A^n)$. Then $\mathcal{D}$ is a dense invariant core fo A and $A\mathcal{D} \subset \mathcal{D}$. Typically, one endows $\mathcal{D}$ with the *graph* topology τ defined by the set of seminorms $f \in \mathcal{D} \mapsto \|A^n f\|$, $n = 0,1,\ldots$. The space $D[\tau]$ is a Fréchet reflexive space. We will call this RHS the *canonical RHS associated with* A (Section 5.4 in [9]).

2.2. Rigged Hilbert Spaces in Quantum Mechanics

For convenience, we follow the recent review by Ref. [10]. The first rigorous formulation of quantum mechanics was that of von Neumann, solely based on Hilbert space concepts. However, most physicists adopted the simpler and more intuitive bra-ket formalism of Dirac and standard textbooks still do the same. Of course, this is not satisfactory from a mathematical perspective. A solution (advocated by Bargmann) is to work in an RHS. This approach was proposed, independently, by one of this paper's authors (JPA) [11–13], Roberts [14,15] and Bohm [16]. See also Bohm-Gadella [17]. One crucial point is how to choose the left-hand space $\mathcal{D}[\tau]$. The elegant solution of Roberts is to start from a family of so-called *labeled observables*, realized by essentially self-adjoint operators having a common dense invariant domain $\mathcal{D}$ and a clear physical significance (position, momentum, energy, angular momentum, etc.). In fact, this idea is older, but not formalized; for instance, Wigner kept asking "What does $1/r$ mean for a harmonic oscillator ?" The key point is that the Gel'fand–Maurin theorem discussed above allows us to combine Dirac's formalism and mathematical rigor. Indeed, a few textbooks have adopted this point of view. In fact, the RHS formulation of quantum mechanics has become an active research theme, focusing on various aspects, such as scattering theory [7] or change of representation, a cornerstone of Dirac's approach [18]. An early review may be found in [19]. Indeed, several versions of axiomatic quantum field theory are advantageously presented from an RHS approach [20]. In addition, the RHS approach allows an interpretation of the extreme spaces: elements of $\mathcal{D}[t]$ may be taken as physically realizable states, whereas $\mathcal{D}^\times[t]$ contains generalized states associated with measurement operations. This is consistent with the dual interpretation of symmetries, namely active vs. passive point of views.

Returning back to the simple case on nonrelativistic QM, the natural choice of RHS is Schwartz's triplet

$$\mathcal{S}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \subset \mathcal{S}^\times(\mathbb{R}^3),$$

where $\mathcal{S}(\mathbb{R}^3)$ is the space of smooth fast decreasing functions and $\mathcal{S}^\times(\mathbb{R}^3)$ is the space of tempered distributions. Then, indeed $\mathcal{S}(\mathbb{R}^3)$ is obtained by choosing as labeled observables the operators $\mathbf{q}$ (position) and $\mathbf{p}$ (momentum) or a suitable polynomial of them, for instance the Hamiltonian of the harmonic oscillator $H = \frac{1}{2}(\mathbf{p}^2 + \mathbf{q}^2)$.

3. Gel'fand Distribution Bases

Let (X, μ) be a measure space with μ a σ-finite positive measure. $x \in X \mapsto \omega_x \in \mathcal{D}^\times$ a weakly measurable map; i.e., we suppose that, for every $f \in \mathcal{D}$, the complex valued function $x \mapsto \langle f|\omega_x\rangle$ is μ-measurable. Since the form which puts $\mathcal{D}$ and $\mathcal{D}^\times$ in conjugate duality is an extension of the inner product of $\mathcal{H}$, we write $\langle f|\omega_x\rangle$ for $\overline{\langle \omega_x|f\rangle}$, $f \in \mathcal{D}$.

We will say that

(i) ω is *total* if, $f \in \mathcal{D}$ and $\langle f|\omega_x\rangle = 0$ for almost every $x \in X$ implies $f = 0$;
(ii) ω is *μ-independent* if the unique measurable function ξ on $\mathbb{R}$ such that $\int_X \xi(x)\langle g|\omega_x\rangle d\mu = 0$, for every $g \in \mathcal{D}$, is $\xi(x) = 0$ μ-a.e.

We recall the following definitions and facts from [5].

Definition 2. *Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space and $\zeta : x \in X \mapsto \zeta_x \in \mathcal{D}^\times$ be a weakly measurable map from $\mathcal{D}$ into $\mathcal{D}^\times$. We say that ζ is a* Gel'fand distribution basis *if ζ is μ-independent and satisfies the Parseval identity: i.e.,*

$$\int_X |\langle f|\zeta_x\rangle|^2 d\mu = \|f\|^2, \quad f \in \mathcal{D}. \tag{4}$$

By (4), it follows that if $g \in \mathcal{H}$ and $\{g_n\}$ is a sequence of elements of $\mathcal{D}$, norm converging to g, then the sequence $\{\eta_n\}$, where $\eta_n(x) = \langle g_n|\zeta_x\rangle$, is convergent in $L^2(X, \mu)$. Put $\eta = \lim_{n\to\infty} \eta_n$. The function $\eta \in L^2(X, \mu)$ depends linearly on g, for each $x \in X$. Thus, a linear functional $\check{\zeta}_x$ on $\mathcal{H}$ can be defined by

$$\langle g|\check{\zeta}_x\rangle = \lim_{n\to\infty} \langle g_n|\zeta_x\rangle, \quad g \in \mathcal{H};\, g_n \to g. \tag{5}$$

For each $x \in X$, $\check{\zeta}_x$ extends ζ_x, but $\check{\zeta}_x$ need not be continuous, as a functional on $\mathcal{H}$.

Moreover, in this case, the sesquilinear form Ω associated with the quadratic form (4); i.e.,

$$\Omega(f, g) = \int_X \langle f|\zeta_x\rangle\langle \zeta_x|g\rangle d\mu, \quad f, g \in \mathcal{D},$$

is well defined on $\mathcal{D} \times \mathcal{D}$, it is bounded with respect to $\| \cdot \|$ and possesses a bounded extension $\widehat{\Omega}$ to $\mathcal{H}$.

The following result describes the behavior of Gel'fand distribution bases.

Corollary 1. *Let ζ be a Gel'fand distribution basis. The following statements hold.*

(i) *For every $f \in \mathcal{H}$, there exists a unique function $\xi_f \in L^2(X, \mu)$ such that*

$$f = \int_X \xi_f(x)\zeta_x d\mu.$$

In particular, if $f \in \mathcal{D}$, then $\xi_f(x) = \langle f|\zeta_x\rangle$ μ-a.e.

(ii) *For every fixed $x \in X$, the map $f \in \mathcal{H} \mapsto \xi_f(x) \in \mathbb{C}$ defines as in (5) a linear functional $\check{\zeta}_x$ on $\mathcal{H}$ and*

$$f = \int_X \langle f|\check{\zeta}_x\rangle \zeta_x d\mu, \quad \forall f \in \mathcal{H}.$$

As proved in [5] (Proposition 3.15) the *synthesis* operator T_ζ defined by

$$T_z g = \int_X g(x)\zeta_x d\mu, \quad g \in L^2(X, \mu)$$

is an isometry of $L^2(X, \mu)$ *onto* $\mathcal{H}$. Its adjoint $T_\zeta^\times : \mathcal{H} \to L^2(X, \mu)$ associates to each vector $f \in \mathcal{H}$ the function $\xi_f \in L^2(X, \mu)$ whose existence is guaranteed by (i) of the previous corollary.

4. Operators Constructed from Gel'fand Distribution Bases

4.1. Construction of the Operators

Let $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ be a rigged Hilbert space. We suppose that the space $\mathcal{D}$ is reflexive under its topology τ. In this section, we consider the measure space $(\mathbb{R}, d\lambda)$, where $d\lambda$ denotes the Lebesgue measure. Let $\zeta : \mathbb{R} \to \mathcal{D}^\times$ be a Gel'fand distribution basis.

Consider the following operator A (see also, [5] (Section 4)):

$$D(A) = \left\{ f \in \mathcal{H} : \int_{\mathbb{R}} \lambda^2 |\langle f|\check{\zeta}_\lambda\rangle|^2 d\lambda < \infty \right\}, \tag{6}$$

$$Af = \int_{\mathbb{R}} \lambda \langle f|\check{\zeta}_\lambda\rangle \zeta_\lambda d\lambda, \quad f \in D(A), \tag{7}$$

where the last equality is intended as a conjugate linear functional on $\mathcal{D}$:

$$\Phi_{A,f}(g) := \langle Af|g\rangle = \int_{\mathbb{R}} \lambda \langle f|\check{\zeta}_\lambda\rangle \langle \zeta_\lambda|g\rangle d\lambda, \quad g \in \mathcal{D}.$$

It is easily seen that $\Phi_{A,f}$ is bounded; thus by a limiting procedure we can extend it to $\mathcal{H}$, where we get

$$\widehat{\Phi}_{A,f}(g) := \langle Af|g\rangle = \int_{\mathbb{R}} \lambda \langle f|\check{\zeta}_\lambda\rangle \langle \check{\zeta}_\lambda|g\rangle d\lambda, \quad g \in \mathcal{H}.$$

Then, we have

Lemma 1. *$(A, D(A))$ is a self-adjoint operator in $\mathcal{H}$.*

Proof. Let $h \in D(A^*)$; then there exists $h^* \in \mathcal{H}$ such that.

$$\langle Af|h\rangle = \int_{\mathbb{R}} \lambda \langle f|\check{\zeta}_\lambda\rangle \langle \check{\zeta}_\lambda|h\rangle d\lambda = \langle f|h^*\rangle,$$

if, and only if, $h^* = \int_{\mathbb{R}} \lambda \langle h|\check{\zeta}_\lambda\rangle \zeta_\lambda d\lambda$ and $\int_{\mathbb{R}} \lambda^2 |\langle h|\check{\zeta}_\lambda\rangle|^2 d\lambda < \infty$; i.e., if, and only if, $h \in D(A)$ and $A^*h = Ah$. $\square$

In a similar way, if u is a Borel function on $\mathbb{R}$, we define

$$D(A_u) = \left\{ f \in \mathcal{H} : \int_{\mathbb{R}} |u(\lambda)|^2 \left| \langle f | \check{\zeta_\lambda} \rangle \right|^2 d\lambda < \infty \right\}, \tag{8}$$

$$A_u f = \int_{\mathbb{R}} u(\lambda) \langle f | \check{\zeta_\lambda} \rangle \zeta_\lambda d\lambda, \quad f \in D(A_u). \tag{9}$$

If $D(A_u)$ is dense, then A_u^* is the operator corresponding to $\overline{u}$, the conjugate of u.

Lemma 2. *The operator A_u is bounded if, and only if, $u \in L^\infty(X, \mu)$.*

Proof. The sufficiency is immediate. The necessity follows from the fact that the analysis operator is also invertible; thus $uh \in L^2(X, \mu)$ for every $h \in L^2(X, \mu)$. This, in turn, implies that $u \in L^\infty(X, \mu)$. □

If u is bounded, then $D(A_u) = \mathcal{H}$ and A_u is bounded.

Let us suppose that $f \in \mathcal{D}$ implies $Af \in \mathcal{D}$. Then, it makes sense to compute $\langle Af | \zeta_\mu \rangle$, which is obviously given by

$$\langle Af | \zeta_\mu \rangle = \left\langle \int_{\mathbb{R}} \lambda \langle f | \zeta_\lambda \rangle \zeta_\lambda d\lambda | \zeta_\mu \right\rangle.$$

On the other hand

$$\langle Af | \zeta_\mu \rangle = \langle f | \hat{A} \zeta_\mu \rangle = \mu \langle f | \zeta_\mu \rangle,$$

by definition of a generalized eigenvector. Thus, we have

$$\left\langle \int_{\mathbb{R}} \lambda \langle f | \zeta_\lambda \rangle \zeta_\lambda d\lambda \middle| \zeta_\mu \right\rangle = \mu \langle f | \zeta_\mu \rangle, \quad \forall f \in \mathcal{D}.$$

Remark 2. *However, we cannot write the lhs as $\int_{\mathbb{R}} \lambda \langle f | \zeta_\lambda \rangle \langle \zeta_\lambda | \zeta_\mu \rangle d\lambda$ since this is undefined, in general.*

Since A is self-adjoint, its spectrum $\sigma(A)$ is real. Using (8) and (9), we find that for every $z \in \mathbb{C} \setminus \mathbb{R}$, the resolvent operator $(A - zI)^{-1}$ is given by

$$(A - zI)^{-1} f = \int_{\mathbb{R}} \frac{1}{\lambda - z} \langle f | \check{\zeta_\lambda} \rangle \zeta_\lambda d\lambda. \tag{10}$$

Remark 3. *By Lemma 1, the operator A is self-adjoint; then it has a spectral decomposition*

$$Af = \int_{\mathbb{R}} \lambda dE(\lambda) f, \quad \forall f \in D(A).$$

The previous equality and (7) provide strong clues to believe that there is a sort of relationship of absolute continuity of the measure defined by $\langle E(\lambda) f | f \rangle$, $f \in \mathcal{H}$ and the Lebesgue measure. This is not too far from truth; however, a more precise analysis must be undertaken.

We begin with proving that, as expected, the operator A defined in (7) has only a continuous spectrum.

Lemma 3. *Let A be the operator defined in (7). Then, for every $\mu \in \mathbb{R}$ the operator $A - \mu I$ is injective and $(A - \mu I)D(A)$ is a proper dense subspace of $\mathcal{H}$; i.e., $\mu \in \sigma_c(A)$.*

Proof. Let $f \in D(A)$ be a solution of $Af - \mu f$=0. Then

$$\langle Af - \mu f | f \rangle = \int_{\mathbb{R}} (\lambda - \mu) |\langle f | \check{\zeta_\lambda} \rangle|^2 d\lambda = 0.$$

Since λ varies in $\mathbb{R}$ one must have $\langle f | \check{\zeta_\lambda} \rangle = 0$ a.e.; this shows that $f = 0$. Moreover, the residual spectrum of A is empty (because A is selfadjoint). Then necessarily $(A - \mu I)D(A)$

is dense in $\mathcal{H}$. The inverse of $A - \mu I$ is given by (10) with $z = \mu$ and by Lemma 2 it follows that $(A - \mu I)^{-1}$ is unbounded if $\mu \in \mathbb{R}$. □

Theorem 1. *Let $\mathcal{D} \subset H \subset \mathcal{D}^\times$ be the canonical RHS associated with a self-adjoint operator A. For every $f, g \in \mathcal{D}$ the function $\lambda \in \mathbb{R} \mapsto \langle E(\lambda) f | g \rangle$ is almost everywhere differentiable and there exists a positive operator $Y_\lambda \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ such that*

$$\frac{d}{d\lambda}\langle E(\lambda) f | g \rangle = \langle Y_\lambda f | g \rangle, \quad \forall f, g \in \mathcal{D}.$$

Proof. Let $f \in \mathcal{D}$. The function $\lambda \in \mathbb{R} \mapsto \langle E(\lambda) f | f \rangle$ is nonnegative and increasing. Then, it is differentiable almost everywhere, by Lebesgue's differentiation theorem; the derivative, of course, depends on f.

Using the polarization identity, we conclude that the function $\lambda \mapsto \langle E(\lambda) f | g \rangle$, $f, g \in \mathcal{D}$, which is bounded variation, is also differentiable a.e. Put

$$u_{f,g}(\lambda) = \frac{d}{d\lambda}\langle E(\lambda) f | g \rangle.$$

It can be easily seen that $u_{f,g}(\lambda)$ is a sesquilinear form in f, g. Now, from the usual spectral calculus one deduces that, for every $\lambda \in \mathbb{R}$, $E(\lambda) f \in D(A^n)$, for each $f \in \mathcal{H}$ and the following inequality holds.

$$\|A^n E(\lambda) f\| = \|E(\lambda) A^n f\| \leq \|A^n f\|, \quad \forall n \in \mathbb{N}, \lambda \in \mathbb{R}, f \in \mathcal{D}.$$

This implies that for each $\lambda \in \mathbb{R}$, $E(\lambda)$ (which maps $\mathcal{D}$ into $\mathcal{D}$) is continuous from $\mathcal{D}[\tau_A]$ into itself. This, in turn implies that, for almost every $\lambda \in \mathbb{R}$, $u_{f,g}(\lambda)$ depends continuously on $f \in \mathcal{D}$. By symmetry, it is separately continuous; however, since $\mathcal{D}[\tau_A]$ is a Fréchet space, it is jointly continuous; i.e., for almost every $\lambda \in \mathbb{R}$ there exists $n \in \mathbb{N}$ and $\gamma_\lambda > 0$ such that

$$|u_{f,g}(\lambda)| = \left| \frac{d}{d\lambda}\langle E(\lambda) f | g \rangle \right| \leq \gamma_\lambda \|A^n f\| \, \|A^n g\|, \quad \forall f, g \in \mathcal{D}.$$

In this case [8] (Ch.10), there exists an operator $Y_\lambda \in \mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ such that

$$u_{f,g}(\lambda) = \langle Y_\lambda f | g \rangle, \quad \forall f, g \in \mathcal{D}.$$

□

On the basis of the previous considerations we expect that in the case of the operator A defined by a Gel'fand distribution basis, we should have

$$\frac{d}{d\lambda}\langle E(\lambda) f | g \rangle = \langle f | \zeta_\lambda \rangle \langle \zeta_\lambda | g \rangle, \quad f, g \in \mathcal{D}.$$

Let us first examine some simple yet significant examples.

Example 1. *In the RHS $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times(\mathbb{R})$, let us consider the operator Q of multiplication by x, i.e.,*

$$(Qf)(x) = xf(x); \quad f \in \mathcal{S}(\mathbb{R}).$$

This operator is the restriction to $\mathcal{S}(\mathbb{R})$ of the multiplication operator by x defined on

$$D(Q) = \{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R})\},$$
$$(Qf)(x) = xf(x), \quad f \in D(Q).$$

This operator has a Gel'fand basis of generalized eigenvectors given by the distributions δ_λ, the Dirac delta centered at λ. Indeed,

$$\langle Q\phi | \delta_\lambda \rangle \langle x\phi(x) | \delta_\lambda \rangle = \lambda \phi(\lambda) = \lambda \langle \phi | \delta_\lambda \rangle, \quad \forall \phi \in \mathcal{S}(\mathbb{R}),$$

which can be read as $Q\delta_\lambda = \lambda\delta_\lambda$.

The spectral resolution of Q*, on the other hand, can be written as*

$$\langle Q\phi|\psi\rangle = \int_{\mathbb{R}} \lambda d\langle E(\lambda)\phi|\psi\rangle, \quad \forall\phi,\psi \in \mathcal{S}(\mathbb{R}),$$

where $E(\lambda)\phi = \chi_\lambda\phi$*, with* χ_λ *the characteristic function of* $(-\infty,\lambda]$.

It is clear that we can also write

$$\langle Q\phi|\psi\rangle = \int_{\mathbb{R}} \lambda\langle\phi|\delta_\lambda\rangle\langle\delta_\lambda|\psi\rangle d\lambda.$$

We have

$$\begin{aligned}\frac{d}{d\lambda}\langle\chi_\lambda\phi|\psi\rangle &= \frac{d}{d\lambda}\int_{\mathbb{R}} \chi_\lambda(x)\phi(x)\overline{\psi(x)}dx \\ &= \frac{d}{d\lambda}\left(\int_{-\infty}^{\lambda} \phi(x)\overline{\psi(x)}dx\right) \\ &= \phi(\lambda)\overline{\psi(\lambda)} = \langle\phi|\delta_\lambda\rangle\langle\delta_\lambda|\psi\rangle.\end{aligned}$$

Example 2. *In the RHS* $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times(\mathbb{R})$ *let us consider the momentum operator* P*, i.e.,*

$$(Pf)(x) = -i\frac{d}{dx}f(x), \quad f \in \mathcal{S}(\mathbb{R}).$$

This operator is the restriction to $\mathcal{S}(\mathbb{R})$ *of the momentum operator defined on* $D(P) = W^{1,2}(\mathbb{R})$*, the familiar Sobolev space.*

This operator has a Gel'fand basis of generalized eigenvectors given by the regular distributions $\zeta_\lambda(x) = \frac{1}{\sqrt{2\pi}}e^{i\lambda x}$*. The use of Fourier transform on the computations of Example 1 shows that*

$$\frac{d}{d\lambda}\langle E(\lambda)\phi|\psi\rangle = \widehat{\phi}(\lambda)\overline{\widehat{\psi}(\lambda)}, \quad \forall\phi,\psi \in \mathcal{S}(\mathbb{R}),$$

where $\widehat{g}$ *denotes the Fourier transform of* g.

The previous example motivates the conjecture that if A is a self-adjoint operator given in terms of some Gel'fand basis ζ as in (7) and $E(\lambda)$ is the spectral family of A, then the spectral family $\{E(\mu);\, \mu \in \mathbb{R}\}$ of A should be given by

$$\langle E(\mu)f|g\rangle = \int_{-\infty}^{\mu} \langle f|\zeta_\lambda\rangle\langle\zeta_\lambda|g\rangle d\lambda, \quad f,g \in \mathcal{D}.$$

In order to study this problem, we begin with considering for $\mu \in \mathbb{R}$ the sesquilinear form

$$\Theta_\mu(f,g) = \int_{\mathbb{R}} \chi_\mu(\lambda)\langle f|\zeta_\lambda\rangle\langle\zeta_\lambda|g\rangle d\lambda,$$

where χ_μ is the characteristic function of $(-\infty,\mu]$. Then Θ is a bounded sesquilinear form on $\mathcal{D} \times \mathcal{D}$, with respect to the norm of $\mathcal{H}$ and hence there exists a bounded operator $B(\mu)$ such that $\Theta_\mu(f,g) = \langle B(\mu)f|g\rangle$. A Gel'fand distribution basis ζ is characterized by the fact that the synthesis operator $T_\zeta : u \in L^2(\mathbb{R}) \mapsto \int_{\mathbb{R}} u(\lambda)\zeta_\lambda d\lambda$ takes values in $\mathcal{H}$ and it is an isometry of $L^2(\mathbb{R})$ onto $\mathcal{H}$ [5] (Proposition 3.15); thus its inverse (or adjoint) $T_\zeta^\times$ (i.e., the analysis operator) is also isometric. Let us denote by $M(\mu)$ the multiplication operator by χ_μ (the characteristic function of $(-\infty,\mu]$), then Θ_μ is represented by the operator $T_\zeta M(\mu)T_\zeta^\times$ which is a projection operator that we denote by $F(\mu)$.

Then the equality

$$\langle F(\mu f)|g\rangle = \int_{-\infty}^{\mu} \langle f|\zeta_\lambda\rangle\langle\zeta_\lambda|g\rangle d\lambda$$

implies that

$$\langle Af|g\rangle = \int_{\mathbb{R}} \lambda \langle f|\zeta_\lambda\rangle\langle\zeta_\lambda|g\rangle d\lambda = \int_{\mathbb{R}} \lambda d\langle F(\lambda)f|g\rangle, \quad \forall f,g \in \mathcal{D}.$$

The uniqueness of the spectral measure implies that $E(\lambda) = F(\lambda)$, for almost every λ. These considerations prove the following theorem.

Theorem 2. *Let A be the self-adjoint operator defined by a Gel'fand distribution basis ζ, in the sense that*

$$Af = \int_{\mathbb{R}} \lambda \langle f|\check{\zeta_\lambda}\rangle \zeta_\lambda d\lambda, \quad f \in D(A).$$

Then, the spectral measure $E(\cdot)$ of A is absolutely continuous with respect to the Lebesgue measure and

$$\langle f|\check{\zeta_\lambda}\rangle\langle\zeta_\lambda|g\rangle = \frac{d}{d\lambda}\langle E(\lambda)f|g\rangle, \quad \forall f,g \in \mathcal{D}.$$

Remark 4. *Replacing ζ_λ with $\check{\zeta_\lambda}$, the previous equality extends to $f,g \in \mathcal{H}$.*

Remark 5. *As a consequence of Theorem 2, in the case under consideration, we conclude that the operators defined in (8), (9) satisfy the usual rules of the functional calculus for unbounded self-adjoint operators. In particular, if u,v are Borel functions on $\mathbb{R}$, one has*

$$A_{u+v} \supset A_u + A_v, \qquad A_{uv} \supset A_u A_v.$$

4.2. Simple Spectrum

In both examples considered above, every generalized eigenvalue has multiplicity 1; i.e., the subspace of generalized eigenvector corresponding to every generalized eigenvalue λ has dimension 1. In finite-dimensional spaces, this is exactly the definition of the operator having simple spectrum.

Let us now discuss in general terms how the notion of simple spectrum can be handled in this context.

Given a self-adjoint operator A, we consider the canonical rigged Hilbert space associated with A. We denote by $\mathcal{D}^\times_\lambda \subset \mathcal{D}^\times$ the subspace consisting of all generalized eigenvectors corresponding to the generalized eigenvalue λ. For all $f \in \mathcal{D}$, one can define a linear functional $\tilde{f}_\lambda$ on $\mathcal{D}^\times_\lambda$ by $\tilde{f}_\lambda(\Phi_\lambda) := \langle\Phi_\lambda|f\rangle$ for all $\Phi_\lambda \in \mathcal{D}^\times_\lambda$. On the other hand, it is easily seen that every continuous functional on $\mathcal{D}^\times_\lambda$ has the form $\tilde{f}_\lambda$ for some $f \in \mathcal{D}$. We denote by $\mathcal{D}^{\times\times}_\lambda$ the space of all these functionals. The correspondence $\mathcal{D} \to \mathcal{D}^{\times\times}_\lambda$ defined by $f \mapsto \tilde{f}_\lambda$ is called the *spectral decomposition of the element f corresponding to A*. If $\tilde{f}_\lambda \equiv 0$ for every generalized eigenvalue λ, implies $f = 0$ then A is sometimes said to have *a complete system of generalized eigenvectors*. However, this condition is not sufficient to guarantee that generalized eigenvectors are a Gelf'and distribution basis and will not be adopted for this reason.

As noted before, in finite dimensional spaces, an operator is said to have a simple spectrum if each eigenvalue has multiplicity 1. In the infinite dimensional case, this notion is useless due to the possible presence of a continuous part of the spectrum. If A is a bounded self-adjoint operator in $\mathcal{H}$, then one says that ξ is *generating* if the set $\{p(A)\xi_0; p \text{ polynomial}\}$ is dense in $\mathcal{H}$. As is known, this is equivalent to say that the von Neumann algebra $\mathcal{A}$ generated by A is maximal abelian (i.e., it coincides with its commutant $\mathcal{A}'$). These facts, in turn, are equivalent to A being unitarily equivalent to a multiplication operator: Assume in fact that A has a generating vector $\xi \in \mathcal{H}$ and put $\mu_\xi(\Delta) = \langle E(\Delta)\xi|\xi\rangle$, $\Delta \in \mathcal{B}(\mathbb{R})$, the σ-algebra of Borel sets on the real line. There then exists a unitary operator $U : \mathcal{H} \to L^2(\sigma(A), d\mu_\xi)$ with

$$(UAU^{-1}f)(\lambda) = \lambda f(\lambda).$$

A similar result can be obtained if A is self-adjoint but not necessarily bounded [2] (Section 5.4). In this case, we prefer to call *cyclic* a vector ξ_0 which lies in the space $\mathcal{D}^\infty(A) = \bigcap_{n=1}^\infty D(A^n)$ and such that the set $\{A^n\xi_0; n \in \mathbb{N}\}$ is total in $\mathcal{H}$ (see also [18]

(Section 3.5); in [3] (Sect. 69) and [2] (Section 5.4) the notion of generating vector is always considered in the context of the von Neumann algebra that the operator generates).

For our purposes, it seems more convenient to strengthen the definition as follows.

Definition 3. *We say that a vector* $\xi_0 \in \mathcal{D}$ *is* strongly cyclic *(for* A*) if the set* $\{A^n\xi_0; n \in \mathbb{N}\}$ *is total in* $\mathcal{D}[\tau]$ *or equivalently if* $\{p(A)\xi_0; p$ *polynomial*$\}$ *is dense in* $\mathcal{D}[\tau]$.

It is easily seen that a strongly cyclic vector is cyclic. By [2] (Proposition 5.20), a generating vector exists if and only if a cyclic vector for A exists.

Theorem 3. *Let* A *be a self-adjoint operator and* $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ *be a RHS such that* $A\mathcal{D} \subset \mathcal{D}$. *If there exists a strongly cyclic vector for* A, *then every subspace* $\mathcal{D}^\times_\lambda$ *is one-dimensional.*

Proof. Suppose that $\mathcal{D}^\times_\lambda$ is not one-dimensional and let Φ_1, Φ_2 be linearly independent functionals in $\mathcal{D}^\times_\lambda$. Then there exists a nonzero vector $f \in \mathcal{D}$ such that $\langle\Phi_1|f\rangle = 1$ and $\langle\Phi_2|f\rangle = 0$. Then if ξ is a cyclic vector for A, there exists a sequence $\{p_n\}$ of polynomials such that $\lim_{n\to\infty} p_n(A)\xi = f$. Then,

$$\lim_{n\to\infty}\langle\Phi_1|p_n(A)\xi\rangle = 1, \qquad \lim_{n\to\infty}\langle\Phi_2|p_n(A)\xi\rangle = 0;$$

this implies that

$$\lim_{n\to\infty}\overline{p_n(\lambda)}\langle\Phi_1|\xi\rangle = 1, \qquad \lim_{n\to\infty}\overline{p_n(\lambda)}\langle\Phi_2|\xi\rangle = 0.$$

The first equality above implies that $\lim_{n\to\infty}\overline{p_n(\lambda)} \neq 0$; thus, the second one entails that $\langle\Phi_2|\xi\rangle = 0$. However, since ξ is strongly cyclyc this, in turn, implies that Φ_2 is identically 0 on $\mathcal{D}$. A contradiction. □

We conjecture that the converse implication holds true, but we could not prove it.

5. Conclusions

Clearly, operators defined by a Gel'fand distribution basis form an interesting class that deserves further study. It could also be extended to more general operators, i.e., quasi-hermitian operators $B = SAS^{-1}$ for some reasonable S. This would correspond to the fact that B is the operator defined by a *Riesz distribution basis* as defined in [5] (Definition 3.19). Roughly speaking, a Riesz distribution basis is in fact obtained from a Gel'fand basis through the action of a continuous operator S having a bounded inverse. We expect that a study of operators defined by a Riesz distribution basis should produce results similar to those obtained in [21,22] for operators defined by (usual) Riesz bases and their generalizations. This sets some goals and directions of research, which we leave for a future work.

Author Contributions: Investigation, J.-P.A. and C.T.; Writing—review & editing, J.-P.A. and C.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: This work has been done within the activities of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni of the INdAM and of the Gruppo UMI Teoria dell'Approssimazione e Applicazioni.

Conflicts of Interest: The authors declare no conflict of interest.

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