Asymptotic Properties of Random Restricted Partition

Tiefeng Jiang1,† and Ke Wang2,*,†

1 School of Statistics, University of Minnesota, 224 Church Street S. E., Minneapolis, MN 55455, USA
2 Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong
* Correspondence: kewang@ust.hk
† These authors contributed equally to this work.

Abstract: We study two types of probability measures on the set of integer partitions of \( n \) with at most \( m \) parts. The first one chooses the partition with a chance related to its largest part only. We obtain the limiting distributions of all of the parts together and that of the largest part as \( n \) tending to infinity for \( m \) fixed or tending to infinity with \( m = o(n^{1/3}) \). In particular, if \( m \) goes to infinity not too fast, the largest part satisfies the central limit theorem. The second measure is very general and includes the Dirichlet and uniform distributions as special cases. The joint asymptotic distributions of the parts are derived by taking limits of \( n \) and \( m \) in the same manner as that in the first probability measure.

Keywords: random partitions; asymptotic distributions; limit laws

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1. Introduction

The partition \( \kappa \) of a positive integer \( n \) is a sequence of positive integers \( k_1 \geq k_2 \geq \cdots \geq k_m \) with \( m \geq 1 \) whose sum is \( n \). We denote \( \kappa = (k_1, \ldots, k_m) \vdash n \) if \( \kappa \) is a partition of \( n \). The number \( m \) is called the length of \( \kappa \) and \( k_i \) the \( i \)th largest part of \( \kappa \). Let \( P_n \) denote the set of partitions of \( n \) and \( P_n(m) \) the set of partitions of \( n \) with length at most \( m \). Thus, \( 1 \leq m \leq n \) and \( P_n(n) = P_n \).

The set of all partitions \( \mathcal{P} = \cup_{n \geq 1} P_n \) is called the macrocanonical ensemble. The partitions of \( n \), \( P_n \), is called the canonical ensemble and the restricted partitions \( P_n(m) \) is the microcanonical ensemble. Integer partitions have a close relationship with statistical physics ([1–3]). To be more precise, a partition \( \kappa \in P_n \) can be interpreted as an assembly of particles with total energy \( n \). The number of particles is the length of \( \kappa \); the number of particles with energy \( l \) is equal to \( \#\{ j : k_j = l \} \). Thus, \( P_n(m) \) is the set of configurations \( \kappa \) with a given number of particles \( m \). It is known that \( P_n(m) \) corresponds to the Bose–Einstein assembly (see Section 3 in [3] for a brief discussion). Therefore, the asymptotic distribution of a probability measure on \( P_n(m) \) as \( n \) tends to infinity is connected to how the total energy of the system is distributed among a given number of particles.

The most natural probability measure on the integer partitions is the uniform measure. The uniform measure on \( P_n(m) \) for \( m = n \) has been well-studied (see [4–6]). However, for the other values of \( m \), to our best knowledge, the whole picture is not clear yet. In [7], as a by-product of studying the eigenvalues of Laplacian–Beltrami operator defined on symmetric polynomials, the limiting distribution of \( (k_1, \ldots, k_m) \) chosen uniformly from \( P_n(m) \) is derived for fixed integer \( m \). This is one of the motivations resulting in this paper. As a special case of a more general measure on \( P_n(m) \) (detailed definition given in Section 1.2 below), we obtain the asymptotic joint distribution of \( (k_1, \ldots, k_m) \in P_n(m) \) imposed with a uniform measure for \( m \to \infty \) and \( m = o(n^{1/3}) \). It would be an intriguing question to understand the uniform measure on \( P_n(m) \) for all values of \( m \). The limiting
shape of the young diagram corresponding to $\mathcal{P}_n(m)$ with respect to uniform measure was studied in [8–11] for $m = n$ and for $m = c\sqrt{n}$ where $c$ is a positive constant.

Another important class of probability measure on the integer partitions is the Plancherel measure which chooses a partition $\kappa \in \mathcal{P}_n$ with probability $\frac{\dim(\kappa)^2}{n!}$. Here, $\dim(\kappa)$ is the degree of the irreducible representation of the symmetric group $S_n$ indexed by $\kappa$. More generally, the $\alpha$-Jack measure (see the detailed definition in [12], for instance), which subsumes the Plancherel measure as a special case when $\alpha = 1$, has also been considered. It is known that both the Plancherel measure (see [13–16], a survey by [17] and the references therein) and $\alpha$-Jack measure (see, for instance, [12, 18, 19]) have a deep connection with random matrix theory.

For a fixed constant $q \in (0, 1)$, the $q$-analog of the Plancherel measure, which is called the $q$-Plancherel measure, on integer partitions has been studied in [20–22]. As explained in Section 2.2 from [21], it is related to a probability measure on $\mathcal{P}_n(\kappa)$, that chooses a partition $\kappa \in \mathcal{P}_n$ and study the asymptotic behavior of the parts of $\kappa$ as $n$ tends to infinity. This probability measure on the microcanonical ensemble $\mathcal{P}_n(m)$ can also be viewed as an analog of a probability measure $\mu(\cdot)$ defined on the macrocanonical ensemble $\mathcal{P}_n$, introduced in [8], where $\mu(\lambda) = cq^{\dim(\lambda)}$ for any $\lambda \in \mathcal{P}$ and $|\lambda|$ is the sum of its parts.

In this paper, we consider two new probability measures on $\mathcal{P}_n(m)$ assuming either $m$ is fixed or $m$ tends to infinity with $n$. We investigate the asymptotic joint distributions of $(k_1, \ldots, k_m)$ as $n$ tends to infinity. This paper is organized as follows. In Section 1.1, we introduce the new probability measure, called the restricted geometric distribution, on $\mathcal{P}_n(m)$. We state the main results, Theorems 1 and 2, obtained under this probability measure assuming $m$ fixed or $m$ tends to infinity with $n$ and $m = o(n^{1/3})$. The overview of the proof of Theorem 2 is explained. In Section 1.2, we first introduce the second probability measure on $\mathcal{P}_n(m)$ and present new results, Theorems 3 and 4, on the joint asymptotic distributions of the parts, by taking limits of $n$ and $m$ in the same manner as that in the previous probability measure. The proofs of the main results and their corollaries are collected in Sections 2 and 3. To be more specific, we prove Theorem 1 and Corollary 1 in Section 2.1 and Theorem 2 in Section 2.2. The proofs of Theorem 3 and two corollaries are presented in Section 3.1 and the proof of Theorem 4 is stated in Section 3.2.

1.1. Restricted Geometric Distribution

The first type of random partitions on $\mathcal{P}_n(m)$ is defined as follows: for $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$, consider the probability measure

$$P(\kappa) = c \cdot q^{k_1}$$

where $0 < q < 1$ and $c = c_{n,m}$ is the normalizing constant that $\sum_{\kappa \in \mathcal{P}_n(m)} P(\kappa) = 1$. We call this probability measure the restricted geometric distribution. This probability measure favors the partitions $\kappa$ with the smallest possible largest part $k_1$. Thus, we concern the
fluctuation of \( k_1 \) around \( \lceil \frac{n}{m} \rceil \). The motivation to work on the measure in (1) has been stated in the Introduction.

When \( m \) is a fixed integer, the main result is the following. Recall that a sequence of random vectors \( X_1, X_2, \ldots \) in \( \mathbb{R}^k \) converges weakly to a random vector \( X \in \mathbb{R}^k \) with distribution function \( F_X \) if the distribution functions \( F_{X_n}(x) \rightarrow F_X(x) \) as \( n \to \infty \) for any continuity point \( x \in \mathbb{R}^k \) of \( F_X \).

**Theorem 1.** For given \( m \geq 2 \), let \( \kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m) \) be chosen with probability \( P(\kappa) \) as in (1). For a subsequence \( n \equiv j_0 \pmod{m} \), define \( j = j_0 \) if \( 1 \leq j_0 \leq m - 1 \) and \( j = m \) if \( j_0 = 0 \). Then as \( n \to \infty \) with \( n \equiv j_0 \pmod{m} \) for a fixed \( j_0 \), we have \( (k_1 - \lceil \frac{n}{m} \rceil, \ldots, k_m - \lceil \frac{n}{m} \rceil) \) converges weakly to a discrete random vector with probability mass function (pmf)

\[
f(l_1, \ldots, l_m) = \frac{q^l}{\sum_{i=0}^{\infty} q^i \cdot |P_{m(i+1)-(m-1)}|}
\]

for all integers \((l_1, \ldots, l_m)\) with \( l_1 \geq 0, l_1 \geq \cdots \geq l_m \) and \( \sum_{i=1}^{m} l_i = j - m \).

**Remark 1.** Note that the summation in the denominator of the pmf \( f(l_1, \ldots, l_m) \) in Theorem 1 starts with \( l = 0 \). To make \( |P_{m(i+1)-(m-1)}| \) non-zero, we have \( m(l + 1) - j \geq m - 1 \Leftrightarrow j \leq m + 1 \). Since \( 1 \leq j \leq m, l = 0 \) enforces \( j = 1 \). Indeed, \( l = 0 \) corresponds to the case when the largest part \( k_1 = \lceil \frac{n}{m} \rceil \). From the constraints on the parts, this happens only when \( j = 1 \) (that is, \( n \equiv 1 \pmod{m} \)) and \( k_1 = \lceil \frac{n}{m} \rceil \), \( k_2 = \cdots = k_m = \lceil \frac{n}{m} \rceil - 1 \). If \( j \neq 1 \), then the case \( l = 0 \) cannot happen and this is guaranteed by \( |P_{m(0+1)-(m-1)}| = 0 \).

From Theorem 1, we immediately obtain the limiting distribution of the largest part \( k_1 \), which fluctuates around its smallest possible value \( \lceil \frac{n}{m} \rceil \). As a consequence, the conditional distribution of \((k_2, \ldots, k_m)\) given the largest part \( k_1 \) is asymptotically a uniform distribution.

**Corollary 1.** Given \( m \geq 2 \), let \( \kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m) \) be chosen with probability \( P(\kappa) \) as in (1). For a subsequence \( n \equiv j_0 \pmod{m} \), define \( j = j_0 \) if \( 1 \leq j_0 \leq m - 1 \) and \( j = m \) if \( j_0 = 0 \). Then as \( n \to \infty \), we have \( k_1 - \lceil \frac{n}{m} \rceil \) converges weakly to a discrete random variable with pmf

\[
f(l) = \frac{q^l \cdot |P_{m(l+1)-(m-1)}|}{\sum_{i=0}^{\infty} q^i \cdot |P_{m(l+1)-(m-1)}|}, \quad l \geq 0.
\]

Furthermore, the conditional distribution of \((k_2 - \lceil \frac{n}{m} \rceil, \ldots, k_m - \lceil \frac{n}{m} \rceil)\) given \( k_1 = \lceil \frac{n}{m} \rceil + l_1 \) \((l_1 \geq 0)\) is asymptotically a uniform distribution on the set \( \{l_2, \ldots, l_m\} \in \mathbb{Z}^{m-1}, l_1 \geq l_2 \geq \ldots \geq l_m \) and \( l_1 + \sum_{i=2}^{m} l_i = j - m \).

We present the proofs of Theorem 1 and Corollary 1 in Section 2.1.

When \( m \) tends to infinity with \( n \) and \( m = o(n^{1/3}) \), we consider the limiting distribution of the largest part \( k_1 \). The main result is that with proper normalization, the largest part \( k_1 \) converges to a normal distribution.

**Theorem 2.** Given \( q \in (0, 1) \), let \( \kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m) \) be chosen with probability \( P(\kappa) \) as in (1). Set \( \lambda = -\log q > 0 \). If \( m = m_n \to \infty \) with \( m = o(n^{1/3}) \), then \( \frac{1}{\sqrt{m}} (k_1 - \lceil \frac{n}{m} \rceil - \gamma m) \) converges weakly to \( N(0, \sigma^2) \) as \( n \to \infty \), where

\[
\gamma = \frac{1}{\lambda^2} \int_0^\lambda \frac{t}{e^t - 1} \, dt \quad \text{and} \quad \sigma^2 = \frac{2}{\lambda^3} \int_0^\lambda \frac{t}{e^t - 1} \, dt \frac{1}{\lambda(e^t - 1)} > 0.
\]

The proof of Theorem 2 is analytic and quite involved. The main technical difficulty in the proof is the estimation of the normalization constant \( c = c_{n,m} \) in (1). We use the Laplace method to estimate \( c_{n,m} \). The same analysis is applied to obtain the asymptotic
distribution of the largest part \( k_1 \). Thanks to the Szekeres formula (see (11)) for the number of restricted partitions, we first approximate \( c_{n,m}^{-1} \) with an integral

\[
c_{n,m}^{-1} \approx C(m) \cdot \int \exp(m\psi(t)) \, dt
\]

for some function \( \psi(t) \) that has a global maximum at \( t_0 > 0 \) and some quantity \( C(m) > 0 \). Thus,

\[
\psi(t) \approx \psi(t_0) - \frac{1}{2} |\psi''(t_0)| t^2
\]

and

\[
c_{n,m}^{-1} \approx C(m) e^{m\psi(t_0)} \cdot \int \exp\left(-\frac{1}{2} m|\psi''(t_0)| t^2\right) dt.
\]

The most significant contribution in the integral on the right hand side of (2) comes from the \( t \) close to \( t_0 \). Indeed, the integral in (2) is reduced to a Gaussian integral as \( n \to \infty \). We prove Theorem 2 in Section 2.2.

It remains to consider the conditional distribution of \( (k_2, \ldots, k_m) \) given the largest part \( k_1 \). It is convenient to work with \( k_1 = \lceil \frac{\gamma}{m} \rceil + l_i \) for \( 1 \leq i \leq m \). In view of Theorem 2, let \( k_1 = \lceil \frac{\gamma}{m} \rceil + l_1 \) with \( l_1 = \gamma m + C \cdot \sqrt{m} \) for an arbitrary positive constant \( C \). Given \( l_1, (l_2, \ldots, l_m) \) has a uniform distribution on the set \( \{(l_2, \ldots, l_m) \in \mathbb{Z}^{m-1}; l_1 \geq l_2 \geq \ldots \geq l_m \text{ and } l_1 + \sum_{i=2}^{m} l_i = j - m\} \). We consider a linear transform \( (j_2, \ldots, j_m) = (l_1 - l_2, \ldots, l_1 - l_m) \). Since uniform distribution is preserved under linear transformations, \( (j_2, \ldots, j_m) \) has the uniform distribution on the set \( \{(j_2, \ldots, j_m) \in \mathbb{N}^{m-1}; j_m \geq \ldots \geq j_2 \geq 0 \text{ and } \sum_{i=2}^{m} j_i = ml_1 + m - j\} \).

In general, the problem is related to understanding the uniform distribution on the set

\[
\left\{(\lambda_2, \ldots, \lambda_m) \in \mathbb{N}^{m-1}; \lambda_2 \geq \ldots \geq \lambda_m \geq 0 \text{ and } \sum_{i=2}^{m} \lambda_i = ml_1\right\}.
\]

To our best knowledge, it is not even clear what the limiting joint distribution of a partition chosen uniformly from \( \mathcal{P}_{m^2}(\gamma m) \) is as \( m \) tends to infinity. We raise the following questions for future projects.

**Question 1.** Given \( q \in (0, 1) \), let \( \kappa = (k_2, \ldots, k_m) \in \mathcal{P}_n(m) \) be chosen with probability \( P(\kappa) \) as in (1). Assume \( m \) tends to infinity with \( n \) and \( m = o(n^{1/3}) \). Determine the asymptotic joint distribution of \( (k_2, \ldots, k_m) \) given \( k_1 \). Furthermore, what is the limiting distribution of \( (k_1, k_2, \ldots, k_m) \) as \( n \) tends to infinity?

We have considered the limiting distribution of \( \kappa \in \mathcal{P}_n(m) \) chosen as in (1) for \( m \) fixed as well as \( m = o(n^{1/3}) \). The requirement of \( m = o(n^{1/3}) \) stems from the technical reason that in this regime, we could provide an asymptotic expression for the normalizing constant \( c \) in (1) (see (21) below) via Lemma 1, which facilitates further fine analysis to identify the limiting distribution of the largest part. It is also interesting to investigate this probability measure for other ranges of \( m \).

**Question 2.** Given \( q \in (0, 1) \), let \( \kappa = (k_2, \ldots, k_m) \in \mathcal{P}_n(m) \) be chosen with probability \( P(\kappa) \) as in (1). Identify the asymptotic distribution of \( \kappa \) for the entire range \( 1 \leq m \leq n \).

### 1.2. A Generalized Distribution

Next we consider a probability measure on \( \mathcal{P}_n(m) \) by choosing a partition \( \kappa = (k_1, \ldots, k_m) \) with chance

\[
P_n(\kappa) = c \cdot f \left( \frac{k_1}{n}, \ldots, \frac{k_m}{n} \right)
\]

\[(3)\]
where $c = c_{n,m} = (\sum_{(k_1, \ldots, k_m) \in \mathcal{P}_n(m)} f(k_1/n, \ldots, k_m/n))^{-1}$ is the normalizing constant and $f(x_1, \ldots, x_m)$ is defined on $\nabla_{m-1}$, the closure of $\nabla_{m-1}$. Here, $\nabla_{m-1}$ is the ordered $(m - 1)$-dimensional simplex defined as

$$\nabla_{m-1} := \{(y_1, \ldots, y_m) \in [0,1]^m; y_1 > y_2 > \ldots > y_{m-1} > y_m \text{ and } y_m = 1 - \sum_{i=1}^{m-1} y_i\}.$$ 

We assume $f$ is a probability density function on $\nabla_{m-1}$ and is either bounded continuous or Lipschitz on $\nabla_{m-1}$.

When $m$ is a fixed integer, we study the limiting joint distribution of the parts of $\kappa$ chosen as in (3). The main result is the following.

**Theorem 3.** Let $m \geq 2$ be a fixed integer. Assume $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ is chosen as in (3), where $f$ is a probability density function on $\nabla_{m-1}$ and $f$ is bounded continuous on $\nabla_{m-1}$. Then $\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)$ converges weakly to a probability measure $\mu$ with density function $f(y_1, \ldots, y_m)$ defined on $\nabla_{m-1}$.

From Theorem 3, we can immediately obtain the limiting convergence to several familiar distributions. We say $(X_1, \ldots, X_m)$ has the symmetric Dirichlet distribution with parameter $\alpha > 0$, denoted by $(X_1, \ldots, X_m) \sim \text{Dir}(\alpha)$, if the distribution has pdf

$$\frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} x_1^{\alpha-1} \cdots x_m^{\alpha-1}$$

on the $(m - 1)$-dimensional simplex

$$W_{m-1} := \{(x_1, \ldots, x_{m-1}, x_m) \in [0,1]^m; \sum_{i=1}^{m-1} x_i = 1\}$$

and zero elsewhere.

**Corollary 2.** Let $m \geq 2$ be a fixed integer. Assume $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ is chosen as in (3) with $f(x_1, \ldots, x_m) = c \cdot x_1^{\alpha-1} \cdots x_m^{\alpha-1}$ for some $\alpha \geq 1$ and $1/c = \int_{\nabla_{m-1}} x_1^{\alpha-1} \cdots x_m^{\alpha-1} dx_1 \cdots dx_m$, then

$$\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right) \rightarrow (X(1), \ldots, X(m))$$

where $(X(1), \ldots, X(m))$ is the decreasing order statistics of $(X_1, \ldots, X_m) \sim \text{Dir}(\alpha)$.

**Corollary 3.** Let $m \geq 2$ be a fixed integer. Assume $\kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m)$ is chosen as in (3) with $f(x_1, \ldots, x_m) = c \cdot x_1^{\alpha-1} \cdots x_m^{\alpha-1}$ for some $\alpha \geq 1$ and $1/c = \int_{\nabla_{m-1}} x_1^{\alpha-1} \cdots x_m^{\alpha-1} dx_1 \cdots dx_m$, then

$$\left(\frac{k_1}{n} \alpha, \ldots, \frac{k_m}{n} \alpha\right) \rightarrow (Y_1, \ldots, Y_m)$$

as $n \rightarrow \infty$, where $(Y_1, \ldots, Y_m)$ has the uniform distribution on

$$\left\{(y_1, \ldots, y_m) \in [0,1]^m; \sum_{i=1}^{m} y_i^{1/\alpha} = 1, y_1 \geq \ldots \geq y_m\right\},$$

or equivalently, $(Y_1, \ldots, Y_m)$ is the decreasing order statistics of the uniform distribution on

$$\left\{(y_1, \ldots, y_m) \in [0,1]^m; \sum_{i=1}^{m} y_i^{1/\alpha} = 1\right\}.$$
Let \( m \) be the set of all real integers. For a set \( A \), the asymptotic distribution of \( \kappa \) is the Erdös–Lehner formula that holds only for \( m \). Let \( \kappa \) distribution of \( \kappa \) be chosen with probability as in (3) and Lipschitz on \( \nabla_{m-1} \). Furthermore, assume the Lipschitz constant \( \| f \|_{\text{Lip}} \leq K \) for an absolute constant \( K > 0 \). Let \( (X_{m,1}, \ldots, X_{m,m}) \) have density function \( f(y_1, \ldots, y_m) \) defined on \( \nabla_{m-1} \). If \( (X_{m,1}, \ldots, X_{m,m}) \) converges weakly to \( X \) as \( n \to \infty \), then \( (\frac{k_1}{n}, \ldots, \frac{k_m}{n}) \) converges weakly to \( X \) as \( n \to \infty \).

We will prove Theorem 4 in Section 3.2. The proof of Theorem 4 follows along the same lines as that of Theorem 3 with modifications. In Theorem 3 where \( m \) is fixed, we only require the function \( f \) in (3) to be bounded continuous on \( \nabla_{m-1} \). This assumption is essentially used to show \( E\left( \psi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)\right) \to E(\psi(x_1, \ldots, x_m)) \) as \( n \to \infty \) for any bounded continuous function \( \psi \) on \( \nabla_{m-1} \) because \( \psi \cdot f \) is still bounded continuous on \( \nabla_{m-1} \). For Theorem 4 where \( m \) depends on \( n \), a stronger assumption on \( f \) with the Lipschitz constant \( \| f \|_{\text{Lip}} \leq K \) is imposed as we need to carefully analyze the difference \( E\left( \psi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)\right) - E(\psi(x_1, \ldots, x_m)) \) in terms of \( m \) and \( n \) for any bounded and Lipschitz function \( \psi \) on \( \nabla_{m-1} \).

We have investigated the limiting distribution of \( \kappa \in \mathcal{P}_n(m) \) chosen as in (3) for both \( m \) fixed and \( m = o(n^{1/3}) \). The assumption \( m = o(n^{1/3}) \) is due to the essential use of the Erdös–Lehner formula \( |\mathcal{P}_n(m)| \sim \frac{n^{(n-1)/m}}{m^n} \) in our proof and it is known that this asymptotic formula holds only for \( m = o(n^{1/3}) \). It would be interesting to understand the limiting distribution of \( \kappa \) for other ranges of \( m \). We leave this as an open question for future research.

Question 3. Let \( \kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m) \) be chosen with probability \( P_n(\kappa) \) as in (3). Identify the asymptotic distribution of \( \kappa \) for the entire range \( 1 \leq m \leq n \).

Notation: For \( x \in \mathbb{R} \), the notation \( \lfloor x \rfloor \) stands for the smallest integer greater than or equal to \( x \). The symbol \( \lceil x \rceil \) denotes the largest integer less than or equal to \( x \). We use \( \mathbb{Z} \) to be the set of all real integers. For a set \( A \), the notation \( \#A \) or \( |A| \) stands for the cardinality of \( A \). We also use \( \sum_{a \in A} 1 \) to represent \( |A| \). We use \( c \cdot A = \{ c \cdot a : a \in A \} \). For \( f(n)g(n) > 0 \), \( f(n) \sim g(n) \) if \( \lim_{n \to \infty} f(n)/g(n) = 1 \).

2. Proofs of Theorems 1 and 2 and Corollary 1

The strategies of deriving Theorems 1 and 2 are different. In addition, the proof of Theorem 2 is relatively lengthy. For clarity, their proofs are given in two sections.
In Section 2.1, we will present the proofs of Theorems 1 and Corollary 1. Theorem 2 will be established in Section 2.2.

2.1. The Proofs of Theorems 1 and Corollary 1

In this section, \( m \) is assumed to be a fixed integer. We start with a lemma concerning the number of restricted partitions \( \mathcal{P}_n(m) \) with the largest part fixed.

**Lemma 1.** Let \( l \geq 0, m \geq 2 \) and \( n \geq 1 \) be integers. Set \( j = m + n - m \left\lceil \frac{n}{m} \right\rceil \). Then \( 1 \leq j \leq m \). If \( 0 \leq l \leq \frac{1}{m-1} \left( \frac{n}{m} - m \right) \), we have

\[
\# \left\{ (k_1, k_2, \ldots, k_m) \in \mathcal{P}_n(m) \mid k_1 = \left\lceil \frac{n}{m} \right\rceil + l \right\} = |\mathcal{P}_{m(l+1)-j}(m-1)|. \tag{4}
\]

If \( \frac{1}{m-1} \left( \frac{n}{m} - m \right) < l \leq n - \left\lceil \frac{n}{m} \right\rceil \), we have

\[
\# \left\{ (k_1, k_2, \ldots, k_m) \in \mathcal{P}_n(m) \mid k_1 = \left\lceil \frac{n}{m} \right\rceil + l \right\} \leq |\mathcal{P}_{m(l+1)-j}(m-1)|. \tag{5}
\]

**Proof.** For \( \kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m) \), let us rewrite \( k_1 = \left\lceil \frac{n}{m} \right\rceil + l_1 \) for \( 1 \leq i \leq m \). By assumption, \( l_1 = l \geq 0 \). Since \( \kappa \vdash n \), we have \( l_1 \geq l_2 \geq \ldots \geq l_m \geq -\left\lceil \frac{n}{m} \right\rceil \) and \( l_1 + \sum_{i=2}^{m} l_i = n - m \left\lceil \frac{n}{m} \right\rceil = j - m \) by assumption. Therefore,

\[
\# \left\{ (k_1, k_2, \ldots, k_m) \in \mathcal{P}_n(m) \mid k_1 = \left\lceil \frac{n}{m} \right\rceil + l \right\} = \# \left\{ (l_2, \ldots, l_m) \in \mathbb{Z}^{m-1} \mid l_1 \geq l_2 \geq \ldots \geq l_m \geq -\left\lceil \frac{n}{m} \right\rceil \text{ and } l_1 + \sum_{i=2}^{m} l_i = j - m \right\} \]

\[
= \# \left\{ (j_2, \ldots, j_m) \in \mathbb{Z}^{m-1} \mid j_m \geq j_2 \geq \ldots \geq 2 \geq 0 \text{ and } \sum_{i=2}^{m} j_i = m(l_1 + 1) - j \right\}
\]

by the transform \( j_i = l_1 - l_i \) for \( 2 \leq i \leq m \).

Assume \( 0 \leq l \leq \frac{1}{m-1} \left( \frac{n}{m} - m \right) \). If \( j_m \geq \ldots \geq j_2 \geq 0 \) and \( \sum_{i=2}^{m} j_i = m(l_1 + 1) - j \), then

\[
j_m \leq \sum_{i=2}^{m} l_i = m(l_1 + 1) - j \leq m(l_1 + 1) \leq 1 + \left\lceil \frac{n}{m} \right\rceil
\]

by assumption, the notation \( l_1 = l \) and the fact \( \lceil x \rceil \geq x \) for any \( x \in \mathbb{R} \). It follows that the left hand side of (4) is identical to

\[
\# \left\{ (j_2, \ldots, j_m) \in \mathbb{Z}^{m-1} \mid j_m \geq \ldots \geq j_2 \geq 0 \text{ and } \sum_{i=2}^{m} j_i = m(l_1 + 1) - j \right\} = |\mathcal{P}_{m(l+1)-j}(m-1)|.
\]

For \( \frac{1}{m-1} \left( \frac{n}{m} - m \right) + 1 \leq l \leq n - \left\lceil \frac{n}{m} \right\rceil \), the upper bound (5) follows directly from the definitions of the sets. \( \Box \)

Now, we are ready to present the proof of Theorem 1.

**Proof.** (Proof of Theorem 1) First, it is easy to check that for the subsequence \( n \equiv j_0 \pmod{m} \), if we define \( j = j_0 \) if \( 1 \leq j_0 \leq m-1 \) and \( j = m \) if \( j_0 = 0 \), then \( j = m + n - m \left\lceil \frac{n}{m} \right\rceil \). Set

\[
M_n = \left\lceil \frac{1}{m-1} \left( \frac{n}{m} - m \right) \right\rceil. \tag{6}
\]
We first estimate the normalizing constant \( c \) in (1).

\[
1 = \sum_{x \in \mathcal{P}_n(m)} P(x) = c \cdot \sum_{k_1=\lceil \frac{n}{m} \rceil}^{n} \sum_{(k_2,\ldots,k_m)+n} q_{k_1}^{n-\lceil \frac{n}{m} \rceil + l}
\]

\[
= c \cdot \sum_{l=0}^{n-\lceil \frac{n}{m} \rceil + l} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} \sum_{(\lceil \frac{n}{m} \rceil + l,k_2,\ldots,k_m)+n} 1.
\]

We first show that, as \( n \) tends to infinity,

\[
\sum_{l=0}^{n-\lceil \frac{n}{m} \rceil + l} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} \sum_{(\lceil \frac{n}{m} \rceil + l,k_2,\ldots,k_m)+n} 1 \sim M_n \sum_{l=0}^{n-\lceil \frac{n}{m} \rceil + l} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} \sum_{(\lceil \frac{n}{m} \rceil + l,k_2,\ldots,k_m)+n} 1.
\]

By Lemma 1,

\[
\frac{\sum_{l=M_n+1}^{\infty} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} \sum_{(\lceil \frac{n}{m} \rceil + l,k_2,\ldots,k_m)+n} 1}{\sum_{l=0}^{M_n} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} \sum_{(\lceil \frac{n}{m} \rceil + l,k_2,\ldots,k_m)+n} 1} \leq \frac{\sum_{l=M_n+1}^{\infty} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} \cdot |\mathcal{P}_{m+1-j}(m-1)|}{\sum_{l=0}^{M_n} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} \cdot |\mathcal{P}_{m+1-j}(m-1)|}
\]

where the last equality follows from (49). Note that the series \( \sum_{l=1}^{\infty} s^{m-2} q^j \) converges for \( 0 < q < 1 \). We have

\[
\frac{\sum_{l=M_n+1}^{\infty} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{m+1-j}{m-1} \rceil}}{\sum_{l=0}^{M_n} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{m+1-j}{m-1} \rceil}} = O\left(\frac{\sum_{l=M_n+1}^{\infty} q_{\lceil \frac{n}{m} \rceil}^{l(m-2)}}{\sum_{l=0}^{M_n} q_{\lceil \frac{n}{m} \rceil}^{l(m-2)}}\right) = o(1). \tag{7}
\]

Therefore, one obtains the normalizing constant

\[
c \sim \frac{1}{q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil} \sum_{l=0}^{M_n} q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil} \cdot |\mathcal{P}_{m+1-j}(m-1)|}. \tag{8}
\]

Now, we study the limiting joint distribution of the parts

\((k_1,k_2,\ldots,k_m) = \left(\lceil \frac{n}{m} \rceil + l_1,\lceil \frac{n}{m} \rceil + l_2,\ldots,\lceil \frac{n}{m} \rceil + l_m\right)\).

First, we claim that it is enough to consider \( l_1 \) to be any fixed integer from \( \{0,1,2,\ldots\} \). Indeed, for any \( L = L(n) \to \infty \) as \( n \to \infty \), it follows from (7), (49) and Lemma 1 that

\[
P\left(k_1 \geq \lceil \frac{n}{m} \rceil + L\right) = \sum_{l=L}^{n-\lceil \frac{n}{m} \rceil} P(k_1 = \lceil \frac{n}{m} \rceil + l) \sim \sum_{l=L}^{M_n} P(k_1 = \lceil \frac{n}{m} \rceil + l) = c \cdot q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} |\mathcal{P}_{m+1-j}(m-1)|.
\]

\[
= c \cdot q_{\lceil \frac{n}{m} \rceil}^{\lceil \frac{n}{m} \rceil + l} \sum_{l=L}^{M_n} (m-2)^l q^j.
\]
Plugging in the normalizing constant $c$ in (8) and letting $L \to \infty$, we have

$$P(k_1 \geq \left\lceil \frac{n}{m} \right\rceil + L) = O\left(\frac{\sum_{j=0}^{M_0} q^j \cdot |P_{ml+m-j}((m-1)|}{\sum_{j=0}^{M_0} q^j \cdot |P_{ml+m-j}((m-1)|}\right)$$

$$= o(1)$$

as $n \to \infty$, where the last equality follows from similar arguments as (7). Likewise, we have as $n$ tends to infinity,

$$c \sim q^{-\left\lceil \frac{n}{m} \right\rceil + l} \sum_{j=0}^{\infty} q^j \cdot |P_{ml+m-j}((m-1)|.$$  \hspace{1cm} (9)

Therefore, for any given $l_1 = 0, 1, 2, \ldots$, we conclude that

$$P\left(k_1 = \left\lceil \frac{n}{m} \right\rceil + l_1, k_2 = \left\lceil \frac{n}{m} \right\rceil + l_2, \ldots, k_m = \left\lceil \frac{n}{m} \right\rceil + l_m \right) = c \cdot q^{-\left\lceil \frac{n}{m} \right\rceil + 1} \sum_{j=0}^{\infty} q^j \cdot |P_{ml+m-j}((m-1)| = f(l_1, \ldots, l_m).$$  \hspace{1cm} (10)

Finally, we show that $f(l_1, \ldots, l_m)$ is indeed a pmf on the set $S := \{ (l_1, \ldots, l_m) \in \mathbb{Z}^m; l_1 \geq 0, l_1 \geq \ldots \geq l_m \text{ and } \sum_{i=1}^m l_i = j - m \}$. To see this, summing over all possible choice of $(l_1, \ldots, l_m)$ from $S$ on both sides of (10), since the number of terms in the sum is finite and independent of $n$, we get the $\sum_{(l_1, \ldots, l_m) \in S} f(l_1, \ldots, l_m) = 1$.

The proof is completed. \hspace{0.5cm} \Box

We continue with the proof of Corollary 1.

**Proof of Corollary 1.** By Theorem 1, it is enough to consider $k_1 = \left\lceil \frac{n}{m} \right\rceil + I$ for $I \in \{ 0, 1, 2, \ldots \}$ in the limiting distribution. From (1), Lemma 1 and (9),

$$P\left(k_1 = \left\lceil \frac{n}{m} \right\rceil + l \right) = c \cdot q^{-\left\lceil \frac{n}{m} \right\rceil + l} \sum_{i=0}^{\infty} q^i \cdot |P_{ml+m-i}((m-1)| = c \cdot q^{-\left\lceil \frac{n}{m} \right\rceil + 1} \cdot |P_{ml+m-j}((m-1)| \rightarrow \frac{q^1 \cdot |P_{ml+m-j}((m-1)|}{\sum_{j=0}^{\infty} q^j \cdot |P_{ml+m-j}((m-1)|}$$

as $n \to \infty$.

Furthermore, since

$$P(k_2 = \left\lceil \frac{n}{m} \right\rceil + l_2, \ldots, k_m = \left\lceil \frac{n}{m} \right\rceil + l_m) \rightarrow \frac{q^1 \cdot |P_{ml+m-j}((m-1)|}{\sum_{j=0}^{\infty} q^j \cdot |P_{ml+m-j}((m-1)|}$$

as $n \to \infty$, it follows immediately the conditional distribution of $(k_2 - \left\lceil \frac{n}{m} \right\rceil, \ldots, k_m - \left\lceil \frac{n}{m} \right\rceil)$ given $k_1 = \left\lceil \frac{n}{m} \right\rceil + L_1 (L_1 \geq 0)$ is asymptotically a uniform distribution on the set $\{ (l_2, \ldots, l_m) \in \mathbb{Z}^{m-1}; l_1 \geq l_2 \geq \ldots \geq l_m \text{ and } \sum_{i=2}^m l_i = j - m \}$. This completes the proof. \hspace{0.5cm} \Box
2.2. The Proof of Theorem 2

Szekeres formula (see [25–28]) says that for any given \( \epsilon > 0 \),

\[
|P_n(k)| = \frac{f(u)}{n} e^{\sqrt{n}g(u) + O(n^{-1/6+\epsilon})} \tag{11}
\]

uniformly for \( k \geq n^{1/6} \), where \( u = k/\sqrt{n} \),

\[
f(u) = \frac{v}{2^{3/2}\pi u} \left(1 - e^{-v} - \frac{1}{2}u^2 e^{-v}\right)^{-1/2}, \tag{12}
\]

\[
g(u) = \frac{2v}{u} - u \log(1 - e^{-v}), \tag{13}
\]

and \( v = v(u) \) is determined implicitly by

\[
u^2 = \int_{0}^{v} \frac{t}{e^t - 1} \, dt. \tag{14}
\]

We start with a technical lemma that will be used in the proof of Theorem 2 later.

**Lemma 2.** Let \( \lambda > 0 \) be given. Define \( \psi(t) = \frac{g(t)}{t} - \frac{\lambda}{t} \) for \( t > 0 \). Then

\[
t_0 := \frac{\lambda}{(\int_{0}^{\lambda} \frac{t}{e^t - 1} \, dt)^{1/2}} \text{ satisfies } \psi''(t_0) = -\frac{2\lambda(e^\lambda - 1)}{t_0^2(e^\lambda - 1 - \frac{1}{2}t_0^2)} < 0.
\]

Furthermore, \( \psi'(t_0) = 0 \), \( \psi(t) \) is strictly increasing on \( (0, t_0] \) and strictly decreasing on \( [t_0, \infty) \).

**Proof.** Trivially, the function \( \frac{t}{e^t - 1} = (\sum_{i=1}^{\infty} \frac{t^{i-1}}{i!})^{-1} \) is positive and decreasing in \( t \in (0, \infty) \). It follows that \( v = v(u) > 0 \) for all \( u \in (0, \infty) \) and

\[
v^2 = \int_{0}^{v} \frac{t}{e^t - 1} \, dt > \frac{v^2}{e^v - 1}.
\]

Thus, \( e^v - 1 - \frac{1}{2}u^2 > 0 \). In particular,

\[
e^v - 1 - \frac{1}{2}u^2 > 0. \tag{15}
\]

By taking derivative from (14), we get

\[
2v \cdot v' = 2u \int_{0}^{v} \frac{t}{e^t - 1} \, dt + u^2 \frac{v \cdot v'}{e^v - 1}.
\]

This implies that \( \frac{v'}{e^v - 1} = \frac{2v'}{e^v} - \frac{2v}{e^v} \), or equivalently,

\[
v' = \frac{v + \frac{u^2}{2(e^v - 1 - \frac{1}{2}u^2)}}{u^2}. \tag{16}
\]

Consequently, \( v' = v'(u) > 0 \) for all \( u > 0 \), and thus \( v(u) \) is strictly increasing on \( (0, \infty) \). Take derivative on \( g(u) \) in (13), and use (14) and (16) to see

\[
g'(u) = -\log(1 - e^{-v});
\]

\[
g''(u) = -\frac{v' e^{-v}}{1 - e^{-v}} = -\frac{v/u}{e^v - 1 - \frac{1}{2}u^2}.
\]
Therefore,

\[
\left( \frac{g(u)}{u} \right)' = \frac{ug'(u) - g(u)}{u^2}
\]  

and

\[
\left( \frac{g(u)}{u} \right)'' = \frac{g''(u)}{u} - 2\frac{g'(u)}{u^2} + \frac{2g(u)}{u^3}
\]

\[
= \frac{v}{u^4} \left( 4 - \frac{u^2}{e^v - 1 - \frac{1}{2}u^2} \right).
\]

With the above preparation, we now study \(\psi(t)\) (we switch the variable “\(u\)” to “\(t\)”).

\[
\psi''(t) = \left( \frac{g(t)}{t} - \frac{\lambda}{t^2} \right)''
\]

\[
= \frac{v}{t^4} \left( 4 - \frac{t^2}{e^v - 1 - \frac{1}{2}t^2} \right) - \frac{6\lambda}{t^4}
\]

\[
= \frac{1}{t^4} \left( 4v - 6\lambda - \frac{v \cdot t^2}{e^v - 1 - \frac{1}{2}t^2} \right). \tag{19}
\]

The assertions in (17) and (18) imply

\[
\left( \frac{g(t)}{t} \right)' = -\frac{t^2 \log(1 - e^{-v}) - tg(t)}{t^3} = -\frac{2v}{t^3}.
\]

Thus, \(\psi'(t) = \frac{2(\lambda - v)}{t}\). Thus, the stable point \(t_0\) of \(\psi(t)\) satisfies that \(v(t_0) = \lambda\). This implies that \(\psi(t)\) is strictly increasing on \((0, t_0]\) and strictly decreasing on \([t_0, \infty)\). It is not difficult to see from (14) that

\[
t_0 = \frac{\lambda}{(\int_0^\lambda \frac{t}{e^t - 1} dt)^{1/2}}.
\]

Plug this into (19) to get

\[
\psi''(t_0) = -\frac{1}{t_0^4} \left( 2\lambda + \frac{\lambda \cdot t_0^2}{e^\lambda - 1 - \frac{1}{2}t_0^2} \right)
\]

\[
= -\frac{2\lambda(e^\lambda - 1)}{t_0^4(e^\lambda - 1 - \frac{1}{2}t_0^2)} < 0
\]

by (15). \(\square\)

Now, we are in a position to prove Theorem 2.

**Proof of Theorem 2.** Let \(M_m = \left[ \frac{1}{m - 1}(\frac{n}{m} - m) \right]\) as in (6). The assumption \(m = o\left(n^{1/3}\right)\) implies

\[
\lim_{n \to \infty} \frac{M_m}{m} = \infty. \tag{20}
\]

Similar to (8), we first claim that the normalization constant

\[
\epsilon \sim \frac{1}{\sum_{j=0}^{M_m} \frac{1}{q^j j! \cdot |\mathcal{P}_{m(j+1)-j}(m-1)|}}. \tag{21}
\]
Indeed, from Lemma 1,
\[
\frac{1}{c} = \sum_{l=0}^{n-\left\lceil \frac{n}{m} \right\rceil + 1} q^\left\lceil \frac{n}{m} \right\rceil + l \sum_{\left\lceil \frac{n}{m} \right\rceil + 1, k_2, \ldots, k_m} 1
\]
\[
= \sum_{l=M_n+1}^{M_n} q^\left\lceil \frac{n}{m} \right\rceil + l \left| \mathcal{P}_{m(l+1)-j}(m-1) \right| + \sum_{l=M_n+1}^{n-\left\lceil \frac{n}{m} \right\rceil + 1} q^\left\lceil \frac{n}{m} \right\rceil + l \sum_{\left\lceil \frac{n}{m} \right\rceil + 1, k_2, \ldots, k_m} 1
\]
and
\[
\sum_{l=M_n+1}^{n-\left\lceil \frac{n}{m} \right\rceil + 1} q^\left\lceil \frac{n}{m} \right\rceil + l \sum_{\left\lceil \frac{n}{m} \right\rceil + 1, k_2, \ldots, k_m} 1 \leq \sum_{l=M_n+1}^{n-\left\lceil \frac{n}{m} \right\rceil + 1} q^\left\lceil \frac{n}{m} \right\rceil + l \left| \mathcal{P}_{m(l+1)-j}(m-1) \right|
\]
\[
= \sum_{l=M_n+2}^{n-\left\lceil \frac{n}{m} \right\rceil + 1} q^\left\lceil \frac{n}{m} \right\rceil + l \left| \mathcal{P}_{lm-j}(m-1) \right|.
\]
Observe that \(\left| \mathcal{P}_{lm-j}(m-1) \right| \leq \left| \mathcal{P}_{lm}(lm) \right| \leq e^{K\sqrt{lm}}\) for some constant \(K > 0\) by the Hardy-Ramanujan formula [29]. Therefore,
\[
\sum_{l=M_n+1}^{n-\left\lceil \frac{n}{m} \right\rceil + 1} q^\left\lceil \frac{n}{m} \right\rceil + l \sum_{\left\lceil \frac{n}{m} \right\rceil + 1, k_2, \ldots, k_m} 1 \leq q^\left\lceil \frac{n}{m} \right\rceil \sum_{l=M_n}^{\infty} e^{-\lambda l + K\sqrt{lm}}
\]
\[
\leq q^\left\lceil \frac{n}{m} \right\rceil \sum_{l=M_n}^{\infty} e^{-\lambda l/2} \leq q^\left\lceil \frac{n}{m} \right\rceil \frac{e^{-\lambda M_n/2}}{1 - e^{-\lambda/2}}
\]
\[
= o\left( \sum_{l=0}^{M_n} q^\left\lceil \frac{n}{m} \right\rceil + l \left| \mathcal{P}_{m(l+1)-j}(m-1) \right| \right)
\]
for \(n\) sufficiently large. This completes the proof of (21).
Hence, following (21) and Lemma 1, without loss of generality, we have
\[
P(k_1 = \left\lceil \frac{n}{m} \right\rceil + l) \sim \frac{q^l \left| \mathcal{P}_{m(l+1)-j}(m-1) \right|}{\sum_{l=0}^{M_n} q^l \left| \mathcal{P}_{m(l+1)-j}(m-1) \right|}
\]
for \(l = 0, 1, 2, \ldots, M_n\), where \(j = m + n - m\left\lceil \frac{n}{m} \right\rceil\) and \(1 \leq j \leq m\). Thus, combined with (20), we arrive at
\[
P(k_1 \leq \left\lceil \frac{n}{m} \right\rceil + m\xi) \sim \frac{\sum_{l=1}^{M_n+1} q^l \left| \mathcal{P}_{lm-j}(m-1) \right|}{\sum_{l=1}^{M_n+1} q^l \left| \mathcal{P}_{lm-j}(m-1) \right|}
\]
for any \(\xi \geq 0\).
In the following, we first apply a fine analysis to estimate the denominator
\[
\sum_{l=1}^{M_n+1} q^l \left| \mathcal{P}_{lm-j}(m-1) \right|.
\]
We divide the range of summation into five parts: \(1 \leq l \leq cm\), \(Cm \leq l \leq M_n\), \(cm \leq l < \gamma m - \sqrt{m} \log m\), \(\gamma m + \sqrt{m} \log m \leq l \leq Cm\) and \(\gamma m - \sqrt{m} \log m \leq l \leq \gamma m + \sqrt{m} \log m\) for some proper constants \(c, C > 0\) and \(\gamma = l_0^{-2}\) (recall \(l_0\) in Lemma 2). The most significant contribution in the summation comes from the range \(\gamma m - \sqrt{m} \log m \leq l \leq \gamma m + \sqrt{m} \log m\) and others are negligible. The estimation for the numerator is similar.
Before we proceed to the technical details, we explain in more detail how the division in (23) is chosen. Following the heuristic explained in (2), the most significant contribution in the summation (23), which is approximated by the integral

$$C(m)e^{\gamma_t(t_0)} \cdot \int \exp \left( -\frac{1}{2} m|\psi''(t)|^2 \right) dt,$$

(24)

comes from the $t$ close to $t_0$ given in Lemma 2. Dividing (23) into five parts can be thought of dividing (24) into five integrals with $t = \sqrt{\frac{m}{C}}$. Indeed, the constants $c, C$ in the division of (23) are chosen (see (30) below) to satisfy $1/\sqrt{C} < t_0 < 1/\sqrt{c}$. Hence, for the parts where $1 \leq l \leq cm$ or $Cm \leq l \leq M_n$, they correspond to the integrals in (24) where $t \geq 1/\sqrt{c}$ or $t \leq 1/\sqrt{C}$ and the contribution is negligible. For the parts in (23) where $Cm \leq l \leq M_n$ or $cm \leq l \leq \gamma m - \sqrt{m}\log m$, they correspond to the integrals in (24) where $t$ is of order $\log m/\sqrt{m}$ away from $t_0$. We show their contribution is also negligible though finer analysis. The main contribution in (23) is essentially from the part where $\gamma m - \sqrt{m}\log m \leq l \leq \gamma m + \sqrt{m}\log m$. This corresponds to the integral in (24) where $t$ is within $O(\log m/\sqrt{m})$ from $t_0$.

**Step 1: Two rough tails are negligible.** First, by the Hardy–Ramanujan formula, there exists a constant $K > 0$ such that

$$|\mathcal{P}_{1m-j}(m-1)| \leq |\mathcal{P}_{im}| \leq e^{K\sqrt{m}},$$

for $l \geq 1$ as $n$ is large. Set $\lambda = -\log q > 0$. It follows that

$$\sum_{l=cm}^{M_n+1} q^l \cdot |\mathcal{P}_{1m-j}(m-1)| \leq \sum_{l=cm}^{\infty} e^{-\lambda l + K\sqrt{m}} \leq \sum_{l=cm}^{\infty} e^{-\lambda l / 2} \leq \frac{1}{1 - e^{-\lambda / 2}} \leq \frac{1}{1 - \frac{\lambda}{2}}$$

(25)

for all $l \geq \left( \frac{4K^2}{\lambda^2} \right)m$ and for $C > \frac{4K^2}{\lambda^2}$. Similarly, for the same $K$ as above,

$$\sum_{l=1}^{cm} q^l \cdot |\mathcal{P}_{1m-j}(m-1)| \leq \sum_{l=1}^{cm} q^l \cdot |\mathcal{P}_{cm}(m)|$$

$$\leq (cm) \cdot |\mathcal{P}_{cm}(m)| \leq (cm) \cdot e^{K\sqrt{cm}}$$

(26)

for all $c > 0$ as $n$ is sufficiently large.

In the rest of the proof, the variable $n$ will be hidden in $m = m_n$ and $j = j_n$. Keep in mind that $m$ is sufficiently large when we say "$n$ is sufficiently large". We set two parameters

$$C = \max \left\{ \frac{8K^2}{\lambda^2}, 2\gamma \right\},$$

(27)

$$c = \min \left\{ \frac{\psi(t_0)^2}{16K^2}, \frac{\gamma}{2} \right\}.$$  

(28)

**Step 2: Two refined tails are negligible.** Recall $t_0$ in Lemma 2. Define $\gamma = t_0^{-2}$ and

$$\Omega_1 = \{ l \in \mathbb{N}; cm \leq l < \gamma m - \sqrt{m}\log m \},$$

$$\Omega_2 = \{ l \in \mathbb{N}; \gamma m - \sqrt{m}\log m \leq l \leq \gamma m + \sqrt{m}\log m \},$$

$$\Omega_3 = \{ l \in \mathbb{N}; \gamma m + \sqrt{m}\log m < l \leq Cm \},$$

(29)

where $c \in (0, \gamma)$ and $C > \gamma$ by (28) and (27). Note that

$$1/\sqrt{C} < t_0 = \gamma^{-1/2} < 1/\sqrt{c}.$$  

(30)
The limit in (20) asserts that \( \Omega_2 \subset \{1, 2, \cdots, M_n\} \) as \( n \) is large. Then
\[
\sum_{l = cm}^{Cm} q^l \cdot |P_{lm-j}(m-1)| = \sum_{i=1}^{3} \sum_{l \in \Omega_{2i}} q^l \cdot |P_{lm-j}(m-1)|.
\]
(31)

Easily,
\[
\sum_{l \in \Omega_1 \cup \Omega_3} q^l \cdot |P_{lm-j}(m-1)| \leq \sum_{l \in \Omega_1 \cup \Omega_3} q^l \cdot |P_{lm}(m)|.
\]
(32)

Take \( n = lm \) and \( k = m \) in (11), we get
\[
|P_{lm}(m)| \sim \frac{f(u)}{lm} e^{\sqrt{m}g(u)}
\]
uniformly for all \( cm \leq l \leq Cm \) where \( u = (\frac{m}{l})^{1/2} \). Notice
\[
q^l \cdot |P_{lm}(m)| \sim \frac{f(u)}{lm} e^{-\lambda l + \sqrt{m}g(u)}.
\]

Consider function \(-\lambda x + \sqrt{mx} \cdot g((mx^{-1})^{1/2})\) for \( x \in [cm, Cm] \). Set \( t = t_x = (mx^{-1})^{1/2} \).

Then
\[
-\lambda x + \sqrt{mx} \cdot g((mx^{-1})^{1/2}) = -\frac{\lambda m}{t^2} + m \frac{g(t)}{t} = m \left( \frac{g(t)}{t} - \frac{1}{t^2} \lambda \right).
\]

By (12) and (13), \( f(x) \) is a continuous function on \([C^{-1/2}, c^{-1/2}]\). Therefore, \( \frac{f((mj^{-1})^{1/2})}{mj} = O(m^{-2}) \) uniformly for all \( j \in \Omega_1 \cup \Omega_3 \), which together with (32) yields
\[
\sum_{l \in \Omega_1 \cup \Omega_3} q^l \cdot |P_{lm-j}(m-1)| \leq O\left( \frac{1}{m^2} \right) \sum_{l \in \Omega_1 \cup \Omega_3} \exp \left[ m \left( \frac{g(t)}{t} - \frac{\lambda}{t^2} \right) \right]
\]
\[
\leq O\left( \frac{1}{m} \right) \cdot \exp \left[ m \max_{l \in \Omega_1 \cup \Omega_3} \left( \frac{g(t)}{t} - \frac{\lambda}{t^2} \right) \right].
\]
(33)

Now,
\[
\max_{l \in \Omega_1 \cup \Omega_3} \left( \frac{g(t)}{t} - \frac{\lambda}{t^2} \right) = \max_{l \in \Omega_1 \cup \Omega_3} \left\{ \psi\left( \sqrt{\frac{m}{t}} \right) \right\}.
\]

Evidently,
\[
\left\{ \sqrt{\frac{m}{T}} \cdots l \in \Omega_1 \right\} \subset \left( \frac{m}{\gamma m - \sqrt{m} \log m} \right)^{1/2}, \quad \frac{1}{1/2} \subset (t_0, \infty); \quad \left\{ \sqrt{\frac{m}{T}} \cdots l \in \Omega_3 \right\} \subset \left[ \frac{1}{\sqrt{c}}, \left( \frac{m}{\gamma m + \sqrt{m} \log m} \right)^{1/2} \right] \subset (0, t_0).
\]
Recall Lemma 2, $\psi(t) = \frac{g(t)}{t} - \frac{\lambda}{t^2}$ is increasing $(0, t_0]$ and decreasing in $[t_0, \infty)$. It follows that

$$\max_{l \in \Omega_1 \cup \Omega_3} \left( \frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2} \right) \leq \max \left\{ \psi \left( \frac{\sqrt{m}}{\sqrt{\gamma m} \pm \sqrt{m} \log m} \right), \psi \left( \frac{\sqrt{m}}{\gamma m + \sqrt{m} \log m} \right) \right\}. $$

Recall that $t_0 = \gamma^{-1/2}$. Notice

$$\left( \frac{\sqrt{m}}{\sqrt{\gamma m} \pm \sqrt{m} \log m} - t_0 \right)^2 = \left[ \frac{1}{\sqrt{\gamma}} \left( 1 \pm \frac{\log m}{\sqrt{\gamma m}} \right)^{-1/2} - t_0 \right]^2 = \frac{(\log m)^2}{4 \gamma^3 m} (1 + o(1)).$$

By Taylor expansion and the fact that $\psi'(t_0) = 0$, we see that

$$\psi \left( \frac{\sqrt{m}}{\sqrt{\gamma m} \pm \sqrt{m} \log m} \right) = \psi(t_0) - L \frac{(\log m)^2}{m} + O(m^{-3/2}(\log m)^3)$$

as $n$ is large, where $L = |\psi''(t_0)| > 0$. This joins (33) to yield that

$$\sum_{l \in \Omega_1 \cup \Omega_3} q^l \cdot |P_{ml-j}(m-1)| \leq \sqrt{m} e^{m\psi(t_0)-(L/2)(\log m)^3} \quad (34)$$

as $n$ is large.

Step 3. The estimate of $\sum_{j \in \Omega_2}$. Take $n = lm - j$ and $k = m - 1$ in (11), we get

$$|P_{ml-j}(m-1)| \sim \frac{f(u)}{ml-j} e^{\sqrt{m} \log m \log m - j}$$

uniformly for all $cm \leq l \leq Cm$ where $u = \frac{m-1}{\sqrt{m} \log m}$.

For $l \in \Omega_2$ from (29),

$$\gamma m^2 - m \sqrt{m} \log m - j \leq ml - j \leq \gamma m^2 + m \sqrt{m} \log m - j.$$

Note that $1 \leq j \leq m$. As $m \to \infty$ with $n \to \infty$, we observe that

$$u = \frac{m-1}{\sqrt{m} \log m} \sim \frac{1}{\sqrt{\gamma}} = t_0$$

and

$$\frac{m^2}{ml-j} \sim \frac{1}{\gamma} = t_0^2.$$

Hence, by continuity,

$$\frac{f(u)}{ml-j} \sim t_0^2 f(t_0) \cdot \frac{1}{m^2}$$
for all \( l \in \Omega_2 \). Consequently,
\[
\sum_{l \in \Omega_2} q^l \cdot |\mathcal{P}_{lm-j}(m-1)|
= (1 + o(1)) \frac{t^2_0 f(t_0)}{m^2} \sum_{l \in \Omega_2} \exp \left\{ -\lambda l + \sqrt{lm-j} \cdot \lambda \left( \frac{m-1}{\sqrt{lm-j}} \right) \right\}
\sim \frac{t^2_0 f(t_0)}{m^2} e^{-\lambda t_j/m} \sum_{l \in \Omega_2} \exp \left\{ -\frac{\lambda (m-1)^2}{m^2 t_j} + \frac{m-1}{t_j} \lambda \left( \frac{l}{t_j} \right) \right\}
\]
by setting \( t_x = \left( m-1 \right) / \sqrt{mx-j} \) for \( x \geq 2 \) (recall \( 1 \leq j \leq m \)), and hence \( x = \frac{t}{m} + \frac{(m-1)^2}{m^2 t_j} \). It is easy to verify that
\[
\max_{l \in \Omega_2} |t_l - t_0| = O\left( \frac{\log m}{\sqrt{m}} \right)
\] (35)
as \( n \to \infty \). We then have
\[
\sum_{l \in \Omega_2} q^l \cdot |\mathcal{P}_{lm-j}(m-1)|
\sim \frac{t^2_0 f(t_0)}{m^2} e^{-\lambda t_j^2} \sum_{l \in \Omega_2} \exp \left\{ (m-1) \left( \frac{\lambda (l)}{t_j} - \frac{\lambda}{t_j^2} \right) \right\}.
\] (36)

Recall Lemma 2. Since \( \psi'(t_0) = 0 \), it is seen from the Taylor’s expansion and (35) that
\[
\psi(t_x) = \psi(t_0) + \frac{1}{2} \psi''(t_0)(t_x - t_0)^2 + O\left( m^{-3/2} (\log m)^3 \right)
\]
uniformly for all \( x \in \Omega_2 \). It follows that
\[
\sum_{l \in \Omega_2} \exp \left[ (m-1) \left( \frac{\lambda (l)}{t_j} - \frac{\lambda}{t_j^2} \right) \right]
= (1 + o(1)) \cdot e^{(m-1)\psi(t_0)} \sum_{l \in \Omega_2} \exp \left[ \frac{1}{2} \psi''(t_0)(t_l - t_0)^2 \right].
\]
It is trivial to check that
\[
\frac{m-1}{\sqrt{mx-j}} = \frac{m-1}{\sqrt{mx}} + \frac{j}{2 \gamma^{1/2} m^2} + O\left( \frac{\log m}{m^2} \right)
\]
uniformly for all \( x \in \Omega_2 \). Therefore,
\[
m\left( \frac{m-1}{\sqrt{mx-j}} - t_0 \right)^2 = m\left( \frac{m-1}{\sqrt{mx}} - t_0 \right)^2 + \frac{j}{2 \gamma^{1/2} m} \left( \frac{m-1}{\sqrt{mx}} - t_0 \right) + O\left( \frac{\log m}{\sqrt{m}} \right)
\]
uniformly for all \( x \in \Omega_2 \) by (35). This tells us that
\[
\sum_{l \in \Omega_2} \exp \left[ (m-1) \left( \frac{\lambda (l)}{t_j} - \frac{\lambda}{t_j^2} \right) \right]
= (1 + o(1)) \cdot e^{(m-1)\psi(t_0)} \sum_{l \in \Omega_2} \exp \left[ \frac{1}{2} \psi''(t_0) \left( \frac{m-1}{\sqrt{mt_j}} - t_0 \right)^2 \right] m \right],
\] (37)
Set \( a_m = \gamma m - \sqrt{m} \log m \), \( b_m = \gamma m + \sqrt{m} \log m \), \( c_m = (m - 1)/\sqrt{m} \) and

\[
\rho(x) = \exp \left[ \frac{1}{2} \psi''(t_0) \left( \frac{c_m}{\sqrt{x}} - t_0 \right)^2 m \right]
\]

(38)

for \( x > 0 \). It is easy to check that there exists an absolute constant \( C_1 > 0 \) such that

\[ \rho(x) \leq e^{-C_1(\log m)^2} \]

(39)

for all \( x \in (a_m, b_m) \setminus ([a_m] + 2, [b_m] - 2) \). Hence,

\[
\int_{a_m}^{b_m} \rho(x) \, dx = \left( \sum_{l=[a_m]}^{[b_m]-1} \int_{l+1}^{l+1} \rho(x) \, dx \right) + \epsilon_m,
\]

(40)

where \( |\epsilon_m| \leq e^{-C_1(\log m)^2} \) for large \( m \). By the expression \( \rho(x) = \exp \left[ \frac{1}{2} \psi''(t_0) \left( \frac{c_m}{\sqrt{x}} - t_0 \right)^2 m \right] \), we get

\[
\rho'(x) = -\frac{1}{2} \rho(x) \psi''(t_0) \left( \frac{c_m}{\sqrt{x}} - t_0 \right) \frac{mc_m}{x^{3/2}}
\]

for \( x > 0 \). Easily \( \frac{mc_m}{x^{3/2}} = O(1) \) and \( \frac{c_m}{\sqrt{x}} - t_0 = O(\frac{\log m}{\sqrt{m}}) \) uniformly for all \([a_m] \leq x \leq [b_m]\). Thus,

\[ |\rho'(x)| \leq \frac{(\log m)^2}{\sqrt{m}} \rho(x) \]

for all \([a_m] \leq x \leq [b_m]\). Therefore, by integration by parts,

\[
\left| \int_{l}^{l+1} \rho(x) \, dx - \rho(l) \right| = \left| \int_{l}^{l+1} \rho'(x)(l + 1 - x) \, dx \right| \leq \int_{l}^{l+1} |\rho'(x)| \, dx \leq \frac{(\log m)^2}{\sqrt{m}} \int_{l}^{l+1} \rho(x) \, dx
\]

as \( m \) is sufficiently large. This, (39) and (40) imply

\[
\left| \sum_{l \in \Omega_2} \rho(l) - \int_{a_m}^{b_m} \rho(x) \, dx \right| \leq \frac{(\log m)^2}{\sqrt{m}} \left( \int_{a_m}^{b_m} \rho(x) \, dx \right) + e^{-C_1(\log m)^2}.
\]

(41)

Set \( \gamma_m = (\log m)\gamma^{-3/2}/2 \). We see from (37) and (38) that

\[
\int_{a_m}^{b_m} \rho(x) \, dx = \frac{2\gamma_m}{\sqrt{m}} \int_{-\gamma_m + o(1)}^{\gamma_m + o(1)} \left( -\frac{u}{\sqrt{m}} + t_0 \right)^{-3} e^{\frac{1}{2} \psi''(t_0) u^2} \, du
\]

\[
= (1 + o(1)) \frac{2\gamma_m}{\sqrt{m}} \int_{-\gamma_m}^{\gamma_m} e^{\frac{1}{2} \psi''(t_0) u^2} \, du
\]

\[
= (1 + o(1)) \frac{2\gamma_m}{\sqrt{m}} \int_{-\infty}^{\infty} e^{\frac{1}{2} \psi''(t_0) u^2} \, du
\]

\[
\sim \frac{\sqrt{m}}{t_0^3} \int_{0}^{\gamma_m} \left( \gamma_m \right)^2 \frac{8\pi}{|\psi''(t_0)|}
\]
by making the transform \( u = - \left( \frac{c_n}{\sqrt{n}} - t_0 \right) \sqrt{m} \). Combining this, (37) and (41), we arrive at

\[
e^{-(m-1)\psi(t_0)} \sum_{l \in \Omega_2} \exp \left[ (m - 1) \left( \frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2} \right) \right] = (1 + o(1)) \sum_{l \in \Omega_2} \rho(l) \\
\sim \sqrt{m} \cdot \frac{1}{t_0^3} \sqrt{\frac{8\pi}{|\psi''(t_0)|}} \tag{42}
\]
as \( n \) is sufficiently large. This and (36) yield

\[
\sum_{l \in \Omega_2} q^l \cdot |P_{lm-j}(m-1)| \\
\sim \frac{t_0^2 f(t_0)}{m^2} e^{\lambda t_0^2 - (\lambda_j/m)} \sum_{l \in \Omega_2} \exp \left\{ (m - 1) \left( \frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2} \right) \right\} \\
\sim \frac{f(t_0)e^{\lambda t_0^2 - \psi(t_0) - (\lambda_j/m)}}{t_0} \cdot \sqrt{\frac{8\pi}{|\psi''(t_0)|}} \cdot e^{m\psi(t_0)} \frac{m^{3/2}}{m^{3/2}} \tag{43}
\]
as \( m \to \infty \).

**Step 4. Wrap-up of the denominator.** By the choice of \( c \) in (28), we have \( \sqrt{c} \leq (4K)^{-1}\psi(t_0) \) in (26). Therefore, we get from (25) and (26) that

\[
\left( \sum_{l=1}^{M_m} + \sum_{l=C_m}^{M_m+1} \right) q^l \cdot |P_{lm-j}(m-1)| \leq e^{\psi(t_0)m/2} \tag{44}
\]
as \( n \) is large. This and (31) imply

\[
\sum_{l=1}^{M_m+1} q^l \cdot |P_{lm}(m-1)| = O(e^{\psi(t_0)m/2}) + \sum_{l=1}^{3} \sum_{l \in \Omega_2} q^l \cdot |P_{lm-j}(m-1)|
\]
as \( m \to \infty \). This identity together with (34) and (43) concludes that

\[
\sum_{l=1}^{M_m+1} q^l \cdot |P_{lm-j}(m-1)| \sim \frac{f(t_0)e^{\lambda t_0^2 - \psi(t_0) - (\lambda_j/m)}}{t_0} \cdot \sqrt{\frac{8\pi}{|\psi''(t_0)|}} \cdot e^{m\psi(t_0)} \frac{m^{3/2}}{m^{3/2}} \tag{45}
\]
as \( m \to \infty \).

**Step 5. Numerator.** We need to show

\[
\lim_{n \to \infty} P\left( \frac{1}{\sqrt{m}} \left( k_1 - \left\lceil \frac{n}{m} \right\rceil - \frac{m}{t_0} \right) \leq x \right) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{x} e^{-\frac{t^2}{2\sigma^2}} dt
\]

for every \( x \in \mathbb{R} \), where \( \sigma = \frac{1}{\sqrt{|\psi'(t_0)|}} \). Recall \( \gamma = t_0^{-2} \). By (22),

\[
P\left( \frac{1}{\sqrt{m}} \left( k_1 - \left\lceil \frac{n}{m} \right\rceil - \frac{m}{t_0} \right) \leq x \right) = \frac{\sum_{j=1}^{b_m} q^l \cdot |P_{ml-j}(m-1)|}{\sum_{l=1}^{M_m+1} q^l \cdot |P_{lm-j}(m-1)|} \tag{46}
\]

where \( b_m' = \gamma m + \sqrt{m} x + 1 \). Recall that \( \sqrt{c} \leq (4K)^{-1}\psi(t_0) \). It is known from (44) that

\[
\sum_{l=1}^{cm} q^l \cdot |P_{lm-j}(m-1)| \leq e^{\psi(t_0)m/2} \tag{47}
\]
as \( n \) is large. Let \( \Omega_1 \) and \( \Omega_2 \) be as in (29). Set \( \Omega'_2 = \{ l \in \mathbb{N}; \gamma m - \sqrt{m} \log m \leq l \leq b'_m \} \). Notice \( \Omega'_2 \subset \Omega_2 \) for large \( m \). By (34), (36) and (47),

\[
\sum_{l=1}^{b'_m} q^l \cdot |P_{ml-j}(m-1)| = O\left( e^{\psi(t_0)/2} + \sqrt{m} e^{\psi(t_0) - (L/2)(\log m)^2} \right) + \sum_{l \in \Omega'_2} q^l \cdot |P_{ml-j}(m-1)|
\]

\[
= O\left( \sqrt{m} \cdot e^{\psi(t_0) - (L/2)(\log m)^2} \right) + \frac{t_0^2 f(t_0)}{m^2} e^{\lambda t_0 - (\lambda j/m)} \sum_{l \in \Omega'_2} \exp \left\{ (m-1) \left( \frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2} \right) \right\}
\]

(48)
as \( m \to \infty \). Review the derivation between (37) and (42) and replace \( b_m \) by \( b'_m \), by the fact \( \Omega'_2 \subset \Omega_2 \) for large \( m \) again, we have

\[
e^{-\gamma m - \sqrt{m} \log m} \sum_{l \in \Omega'_2} \exp \left\{ (m-1) \left( \frac{g(t_l)}{t_l} - \frac{\lambda}{t_l^2} \right) \right\}
\]

where, as mentioned before, \( a_m = \gamma m - \sqrt{m} \log m \) and \( |e_m| \leq e^{-C_1(\log m)^2} \) for large \( m \). Let us evaluate the integral above. In fact, from (38) we see that

\[
\int_{a_m}^{b'_m} \rho(x) \, dx = \int_{a_m}^{b'_m} \exp \left\{ \frac{1}{2} \psi''(t_0) \left( \frac{e_m}{\sqrt{x}} - t_0 \right)^2 m \right\} \, dx.
\]

Recall the fact \( \gamma = t_0^{-2} \). Set \( w = -\left( \frac{e_m}{\sqrt{x}} - t_0 \right) \sqrt{m} \). Then

\[
\int_{a_m}^{b'_m} \rho(x) \, dx = \frac{2 \gamma^2}{\sqrt{m}} \int_{-\gamma m + o(1)}^{\gamma m + o(1)} \left( - \frac{w}{\sqrt{m}} + t_0 \right)^{-3} e^{-\gamma \psi''(t_0) |w|^2} \, dw
\]

\[
= (1 + o(1)) \frac{2 \gamma^2}{t_0^{3/2}} \int_{-\infty}^{\infty} e^{-\gamma \psi''(t_0) |w|^2} \, dw
\]

\[
= (1 + o(1)) \frac{\sqrt{m}}{t_0^{-3/2}} \frac{1}{\sqrt{\gamma}} \int_{-\infty}^{\infty} e^{-u^2/(2\sigma^2)} \, du = (1 + o(1)) \sqrt{m} \int_{-\infty}^{\infty} e^{-u^2/(2\sigma^2)} \, du
\]

where \( \gamma_m = (\log m)^{-3/2} / 2 \) and \( \sigma^2 = \frac{\gamma^2}{\psi''(t_0)} \). Collect the assertions from (48) to the above to obtain

\[
\sum_{l=1}^{b'_m} q^l \cdot |P_{ml-j}(m-1)| = (1 + o(1)) \frac{t_0^2 f(t_0)}{m^2} e^{\lambda t_0 - (\lambda j/m)} \cdot e^{(m-1)\psi(t_0)} \cdot \sqrt{m} \int_{-\infty}^{\infty} e^{-u^2/(2\sigma^2)} \, du
\]

\[
\sim \frac{t_0^2 f(t_0)}{m^2} e^{\lambda t_0 - (\lambda j/m)} \cdot e^{\psi(t_0)} \cdot \frac{e^{m\psi(t_0)}}{m^{3/2}} \int_{-\infty}^{\infty} e^{-u^2/(2\sigma^2)} \, du
\]

as \( m \to \infty \). Join this with (45) and (46) to conclude that

\[
P\left( \frac{1}{\sqrt{m}} \left( k_1 - \left\lfloor \frac{n}{m} \right\rfloor - \frac{m}{t_0^2} \right) \leq x \right) \to \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{x} e^{-u^2/(2\sigma^2)} \, du
\]
as $m \to \infty$. Notice that $\sigma^2 = \frac{4}{|\psi''(0)|^2}$. The proof is completed by using Lemma 2 and the fact $\gamma = t_0^{-2}$.

**3. Proofs of Theorems 3 and 4 and Corollaries 2 and 3**

In Section 3.1 below, we will prove Theorem 3, Corollaries 2 and 3 where $m$ is assumed to be a fixed integer. Theorem 4 studies the case when $m$ tends to infinity with $n$ and $m = o(n^{1/3})$. Its proof is given in Section 3.2.

**3.1. The Proofs of Theorem 3 and Corollaries 2 and 3**

From [4], we have

$$|\mathcal{P}_n(m)| \sim \frac{(n-1)!}{m!}$$

(49)

uniformly for $m = o(n^{1/3})$ in the sense that for any $\epsilon > 0$ and $0 < m^3/n < \epsilon$, the ratio of $|\mathcal{P}_n(m)|$ to $(\frac{n-1}{m!})$ remains between $1 \pm \epsilon$ as $n \to \infty$. We start with the proof of Theorem 3.

**Proof of Theorem 3.** To prove the conclusion, it suffices to show that for any bounded continuous function $\psi$ on $\mathbb{V}_{m-1}$,

$$\mathbb{E}\left(\psi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)\right) \to \mathbb{E}(\psi(x_1, \ldots, x_m))$$

as $n$ tends to infinity, where $(x_1, \ldots, x_m) \sim \mu$. By definition,

$$\mathbb{E}\left(\psi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)\right) = \frac{\sum_{(k_1, \ldots, k_m) \in \mathcal{P}_n(m)} \Psi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right) f\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)}{\sum_{(k_1, \ldots, k_m) \in \mathcal{P}_n(m)} f\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)}$$

(50)

$$= \frac{n^{-(m-1)} \sum_{(k_1, \ldots, k_m) \in \mathcal{R}_n(m)} \Psi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right) f\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)}{n^{-(m-1)} \sum_{(k_1, \ldots, k_m) \in \mathcal{P}_n(m)} f\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)} + \mathcal{E}_{n,m},$$

where the set

$$\mathcal{R}_n(m) := \{(k_1, \ldots, k_m) \in \mathcal{P}_n(m) : k_1 \geq \ldots \geq k_m > 0\}$$

and

$$\mathcal{E}_{n,m} := \frac{\sum_{(k_1, \ldots, k_m) \in \mathcal{P}_n(m) \setminus \mathcal{R}_n(m)} \Psi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right) f\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)}{\sum_{(k_1, \ldots, k_m) \in \mathcal{P}_n(m)} f\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)}.$$

On the other hand,

$$\mathbb{E}(\psi(x_1, \ldots, x_m)) = \int_{\mathbb{V}_{m-1}} \psi(y_1, \ldots, y_m) f(y_1, \ldots, y_m) \, dy_1 \ldots dy_{m-1}$$

(51)

$$= \frac{\int_{\mathbb{V}_{m-1}} \psi(y_1, \ldots, y_m) f(y_1, \ldots, y_m) \, dy_1 \ldots dy_{m-1}}{f(y_1, \ldots, y_m) \, dy_1 \ldots dy_{m-1}}.$$

In order to compare (50) and (51), we divide the proof into a few steps.

**Step 1: Estimate of $|\mathcal{E}_{n,m}|$.** We claim that the term $\mathcal{E}_{n,m}$ is negligible as $n \to \infty$. We first estimate the size of $\mathcal{R}_n(m)$. For any $(k_1, \ldots, k_m) \in \mathcal{R}_n(m)$, set $j_i = k_i - (m - i + 1)$ for $1 \leq i \leq m$. It is easy to verify that $j_i = k_i - (m - i + 1)$ for $2 \leq i \leq m$. Thus,

$$j_1 + \cdots + j_m = n - \left(\frac{m+1}{2}\right)$$

for any $(k_1, \ldots, k_m) \in \mathcal{R}_n(m)$. This implies that $\mathcal{R}_n(m)$ is negligible as $n \to \infty$. Therefore, $|\mathcal{E}_{n,m}|$ is negligible as $n \to \infty$. The proof is completed by using Lemma 2 and the fact $\gamma = t_0^{-2}$. □
and $j_1 \geq \cdots \geq j_m \geq 0$. Therefore, $(j_1, \ldots, j_m) \in \mathcal{P}_{n-(m+1)}(m)$. Indeed, this transform is a bijection between $\mathcal{R}_n(m)$ and $\mathcal{P}_{n-(m+1)}(m)$, which implies

$$|\mathcal{R}_n(m)| = |\mathcal{P}_{n-(m+1)}(m)|.$$

On the other hand, we know from (49),

$$|\mathcal{P}_N(m)| \sim \frac{(N-1)!}{m!}$$

as $N \to \infty$. Thus, by Stirling’s formula,

$$\frac{|\mathcal{R}_n(m)|}{|\mathcal{P}_n(m)|} \sim \frac{(n-(m+1)/2)^{n-1}}{(n-m)^{n-m!}((n-(m+1)/2)-m)!} \sim \frac{n!(n-(m+1)/2)-m)!}{(1-m/2)^{1/2}} \sim \frac{(1-m/2)^n(1-(m+1)/2)^m}{(1-m/2)^n}$$

as $n \to \infty$. By assumption $m = o(\sqrt{n})$, we have $n - (m+1)/2 \to \infty$ with $n$. Using the fact that $\lim_{N \to \infty} (1 + \frac{3}{N})^N = \exp(x)$, we obtain

$$\frac{|\mathcal{R}_n(m)|}{|\mathcal{P}_n(m)|} \sim \exp\left(-\frac{m(m+1)}{2n-m}\right).$$

Thus, as long as $m = o(n^{1/3})$,

$$|\mathcal{R}_n(m)| \sim |\mathcal{P}_n(m)| \quad \text{and} \quad |\mathcal{P}_n(m) \setminus \mathcal{R}_n(m)| = o(|\mathcal{P}_n(m)|)$$

as $n \to \infty$. Further, since $\int_{\mathcal{V}_{m-1}} f(y_1, \ldots, y_m) \, dy_1 \cdots dy_{m-1} = 1$, there exists a region $S$ on $\mathcal{V}_{m-1}$ whose measure $|S| \geq \mu |\mathcal{V}_{m-1}|$ for some constant $\mu > 0$ such that $f(y_1, \ldots, y_m) > c$ on $S$ for some $c > 0$. Thus, for $n$ sufficiently large, $f(k_1/n, \ldots, k_m/n) > c_0 > 0$ for $(k_1, \ldots, k_m)$ in a subset of $\mathcal{P}_n(m)$ with cardinality at least a small fraction of $|\mathcal{P}_n(m)|$. Moreover, since the functions $\psi$ and $f$ are bounded on $\mathcal{V}_{m-1}$, we conclude

$$|\mathcal{E}_{n,m}| = O\left(\frac{|\mathcal{P}_n(m) \setminus \mathcal{R}_n(m)|}{|\mathcal{P}_n(m)|}\right) = o(1) \quad (52)$$

as $n \to \infty$, as long as $m = o(n^{1/3})$.

**Step 2: Compare the numerators of (50) and (51).** For convenience, denote

$$G(y_1, \ldots, y_{m-1}) = \psi\left(y_1, \ldots, y_{m-1}, 1 - \sum_{i=1}^{m-1} y_i\right) f\left(y_1, \ldots, y_{m-1}, 1 - \sum_{i=1}^{m-1} y_i\right). \quad (53)$$

Since $\psi, f$ are bounded continuous functions on $\mathcal{V}_{m-1}$, it is easy to check that $G$ is also bounded and continuous on $\mathcal{V}_{m-1}$. We can rewrite the numerator in (50) as follows.
\[ I_1 := \frac{1}{n^{m-1}} \sum_{k_1, \ldots, k_m > 0 \atop k_1 + \ldots + k_m = n} G\left(\frac{k_1}{n}, \ldots, \frac{k_m - 1}{n}\right) \]

\[ = \frac{1}{n^{m-1}} \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} G\left(\frac{k_1}{n}, \ldots, \frac{k_{m-1}}{n}\right) I_{A_n} \]

\[ = \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} \int_{k_1/n}^{k_1} \cdots \int_{k_{m-1}/n}^{k_{m-1}} G\left(\frac{k_1}{n}, \ldots, \frac{k_{m-1}}{n}\right) I_{A_n} \, dy_1 \cdots dy_{m-1}, \]

where \( I_{A_n} \) is the indicator function of set \( A_n \) defined as below

\[ A_n = \frac{1}{n} \{ (k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1} ; k_1 > \ldots > k_{m-1} > 1 - \sum_{i=1}^{m-1} k_i > 0 \}. \]

Similarly,

\[ I_2 := \int_{\{0,1\}^{m-1}} G(y_1, \ldots, y_{m-1}) \, dy_1 \cdots dy_{m-1} \]

\[ = \int_{[0,1]^{m-1}} G(y_1, \ldots, y_{m-1}) I_A \, dy_1 \cdots dy_{m-1} \]

\[ = \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} \int_{k_1/n}^{1} \cdots \int_{k_{m-1}/n}^{1} G(y_1, \ldots, y_{m-1}) I_A \, dy_1 \cdots dy_{m-1}, \]

where the \( I_A \) is the indicator function of set \( A \) denoted by

\[ A = \{ (x_1, \ldots, x_{m-1}) \in [0,1]^{m-1} ; x_1 > \ldots > x_{m-1} > 1 - \sum_{i=1}^{m-1} x_i \geq 0 \}. \]

Now, we estimate the difference between the numerators in (50) and (51).

\[ I_1 - I_2 = \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} \int_{k_1/n}^{k_1} \cdots \int_{k_{m-1}/n}^{k_{m-1}} G\left(\frac{k_1}{n}, \ldots, \frac{k_{m-1}}{n}\right) I_{A_n} - G(y_1, \ldots, y_{m-1}) I_A \, dy_1 \cdots dy_{m-1} \]

which is identical to

\[ \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} \int_{k_1/n}^{k_1} \cdots \int_{k_{m-1}/n}^{k_{m-1}} \left( G\left(\frac{k_1}{n}, \ldots, \frac{k_{m-1}}{n}\right) - G(y_1, \ldots, y_{m-1}) \right) I_{A_n} \, dy_1 \cdots dy_{m-1} \]

\[ + \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} \int_{k_1/n}^{k_1} \cdots \int_{k_{m-1}/n}^{k_{m-1}} G(y_1, \ldots, y_{m-1}) (I_{A_n} - I_A) \, dy_1 \cdots dy_{m-1} \]

\[ := S_1 + S_2. \]
Step 3: Estimate $S_1$. Since $G$ is uniformly continuous on $\nabla_{m-1}$, for any $\epsilon > 0$ and any $y_i \in \left[\frac{k_{i-1}}{n}, \frac{k_i}{n}\right]$ (1 $\leq$ $i$ $\leq$ $m-1$),

$$\left|G\left(\frac{k_1}{n}, \ldots, \frac{k_{m-1}}{n}\right) - G(y_1, \ldots, y_{m-1})\right| < \epsilon,$$  \hspace{1cm} (54)

when $n$ is sufficiently large. Thus,

$$|S_1| \leq \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \cdots \int_{\frac{k_{m-1}}{n}}^{\frac{k_{m-1}+1}{n}} \left|G\left(\frac{k_1}{n}, \ldots, \frac{k_{m-1}}{n}\right) - G(y_1, \ldots, y_{m-1})\right| dy_1 \cdots dy_{m-1}$$

$$\leq \epsilon \left(\frac{1}{n}\right)^{m-1} n^{m-1}$$

$$= \epsilon$$

for $n$ sufficiently large.

Step 4: Estimate $S_2$. Since $G$ is bounded on $\nabla_{m-1}$, $\|G\|_\infty := \sup_{x \in \nabla_{m-1}} |G(x)| < \infty$ and thus,

$$|S_2| \leq \|G\|_\infty \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \cdots \int_{\frac{k_{m-1}}{n}}^{\frac{k_{m-1}+1}{n}} |I_{A_n} - I_A| dy_1 \cdots dy_{m-1}.$$  \hspace{1cm} (55)

Now, we control $|I_{A_n} - I_A|$ provided $\frac{k_i}{n} < y_i < \frac{k_i+1}{n}$ for $1 \leq i \leq m-1$. By definition,

$$I_{A_n} = \begin{cases} 1, & \text{if } \frac{k_1}{n} > \cdots > \frac{k_{m-1}}{n} > 1 - \sum_{i=1}^{m-1} \frac{k_i}{n} > 0 \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (56)

and

$$I_A = \begin{cases} 1, & \text{if } y_1 > \cdots > y_{m-1} > 1 - \sum_{i=1}^{m-1} y_i \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (57)

Let $B_n$ be a subset of $A_n$ such that

$$B_n = A_n \cap \left\{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1} : \frac{k_{m-1}}{n} + \sum_{i=1}^{m-1} \frac{k_i}{n} > \frac{m}{n} + 1\right\}.$$  

Given $(k_1, \ldots, k_{m-1}) \in B_n$, for any

$$\frac{k_1-1}{n} < y_1 < \frac{k_1}{n}, \ldots, \frac{k_{m-1}-1}{n} < y_{m-1} < \frac{k_{m-1}}{n},$$  \hspace{1cm} (58)

it is easy to verify from (57) and (56) that $I_A = 1$. Hence,

$$I_{A_n} = I_{B_n} + I_{A_n \setminus B_n} \leq I_A + I_{A_n \cap \left\{(k_{m-1}+\sum_{i=1}^{m-1} k_i \leq n+m\}\right\}}$$

$$= I_A + \sum_{j=n+1}^{n+m} I_{E_j}$$  \hspace{1cm} (59)
where

\[ E_j = \left\{ (k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}; k_1 > \ldots > k_{m-1} \geq 1, \right. \]
\[ \left. k_{m-1} + \sum_{i=1}^{m-1} k_i = j, \sum_{i=1}^{m-1} k_i < n \right\} \]

for \( n + 1 \leq j \leq m + n \). Let us estimate the size of \( |E_j| \). From the last two restrictions, we obtain \( k_{m-1} > j - n \). Since \( \sum_{i=1}^{m-1} k_i < n \) and \( k_i > k_{m-1} \) for \( 1 \leq i \leq m - 2 \), we have \( j - n + 1 \leq k_{m-1} \leq \frac{n}{m} \).

For each fixed \( k_{m-1} \), since \( k_1 > \ldots > k_{m-2} \) is the ordered positive integer solution to the linear equation \( \sum_{i=1}^{m-2} k_i = j - 2k_{m-1} \), thus,

\[ |E_j| \leq \sum_{j-n+1 \leq k_{m-1} \leq \frac{n}{m}} \left( \frac{j-2(j-n)}{m-3} n \left( \frac{n}{m-1} - n - j \right) \left( \frac{2n-j-3}{m-3} \right) \right) \leq \left( \frac{m \cdot n^{m-2}}{(m-1)!(m-3)!} \right). \]

As a result, we obtain the crude upper bound

\[ \sum_{j=n+1}^{n+m} |E_j| \leq \sum_{j=n+1}^{n+m} \left( \frac{n}{m-1} + n - j \right) \left( \frac{2n-j-3}{m-3} \right) \leq \frac{m \cdot n^{m-2}}{(m-1)!(m-3)!}. \] (60)

On the other hand, consider a subset of \( A_n^* := \{ \frac{k}{n}, \frac{2}{n}, \ldots, 1 \}^{m-1} \setminus A_n \) defined by

\[ C_n = 1 \left\{ (k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}; \right. \]
\[ \left. \text{either } k_i \leq k_{i+1} - 1 \text{ for some } 1 \leq i \leq m - 2, \right. \]
\[ \text{or } k_1 + \ldots + k_{m-2} + 2k_{m-1} \leq n, \text{ or } k_1 + \ldots + k_{m-1} \geq m + n - 1 \left\} \right. \}

Set \( A^c = [0, 1]^{m-1} \setminus A \). Given \( (k_1, \ldots, k_{m-1}) \in C_n \), for any \( k_i \)'s and \( y_i \)'s satisfying (58), it is not difficult to check that \( I_{A^c} = 1 \). Consequently,

\[ I_{A_n^*} = I_{C_n} + I \left( \left\{ \frac{k_1}{n}, \ldots, \frac{k_{m-1}}{n} \in A_n^*; k_i > k_{i+1} - 1 \text{ for all } 1 \leq i \leq m - 2, \right. \right. \]
\[ \left. \left. k_1 + \ldots + k_{m-2} + 2k_{m-1} > n, \text{ and } k_1 + \ldots + k_{m-1} \geq m + n - 1 \right\} \right. \right. \]
\[ \leq I_{A^c} + I_{D_{n,m,1}} + I_{D_{n,m,2}} \]

or equivalently,

\[ I_{A_n} \geq I_A - I_{D_{n,m,1}} - I_{D_{n,m,2}}, \] (61)

where

\[ D_{n,m,1} = \bigcup_{l=n}^{n+m-2} \frac{1}{n} \left\{ (k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}; \sum_{i=1}^{m-1} k_i = l, k_1 \geq \ldots \geq k_{m-1} \right\}; \]
\[ D_{n,m,2} = \bigcup_{l=1}^{m-2} \frac{1}{n} \left\{ (k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}; k_1 = k_{l+1}, k_1 \geq \ldots \geq k_{m-1}, \right. \]
\[ \left. \sum_{i=1}^{m-1} k_i + k_{m-1} \geq n + 1, \sum_{i=1}^{m-1} k_i \leq n + m - 2 \right\} \].
By the definition of partitions and (49), we have the following bound on $|D_{n,m,1}|$.

$$
|D_{n,m,1}| \leq \sum_{l=n}^{n+m-2} |\mathcal{P}_l(m-1)| \sim \sum_{l=n}^{n+m-2} (m-1)! \frac{(l-1)!}{(m-1)!} \\
\leq (m-1) \frac{(n+m-2)}{(m-1)!} \leq (n + m - 2)^{m-2} \frac{1}{(m-2)!^2}
$$

(62)

as $n \to \infty$.

The estimation of $|D_{n,m,2}|$ is the same argument as in (60). For the cases $m = 3$ or $m = 4$, it is easy to verify that $|D_{n,m,2}| = O(n^{m-2})$. Now, we assume $m \geq 5$. First, from the decreasing order of $k_i$ and $\sum_{i=1}^{m-1} k_i \leq n + m - 2$, we determine the range of $k_{m-1}$,

$$
1 \leq k_{m-1} \leq \frac{n + m - 2}{m - 1}.
$$

On the other hand, $n + 1 - 2k_{m-1} \leq \sum_{i=1}^{m-2} k_i \leq n + m - 2 - k_{m-1}$. If $l \neq m - 2$, from the restriction $k_l = k_l$, we see $k_1 + \ldots + k_{l-1} + k_{l+2} + \ldots + k_{m-2} = s - 2k_l$ is the ordered positive integer solutions to the equation $j_1 + \ldots + j_{m-4} = s - 2k_l$, where $n + 1 - 2k_{m-1} \leq s \leq n + m - 2 - k_{m-1}$. If $l = m - 2$, then $k_1 + \ldots + k_{m-3} = s - 2k_{m-1}$ and $n + 1 - 3k_{m-1} \leq s - 2k_{m-1} \leq n + m - 2 - 2k_{m-1}$. Therefore, we have the following crude upper bound

$$
|D_{n,m,2}| \leq \sum_{l=n}^{m-3} \sum_{k_{m-1}=1}^{n+m-2-k_{m-1}} \sum_{s=n+1-2k_{m-1}}^{n+m-2} \sum_{k_l \leq s/2}^{(s-k_l-1)} (m-1)! \\
+ \sum_{k_{m-1}=1}^{n+m-2} \sum_{s=n+1-3k_{m-1}}^{n+m-2} \sum_{k_l \leq s/2}^{(s-k_l-1)} (m-3)! \\
= O\left(\frac{n^2(m-3)}{m^2(m-4)!} \frac{(n + m - 6)}{m - 5} + \frac{n^2}{m^2(m-3)!} \frac{(n + m - 6)}{m - 4}\right) \\
= O\left(\frac{n^2(n + m)^{m-4}}{m(m-4)!(m-5)!}\right).
$$

(63)

Joining (59) and (61), and assuming (58) holds, we arrive at

$$
|I_{A_n} - I_A| \leq I_{D_{n,m,1}} + I_{D_{n,m,2}} + \sum_{i=n+1}^{n+m} I_{E_i}.
$$

Observe that $D_{n,m,i}$’s and $E_i$’s do not depend on $y_i$’s, we obtain from (55) that

$$
|S_2| \leq ||G|| \sum_{i=1}^{n} I_{D_{n,m,i}} + \sum_{i=n+1}^{n+m} I_{E_i} \int_{1/m-1}^{1} \cdots \int_{1/m-1}^{1} dy_1 \cdots dy_m.
$$

For $2 \leq m \leq 4$,

$$
|S_2| = O(n^{-1}).
$$

For $m \geq 5$, by (60), (62) and (63),

$$
|S_2| = O\left(\frac{m \cdot n^{m-2}}{(m-1)!^2} \frac{1}{(m-2)!^2} + \frac{n^2(n + m)^{m-4}}{m(m-4)!^2(m-5)!}\right) \cdot \frac{1}{n^{m-1}}
$$

$$
= O\left(\frac{1 + \frac{m}{n} \frac{n}{m}}{n}\right).
$$
as \( n \to \infty \).

**Step 5: Difference between the expectations (50) and (51).** For any \( \varepsilon > 0 \), from Step 3 and Step 4, we obtain the difference between the numerators in (50) and (51)

\[
|I_1 - I_2| \leq |S_1| + |S_2| \leq \varepsilon + O\left(\frac{(1 + \frac{m}{n})^m}{n}\right) < 2\varepsilon
\]

(64)

for \( n \) sufficiently large. Choosing \( \psi \) to be identity on \( \nabla_{m-1} \), we obtain the difference between the denominators in (50) and (51) as follows:

\[
\left| n^{-(m-1)} \sum_{(k_1, \ldots, k_m) \in P_n(m)} f\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right) - \int_{\nabla_{m-1}} f(y_1, \ldots, y_m) \, dy_1 \ldots dy_{m-1} \right| < 2\varepsilon
\]

for \( n \) sufficiently large.

Finally, we estimate the expectations (50) and (51). Since \( m \) is fixed, by (52), (64), and the triangle inequality,

\[
\left| \mathbb{E}\left( \psi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right) \right) - \mathbb{E}(\psi(x_1, \ldots, x_m)) \right| \to 0
\]

as \( n \to \infty \). This completes the proof. \( \square \)

Next, we provide the proof of Corollary 2.

**Proof of Corollary 2.** By Theorem 3,

\[
\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right) \to (x_1, \ldots, x_m) \sim \mu
\]

as \( n \to \infty \), where \( \mu \) has pdf

\[
g(y_1, \ldots, y_m) = \frac{y_1^{a-1} \cdots y_m^{a-1}}{\int_{\nabla_{m-1}} y_1^{a-1} \cdots y_m^{a-1} \, dy_1 \ldots dy_{m-1}}.
\]

(65)

It suffices to show the order statistics \((X_{(1)}, \ldots, X_{(m)})\) of \((X_1, \ldots, X_m) \sim \text{Dir}(\alpha)\) has the same pdf on \( \nabla_{m-1} \). For any continuous function \( \psi \) defined on \( \nabla_{m-1} \), by symmetry,

\[
\mathbb{E}\psi(X_{(1)}, \ldots, X_{(m)})
\]

\[
= \int_{W_{m-1}} \psi(y_1, \ldots, y_{(m)}) \cdot \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} \cdot \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} \cdot \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} \cdot \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} \cdot \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} \cdot \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} \cdot \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} \cdot \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} \quad dy_1 \ldots dy_{m-1}
\]

Therefore, the pdf of \((X_{(1)}, \ldots, X_{(m)})\) is

\[
\frac{m! \Gamma(m\alpha)}{\Gamma(\alpha)^m} y_1^{a-1} \cdots y_m^{a-1}
\]

(66)

on the set \( \nabla_{m-1} \). Similarly, by the definition of pdf we have

\[
\int_{W_{m-1}} \frac{\Gamma(m\alpha)}{\Gamma(\alpha)^m} y_1^{a-1} \cdots y_m^{a-1} = 1.
\]
By symmetry, we obtain
\[ \int_{\nabla_{m-1}} y_1^{a-1} \cdots y_m^{a-1} \, dy_1 \cdots dy_{m-1} = \frac{\Gamma(a)^m}{m! \Gamma(ma)}. \]
Comparing the above with (66) and (65), we complete the proof. \(\square\)
We conclude this subsection with the proof of Corollary 3.

**Proof of Corollary 3.** By Theorem 3 or Corollary 2,
\[ \left( \frac{k_1}{n}, \ldots, \frac{k_m}{n} \right) \to (\tilde{Y}_1, \ldots, \tilde{Y}_m) \sim \mu \]
as \(n \to \infty\), where \(\mu\) has pdf
\[ \frac{m! \cdot \Gamma(m)}{\Gamma(\frac{m}{a})^m} (y_1 \cdots y_m)^{a-1} \]
on \(\nabla_{m-1}\) and zero elsewhere. Since \(f(x) = x^a\) is continuous,
\[ \left( \left( \frac{k_1}{n} \right)^a, \ldots, \left( \frac{k_m}{n} \right)^a \right) \to (\tilde{Y}_1^a, \ldots, \tilde{Y}_m^a) \]
as \(n \to \infty\).
Now, it suffices to show \( (\tilde{Y}_1^a, \ldots, \tilde{Y}_m^a) \) has the uniform distribution on the set
\[ \mathcal{U}_{m-1} = \left\{ (x_1, \ldots, x_m) \in [0,1]^m \mid \sum_{i=1}^m x_i^a = 1, x_1 \geq \ldots \geq x_m \right\}. \]
This can be seen by change of variables. For any continuous function \(\psi\) defined on \(\nabla_{m-1}\),
\[ \mathbb{E}\psi(\tilde{Y}_1^a, \ldots, \tilde{Y}_m^a) = \int_{\nabla_{m-1}} \psi(y_1^a, \ldots, y_m^a) \frac{m! \cdot \Gamma(m)}{\Gamma(\frac{m}{a})^m} y_1^{a-1} \cdots y_m^{a-1} \, dy_1 \cdots dy_{m-1} \]
\[ = \int_{\mathcal{U}_{m-1}} \psi(x_1, \ldots, x_m) \frac{m! \cdot \Gamma(m)}{\Gamma(ma-1) \Gamma(\frac{m}{a})^m}\, dx_1 \cdots dx_{m-1}. \]
In the last equality, we set \(y_i = x_i^a\) for \(1 \leq i \leq m\). Therefore, we can see the pdf of \( (\tilde{Y}_1^a, \ldots, \tilde{Y}_m^a) \) is a constant on \(\mathcal{U}_{m-1}\), which is the uniform distribution on \(\mathcal{U}_{m-1}\). The proof is complete. \(\square\)

### 3.2. The Proof of Theorem 4

In Section 3.1 we have studied the asymptotic distribution of \( (\frac{k_1}{n}, \ldots, \frac{k_m}{n}) \) as \(m\) is fixed. Now, we consider the case that \(m\) depends on \(n\). Note that the Formula (49) holds as long as \(m = o(n^{1/3})\).

Let \(\mu\) and \(\nu\) be two Borel probability measures on a Polish space \(S\) with the Borel \(\sigma\)-algebra \(B(S)\). Define
\[ \rho(\mu, \nu) = \sup_{\|\varphi\|_L \leq 1} \left| \int_S \varphi(x) \mu(dx) - \int_S \varphi(x) \nu(dx) \right|, \tag{67} \]
where \(\varphi\) is a bounded Lipschitz function defined on \(S\) with \(\|\varphi\| = \sup_{x \in S} |\varphi(x)|\), and \(\|\varphi\|_L = \|\varphi\| + \sup_{x \neq y} |\varphi(x) - \varphi(y)|/|x - y|\). It is known that \(\mu_n\) converges to \(\mu\) weakly if and only if \(\lim_{n \to \infty} \int \varphi(x) \mu_n(dx) = \int \varphi(x) \mu(dx)\) for every bounded and Lipschitz continuous function \(\varphi(x)\) defined on \(\mathbb{R}^m\), and if and only if \(\lim_{n \to \infty} \rho(\mu_n, \mu) = 0\); see, e.g., Chapter 11 from [30].
Let \( \{X_i, X_n, i \geq 1; n \geq 1\} \) be random variables taking values in \([0,1]\). Set \( X_n = (X_{n1}, X_{n2}, \cdots) \in [0,1]^\infty \). If \( X_n = 0 \) for \( i > m \), we simply write \( X_n = (X_{n1}, \cdots, X_{nm}) \). We say that \( X_n \) converges weakly to \( X := (X_1, X_2, \cdots) \) as \( n \to \infty \) if, for any \( r \geq 1 \), \((X_{n1}, \cdots, X_{nr})\) converges weakly to \( X = (X_1, \cdots, X_r) \) as \( n \to \infty \). This convergence actually is the same as the weak convergence of random variables in \([0,1]^\infty, d\) where

\[
d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}
\]

for \( x = (x_1, x_2, \cdots) \in [0,1]^\infty \) and \( y = (y_1, y_2, \cdots) \in [0,1]^\infty \). The topology generated by this metric is the same as the product topology.

**Lemma 3.** Let \( m = m_n \to \infty \) as \( n \to \infty \). Let \( \kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m) \) be chosen with probability as in (3) under the assumption of Theorem 4. Let \((X_{m1}, \cdots, X_{mm})\) and \( X = (X_1, X_2, \cdots) \) be random variables taking values in \( \nabla_{m-1} \) and \( \nabla \), respectively. If

\[
\sup_{\|\varphi\|_L \leq 1} \left| \mathbb{E}\varphi\left(\frac{k_1}{n}, \cdots, \frac{k_m}{n}\right) - \mathbb{E}\varphi\left(X_{m1}, \cdots, X_{mm}\right) \right| \to 0 \quad (69)
\]

as \( n \to \infty \), and \((X_{m1}, \cdots, X_{mm})\) converges weakly to \( X \) as \( n \to \infty \), then \((\frac{k_1}{n}, \cdots, \frac{k_m}{n})\) converges weakly to \( X \) as \( n \to \infty \).

**Proof.** Given integer \( r \geq 1 \), to prove the theorem, it is enough to show \((\frac{k_1}{n}, \cdots, \frac{k_r}{n})\) converges weakly to \((X_1, \cdots, X_r)\) as \( n \to \infty \). Since \( m = m_n \to \infty \) as \( n \to \infty \), without loss of generality, we assume \( r < m \) in the rest of discussion. For any random vector \( Z \), let \( \mathcal{L}(Z) \) denote its probability distribution. Review (67). By the triangle inequality,

\[
\rho\left(\mathcal{L}\left(\frac{k_1}{n}, \cdots, \frac{k_r}{n}\right), \mathcal{L}(X_1, \cdots, X_r)\right)
\leq \rho\left(\mathcal{L}\left(\frac{k_1}{n}, \cdots, \frac{k_r}{n}\right), \mathcal{L}(X_{m1}, \cdots, X_{mr})\right) + \rho\left(\mathcal{L}(X_{m1}, \cdots, X_{mr}), \mathcal{L}(X_1, \cdots, X_r)\right) \quad (70)
\]

For any function \( \varphi(x_1, \cdots, x_r) \) defined on \([0,1]^r\) with \( \|\varphi\|_L \leq 1 \), set \( \tilde{\varphi}(x_1, \cdots, x_m) = \varphi(x_1, \cdots, x_r) \) for all \((x_1, \cdots, x_m) \in \mathbb{R}^m\). Then \( \|\tilde{\varphi}\|_L \leq 1 \). Condition (69) implies that the middle one among the three distances in (70) goes to zero. Further, the assumption that \((X_{m1}, \cdots, X_{mm})\) converges weakly to \( X \) implies the third distance in (70) also goes to zero. Hence, the first distance goes to zero. The proof is completed. \( \Box \)

With Lemma 3 and the estimation in Theorem 3, we obtain the proof of Theorem 4.

**Proof of Theorem 4.** Assume \( \kappa = (k_1, \ldots, k_m) \in \mathcal{P}_n(m) \) is chosen with probability as in (3). The proof is almost identical to the proof of Theorem 3. We only mention the difference and modifications. Instead of choosing the test function \( \psi \) to be bounded and continuous as in the beginning of Theorem 3, we select \( \psi = \varphi \) to be bounded and Lipschitz. Following the proof of Theorem 3, the function \( G \) defined in (53) in Step 2 is now bounded and Lipschitz on \( \nabla_{m-1} \). The major change happens in Step 3, where we replace the estimation in (54) by

\[
\left| G\left(\frac{k_1}{n}, \cdots, \frac{k_{m-1}}{n}\right) - G(y_1, \ldots, y_{m-1}) \right| \leq C \cdot \sqrt{\frac{m}{n}}
\]

for some constant $C$ depending only on the Lipschitz constant of $G$, where $y_i \in [\frac{k_i-1}{n}, \frac{k_i}{n}]$ for $1 \leq i \leq m - 1$. Consequently, the term $S_1$ defined in the end of Step 2 is now bounded as follows:

$$|S_1| \leq \sum_{(k_1, \ldots, k_{m-1}) \in \{1, \ldots, n\}^{m-1}} \int_{\frac{k_1}{n}}^{\frac{k_2}{n}} \cdots \int_{\frac{k_{m-1}}{n}}^{\frac{k_{m-1}}{n}} |G\left(\frac{k_1}{n}, \ldots, \frac{k_{m-1}}{n}\right) - G(y_1, \ldots, y_{m-1})| \, dy_1 \cdots dy_{m-1} \leq C \cdot \sqrt{m} \left(\frac{1}{n}\right)^{m-1} n^{m-1} = C \sqrt{m} n.$$ 

**Step 4** remains the same and we modify **Step 5** using the changes mentioned above. The difference between the numerators in (50) and (51) now becomes

$$|I_1 - I_2| \leq |S_1| + |S_2| \leq C_1 \left(\frac{\sqrt{m}}{n} + \left(\frac{1 + \frac{m}{n}\right)^m\right)$$

as $n \to \infty$ for some constant $C_1$ depending only on the Lipschitz constants of $\varphi$ and $f$ and the upper bounds of $\varphi$ and $f$ on the compact set $\nabla_{m-1}$. Using the same argument in the end of the proof of Theorem 3 and the assumption that $\|f\|_{L^\infty} \leq K$, we have for any $\varphi$ defined on $\nabla_{m-1}$ satisfying $\|\varphi\|_{L^1} \leq 1$,

$$\sup_{\|\varphi\|_{L^1} \leq 1} |\mathbb{E}\left(\varphi\left(\frac{k_1}{n}, \ldots, \frac{k_m}{n}\right)\right) - \mathbb{E}(\varphi(X_{m,1}, \ldots, X_{m,m}))| = O\left(\frac{\sqrt{m}}{n} + \left(\frac{1 + \frac{m}{n}\right)^m\right) + |E_{n,m}| \to 0.$$ 

as $n \to \infty$. Recall in (52), we have $|E_{n,m}| \to 0$ as long as $m = o(n^{1/3})$. Therefore, by Lemma 3, we conclude that $(\frac{k_1}{n}, \ldots, \frac{k_m}{n})$ converges weakly to $X$ as $n \to \infty$. □

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**References**


2. Van Lier, C.; Uhlenbeck, G. On the statistical calculation of the density of the energy levels of the nuclei. *Physica* 1937, 4, 531–542. [CrossRef]


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