Statistical Analysis of Inverse Weibull Constant-Stress Partially Accelerated Life Tests with Adaptive Progressively Type I Censored Data

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Abstract: In life-testing investigations, accelerated life testing is crucial since it reduces both time and costs. In this study, constant-stress partially accelerated life tests using adaptive progressively Type I censored samples are taken into account. This is accomplished under the assumption that the lifespan of products under normal use conditions follows the inverse Weibull distribution. In addition to using the maximum likelihood approach, the maximum product of the spacing procedure is utilized to obtain the point and interval estimates of the model parameters as well as the acceleration factor. Employing the premise of independent gamma priors, the Bayes point estimates using the squared error loss function and the Bayes credible intervals are obtained based on both the likelihood and product of spacing functions via the Markov chain Monte Carlo technique. To assess the effectiveness of the various approaches, a simulation study is used because it is not possible to compare the findings theoretically. To demonstrate the applicability of the various approaches, two real datasets for the lifetime of micro-droplets in the ambient environment and light-emitting diode failure data are investigated. Based on the numerical results, to estimate the parameters and acceleration factor of the inverse Weibull distribution based on the suggested scheme with constant-stress partially accelerated life tests, it is recommended to utilize the Bayesian estimation approach.

Keywords: accelerated life test; inverse Weibull distribution; maximum product of spacing estimation; squared error loss; Bayesian estimation

MSC: 62F10; 62F15; 62N01; 62N02; 62N05

1. Introduction

Due to the powerful competition among producers, many modern-made goods are exceptionally reliable. Therefore, experimenters looking into various potential scenarios regarding the dependability and quality of such items encounter the difficulty of not knowing enough about the failure of such products under usual working situations. As a result, experimenters have offered accelerated life tests (ALTs) in the literature as a methodology for effectively measuring the lifetimes of extremely reliable products. To induce early failure, such investigations subject the test objects to higher-than-normal levels of stress, including, but not restricted to, increased levels of voltage, weight, and temperature. There are many ALT models. However, the two most frequently used models are the constant-stress and step-stress models; see Ahmad et al. [1] and Dey et al. [2]. Each unit being tested is kept under ongoing stress until the test is finished or all units fail under a constant-stress accelerated life test scenario. In the literature, constant-stress ALT models were discussed by many writers; see, for example, Wang et al. [3], Wang [4], El-Din et al. [5], Dey and Nassar [6], Sief et al. [7], and Kumar et al. [8]. In contrast, in a step-stress ALT experiment, the stress is increased gradually at prescribed intervals or after a predetermined number of failures.
Numerous academics have examined this model in the literature; see, for example, Abdel-Hamid and Al-Hussaini [9], Hamada [10], Nassar et al. [11], and Amleh and Raqab [12]. One of the primary priorities of ALT analysis is to assess product performance under normal use circumstances using the data acquired during accelerated stress stages. To accomplish this, it is necessary to comprehend the life–stress model, which defines the linkage between a product’s lifetime and stress conditions. Life–stress models are not always known or taken for granted. Consequently, a partially accelerated life test (PALT) is one way to evaluate a product’s reliability under normal use situations. One of the types of PALTs, which is the focus of this research, is the constant-stress PALT (CSPALT), in which we only examine each item under accelerated or normal conditions. Many studies have also taken CSPALTs into consideration, including, for instance, Hyun and Lee [13], Ahmadini et al. [14], Mohamed [15], and Nassar and Alam [16].

In reliability studies and life-testing investigations, censoring is a highly frequent phenomenon. Overall, censoring simply means that precise failure times are only identified for a part of the items under investigation, where the items are lost or eliminated from the experiment before failure because of financial and time constraints. The data observed from such studies are referred to as censored data. Various censoring plans are presented in the literature, including Type I and Type II censoring, as single-stage censoring schemes, but they are not flexible enough to withdraw units at any time other than when the experiment is over. The removal of live units from the experiment at times other than the terminal point is possible using progressive Type II censoring, which is a more general censoring scheme. Failure-censored schemes, such as Type II and progressive Type II censoring plans, are no longer appropriate in many types of products due to the short product development time frames and the strict time limitations that must be placed on reliability tests. A new censoring method known as an adaptive progressive Type I censoring (APT-IC) scheme was thus presented by Lin and Huang [17] as a result. This plan ensures that the experiment will end at a specific time and yields a higher efficiency in estimation. The next section provides a thorough explanation of the APT-IC scheme. Little work has been performed by considering this scheme; see, for example, Lin et al. [18], Ismail [19], Okasha and Mustafa [20], and Nassar and Dobbah [21].

The inverse Weibull (IW) distribution plays a crucial role in describing the lifespan of components with some monotone failure rates, including decreasing and unimodal shapes. Nelson [22] reported that the IW distribution offers a good fit for a variety of engineering datasets. If an object’s lifetime, say, $Z$, follows the IW distribution under normal conditions, then its probability density function (PDF) can be expressed as

$$f_1(z) = \alpha \beta z^{-(\beta+1)} e^{-\alpha z^{-\beta}}; z > 0, \alpha, \beta > 0,$$  \hspace{1cm} (1)

where $\alpha$ and $\beta$ are the scale and shape parameters, respectively. The corresponding cumulative distribution function (CDF) is given by

$$F_1(z) = e^{-\alpha z^{-\beta}}.$$  \hspace{1cm} (2)

The reliability function (RF) and hazard rate function (HRF) are, respectively, given by

$$R_1(z) = 1 - e^{-\alpha z^{-\beta}}$$  \hspace{1cm} (3)

and

$$H_1(z) = \frac{\alpha \beta z^{-(\beta+1)}}{e^\alpha z^{-\beta} - 1}.$$  \hspace{1cm} (4)

Despite the flexibility and adaptability of the APT-I censoring scheme and the popularity of the IW distribution in modelling lifetime data, no work has looked at how to acquire the unknown parameters in the presence of CSPALT in this case. Moreover, it is the first time that Bayesian estimation has been incorporated using the product of spacing function (PSF) based on APT-IC samples with CSPALT. As a result, our main purposes in this paper
are: (1) to investigate the point and interval estimators for the model parameters using two classical procedures, namely maximum likelihood (ML) and maximum product of spacing (MPS); (2) to study the Bayes point and credible intervals based on the two aforementioned methods; (3) to compare the efficiency of the derived estimators by means of a simulation study; and (4) to demonstrate the relevance of the offered estimators by exploring two real datasets.

The rest of the article is arranged as follows: Section 2 addresses the model’s description. Section 3 offers the ML estimators (MLEs) as well as the approximate confidence intervals (ACIs) of the model parameters. The point estimators and ACIs using the MPS method are derived in Section 4. The Bayesian estimations utilizing the likelihood function (LF) and PSF are presented in Section 5. The findings of the simulation investigation are outlined in Section 6. In Section 7, two engineering applications are considered. Finally, in Section 8, a few concluding observations are addressed.

2. Model Description

Assuming that we own \( n \) items, these items are split into two sets: the first one includes \( n_1 \) items, which are randomly selected from \( n \) test items and subjected to normal operating conditions, and the other set contains \( n_2 = n - n_1 \) remaining items, which are subjected to accelerated operating conditions. For each group, the items are tested using APT-IC with a predetermined number of failure \( m_r \), a progressive censoring scheme \( S_{r1}, \ldots, S_{rm_r} \) and a prefixed time \( T_r, r = 1, 2 \). At the time of the \( i \)th failure, denoted by \( Z_{r1;m_r,n_r}, i = 1, \ldots, m_r, S_{ri} \) items are randomly discarded from the remaining items. If the prefixed time \( T_r \) occurs before \( Z_{rm_r;m_r,n_r} \), the experiment stops at \( T_r, r = 1, 2 \). Otherwise, if \( Z_{rm_r;m_r,n_r} \) occurs before \( T_r \), the test will not end but will continue to acquire failures without any additional withdrawals until reach the prefixed time \( T_r, r = 1, 2 \). At the prefixed time \( T_r \), the experiment is terminated, and all the remaining items \( S_r^* = n_r - J_r - \sum_{i=1}^{m_r-1} S_{ri} \) are discarded, where \( J_r \) is the number of failures that occurred before time \( T_r, r = 1, 2 \). In this case, we have the observations \( (Z_{r1;m_r,n_r}, Z_{r2;m_r,n_r}, \ldots, Z_{rJ_r;m_r,n_r}) \) with the progressive censoring scheme \( S_{r1}, \ldots, S_{rm_r}, \ldots, S_{rJ_r} \), where \( S_{rm_r} = \cdots = S_{rJ_r} = 0 \).

The lifespan of an item tested under normal operating conditions is assumed to follow the IW distribution with a PDF, CDF, RF and HRF provided by (1)–(4). On the other hand, the HRF of a tested item under accelerated conditions is

\[
H_2(z) = \frac{\lambda \alpha \beta z^{-(\beta+1)}}{e^{\alpha z^\beta} - 1}.
\]  

(5)

Employing the connection \( R_2(z) = \exp[ - \int_0^z H_2(y)dy ] \), the RF under the accelerated conditions can be acquired as

\[
R_2(z) = \left( 1 - e^{-\alpha z^\beta} \right)^\lambda.
\]  

(6)

The associated CDF and PDF are expressed, respectively, by

\[
F_2(z) = 1 - \left( 1 - e^{-\alpha z^\beta} \right)^\lambda
\]  

(7)

and

\[
f_2(z) = \lambda \alpha \beta z^{-(\beta+1)} e^{-\alpha z^\beta} \left( 1 - e^{-\alpha z^\beta} \right)^{\lambda-1}.
\]  

(8)
Based on the realizations of the two APT-IC samples in the presence of CSPALT, the joint LF, without the constant term, can be expressed as follows:

\[
L(\omega | z) = \prod_{r=1}^{J} \left\{ \prod_{i=1}^{l_r} f_r(z_{ri}) [1 - F_r(z_{ri})]^{S_{ri}} [1 - F_r(T_r)]^{S^*} \right\},
\]

where \( z_{ri} = z_{ri(m_r,n_r)} \) for simplicity, \( \omega = (\alpha, \beta, \lambda)^\top \) and \( z = (z_{r1}, \ldots, z_{rJ}), r = 1, 2 \).

In recent years, a very competitive estimation method called the MPS method has been widely used as an alternative to the conventional ML procedure. Cheng and Amin [23] introduced the MPS method to estimate unknown parameters, particularly for models with an anonymous scale and shifted threshold. They reported that the asymptotic sufficiency, consistency, and efficiency properties of the MPS and ML estimators are equivalent. The MPS estimators (MPSEs) are acquired by maximizing the product of the discrepancies between the CDF values at nearby ordered points. Anatolyev and Kosenok [24] explored the invariance property of the MPSEs and indicated that it has the exact property of the MLEs. Based on two APT-IC samples with CSPALT, we can write the joint PSF, ignoring the constant term, to be maximized using the same approach of Ng et al. [25] as

\[
P(\omega | z) = \prod_{r=1}^{J} \left\{ \prod_{i=1}^{l_r+1} \Delta_{ri} \prod_{i=1}^{l_r} [1 - F_r(z_{ri})]^{S_{ri}} [1 - F_r(T_r)]^{S^*} \right\},
\]

where \( \Delta_{ri} = F_r(z_{ri}) - F_r(z_{ri-1}) \). Before progressing further, and for clarity, Figure 1 shows the various point and interval estimations discussed in this study.

**Figure 1.** Flowchart of the proposed estimation methods.

### 3. Maximum Likelihood Estimation

In this section, the ML approach is considered to obtain the point and interval estimations of \( \alpha, \beta \) and \( \lambda \) by employing the data gathered beneath the proposed censoring with CSPALT. Suppose that \( Z_{r1} < Z_{r2} < \ldots < Z_{rJ} \) is two APT-IC samples with CSPALT with a progressive censoring scheme \( (S_{r1}, \ldots, S_{r_{m-1}}, 0, \ldots, 0) \) taken from two IW populations; the first one has a PDF and CDF given by (1) and (2), respectively, while the second population has a CDF and PDF as expressed in (7) and (8), respectively. Based on these assumptions, and from (9), we can write the LF of the obtained samples given \( J_r \geq 1, r = 1, 2 \) as follows:

\[
L(\omega | z) = (a\beta)^{J^2} \lambda^{J^2} \exp \left[ -a \sum_{r=1}^{J} \sum_{i=1}^{l_r} z_{ri}^{-\beta} - (\beta + 1) \sum_{r=1}^{J} \sum_{i=1}^{l_r} \log(z_{ri}) \right] \times \prod_{r=1}^{J} \prod_{i=1}^{l_r} \left( 1 - e^{-a z_{ri}^{-\beta}} \right)^{\lambda^r - 1} R_{ri}^{\lambda^r - 1} \prod_{r=1}^{J} \prod_{i=1}^{2} \left( 1 - e^{-a T_r^{-\beta}} \right)^{\lambda^r - 1} S^*_r,
\]
where \( J = J_1 + J_2 \) and \( R_{ri} = 1 + S_{ri}, r = 1, 2, i = 1, \ldots, J \). Working with the LF’s natural logarithm rather than the LF itself is more practical. As a result, the natural logarithm of (11), denoted by \( \ell(\omega | z) \), can be expressed as follows:

\[
\ell(\omega | z) = J \log(\alpha \beta) + J_2 \log(\lambda) - \alpha^2 \sum_{r=1}^{J} \sum_{i=1}^{J_r} z_{ri}^{-\beta} - (\beta + 1) \sum_{r=1}^{J} \sum_{i=1}^{J_r} \log(z_{ri}) + \sum_{r=1}^{J} \sum_{i=1}^{J_r} [\lambda^{-1} R_{ri} - 1] \log \left( 1 - e^{-\alpha z_{ri}^{-\beta}} \right) + \sum_{r=1}^{J} \lambda^{-1} S_{ri}^2 \log \left( 1 - e^{-\lambda T_r^{-\beta}} \right). \tag{12}
\]

By differentiating (12) with respect to \( \alpha, \beta \) and \( \lambda \), the MLEs of these parameters can be acquired. Another simple approach is to simultaneously solve the following three normal equations:

\[
\frac{\partial \ell(\omega | z)}{\partial \alpha} = \frac{J}{\alpha} - \sum_{r=1}^{J} \sum_{i=1}^{J_r} z_{ri}^{-\beta} + \sum_{r=1}^{J} \sum_{i=1}^{J_r} \lambda^{-1} R_{ri} - 1 \sum_{r=1}^{J} \sum_{i=1}^{J_r} \lambda^{-1} S_{ri}^2 \left( e^{\alpha z_{ri}^{-\beta}} - 1 \right) = 0, \tag{13}
\]

\[
\frac{\partial \ell(\omega | z)}{\partial \beta} = \frac{J}{\beta} + \sum_{r=1}^{J} \sum_{i=1}^{J_r} z_{ri}^{-\beta} \log(z_{ri}) - \sum_{r=1}^{J} \sum_{i=1}^{J_r} \log(z_{ri}) - \sum_{r=1}^{J} \sum_{i=1}^{J_r} [\lambda^{-1} R_{ri} - 1] \log(z_{ri}) - \sum_{r=1}^{J} \sum_{i=1}^{J_r} \lambda^{-1} S_{ri}^2 \left( e^{\alpha z_{ri}^{-\beta}} - 1 \right) = 0. \tag{14}
\]

and

\[
\frac{\partial \ell(\omega | z)}{\partial \lambda} = \frac{J_2}{\lambda} + \sum_{r=1}^{J} R_{2r} \log \left( 1 - e^{-\alpha z_{ri}^{-\beta}} \right) + S_{r}^2 \log \left( 1 - e^{-\lambda T_r^{-\beta}} \right) = 0. \tag{15}
\]

For a fixed \( \alpha \) and \( \beta \), the MLE of the acceleration factor \( \lambda \) can be obtained from (15) as

\[
\hat{\lambda}(\alpha, \beta) = \frac{J_2}{\sum_{i=1}^{J} R_{2i} \log \left( 1 - e^{-\alpha z_{ri}^{-\beta}} \right) + S_{r}^2 \log \left( 1 - e^{-\lambda T_r^{-\beta}} \right)}. \tag{16}
\]

By replacing \( \lambda \) in (13) and (14) by its MLE acquired in (16), the MLEs of \( \alpha \) and \( \beta \), denoted by \( \hat{\alpha} \) and \( \hat{\beta} \), are the solutions of the following nonlinear equations:

\[
\frac{J}{\hat{\alpha}} - \sum_{r=1}^{J} \sum_{i=1}^{J_r} z_{ri}^{-\beta} + \sum_{r=1}^{J} \sum_{i=1}^{J_r} \hat{\lambda}(\alpha, \beta) R_{ri}^{-1} - 1 \sum_{r=1}^{J} \sum_{i=1}^{J_r} \hat{\lambda}(\alpha, \beta) S_{ri}^2 \left( e^{\alpha z_{ri}^{-\beta}} - 1 \right) = 0, \tag{17}
\]

and

\[
\frac{J}{\hat{\beta}} + \sum_{r=1}^{J} \sum_{i=1}^{J_r} z_{ri}^{-\beta} \log(z_{ri}) - \sum_{r=1}^{J} \sum_{i=1}^{J_r} \log(z_{ri}) - \sum_{r=1}^{J} \sum_{i=1}^{J_r} \hat{\lambda}(\alpha, \beta) R_{ri}^{-1} - 1 \sum_{r=1}^{J} \sum_{i=1}^{J_r} \hat{\lambda}(\alpha, \beta) S_{ri}^2 \left( e^{\alpha z_{ri}^{-\beta}} - 1 \right) - \sum_{r=1}^{J} \sum_{i=1}^{J_r} \hat{\lambda}(\alpha, \beta) S_{ri}^2 \left( e^{\alpha z_{ri}^{-\beta}} - 1 \right) = 0. \tag{18}
\]

It is noted that Equations (17) and (18) cannot be solved analytically. Thus, the MLEs \( \hat{\alpha} \) and \( \hat{\beta} \) cannot be obtained in their explicit forms. To overcome this dilemma, some numerical techniques such as the Newton–Raphson method can be utilized to obtain the required
estimates; for more detail about the ML approach, see [26]. Once the MLEs $\hat{\alpha}$ and $\hat{\beta}$ are obtained, the MLE $\hat{\lambda} = \hat{\lambda}(\alpha, \beta)$ can be computed from (16) by superseding $\alpha$ and $\beta$ by their MLEs.

Instead of obtaining point estimates for the unknown parameters, one may also be interested to obtain a range of values that may contain these parameters with a certain probability. These ranges are known as interval estimations. Here, we construct the ACIs of matrix. In this case, the AVCM takes the form

\[ I^{-1}(\hat{\omega}) = \left( \begin{array}{ccc} -\frac{\partial^2 I(\omega|z)}{\partial \alpha^2} & -\frac{\partial^2 I(\omega|z)}{\partial \alpha \partial \beta} & -\frac{\partial^2 I(\omega|z)}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 I(\omega|z)}{\partial \alpha \partial \beta} & -\frac{\partial^2 I(\omega|z)}{\partial \beta^2} & -\frac{\partial^2 I(\omega|z)}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 I(\omega|z)}{\partial \alpha \partial \lambda} & -\frac{\partial^2 I(\omega|z)}{\partial \beta \partial \lambda} & -\frac{\partial^2 I(\omega|z)}{\partial \lambda^2} \end{array} \right)^{-1} = \left( \begin{array}{ccc} \hat{\sigma}_{11} & \hat{\sigma}_{12} & \hat{\sigma}_{13} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} & \hat{\sigma}_{23} \\ \hat{\sigma}_{31} & \hat{\sigma}_{32} & \hat{\sigma}_{33} \end{array} \right), \quad (19) \]

where the hat implies that the derivatives are evaluated at $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ and

\[
\frac{\partial^2 I(\omega|z)}{\partial \alpha^2} = -I^{-1} - \sum_{r=1}^{l_r} \left( \lambda^{r-1} R_{ri} - 1 \right) e^{az_r} \beta^{2} \left( e^{az_r} - 1 \right)^2 - \sum_{r=1}^{l_r} \frac{\lambda^{r-1} S_r \left( e^{aT_r} - 1 \right)^2}{T_r^2} \left( e^{aT_r} - 1 \right)^2 \]

\[
\frac{\partial^2 I(\omega|z)}{\partial \beta^2} = -I^{-1} - \alpha \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} z_r^{-\beta} \log(z_r) - \alpha \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} \left( \lambda^{r-1} R_{ri} - 1 \right) \log^2(z_r) \psi_{ri} \sum_{r=1}^{l_r} \frac{\lambda^{r-1} S_r \log^2(T_r) \psi_{Tr}}{T_r^2 \left( e^{aT_r} - 1 \right)^2} - \frac{\partial^2 I(\omega|z)}{\partial \alpha \partial \beta} = \frac{I^{-1}}{\hat{\lambda}^2} \]

\[
\frac{\partial^2 I(\omega|z)}{\partial \alpha \partial \lambda} = \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} \frac{R_{2i} \beta}{z_r^{\beta} \left( e^{az_r} - 1 \right)} + \frac{S_r^2}{T_r^2} \left( e^{aT_r} - 1 \right) \]

and

\[
\frac{\partial^2 I(\omega|z)}{\partial \beta \partial \lambda} = -a \sum_{i=1}^{l_r} \frac{R_{2i} \log(z_{2i})}{z_r^{\beta} \left( e^{az_r} - 1 \right)} - \frac{a S_r^2 \log(T_r)}{T_r^2 \left( e^{aT_r} - 1 \right)},
\]
where \( \psi_{ri} \equiv \psi_{ri}(z_{ri}; \alpha, \beta) = (\alpha - z_{ri}^\beta) e^{\alpha z_{ri}^\beta} + z_{ri}^\beta \) and \( \psi_{Tr} \equiv \psi_{Tr}(T_{ri}; \alpha, \beta) \). Utilizing the asymptotic normality of the MLEs, the 100(1 - \( \epsilon \))% ACIs of \( \alpha, \beta \), and \( \lambda \) can be constructed, respectively, as:

\[
\hat{\alpha} \pm y_{e/2} \hat{\sigma}_{11}, \quad \hat{\beta} \pm y_{e/2} \hat{\sigma}_{22} \quad \text{and} \quad \hat{\lambda} \pm y_{e/2} \hat{\sigma}_{33},
\]

where \( y_{e/2} \) is the upper \( (\epsilon/2) \)th percentile point of the standard normal distribution.

### 4. Maximum Product of Spacing Estimation

The MPSEs are obtained by choosing parameter values that maximize the product of the distances between the values of the CDF at neighbouring ordered points. According to Anatolyev and Kosenok [24], the MPSEs are more effective for small sample sizes than the MLEs, making the MPS approach even more appealing in life testing and reliability investigations. Many authors considered the MPS method to estimate the unknown parameters of lifetime models; see, for example, Basu et al. [27,28] and Okasha and Nassar [29]. In this section, the MPS procedure is proposed to obtain the point estimates and the ACIs of the IW distribution when APT-IC samples with CSPALT are obtained. By working with the same notation in the earlier sections, from (1), (2), (7), (8) and (10), we can write the PSF given \( J_r \geq 1, r = 1, 2 \), as

\[
P(\omega|z) = \prod_{r=1}^{l_r} \left\{ \prod_{i=1}^{l_r} \left[ \left( 1 - e^{-\alpha z_{ri}^\beta} \right)^{\lambda - 1} - \left( 1 - e^{-\alpha z_{ri}^\beta} \right)^{\lambda - 1} \right] \right\} \cdot \prod_{r=1}^{l_r} \left( 1 - e^{-\alpha \lambda r^{-1}} \right)^{\lambda - 1} S_r^*.
\]

Let \( p(\omega|z) \) be the natural logarithm of (20), which can be expressed as follows:

\[
p(\omega|z) = 2 \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} \log \left[ \left( 1 - e^{-\alpha z_{ri}^\beta} \right)^{\lambda - 1} - \left( 1 - e^{-\alpha z_{ri}^\beta} \right)^{\lambda - 1} \right] + 2 \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} \lambda^{r-1} S_r \log \left( 1 - e^{-\alpha z_{ri}^\beta} \right) + 2 \sum_{r=1}^{l_r} \lambda^{r-1} S_r^* \log \left( 1 - e^{-\alpha \lambda r^{-1}} \right).
\]

The MPSEs, denoted by \( \hat{\alpha}, \hat{\beta}, \) and \( \hat{\lambda} \), can be acquire by maximizing the objective Function (21) with respect to \( \alpha, \beta, \) and \( \lambda \). By simultaneously solving the following three nonlinear equations, we can also obtain the necessary estimators:

\[
\frac{\partial p(\omega|z)}{\partial \alpha} = 2 \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} \frac{\phi_{ri} - \psi_{ri}}{\Delta_{ri}} + 2 \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} \frac{\lambda^{r-1} S_r}{\alpha^2 z_{ri}^\beta \left( e^{\alpha z_{ri}^\beta} - 1 \right)} + 2 \sum_{r=1}^{l_r} \frac{\lambda^{r-1} S_r^*}{T_r^\beta \left( e^{\alpha \lambda r^{-1}} - 1 \right)} = 0,
\]

\[
\frac{\partial p(\omega|z)}{\partial \beta} = 2 \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} \frac{\xi_{ri} - \xi_{ri}}{\Delta_{ri}} - \alpha \sum_{r=1}^{l_r} \sum_{i=1}^{l_r} \frac{\lambda^{r-1} S_r \log(z_{ri})}{\alpha z_{ri}^\beta \left( e^{\alpha z_{ri}^\beta} - 1 \right)} + 2 \sum_{r=1}^{l_r} \frac{\lambda^{r-1} S_r^* \log(T_r)}{T_r^\beta \left( e^{\alpha \lambda r^{-1}} - 1 \right)} = 0
\]

and

\[
\frac{\partial p(\omega|z)}{\partial \lambda} = \frac{\xi_{2i} - \xi_{2i}}{\Delta_{2i}} + \frac{\lambda_{2i}}{\Delta_{2i}} \log \left( 1 - e^{-\alpha z_{2i}^\beta} \right) + S_2^* \log \left( 1 - e^{-\alpha T_2^\beta} \right) = 0,
\]

where
\[ \Delta_{ri} = \left( 1 - e^{-\alpha z_{ri}} \right)^{\lambda r - 1} - \left( 1 - e^{-\alpha z_{ri}} \right)^{\lambda r - 1} \]

\[ \zeta_{ri} = -\alpha \phi_{ri} \log (z_{ri}) \]

\[ \zeta_{2i} = \lambda \left( 1 - e^{-\alpha z_{ri}} \right)^{\lambda} \log \left( 1 - e^{-\alpha z_{ri}} \right) \]

One can observe that the MPSEs \( \hat{\alpha}, \hat{\beta}, \) and \( \hat{\lambda} \) cannot be acquired in closed expressions. Thus, any numerical technique can be employed to solve (22)–(24). We can obtain the ACIs for the unknown parameters by employing the asymptotic properties of MPSEs, just as we did in the case of the MLEs. The asymptotic distribution of the MPSEs \( \hat{w} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}) \) is a normal distribution with a mean of \( \alpha \) and an AVCM \( I^{-1}(\alpha) \). Here, we consider \( I^{-1}(\alpha) \) to estimate \( I^{-1}(\alpha) \), where the elements of \( I(\alpha) \) are as follows:

\[
\frac{\partial^2 p(\alpha | z)}{\partial \alpha^2} = \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{\phi_{ri-1} - \phi_{ri}}{\Delta_{ri}} - \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{(\phi_{ri-1} - \phi_{ri})^2}{\Delta_{ri}^2} - \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{\lambda^{r-1} S_{ri} \phi e^{\alpha z_{ri}}}{\Delta_{ri}^2 (e^{\alpha z_{ri}} - 1)^2}
\]

\[
\frac{\partial^2 p(\alpha | z)}{\partial \beta^2} = \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{\xi_{ri-1} - \xi_{ri}}{\Delta_{ri}} - \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{(\xi_{ri-1} - \xi_{ri})^2}{\Delta_{ri}^2} - \alpha \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{\lambda^{r-1} S_{ri} \log^2 (z_{ri}) \psi_{ri}}{\Delta_{ri}^2 (e^{\alpha z_{ri}} - 1)^2}
\]

\[
\frac{\partial^2 p(\alpha | z)}{\partial \lambda^2} = \sum_{i=1}^{l+1} \frac{\xi_{2i-1} - \xi_{2i}}{\Delta_{2i}} - \sum_{i=1}^{l+1} \frac{(\xi_{2i-1} - \xi_{2i})^2}{\Delta_{2i}^2}
\]

\[
\frac{\partial^2 p(\alpha | z)}{\partial \alpha \partial \beta} = \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{\phi_{ri-1} - \phi_{ri}}{\Delta_{ri}} - \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{(\phi_{ri-1} - \phi_{ri}) (\xi_{ri-1} - \xi_{ri})}{\Delta_{ri}^2} + \sum_{r=1}^{l} \sum_{i=1}^{l+1} \frac{\lambda^{r-1} S_{ri} \log (z_{ri}) \psi_{ri}}{\Delta_{ri}^2 (e^{\alpha z_{ri}} - 1)^2}
\]

\[
\frac{\partial^2 p(\alpha | z)}{\partial \alpha \partial \lambda} = \sum_{i=1}^{l+1} \frac{\xi_{2i-1} - \xi_{2i}}{\Delta_{2i}} - \sum_{i=1}^{l+1} \frac{(\phi_{2i-1} - \phi_{2i}) (\xi_{2i-1} - \xi_{2i})}{\Delta_{2i}^2} + \sum_{i=1}^{l} \frac{S_{2i}}{\Delta_{2i}^2 (e^{\alpha z_{2i}} - 1)}
\]

\[
\frac{\partial^2 p(\alpha | z)}{\partial \beta \partial \lambda} = \sum_{i=1}^{l+1} \frac{\xi_{2i-1} - \xi_{2i}}{\Delta_{2i}} - \sum_{i=1}^{l+1} \frac{(\phi_{2i-1} - \phi_{2i}) (\xi_{2i-1} - \xi_{2i})}{\Delta_{2i}^2} + \sum_{i=1}^{l} \frac{S_{2i}}{\Delta_{2i}^2 (e^{\alpha z_{2i}} - 1)}
\]
and

\[
\frac{\partial^2 p(\omega|z)}{\partial \beta \partial \lambda} = \sum_{i=1}^{j_2+1} v_{2i-1} - v_{2i} - \sum_{i=1}^{j_2+1} \left( \tilde{\zeta}_{2i-1} - \tilde{\zeta}_{2i} \right) \frac{\Delta_i}{\Delta_i^2} - \alpha \sum_{i=1}^{j_2} \frac{S_{2i} \log(z_{2i})}{z_{2i}^\beta \left(e^{z_{2i}^\beta} - 1 \right)} - \frac{\alpha S_{1}^2 \log(T_2)}{T_2^\beta \left(e^{T_2^\beta} - 1 \right)} ,
\]

where

\[
\begin{align*}
\phi_{ri} &= \phi_{ri} z_{ri}^{-\beta} \left[ \left( \lambda r^{-1} - 1 \right) \left( e^{az_{ri}^{-\beta}} - 1 \right) \right]^{-1} - 1 , \\
\xi_{ri} &= \xi_{ri} \log(z_{ri}) \left\{ \alpha z_{ri}^{-\beta} \left[ 1 - \left( \lambda r^{-1} - 1 \right) \left( e^{az_{ri}^{-\beta}} - 1 \right) \right]^{-1} - 1 \right\} , \\
\xi_{2i} &= \log \left( 1 - e^{-az_{2i}^{-\beta}} \right) \left[ \xi_{2i} + \left( 1 - e^{-az_{2i}^{-\beta}} \right)^\lambda \right] , \\
\hat{\xi}_{2i} &= \lambda \left[ \xi_{2i} + \left( 1 - e^{-az_{2i}^{-\beta}} \right)^\lambda \right] , \\
v_{2i} &= \alpha \hat{\xi}_{2i} \log(z_{2i}) .
\end{align*}
\]

After obtaining the estimated variances of \( \hat{\alpha}, \hat{\beta}, \) and \( \hat{\lambda} \), denoted by \( \hat{\sigma}_{ii}, i = 1, 2, 3 \), which are the main diagonal elements of \( I^{-1}(\hat{\omega}) \), the 100\( (1 - \epsilon) \)% ACIs of \( \alpha, \beta \) and \( \lambda \) can be obtained, respectively, as follows:

\[
\begin{align*}
\hat{\alpha} \pm y_{\epsilon/2} \hat{\sigma}_{11} , \\
\hat{\beta} \pm y_{\epsilon/2} \hat{\sigma}_{22} , \\
\hat{\lambda} \pm y_{\epsilon/2} \hat{\sigma}_{33} .
\end{align*}
\]

5. Bayesian Estimation

In this section, the Bayesian estimations of \( \alpha, \beta \) and \( \lambda \) are considered, as well as credible interval constructions. For analyzing failure time data, the Bayesian estimation approach has attracted a lot of attention. It utilizes one’s past knowledge about the parameters and also takes into account the information that is readily available. In this study, the Bayesian estimations are developed based on both the LF and PSF to obtain the point and interval estimations of the unknown parameters. The Bayes estimators are obtained by employing the SE loss function and by assuming that the random variables \( \alpha, \beta \) and \( \lambda \) can be considered, as well as independent gamma (G) priors. On the other hand, many authors have assumed that the acceleration factor follows a non-informative prior, which delivers little information relative to the parameter; see, for example, Abushal and Soliman [30], Ahmad et al. [1], and Mahmoud et al. [31]. In our case, we assume that the independent parameter \( \lambda \) has a three-parameter \( G \) distribution with a location parameter equal to one. The use of the \( G \) distribution is more practical because it has flexibility and gives a wide range of knowledge about the unknown parameters. Consequently, we have \( \alpha \sim G(c_1, d_1), \beta \sim G(c_2, d_2), \) and \( \lambda \sim G^*(c_3, d_3, 1) \), where \( G^*(.) \) denotes the three-parameter \( G \) distribution. Based on the aforementioned assumptions, we can write the joint prior distribution of the unknown parameters as

\[
\pi(\omega) \propto \alpha^{c_1-1} \beta^{c_2-1} (\lambda - 1)^{c_3-1} e^{-\left| d_1 \alpha + d_2 \beta + d_3 (\lambda - 1) \right|}, \alpha, \beta > 0, \lambda > 1 ,
\]

(25)

where \( c_i > 0 \) and \( d_i > 0, i = 1, 2, 3 \) are the hyper-parameters. The most important part of the Bayesian analysis is the posterior distribution. After owning the observed data, it preserves all the knowledge attainable regarding the unknown parameters. Here, we
derive the joint posterior distribution of the unknown parameters based on both LF and PSF as explained in the next subsections.

5.1. Bayesian Estimation Using LF-Based

Combining the observed data given by the LF in (11) with the joint prior distribution as expressed by (25), the joint posterior distribution obtained via the LF-based method can be formulated as follows:

\[
g(\omega|z) = A(\lambda - 1)^{\frac{l_f + c_1 - 1}{2}} \beta^{l_f + c_2 - 1} \exp\left\{ -\alpha \left( \sum_{i=1}^{l_f} \sum_{r=1}^{l_i} z_{ri}^{-\beta} + d_1 \right) - \beta \left( \sum_{i=1}^{l_f} \sum_{r=1}^{l_i} \log(z_{ri}) + d_2 \right) \right\} \times \exp\left\{ \sum_{i=1}^{l_f} [\lambda^{r_i} - 1] R_{ri} - 1] \log\left( 1 - e^{-aC_i^2} \right) \right\},
\]

where \(A\) is the normalized constant. Based on the SE loss function, the Bayes estimator of any function of the unknown parameters, say, \(\eta(\omega)\), can be acquired as follows:

\[
\hat{\eta}(\omega) = \int_0^\infty \int_0^\infty \int_0^\infty \eta(\omega) \pi(\omega) L(\omega|z) \pi(\omega) \ d\omega d\beta d\lambda.
\]

In general, it is impossible to obtain the ratio of the three integrals furnished by (27) in a closed form. In this situation, we generate samples from the posterior distributions using the MCMC procedure, and after that, we calculate the Bayes estimates for the unknown parameters as well as the associated credible intervals. To utilize the MCMC procedure, it is required to derive the full conditional distributions (FCDs) of the different unknown parameters. From the joint posterior distribution in (26), the FCDs of \(\alpha, \beta\) and \(\lambda\) can be expressed, respectively, as follows:

\[
g(\alpha|\omega_{\alpha}, z) \propto (\lambda - 1)^{\frac{l_f + c_1 - 1}{2}} \exp\left\{ -\alpha \left( \sum_{i=1}^{l_f} \sum_{r=1}^{l_i} z_{ri}^{-\beta} + d_1 \right) - \beta \left( \sum_{i=1}^{l_f} \sum_{r=1}^{l_i} \log(z_{ri}) + d_2 \right) \right\} \times \exp\left\{ \sum_{i=1}^{l_f} [\lambda^{r_i} - 1] R_{ri} - 1] \log\left( 1 - e^{-aC_i^2} \right) \right\},
\]

\[
g(\beta|\omega_{\beta}, z) = \beta^{l_f + c_2 - 1} \exp\left\{ -\alpha \left( \sum_{i=1}^{l_f} \sum_{r=1}^{l_i} z_{ri}^{-\beta} + d_1 \right) - \beta \left( \sum_{i=1}^{l_f} \sum_{r=1}^{l_i} \log(z_{ri}) + d_2 \right) \right\} \times \exp\left\{ \sum_{i=1}^{l_f} [\lambda^{r_i} - 1] R_{ri} - 1] \log\left( 1 - e^{-aC_i^2} \right) \right\},
\]

and

\[
g(\lambda|\omega_{\lambda}, z) = (\lambda - 1)^{\frac{l_f + c_3 - 1}{2}} \exp\left\{ -\alpha \left[ \sum_{i=1}^{l_f} \sum_{r=1}^{l_i} R_{ri} \log\left( 1 - e^{-aC_i^2} \right) - \lambda \right] \right\},
\]

where \(\omega_{\alpha}\) denotes the vector of the unknown parameters except the parameter \(\alpha\). It can be easily seen from (30) that the FCD of the parameter \(\lambda\) is a three-parameter gamma
distribution with a location parameter equal to one; then, any of the gamma-generating routines can be used to make samples of \( \lambda \) with ease. As a result, we can write \( \lambda \sim G^*(c^*, d^*, 1) \), where \( c^* \) and \( d^* \) are the shape and scale parameters, respectively, and are given by

\[
c^* = I_2 + c_3 \quad \text{and} \quad d^* = d_3 - \sum_{i=1}^{J_3} R_{2i} \log \left(1 - e^{-\alpha_2^\beta} \right) - S_2^* \log \left(1 - e^{-\alpha T_2^\beta} \right).
\]

On the other hand, although the FCDs of the two parameters \( \alpha \) and \( \beta \) given by (28) and (29), respectively, are unknown, their plots demonstrate that they are similar to the normal distribution. Therefore, we consider using the Metropolis–Hastings (MH) technique with the normal proposal distribution (NPD) to generate samples from these distributions. The following MH-within-Gibbs sampling steps can be used to obtain samples of \( \alpha \) and \( \beta \) and \( \lambda \):

**Step 1.** Put \( t = 1 \) and determine the start values as \((\alpha^{(0)}, \beta^{(0)}, \lambda^{(0)}) = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})\).

**Step 2.** Generate \( \lambda^{(i)} \) from \( G^*(c^*, d^*, 1) \) evaluated at \( \alpha^{(i-1)} \), and \( \beta^{(i-1)} \).

**Step 3.** Employ MH steps to obtain \( \alpha^{(i)} \) from \( g(\alpha|\omega_{1}, z) \) with NPD \( N(\alpha^{(i-1)}, \tilde{\sigma}_{11}) \).

**Step 4.** Use MH steps to acquire \( \beta^{(i)} \) from \( g(\beta|\omega_{2}, z) \) with NPD \( N(\beta^{(i-1)}, \tilde{\sigma}_{22}) \).

**Step 5.** Set \( t = t + 1 \).

**Step 6.** Redo steps 2–5 \( M \) times to obtain \((\alpha^{(l)}, \beta^{(l)}, \lambda^{(l)}), l = 1, \ldots, M\).

Let \( \omega_{l}, l = 1, 2, 3 \) be the unknown parameter to be estimated, where \( \omega_{1} = \alpha, \omega_{2} = \beta \), and \( \omega_{3} = \lambda \); then, the Bayes estimate of \( \omega_{l} \) based on the SE loss function can be obtained as

\[
\hat{\omega}_{Bl} = \frac{\sum_{l=B+1}^{M} \omega_{l}^{(l)}}{M - B}, l = 1, 2, 3,
\]

where \( B \) is the burn-in period. To acquire the Bayes credible intervals (BCIs) of \( \omega_{l}, l = 1, 2, 3 \), we first order \( \omega_{l}^{(l)}, l = 1, 2, 3 \) and \( t = B + 1, \ldots, M \) as \( \omega_{l}^{[B+1]}, \omega_{l}^{[B+2]}, \ldots, \omega_{l}^{[M]} \). Then, the 100(1 – \( \epsilon \))% BCI of the parameter \( \omega_{l} \) can be obtained as follows:

\[
\{\alpha_{l}^{[(M-B)/2]}, \alpha_{l}^{[(M-B)/(1-\epsilon/2)]}\}, l = 1, 2, 3.
\]

### 5.2. Bayesian Estimation Using PSF-Based

The MPS approach was first introduced by Cheng and Amin [23] to replace the LF with the PSF while retaining as many of the beneficial characteristics of the method of ML. When the PSF is used in place of the LF function in Bayesian estimation, Coolen and Newby [32] demonstrated that there are no significant practical concerns. They declared that even though the posterior distribution obtained using the PSF differs from the one obtained using the LF, it is asymptotically identical to the posterior distribution received utilizing the conventional method. On can also see for more details the work of Singh et al. [33] and Nassar et al. [34]. Here, the joint posterior distribution is derived by using the PSF function given by (20) rather than the LF. Combining the PSF in (20) with the joint prior distribution in (25), we can write the joint posterior distribution of the unknown parameters as follows:

\[
g^* (\omega | z) = A^{*} \alpha^{c_{1}^{-1}} \beta^{c_{2}^{-1}} (\lambda - 1)^{c_{3}^{-1}} e^{-(d_{1}^{\lambda} + d_{2}^{\beta})} \prod_{r=1}^{J_3} \prod_{i=1}^{J_r} \left(1 - e^{-\alpha z_{r}^{\beta}} \right)^{-1} - \left(1 - e^{-\alpha T_{r}^{\beta}} \right)^{-1}] 
\]

\[
\times e^{-d_{3}(\lambda - 1)} \exp \left[ \sum_{r=1}^{J_3} \sum_{i=1}^{J_r} \lambda^{r-1} S_{ri} \log \left(1 - e^{-\alpha z_{r}^{\beta}} \right) + \sum_{r=1}^{J_3} \lambda^{r-1} S_{ri} \log \left(1 - e^{-\alpha T_{r}^{\beta}} \right) \right],
\]

(31)
where $A^*$ is the normalized constant. Based on the SE loss function, the Bayes estimator of $\eta(\omega)$ using the PSF-based method can be expressed as follows:

$$\hat{\eta}_B(\omega) = \frac{\int_1^\infty \int_0^\infty \int_0^\infty \eta(\omega) P(\omega|z) \pi(\omega) \, d\omega \, dz \, d\lambda}{\int_1^\infty \int_0^\infty \int_0^\infty P(\omega|z) \pi(\omega) \, d\omega \, dz \, d\lambda}. \tag{32}$$

It is obvious that the ratio of the integrals in (32) cannot be reduced to a closed form. The Bayes estimates of the unknown parameters as well as the accompanying BCIs are therefore computed using samples from the posterior distribution in (31) generated by the MCMC approach. To accomplish this, we can write the FCDs of $\alpha, \beta$ and $\lambda$ as follows:

$$g^*(\alpha|\omega_{-\alpha}, z) \propto a^{t-1}e^{-a^2} \prod_{r=1}^{l_2+1} \left( 1 - e^{-a z_{n-1}^r} \right)^{\lambda-1} - \left( 1 - e^{-a z_{n-1}^r} \right)^{\lambda-1} \right]$$

$$\times \exp \left[ \sum_{r=1}^{l_2} \sum_{i=1}^{l_1} \lambda^{r-1} S_n \log \left( 1 - e^{-a z_{n-1}^r} \right) + \sum_{r=1}^{l_2} \lambda^{r-1} S^*_n \log \left( 1 - e^{-a T_r^\beta} \right) \right], \tag{33}$$

$$g^*(\beta|\omega_{-\beta}, z) \propto \beta^{t-1}e^{-b^2} \prod_{r=1}^{l_2+1} \left( 1 - e^{-a z_{n-1}^r} \right)^{\lambda-1} - \left( 1 - e^{-a z_{n-1}^r} \right)^{\lambda-1} \right]$$

$$\times \exp \left[ \sum_{r=1}^{l_2} \sum_{i=1}^{l_1} \lambda^{r-1} S_n \log \left( 1 - e^{-a z_{n-1}^r} \right) + \sum_{r=1}^{l_2} \lambda^{r-1} S^*_n \log \left( 1 - e^{-a T_r^\beta} \right) \right] \tag{34}$$

and

$$g^*(\lambda|\omega_{-\lambda}, z) \propto (\lambda - 1)^{t-1}e^{-a^2} \prod_{i=1}^{l_2+1} \left( 1 - e^{-a z_{2i-1}^r} \right)^{\lambda} - \left( 1 - e^{-a z_{2i-1}^r} \right)^{\lambda} \right]$$

$$\times \exp \left[ \lambda \sum_{i=1}^{l_2} S_{2i} \log \left( 1 - e^{-a z_{2i}^r} \right) + \lambda S^*_2 \log \left( 1 - e^{-a T_r^\beta} \right) \right]. \tag{35}$$

It is clear that the FCSs in (33)–(35) cannot be reduced to any well-known distributions. However, their plots are similar to the normal distribution; consequently, as in the case of Bayesian estimation using the LF-based method, the MH technique with NPD is considered to yield samples from these distributions. The following steps can be employed to acquire the required samples of $\alpha, \beta$, and $\lambda$:

**Step 1.** Set $t = 1$ and determine the initial values of $(\alpha^{(0)}, \beta^{(0)}, \lambda^{(0)}) = (\hat{\alpha}, \hat{\beta}, \lambda)$.

**Step 2.** Generate $\alpha^{(t)}$ from $g^*(\alpha|\omega_{-\alpha}, z)$ using the MH steps using $N(\alpha^{(t-1)}, \hat{\sigma}_{11})$.

**Step 3.** Generate $\beta^{(t)}$ from $g^*(\beta|\omega_{-\beta}, z)$ using the MH steps using $N(\beta^{(t-1)}, \hat{\sigma}_{22})$.

**Step 4.** Generate $\lambda^{(t)}$ from $g^*(\lambda|\omega_{-\lambda}, z)$ using the MH steps using $N(\lambda^{(t-1)}, \hat{\sigma}_{33})$.

**Step 5.** Set $t = t + 1$.

**Step 6.** Redo steps 2–5 $M$ times to acquire $(\alpha^{(t)}, \beta^{(t)}, \lambda^{(t)}), t = 1, \ldots, M$.

For the parameter $\omega_l, l = 1, 2, 3$, the Bayes estimate based on the SE loss function using the PSF-based method can be computed as

$$\hat{\omega}_B = \frac{\sum_{l=B+1}^M \omega_l^{(t)}}{M - B}, l = 1, 2, 3.$$

The BCI of the unknown parameter $\omega_l, l = 1, 2, 3$ using the PSF-based method can be obtained using the same approach discussed in the previous subsection.
6. Monte Carlo Simulations

To compare the behavior of the proposed estimators of the IW parameters \((\alpha, \beta)\) and the acceleration factor \(\lambda\), based on two sets of the true parameter values of \((\alpha, \beta, \lambda)\), namely Set 1: \((0.4,0.8,1.5)\) and Set 2: \((1.5,1.2,2)\), we generate 1000 APT-IC samples based on various choices of \(n_r\) (group size), \(m_r\) (effective sample size), \(T_r\) (threshold time) and \((S_1,\ldots,S_m,\ldots)\) (progressive censoring). Table 1 shows several combinations of \(n_r, m_r\) and \(S_{ri}, i = 1, 2,\ldots, m_r\). For brevity, the censoring scheme \((1,1,1,0,0,0,0,0)\) is denoted by \((1^3,0^5)\). In addition, to demonstrate the influence of the ideal times on the derived estimates, two different choices of \(T_r, r = 1, 2\) are also considered, such as \((T_1, T_2) = (0.5,0.8)\) and \((1.2,1.5)\) for Set 1, as well as \((1,1.5)\) and \((2.5,2)\) for Set 2. To be clear, for each set of \((n_1,n_2)\) in Table 1, the proposed tests \((1,5)\) used uniform censoring; tests \((2,6)\) used left censoring; tests \((3,7)\) used middle censoring; and tests \((4,8)\) used right censoring.

<table>
<thead>
<tr>
<th>((n_1,n_2))</th>
<th>Test</th>
<th>((m_1,m_2))</th>
<th>({S_{1i},S_{2i}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(30,40)</td>
<td>1</td>
<td>(15,20)</td>
<td>{(1^{15}),{1^{20}}}</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td>{(15,0^{14}),{20,0^{19}}}</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>{(0^7,15,0^7),(0^9,20,0^{10})}</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td></td>
<td>{(0^{14},15),(0^{19},20)}</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
<td>{(16,0^{18}),{18,0^{24}}}</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td></td>
<td>{(6,0^{23}),(8,0^{31})}</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td></td>
<td>{(0^{11},6,0^{12}),(0^{15},8,0^{16})}</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td></td>
<td>{(0^{23},6),(0^{31},8)}</td>
</tr>
<tr>
<td>(80,70)</td>
<td>1</td>
<td>(40,35)</td>
<td>{(1^{40}),(1^{35})}</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td>{(40,0^{39}),(35,0^{34})}</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td>{(0^{39},40),(0^{35},35,0^{17})}</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td></td>
<td>{(0^{39},40),(0^{34},35)}</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
<td>{(116,0^{48}),(114,0^{42})}</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td></td>
<td>{(16,0^{65}),(14,0^{55})}</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td></td>
<td>{(0^{31},16,0^{32}),(0^{27},14,0^{28})}</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td></td>
<td>{(0^{63},16),(0^{55},14)}</td>
</tr>
</tbody>
</table>

Once the desired samples were collected, using R 4.1.2 software with the ‘maxLik’ package introduced by Henningsen and Toomet [35], the MLEs and MPSEs along their 95% ACIs of \(\alpha\), \(\beta\), and \(\lambda\) were calculated via the Newton–Raphson method. Additionally, to carry out the Bayesian inferences, the associated values of the hyper-parameters \((c_1,c_2,c_3,d_1,d_2,d_3)\) were taken as \((0.8,1,6,1.5,2,2,3)\) and \((3.2,4,3,2,2,2.3)\) for the given Sets 1 and 2, respectively. These hyper-parameter values are specified in such a way that the prior mean satisfies the sample mean of the target parameter. Since the posterior distribution using the LF-based (or PSF-based) method is reduced to the corresponding LF (or PSF) when the prior information of \(\alpha\), \(\beta\), and \(\lambda\) is not available, we therefore omitted the results under noninformative priors and only considered informative priors. To generate the posterior samples from \((26)\) and \((31)\), using the ‘coda’ package proposed by Plummer et al. [36], 12,000 MCMC samples were generated, and the first 2000 variants were ignored as burn-in. Then, using the remaining 10,000 MCMC samples, the computations of the proposed Bays point and interval estimates of \(\alpha\), \(\beta\) and \(\lambda\) using both LF and PSF approaches were obtained.

The Brooks–Gelman–Rubin (BGR) diagnostic statistic evaluates the convergence of Markovian chains by analyzing the difference between the variance-within chains and the variance-between chains for each model parameter. Using this diagnostic, the posterior distribution is judged to have converged if the ratio of variance-between to -within is close to one; see Gelman and Rubin [37] and Brooks and Gelman [38] for more details. To check the convergence of the simulated MCMC draws of the unknown parameters \(\alpha\), \(\beta\), and \(\lambda\) developed from Bayesian LF-based (or PSF-based) methods, the trace, autocorrelation and BGR convergence diagnostic plots when \((T_1, T_2) = (0.5,0.8), (n_1,n_2) = (30,40), \)
\((m_1, m_2) = (15, 20)\) and \(\{S_{1i}, S_{2i}\} = \{(1^{15}), (1^{20})\}\) based on Set 1 (as an example) are shown in Figure 2. It shows that the Markov chain draws of \(\alpha\), \(\beta\), and \(\lambda\) are mixed well, and thus, the calculated estimates are satisfactory. The BGR statistic indicates that there is no significant difference between the simulated chains and proves that the burn-in sample has an efficient size to ignore the effect of starting point values.

\[
\begin{align*}
\text{(a) Bayesian LF-based method} & \quad \text{(b) Bayesian PSF-based method} \\
\end{align*}
\]

Figure 2. Trace (top), autocorrelation (center) and BGR diagnostic (bottom) plots for MCMC draws of \(\alpha\), \(\beta\), and \(\lambda\) in Monte Carlo simulation.

A comparison of several point estimations of \(\bar{\omega}_l\), \(l = 1, 2, 3\), where \(\bar{\omega}_1 = \alpha\), \(\bar{\omega}_2 = \beta\), and \(\bar{\omega}_3 = \lambda\), is then made based on two different criteria, namely root-mean-square errors (RMSEs) and mean relative absolute biases (MRABs), by the following formulae:

\[
\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( \bar{\omega}_l^{(i)} - \bar{\omega}_l \right)^2}
\]

and

\[
\text{MRAB} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\bar{\omega}_l} \left| \bar{\omega}_l^{(i)} - \bar{\omega}_l \right|, \quad l = 1, 2, 3,
\]

respectively, where \(N\) is the number of generated sequence data, and \(\bar{\omega}_l^{(i)}\) denotes the calculated estimate at the \(i\)th sample of \(\bar{\omega}_l\), \(l = 1, 2, 3\). Additionally, the evaluation of various interval estimates of \(\bar{\omega}_l\), \(l = 1, 2, 3\) is determined by two other standards, namely the average confidence lengths (ACLs) and coverage percentages (CPs), by the following formulae:

\[
\text{ACL}_{(1-\epsilon)}\% (\bar{\omega}_l) = \frac{1}{N} \sum_{i=1}^{N} \left( \mathcal{U}_{\bar{\omega}_l^{(i)}} - \mathcal{L}_{\bar{\omega}_l^{(i)}} \right),
\]

and

\[
\text{CP}_{(1-\epsilon)}\% (\bar{\omega}_l) = \frac{1}{N} \sum_{i=1}^{N} \left( \mathcal{L}_{\bar{\omega}_l^{(i)}} \mathcal{U}_{\bar{\omega}_l^{(i)}} \right) (\bar{\omega}_l), \quad l = 1, 2, 3,
\]

respectively, where \(I(\cdot)\) is the indicator function, and \(\mathcal{L}(\cdot)\) is the lower bound and \(\mathcal{U}(\cdot)\) is the upper bound of asymptotic (or credible) interval estimates.
All simulated outcomes are displayed graphically by using the heatmap approach for the RMSEs, MRABs, ACLs and CPs of $\alpha$, $\beta$, and $\lambda$. The simulation findings are shown in Figures 3–8, respectively. Each heatmap has some notation for clarity, such as (i) Bayes estimates via the LF-based method, abbreviated as “BE-LF”; (ii) Bayes estimates via the PSF-based method, abbreviated as “BE-PSF”; (iii) ACI estimates via the LF-based method, abbreviated as “ACI-LF”; (iv) ACI estimates via the PSF-based method, abbreviated as “ACI-PSF”; (v) BCI estimates via the LF-based method, abbreviated as “BCI-LF”; and (vi) BCI estimates via the PSF-based method, abbreviated as “BCI-PSF”. From Figures 3–6, we provide the following key findings in terms of the lowest RMSE, MRAB, and ACL values, as well as the highest CP values:

- The proposed point (or interval) estimates of $\alpha$, $\beta$, and $\lambda$ have shown good performance based on both given parameter sets.
- As $n_r$ (or $m_r$) increases, all suggested estimates function satisfactorily, which satisfies the consistency feature of the acquired estimates. Equivalent behavior is also noted when $S_{ir}$, $i = 1, 2, \ldots, m_r$ decrease.
- The Bayes estimates developed by LF-based (or PSF-based) methods provide higher performance compared to the frequentist estimates of all unknown parameters because the Bayesian point (or interval) estimates involve more priority information on the unknown parameters than the classical estimates.
- The RMSEs and MRABs of all estimates of $\alpha$ and $\beta$ grow as $T_r, r = 1, 2$ increase under Set 1, but those linked to the acceleration factor $\lambda$ decrease (in the case of frequentist estimation) and increase (in the case of Bayesian estimation).
- The RMSEs and MRABs of all estimates for $\alpha$ decrease, while those of $\beta$ grow as $T_r, r = 1, 2$ increase under Set 2. In the case of Set 1, the same pattern of $\lambda$ as in Set 2 is shown.
- The ACLs of all estimates of $\alpha$ and $\lambda$ grow, while those connected with $\beta$ decrease, and the opposite tendency is shown in terms of their CPs as $T_r, r = 1, 2$ increase under Set 1.
- As $T_r, r = 1, 2$ increase under Set 2, in most cases, the ACLs of $\beta$ and $\lambda$ decrease (in the case of frequentist estimation) and increase (in the case of Bayesian estimation), while those associated with $\alpha$ decrease based on all proposed methods. The opposite behavior is also observed in terms of their CPs.
- As $(\alpha, \beta, \lambda)$ increase, for each $T_r, r = 1, 2$, the RMSEs, MRABs and ACLs of $\beta$ and $\lambda$ increase, while those values associated with $\alpha$ increase (in the case of frequentist estimation) and decrease (in the case of Bayesian estimation). Similarly, the opposite behavior is also noted in terms of their CPs.
- It is evident from comparing the four different estimation techniques, for both Sets 1 and 2, that the point/interval estimates of $\alpha$ derived from the ML and BE-LF approaches behave better than the other estimates, while the estimates of $\beta$ and $\lambda$ derived from the MPS and BE-PSF methods behave better than the other estimates.
- The point and interval estimates of the unknown parameters in the case of uniform (or left) censoring perform better than the others when comparing the effects of different progressive censoring plans.
- In conclusion, the simulation results suggested that the Bayes LF-based approach via the MH-within-Gibbs algorithm is the best for estimating the unknown shape parameter $\alpha$, while the Bayes PSF-based approach via the MH algorithm is the best for estimating the unknown parameters $\beta$ and $\lambda$. 
Figure 3. Heatmap for the estimation results of $\alpha$ using Set 1.

Figure 4. Cont.
Heatmap for the estimation results of $\alpha$ using Set 2.

Heatmap for the estimation results of $\beta$ using Set 1.
Figure 6. Heatmap for the estimation results of $\beta$ using Set 2.

Figure 7. Cont.
Figure 7. Heatmap for the estimation results of $\lambda$ using Set 1.

Figure 8. Heatmap for the estimation results of $\lambda$ using Set 2.
7. Real Data Applications

In this section, two separate accelerated datasets are examined in order to investigate how the estimating approaches suggested in the preceding sections operate in reality.

7.1. Micro-Droplets

Micro-droplets can transmit infectious disease pathogens such as fatal respiratory diseases when an individual comes into contact with them. Sneezing causes extremely small particles of about 10 µm to float in the air. These particles carry viruses and disease-causing microorganisms. When they are suspended in the air, people inhale them and the pathogens are spread. Therefore, people should not touch an unknown object, stay close to a person or stay in an infected room. Controlling the airflow speed increases (or decreases) the persistence of micro-droplets in the air; see Aliabadi et al. [39]. This problem leads us, in this subsection, to study the lifetime of micro-droplets in the ambient environment as an application of the partially accelerated life test model. Average particle diameters in meters/second using different air velocities for underused and accelerated conditions are used. According to Asadi et al. [40], the average particle diameters for an air velocity of 0.35 m/s with \( n_1 = 15 \) are taken as the data under normal use conditions, while the data under an air velocity of 0.20 m/s with \( n_2 = 19 \) are taken as the accelerated data. These authors also stated that the Gompertz distribution provides a good fit to the micro-droplets data. For computational purposes, for both datasets given by 0.35 m/s and 0.20 m/s, we multiplied each time point by two. In Table 2, the newly transformed micro-droplet datasets are listed.

To check if the IW distribution is an appropriate model to fit the micro-droplets data, following Asadi et al. [40], the Kolmogorov–Smirnov (KS) distance and the associated \( p \)-value at a 5% significance level are obtained. Other tests can be easily incorporated to check the suitability of the model. Using Table 2, the MLEs with their standard errors (SEs) of \( \alpha \) and \( \beta \), as well as the associated KS (\( p \)-value), are computed and presented in Table 3. These results show that the IW distribution fits the micro-droplet data well. On the other hand, using datasets for both normal and accelerated conditions, the estimated/empirical RF and probability-probability (PP) plots are shown in Figure 9. It is evident, for both given datasets, that the considered IW distribution gives a suitable fit to the datasets as well.

### Table 2. Times of micro-droplets in the air.

<table>
<thead>
<tr>
<th>Normal use condition (0.35 m/s)</th>
<th>0.94</th>
<th>1.08</th>
<th>1.10</th>
<th>1.60</th>
<th>1.92</th>
<th>2.28</th>
<th>2.48</th>
<th>2.60</th>
<th>2.76</th>
<th>3.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accelerated stress condition (0.20 m/s)</td>
<td>1.60</td>
<td>1.80</td>
<td>1.94</td>
<td>2.02</td>
<td>2.18</td>
<td>2.30</td>
<td>2.30</td>
<td>2.30</td>
<td>2.36</td>
<td>2.44</td>
</tr>
<tr>
<td>2.50</td>
<td>2.54</td>
<td>2.58</td>
<td>2.60</td>
<td>2.62</td>
<td>2.68</td>
<td>2.72</td>
<td>2.74</td>
<td>2.80</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 3. MLEs and KS (\( p \)-value) for micro-droplets data.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Par.</th>
<th>MLE (SE)</th>
<th>KS (( p )-Value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal use</td>
<td>( \alpha )</td>
<td>3.3465 (0.9741)</td>
<td>0.2222 (0.449)</td>
</tr>
<tr>
<td></td>
<td>( \beta )</td>
<td>2.1103 (0.3994)</td>
<td></td>
</tr>
<tr>
<td>Accelerated stress</td>
<td>( \alpha )</td>
<td>85.363 (56.411)</td>
<td>0.2323 (0.253)</td>
</tr>
<tr>
<td></td>
<td>( \beta )</td>
<td>5.7663 (0.9130)</td>
<td></td>
</tr>
</tbody>
</table>

Now, to demonstrate the feasibility of the proposed estimation methodologies, several APT-IC samples from micro-droplet data for normal use and accelerated stress conditions are generated. From Table 2, taking \( m_1 = 10 \) and \( m_2 = 15 \), three APT-IC samples using different removal patterns \( S_{ri}, \ i = 1, 2, \ldots, m_r, \ r = 1, 2 \) are generated and reported in
Table 4. Since there is no available prior information, the hyper-parameters of $\alpha$, $\beta$ and $\lambda$ are set to be 0.001; although the prior densities for all parameters are considered proper, this setting implies that the prior densities are almost noninformative. Employing the MCMC procedures presented in Section 5, we reproduced 50,000 Markov chain variations. The initial guesses of $\alpha$, $\beta$, and $\lambda$ for running the MCMC sampler were assumed to be their frequentist estimates. To discard the impact of initial guesses, the first 10,000 variants of each generated chain were removed. In Table 5, the acquired point and interval estimates of $\alpha$, $\beta$, and $\lambda$ obtained by the frequentist approaches (including ML and MPS methods) and Bayes procedures (including LF-based and PSF-based methods) are computed and declared. It can be seen from Table 5 that the point estimates for all unknown parameters act similarly and seem close to each other, as anticipated. An identical performance is also seen in the case of interval estimates. It also shows, in terms of the smallest SEs and shortest interval length, that the Bayesian point (or interval) estimates developed from PSF-based methods perform well compared to those obtained from LF-based methods, and both are more satisfactory than the frequentist estimates. The consequences detailed in Table 5 support the same conclusions as in Section 6.

![Figure 9. Estimated RF and PP plots for (a) Normal use condition and (b) Accelerated stress condition from micro-droplet data.](image)

Table 4. Different artificial APT-IC samples from micro-droplet data.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$S_{1i}$</th>
<th>$S_{2i}$</th>
<th>$T_1(J_1)$</th>
<th>$T_2(J_2)$</th>
<th>$S^*_1$</th>
<th>$S^*_2$</th>
<th>Generated Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(5, 0$^9$)</td>
<td>(4, 0$^{14}$)</td>
<td>2.50(6)</td>
<td>2.45(8)</td>
<td>4</td>
<td>7</td>
<td>0.94, 1.08, 1.10, 1.60, 1.92, 2.48</td>
</tr>
<tr>
<td>2</td>
<td>(3, 0$^8$, 2)</td>
<td>(2, 0$^{13}$, 2)</td>
<td>3.10(9)</td>
<td>2.65(13)</td>
<td>3</td>
<td>4</td>
<td>0.94, 1.08, 1.10, 1.60, 1.92, 2.28, 2.48, 2.76, 3.00</td>
</tr>
<tr>
<td>3</td>
<td>(0$^5$, 1$^5$)</td>
<td>(0$^{11}$, 1$^{14}$)</td>
<td>3.60(10)</td>
<td>2.75(15)</td>
<td>1</td>
<td>1</td>
<td>0.94, 1.08, 1.10, 1.60, 1.92, 2.28, 2.60, 3.00, 3.22, 3.58</td>
</tr>
</tbody>
</table>
Table 5. Point and interval estimates of $\alpha$, $\beta$, and $\lambda$ from micro-droplet data.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Par.</th>
<th>MLE BE-LF</th>
<th>BE-PSF</th>
<th>ACI-LF BCI-LF</th>
<th>ACI-PSF</th>
<th>BCI-PSF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Est. SE.</td>
<td>Est. SE.</td>
<td>Lower Upper Length</td>
<td>Lower Upper Length</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\alpha$</td>
<td>5.5147 0.9999</td>
<td>5.3973 0.1516</td>
<td>3.5549 7.4745 3.9196</td>
<td>5.2100 5.5896 0.3796</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>1.1620 0.2567</td>
<td>1.1390 0.0874</td>
<td>0.6589 1.6651 1.0062</td>
<td>0.9782 1.3031 0.3249</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda$</td>
<td>4.2236 2.3062</td>
<td>6.2978 3.5571</td>
<td>0.0000 8.7437 8.7437</td>
<td>1.8902 13.093 11.203</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\alpha$</td>
<td>6.2496 1.1053</td>
<td>6.1308 0.1548</td>
<td>4.0833 8.4159 4.3326</td>
<td>5.9275 6.3234 0.3960</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>1.3490 0.2274</td>
<td>1.3408 0.0798</td>
<td>0.9033 1.7947 0.8914</td>
<td>1.1857 1.4957 0.3101</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\alpha$</td>
<td>6.2349 1.1476</td>
<td>6.1303 0.1434</td>
<td>3.9857 8.4841 4.4983</td>
<td>5.9373 6.3230 0.3857</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>1.5898 0.2480</td>
<td>1.4999 0.1198</td>
<td>1.1038 2.0799 0.9722</td>
<td>1.3500 1.6569 0.3069</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda$</td>
<td>4.2551 1.8731</td>
<td>6.0161 3.7458</td>
<td>0.5839 7.9262 7.3423</td>
<td>1.2022 13.897 12.695</td>
<td></td>
</tr>
</tbody>
</table>

To investigate the convergence of the offered MCMC procedures, for each Markovian chain of $\alpha$, $\beta$, and $\lambda$ after burn-in from Sample 1 as an example, a trace plot (which furnishes an essential tool for evaluating the mixing of a chain) and density plot (which provides a smoothed histogram of outputs) are depicted in Figure 10. In each trace plot, for specification, the sample mean and the two Bayes credible bounds of $\alpha$, $\beta$, and $\lambda$ are defined by soled (–) and dashed (- - -) lines, respectively. Additionally, the sample mean in each density plot is represented by a solid (–) line. It reveals that the MCMC draws yielded from the suggested conditional posterior distributions using LF-based (or PSF-based) methods of $\alpha$, $\beta$, or $\lambda$ converge adequately. Moreover, to demonstrate whether the MCMC samples are sufficiently close to the target conditional posterior distributions, the BGR diagnostic statistic based on Sample 1 (as an example) from micro-droplet data is plotted and displayed in Figure 11. It is evident that there is no significant difference between the variance within chains and the variance between chains. Figure 11 also shows that the burn-in sample has a suitable size to neglect the influence of the starting points. It also verifies that the densities of all unknown parameters generated by the MH sampler are fairly symmetrical, except that the density of $\alpha$ generated by the MH-within-Gibbs sampler using the LF-based method is positively skewed.

Figure 10. Cont.
7.2. Light-Emitting Diodes

Light-emitting diodes (LEDs) have been widely used in all the different sorts of semiconductor diodes known today and are typically employed in television and color displays. An LED is produced from a very thin layer of fairly heavily doped semiconductor fabric depending on the semiconductor material used and the amount of doping when the forward-biased LED emits colored light at a particular spectral wavelength. In this application, following Dey et al. [2], we shall consider the observed failure samples (at 1000 h) generated under normal use (with \( n_1 = 58 \)) and accelerated stress (with \( n_2 = 58 \)) conditions; see Table 6. These authors also reported that the Nadarajah–Haghighi distribution gives an appropriate fit to the LED data. To see whether the IW distribution is a suitable model to fit the LED data or not, according to Dey et al. [2], we obtain the KS distance along...
its p-value. Using complete LED datasets, the MLEs (with their SEs) of \( \alpha \) and \( \beta \), as well as the associated KS (p-value), are calculated and presented in Table 7. This table shows that the IW distribution fits the LED data satisfactorily. The estimated/empirical RF and PP plots are shown in Figure 12. This figure indicates that the IW distribution provides an adequate fit to the LED datasets.

**Table 6.** Light-emitting diode failure data.

| Normal use condition | 0.18, 0.19, 0.19, 0.34, 0.36, 0.40, 0.44, 0.44, 0.45, 0.46, 0.47, 0.53, 0.57, 0.63, 0.65, 0.70, 0.71, 0.71, 0.75, 0.76, 0.79, 0.80, 0.85, 0.98, 1.01, 1.07, 1.12, 1.14, 1.15, 1.17, 1.20, 1.23, 1.24, 1.25, 1.26, 1.32, 1.33, 1.33, 1.39, 1.42, 1.50, 1.55, 1.58, 1.62, 1.68, 1.70, 1.79, 2.00, 2.01, 2.04, 2.54, 3.61, 3.76, 4.65, 8.97 |
| Accelerated stress condition | 0.13, 0.16, 0.20, 0.20, 0.21, 0.25, 0.26, 0.28, 0.28, 0.30, 0.31, 0.33, 0.35, 0.35, 0.35, 0.39, 0.50, 0.52, 0.58, 0.60, 0.60, 0.62, 0.63, 0.67, 0.71, 0.73, 0.75, 0.75, 0.78, 0.80, 0.80, 0.86, 0.90, 0.91, 0.93, 0.93, 0.94, 0.98, 0.99, 1.01, 1.03, 1.06, 1.06, 1.10, 1.22, 1.24, 1.28, 1.38, 1.39, 1.46, 1.48, 1.52, 1.74, 1.95, 2.46, 3.02, 5.16 |

**Table 7.** MLEs and KS (p-value) for LED data.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Par.</th>
<th>MLE (SE)</th>
<th>KS (p-Value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal use</td>
<td>( \alpha )</td>
<td>0.5960 (0.0967)</td>
<td>0.1189 (0.385)</td>
</tr>
<tr>
<td></td>
<td>( \beta )</td>
<td>1.3385 (0.1253)</td>
<td></td>
</tr>
<tr>
<td>Accelerated stress</td>
<td>( \alpha )</td>
<td>0.3709 (0.0718)</td>
<td>0.1497 (0.149)</td>
</tr>
<tr>
<td></td>
<td>( \beta )</td>
<td>1.3563 (0.1305)</td>
<td></td>
</tr>
</tbody>
</table>

From the original datasets, three APT-IC samples are generated in Table 8 when \( m_1 = m_2 = 29 \) with different choices of \( S_{ri}, i = 1, 2, \ldots, m_r, r = 1, 2 \). To develop the Bayes estimates, when the values of all hyper-parameters are considered to be 0.001, the first 10,000 of the 50,000 MCMC samples are removed as burn-in. However, for each sample in Table 8, both the point/interval estimators of \( \alpha \), \( \beta \), or \( \lambda \) are obtained and shown in Table 9. It is obvious from Table 9 that the estimates of \( \alpha \), \( \beta \) and \( \lambda \) obtained via the frequentist (or Bayesian) approach are not significantly different from each other. Additionally, these results support the same findings investigated for the micro-droplet data.

Based on Sample 1 as an example, Figure 13 displays the trace and density plots of \( \alpha \), \( \beta \), and \( \lambda \) using their MCMC draws after burn-in. It shows that the simulated MCMC variates of \( \alpha \), \( \beta \) or \( \lambda \) are mixed adequately. In all cases, the associated density of each unknown parameter is almost symmetrical. Furthermore, using Sample 1 (as an example) from LED data, the BGR diagnostic statistic is shown in Figure 14. The simulated Markov chains are near 1, as can be shown in Figure 14; hence, this finding indicates a good convergence. To summarize, the results of the micro-droplet and LED data showed that the proposed model is good for exploring engineering problems and displayed the practical usefulness of the proposed methodologies in real-life phenomena.
Figure 12. Estimated RF and PP plots for (a) Normal use condition and (b) Accelerated stress condition from LED data.

Table 8. Different artificial APT-IC samples from LED data.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$S_{1i}$</th>
<th>$T_1(J_1)$</th>
<th>$S_{2i}$</th>
<th>$T_2(J_2)$</th>
<th>$S_{1i}^*$</th>
<th>$S_{2i}^*$</th>
<th>Generated Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(29, 0)</td>
<td>1.55(25)</td>
<td>4</td>
<td>(29, 0)</td>
<td>0.95(23)</td>
<td>6</td>
<td>0.18, 0.19, 0.34, 0.40, 0.45, 0.47, 0.53, 0.57, 0.63, 0.71, 0.75, 0.76, 0.79, 0.80, 0.85, 0.98, 1.01, 1.14, 1.15, 1.20, 1.26, 1.32, 1.33, 1.39, 1.50</td>
</tr>
<tr>
<td></td>
<td>(1)</td>
<td>0.13, 0.16, 0.20, 0.25, 0.28, 0.30, 0.31, 0.33, 0.35, 0.39, 0.50, 0.52, 0.58, 0.60, 0.62, 0.71, 0.75, 0.80, 0.80, 0.93, 0.94</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(1)</td>
<td>1.75(25)</td>
<td>8</td>
<td>(1)</td>
<td>1.25(24)</td>
<td>10</td>
<td>0.18, 0.19, 0.36, 0.44, 0.45, 0.47, 0.57, 0.63, 0.70, 0.71, 0.76, 0.79, 0.85, 1.01, 1.12, 1.15, 1.20, 1.24, 1.26, 1.33, 1.39, 1.50, 1.58, 1.62, 1.70</td>
</tr>
<tr>
<td></td>
<td>(0)</td>
<td>0.13, 0.20, 0.21, 0.26, 0.28, 0.31, 0.35, 0.35, 0.50, 0.58, 0.60, 0.63, 0.71, 0.75, 0.78, 0.80, 0.90, 0.93, 0.94, 0.99, 1.03, 1.06, 1.22, 1.24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(0, 4, 5)</td>
<td>1.25(29)</td>
<td>0</td>
<td>(0, 4, 5)</td>
<td>0.95(29)</td>
<td>0</td>
<td>0.18, 0.19, 0.19, 0.34, 0.36, 0.40, 0.44, 0.44, 0.45, 0.46, 0.47, 0.53, 0.57, 0.63, 0.65, 0.70, 0.71, 0.71, 0.75, 0.76, 0.76, 0.79, 0.80, 0.85, 0.98, 1.01, 1.14, 1.23</td>
</tr>
<tr>
<td></td>
<td>(0, 4, 5)</td>
<td>0.13, 0.16, 0.20, 0.20, 0.21, 0.25, 0.26, 0.28, 0.28, 0.30, 0.31, 0.33, 0.35, 0.35, 0.39, 0.50, 0.52, 0.58, 0.60, 0.60, 0.62, 0.63, 0.67, 0.75, 0.78, 0.80, 0.86, 0.94</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 9. Point and interval estimates of $\alpha$, $\beta$, and $\lambda$ from LED data.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Par.</th>
<th>MLE</th>
<th>BE-LF</th>
<th>ACI-LF</th>
<th>BCI-LF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>MPSE</td>
<td>BE-PSE</td>
<td>ACI-PSE</td>
<td>BCI-PSE</td>
</tr>
<tr>
<td></td>
<td>Est.</td>
<td>SE.</td>
<td>Est.</td>
<td>SE.</td>
<td>Lower</td>
</tr>
<tr>
<td>1</td>
<td>$\alpha$</td>
<td>3.5146</td>
<td>0.1635</td>
<td>3.4394</td>
<td>0.0879</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>0.4510</td>
<td>0.0465</td>
<td>0.4326</td>
<td>0.0368</td>
</tr>
<tr>
<td></td>
<td>$\lambda$</td>
<td>59.639</td>
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<td>59.614</td>
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<td>2</td>
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</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>0.7949</td>
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<tr>
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</table>

(a) Bayes estimation via LF-based method  (b) Bayes estimation via PSF-based method

Figure 13. Density (left) and trace (right) plots of $\alpha$, $\beta$, and $\lambda$ from LED data.
8. Concluding Remarks

The analysis of a constant-stress partially accelerated life test is covered in this article when the testing products’ lifetimes have an inverse Weibull distribution. The maximum likelihood and maximum product of spacing methods as conventional methods are taken into consideration to provide the point and interval estimates of the interesting parameters under an adaptive, progressive Type I censoring scheme. Utilizing the asymptotic properties of the classical estimates, the approximate confidence intervals are constructed. Furthermore, the Bayesian technique is employed to obtain the point as well as the interval estimates of the unknown parameters. The joint posterior distribution is derived using both the likelihood and the product of spacing functions, and the Bayes estimates are computed using the squared error loss function. By presenting alternative scenarios for sample size, progressive censoring schemes, and cutoff times, simulation studies are used to test the effectiveness of the different point and interval estimators. To prove the practicality of the techniques used in this work, two applications to actual datasets are taken into consideration. The numerical results demonstrate that the point and interval estimates are more accurate when using the Bayesian estimation method with the product of the spacing function than the other methods. One of the main advantages when considering the maximum product of the spacing estimation method is that it needs less running time for computations than the maximum likelihood method. As future work, the methods discussed in this paper can be extended to obtain the Bayes estimators using other loss functions rather than the squared error loss function. It is also of interest to investigate the same procedures offered in this study in the case of an improved adaptive progressive Type II censoring scheme.

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