

Article

Parameter Estimation and Hypothesis Testing of The Bivariate Polynomial Ordinal Logistic Regression Model

Marisa Rifada ^{1,2} , Vita Ratnasari ^{1,*} and Purhadi Purhadi ¹ 

¹ Department of Statistics, Faculty of Science and Data Analytics, Institut Teknologi Sepuluh Nopember, Surabaya 60111, Indonesia

² Department of Mathematics, Faculty of Science and Technology, Universitas Airlangga, Surabaya 60115, Indonesia

* Correspondence: vita_ratna@statistika.its.ac.id

Abstract: Logistic regression is one of statistical methods that used to analyze the correlation between categorical response variables and predictor variables which are categorical or continuous. Many studies on logistic regression have been carried out by assuming that the predictor variable and its logit link function have a linear relationship. However, in several cases it was found that the relationship was not always linear, but could be quadratic, cubic, or in the form of other curves, so that the assumption of linearity was incorrect. Therefore, this study will develop a bivariate polynomial ordinal logistic regression (BPOLR) model which is an extension of ordinal logistic regression, with two correlated response variables in which the relationship between the continuous predictor variable and its logit is modeled as a polynomial form. There are commonly two correlated response variables in scientific research. In this study, each response variable used consisted of three categories. This study aims to obtain parameter estimators of the BPOLR model using the maximum likelihood estimation (MLE) method, obtain test statistics of parameters using the maximum likelihood ratio test (MLRT) method, and obtain algorithms of estimating and hypothesis testing for parameters of the BPOLR model. The results of the first partial derivatives are not closed-form, thus, a numerical optimization such as the Berndt–Hall–Hall–Hausman (BHHH) method is needed to obtain the maximum likelihood estimator. The distribution statistically test is followed the Chi-square limit distribution, asymptotically.

Keywords: bivariate; ordinal logistic regression; polynomial; scientific research

MSC: 62F03; 62F10



Citation: Rifada, M.; Ratnasari, V.; Purhadi, P. Parameter Estimation and Hypothesis Testing of The Bivariate Polynomial Ordinal Logistic Regression Model. *Mathematics* **2023**, *11*, 579. <https://doi.org/10.3390/math11030579>

Academic Editors: Maurizio Brizzi and Christophe Chesneau

Received: 22 November 2022

Revised: 7 January 2023

Accepted: 18 January 2023

Published: 21 January 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The logistic regression model is one of statistical methods which is used to analyze the correlation between categorical response variables and predictor variables that are categorical or continuous. If the response variable has more than two categories and there are levels in that category (ordinal scale), then an ordinal logistic regression model is used [1]. The logistic regression modeling often does not involve only one response variable because of a phenomenon involving multiple response variables. Logistic regression involving one response variable is called univariate logistic regression. Meanwhile, logistic regression that involves two or more response variables and correlates between response variables is called multivariate logistic regression. Especially for multivariate logistic regression involving two correlated response variables, it is called the bivariate logistic regression.

Several studies on ordinal logistic regression have been carried out for bivariate cases. Dale [2] has studied parameter estimation using the MLE method. Williamson et al. [3] used the generalized estimating equation (GEE) method, whereas Enea and Lovison [4] used penalized MLE for the parameter estimation of bivariate ordinal logistic regression

models. The previous studies examined logistic regression modeling by assuming that the continuous predictor variable and its logit link function have a linear relationship. However, the reality is that in some cases it is found that the relationship is not always linear, but can be quadratic, cubic, or other curves, so that the assumption of linearity is incorrect.

Royston and Altman [5] introduced the fractional polynomial approach, which is one of the development of a polynomial model, to understand the functional form of a continuous predictor variable. This model involves a more flexible relationship that can be linear or non-linear and has been used by several researchers [6–9]. In its development, several studies have examined the use of fractional polynomials in logistic regression analysis [10–12]. However, these previous studies were limited to univariate binary logistic regression. Furthermore, the bivariate binary logistic regression has been developed but limited to the second order [13]. Therefore, we propose an extension of ordinal logistic regression with two correlated response variables in which the relationship between the continuous predictor variable and its logit is modeled as a polynomial form. The proposed model is called the bivariate polynomial ordinal logistic regression (BPOLR) model. This study aimed to obtain parameter estimators of the BPOLR model using the MLE method, obtain test statistics of parameters using the MLRT method, and obtain algorithms of estimating and hypothesis testing for parameters of the BPOLR model.

An example of applying the BPOLR model can be applied to modeling the nutritional status of toddlers. Based on Narendra et al. [14], children from birth to about one year old have growth that increases rapidly and then decreases slowly as the child gets older. Therefore, the polynomial approach is more suitable for modeling child growth [15]. Tilling et al. [16] modeled child growth using the fractional polynomial approach and concluded that the model with the fractional polynomial approach shows a growth rate that is initially fast and then slows down over time.

The following discussion in this paper is divided into several main topics. In Section 2, we introduce the BPOLR model followed by parameter estimation using the MLE method in Section 3. We present simultaneous hypothesis testing in Section 4 to test the significance of all parameters together using the MLRT method and we present a simulation study in Section 5. Furthermore, Section 6 contains conclusions.

2. The Bivariate Polynomial Ordinal Logistic Regression (BPOLR) Model

The BPOLR model is an expansion of the ordinal logistic regression model when there are two correlated ordinal response variables and the relationship between the continuous predictor variable and its logit is modeled as a polynomial form. In this study, the response variables used each have 3 categories. Let Y_1 and Y_2 be response variables which have a value of 1; 2, or 3, then $Y_{ab} = (Y_1 = a, Y_2 = b)$; $a, b = 1, 2, 3$ are random variables that have their respective probabilities π_{ab} , as presented in Table 1 below.

Table 1. Probabilty of the response variables.

Y_1	Y_2			Total
	1	2	3	
1	π_{11}	π_{12}	π_{13}	$\pi_{1\bullet}$
2	π_{21}	π_{22}	π_{23}	$\pi_{2\bullet}$
3	π_{31}	π_{32}	π_{33}	$1 - \pi_{1\bullet} - \pi_{2\bullet}$
Total	$\pi_{\bullet 1}$	$\pi_{\bullet 2}$	$1 - \pi_{\bullet 1} - \pi_{\bullet 2}$	1

Based on Table 1, the random vector $\mathbf{y} = [Y_{11} \ Y_{12} \ Y_{13} \ Y_{21} \ Y_{22} \ Y_{23} \ Y_{31} \ Y_{32}]$ have a multinomial distribution so it has a joint probability density function as follows:

$$P(Y_{11} = y_{11}, Y_{12} = y_{12}, \dots, Y_{32} = y_{32}) = \prod_{a=1}^3 \prod_{b=1}^3 \pi_{ab}^{y_{ab}} \tag{1}$$

where $0 < \pi_{ab} < 1$; $a, b = 1, 2, 3; y_{ab} = 0, 1$; $\sum_{a=1}^3 \sum_{b=1}^3 Y_{ab} = 1$; and $\sum_{a=1}^3 \sum_{b=1}^3 \pi_{ab} = 1$. $\pi_{ab} = P(Y_1 = a, Y_2 = b)$ is the joint probability of the response variables. $\pi_{a\bullet} = \sum_{b=1}^3 P(Y_1 = a, Y_2 = b)$ is the marginal probability for the variable Y_1 and $\pi_{\bullet b} = \sum_{a=1}^3 P(Y_1 = a, Y_2 = b)$ is the marginal probability for the variable Y_2 .

The model that can be used for ordinal logistic regression is the cumulative logit model. In this study, the BPOLR model has two ordinal-scale response variables that are correlated with each other and each has 3 categories, so we have the cumulative probability for Y_1 and Y_2 , i.e., $P(Y_1 \leq a | \mathbf{x})$; $a = 1, 2$ and $P(Y_2 \leq b | \mathbf{x})$; $b = 1, 2$. Let $F_{a\bullet}(\mathbf{x}) = P(Y_1 \leq a | \mathbf{x})$; $a = 1, 2$ is the marginal cumulative probability of variable Y_1 given \mathbf{x} , $F_{\bullet b}(\mathbf{x}) = P(Y_2 \leq b | \mathbf{x})$; $b = 1, 2$ is the marginal cumulative probability of variable Y_2 given \mathbf{x} and $F_{ab}(\mathbf{x}) = P(Y_1 \leq a, Y_2 \leq b | \mathbf{x})$; $a, b = 1, 2$ is the joint cumulative probability between variables Y_1 and Y_2 given \mathbf{x} , so we have the BPOLR model as follows:

- The cumulative logit model for Y_1

$$g_1 = \text{logit}(P(Y_1 \leq 1 | \mathbf{x})) = \text{logit}(F_{1\bullet}(\mathbf{x})) = \ln\left(\frac{F_{1\bullet}(\mathbf{x})}{1 - F_{1\bullet}(\mathbf{x})}\right) = \alpha_{11} + \sum_{j=1}^k \beta_{1j}^T \mathbf{x}_{j(r_j)}^* \tag{2}$$

$$g_2 = \text{logit}(P(Y_1 \leq 2 | \mathbf{x})) = \text{logit}(F_{2\bullet}(\mathbf{x})) = \ln\left(\frac{F_{2\bullet}(\mathbf{x})}{1 - F_{2\bullet}(\mathbf{x})}\right) = \alpha_{12} + \sum_{j=1}^k \beta_{1j}^T \mathbf{x}_{j(r_j)}^* \tag{3}$$

- The cumulative logit model for Y_2

$$g_3 = \text{logit}(P(Y_2 \leq 1 | \mathbf{x})) = \text{logit}(F_{\bullet 1}(\mathbf{x})) = \ln\left(\frac{F_{\bullet 1}(\mathbf{x})}{1 - F_{\bullet 1}(\mathbf{x})}\right) = \alpha_{21} + \sum_{j=1}^k \beta_{2j}^T \mathbf{x}_{j(r_j)}^* \tag{4}$$

$$g_4 = \text{logit}(P(Y_2 \leq 2 | \mathbf{x})) = \text{logit}(F_{\bullet 2}(\mathbf{x})) = \ln\left(\frac{F_{\bullet 2}(\mathbf{x})}{1 - F_{\bullet 2}(\mathbf{x})}\right) = \alpha_{22} + \sum_{j=1}^k \beta_{2j}^T \mathbf{x}_{j(r_j)}^* \tag{5}$$

- The odds ratio transformation model for Y_1 and Y_2

$$g_5 = \ln(\psi_{11}(\mathbf{x})) = \ln\left(\frac{F_{11}(\mathbf{x})[1 - F_{1\bullet}(\mathbf{x}) - F_{\bullet 1}(\mathbf{x}) + F_{11}(\mathbf{x})]}{[F_{1\bullet}(\mathbf{x}) - F_{11}(\mathbf{x})][F_{\bullet 1}(\mathbf{x}) - F_{11}(\mathbf{x})]}\right) = \Delta_{11} + \sum_{j=1}^k \gamma_j^T \mathbf{x}_{j(r_j)}^* \tag{6}$$

$$g_6 = \ln(\psi_{12}(\mathbf{x})) = \ln\left(\frac{F_{12}(\mathbf{x})[1 - F_{1\bullet}(\mathbf{x}) - F_{\bullet 2}(\mathbf{x}) + F_{12}(\mathbf{x})]}{[F_{1\bullet}(\mathbf{x}) - F_{12}(\mathbf{x})][F_{\bullet 2}(\mathbf{x}) - F_{12}(\mathbf{x})]}\right) = \Delta_{12} + \sum_{j=1}^k \gamma_j^T \mathbf{x}_{j(r_j)}^* \tag{7}$$

$$g_7 = \ln(\psi_{21}(\mathbf{x})) = \ln\left(\frac{F_{21}(\mathbf{x})[1 - F_{2\bullet}(\mathbf{x}) - F_{\bullet 1}(\mathbf{x}) + F_{21}(\mathbf{x})]}{[F_{2\bullet}(\mathbf{x}) - F_{21}(\mathbf{x})][F_{\bullet 1}(\mathbf{x}) - F_{21}(\mathbf{x})]}\right) = \Delta_{21} + \sum_{j=1}^k \gamma_j^T \mathbf{x}_{j(r_j)}^* \tag{8}$$

$$g_8 = \ln(\psi_{22}(\mathbf{x})) = \ln\left(\frac{F_{22}(\mathbf{x})[1 - F_{2\bullet}(\mathbf{x}) - F_{\bullet 2}(\mathbf{x}) + F_{22}(\mathbf{x})]}{[F_{2\bullet}(\mathbf{x}) - F_{22}(\mathbf{x})][F_{\bullet 2}(\mathbf{x}) - F_{22}(\mathbf{x})]}\right) = \Delta_{22} + \sum_{j=1}^k \gamma_j^T \mathbf{x}_{j(r_j)}^* \tag{9}$$

where $\{\alpha_{1a}, \alpha_{2b}, \Delta_{ab}\}$; $a, b = 1, 2$ are the intercept parameters with $\alpha_{11} < \alpha_{12}$ and $\alpha_{21} < \alpha_{22}$; β_{1j}, β_{2j} and γ_j are vector of parameters for the j -th predictor variable, which are symbolized by $\beta_{1j} = [\beta_{01j} \ \beta_{11j} \ \dots \ \beta_{r1j}]^T$, $\beta_{2j} = [\beta_{02j} \ \beta_{12j} \ \dots \ \beta_{r2j}]^T$, $\gamma_j = [\gamma_{0j} \ \gamma_{1j} \ \dots \ \gamma_{rj}]^T$. \mathbf{x} is vector of predictor variable with $\mathbf{x} = [1 \ \mathbf{x}_{1(r_1)}^* \ \dots \ \mathbf{x}_{k(r_k)}^*]^T$ where $\mathbf{x}_{j(r_j)}^* = [1 \ x_j \ x_j^2 \ \dots \ x_j^{r_j}]^T$ is vector of the j -th predictor variable with the r -th degree of polynomial.

Based on Equations (2)–(5), the marginal cumulative probability of variable Y_1 and Y_2 can be obtained from the following equation:

$$P(Y_1 \leq a | \mathbf{x}) = F_{a\bullet}(\mathbf{x}) = \frac{\exp\left(\alpha_{1a} + \sum_{j=1}^k \beta_{1j}^T \mathbf{x}_{j(r_j)}^*\right)}{1 + \exp\left(\alpha_{1a} + \sum_{j=1}^k \beta_{1j}^T \mathbf{x}_{j(r_j)}^*\right)}; \quad a = 1, 2 \tag{10}$$

$$P(Y_2 \leq b | \mathbf{x}) = F_{b\bullet}(\mathbf{x}) = \frac{\exp\left(\alpha_{2b} + \sum_{j=1}^k \beta_{2j}^T \mathbf{x}_{j(r_j)}^*\right)}{1 + \exp\left(\alpha_{2b} + \sum_{j=1}^k \beta_{2j}^T \mathbf{x}_{j(r_j)}^*\right)}; \quad b = 1, 2 \tag{11}$$

$\psi_{ab}(\mathbf{x})$ is the odds ratio (OR), a measure of association that can indicate a correlation between the response variables. The OR value is formulated as follows:

$$\psi_{ab}(\mathbf{x}) = \frac{P(Y_1 \leq a, Y_2 \leq b | \mathbf{x})P(Y_1 > a, Y_2 > b | \mathbf{x})}{P(Y_1 > a, Y_2 \leq b | \mathbf{x})P(Y_1 \leq a, Y_2 > b | \mathbf{x})} = \frac{F_{ab}(\mathbf{x})[1 - F_{a\bullet}(\mathbf{x}) - F_{b\bullet}(\mathbf{x}) + F_{ab}(\mathbf{x})]}{[F_{a\bullet}(\mathbf{x}) - F_{ab}(\mathbf{x})][F_{b\bullet}(\mathbf{x}) - F_{ab}(\mathbf{x})]} \tag{12}$$

where $F_{ab}(\mathbf{x})$ can be obtained from:

$$F_{ab}(\mathbf{x}) = \begin{cases} \frac{c - \sqrt{c^2 + d}}{2(\psi_{ab}(\mathbf{x}) - 1)} & , \psi_{ab}(\mathbf{x}) \neq 1 \\ F_{a\bullet}(\mathbf{x})F_{b\bullet}(\mathbf{x}) & , \psi_{ab}(\mathbf{x}) = 1 \end{cases} \tag{13}$$

where $a, b = 1, 2$; $c = 1 + (\psi_{ab}(\mathbf{x}) - 1)(F_{a\bullet}(\mathbf{x}) + F_{b\bullet}(\mathbf{x}))$ and $d = -4\psi_{ab}(\mathbf{x})(\psi_{ab}(\mathbf{x}) - 1)F_{a\bullet}(\mathbf{x})F_{b\bullet}(\mathbf{x})$.

Next, the joint probability of the response variables, π_{ab} , can be obtained as follows:

$$\pi_{11} = F_{11}; \pi_{12} = F_{12} - F_{11}; \pi_{13} = F_{1\bullet} - F_{12}; \pi_{21} = F_{21} - F_{11}; \pi_{22} = F_{22} - F_{12} - F_{21} + F_{11}; \pi_{23} = F_{2\bullet} - F_{22} - F_{1\bullet} + F_{12}; \pi_{31} = F_{\bullet 1} - F_{21}; \pi_{32} = F_{\bullet 2} - F_{22} - F_{\bullet 1} + F_{21}; \pi_{33} = 1 - F_{2\bullet} - F_{\bullet 2} + F_{22}.$$

3. Parameter Estimation of The BPOLR Model

In this study, the parameter estimation of the BPOLR model was carried out using the maximum likelihood estimation (MLE) method. The principle of the MLE method is to estimate the parameters of the BPOLR model, namely:

$$\theta = [\alpha_{11} \quad \alpha_{12} \quad \beta_{011} \quad \beta_{111} \quad \dots \quad \beta_{r1k} \quad \alpha_{21} \quad \alpha_{22} \quad \beta_{021} \quad \beta_{121} \quad \dots \quad \beta_{r2k} \quad \Delta_{11} \quad \Delta_{12} \quad \Delta_{21} \quad \Delta_{22} \quad \gamma_{01} \quad \gamma_{11} \quad \dots \quad \gamma_{rk}]^T \tag{14}$$

obtained by maximizing the likelihood function. Based on Equation (1), the likelihood function is obtained as follows:

$$L(\theta) = \prod_{i=1}^n \prod_{a=1}^3 \prod_{b=1}^3 \pi_{abi}^{y_{abi}} \tag{15}$$

To simplify the calculation, an ln transformation is carried out on the likelihood function so that the ln-likelihood function is formed as follows:

$$Q = \ln L(\theta) = \sum_{i=1}^n \sum_{a=1}^3 \sum_{b=1}^3 y_{abi} \ln(\pi_{abi}) = \sum_{i=1}^n [y_{11i} \ln \pi_{11i} + y_{12i} \ln \pi_{12i} + \dots + y_{33i} \ln \pi_{33i}] \tag{16}$$

the next step of the ln-likelihood function is the first partial derivative of the parameter to be estimated and then equated with zero. The results of the first partial derivative of the ln-likelihood function with respect to its parameters are as follows:

- The first partial derivative of the ln-likelihood function to the parameter α_{11}

$$\frac{\partial Q}{\partial \alpha_{11}} = \sum_{i=1}^n \left[\left(\frac{y_{11i}}{\pi_{11i}} - \frac{y_{12i}}{\pi_{12i}} - \frac{y_{21i}}{\pi_{21i}} + \frac{y_{22i}}{\pi_{22i}} \right) \delta_{11i} + \left(\frac{y_{12i}}{\pi_{12i}} - \frac{y_{13i}}{\pi_{13i}} - \frac{y_{22i}}{\pi_{22i}} + \frac{y_{23i}}{\pi_{23i}} \right) \delta_{12i} + \left(\frac{y_{13i}}{\pi_{13i}} - \frac{y_{23i}}{\pi_{23i}} \right) \right] \xi_{1 \bullet i} \quad (17)$$

- The first partial derivative of the ln-likelihood function to the parameter α_{12}

$$\frac{\partial Q}{\partial \alpha_{12}} = \sum_{i=1}^n \left[\left(\frac{y_{21i}}{\pi_{21i}} - \frac{y_{22i}}{\pi_{22i}} - \frac{y_{31i}}{\pi_{31i}} + \frac{y_{32i}}{\pi_{32i}} \right) \delta_{21i} + \left(\frac{y_{22i}}{\pi_{22i}} - \frac{y_{23i}}{\pi_{23i}} - \frac{y_{32i}}{\pi_{32i}} + \frac{y_{33i}}{\pi_{33i}} \right) \delta_{22i} + \left(\frac{y_{23i}}{\pi_{23i}} - \frac{y_{33i}}{\pi_{33i}} \right) \right] \xi_{2 \bullet i} \quad (18)$$

- The first partial derivative of the ln-likelihood function to the parameter $\beta_{1j}; j = 1, 2, \dots, k$

$$\frac{\partial Q}{\partial \beta_{1j}} = \sum_{i=1}^n \left[\left\{ \left(\frac{y_{11i}}{\pi_{11i}} - \frac{y_{12i}}{\pi_{12i}} - \frac{y_{21i}}{\pi_{21i}} + \frac{y_{22i}}{\pi_{22i}} \right) \delta_{11i} + \left(\frac{y_{12i}}{\pi_{12i}} - \frac{y_{13i}}{\pi_{13i}} - \frac{y_{22i}}{\pi_{22i}} + \frac{y_{23i}}{\pi_{23i}} \right) \delta_{12i} + \left(\frac{y_{13i}}{\pi_{13i}} - \frac{y_{23i}}{\pi_{23i}} \right) \right\} \xi_{1 \bullet i} \mathbf{x}_{j(r_j)}^* + \left\{ \left(\frac{y_{21i}}{\pi_{21i}} - \frac{y_{22i}}{\pi_{22i}} - \frac{y_{31i}}{\pi_{31i}} + \frac{y_{32i}}{\pi_{32i}} \right) \delta_{21i} + \left(\frac{y_{22i}}{\pi_{22i}} - \frac{y_{23i}}{\pi_{23i}} - \frac{y_{32i}}{\pi_{32i}} + \frac{y_{33i}}{\pi_{33i}} \right) \delta_{22i} + \left(\frac{y_{23i}}{\pi_{23i}} - \frac{y_{33i}}{\pi_{33i}} \right) \right\} \xi_{2 \bullet i} \mathbf{x}_{j(r_j)}^* \right] \quad (19)$$

- The first partial derivative of the ln-likelihood function to the parameter α_{21}

$$\frac{\partial Q}{\partial \alpha_{21}} = \sum_{i=1}^n \left[\left(\frac{y_{11i}}{\pi_{11i}} - \frac{y_{12i}}{\pi_{12i}} - \frac{y_{21i}}{\pi_{21i}} + \frac{y_{22i}}{\pi_{22i}} \right) \mu_{11i} + \left(\frac{y_{21i}}{\pi_{21i}} - \frac{y_{22i}}{\pi_{22i}} - \frac{y_{31i}}{\pi_{31i}} + \frac{y_{32i}}{\pi_{32i}} \right) \mu_{21i} + \left(\frac{y_{31i}}{\pi_{31i}} - \frac{y_{32i}}{\pi_{32i}} \right) \right] \xi_{\bullet 1i} \quad (20)$$

- The first partial derivative of the ln-likelihood function to the parameter α_{22}

$$\frac{\partial Q}{\partial \alpha_{22}} = \sum_{i=1}^n \left[\left(\frac{y_{12i}}{\pi_{12i}} - \frac{y_{13i}}{\pi_{13i}} - \frac{y_{22i}}{\pi_{22i}} + \frac{y_{23i}}{\pi_{23i}} \right) \mu_{12i} + \left(\frac{y_{22i}}{\pi_{22i}} - \frac{y_{23i}}{\pi_{23i}} - \frac{y_{32i}}{\pi_{32i}} + \frac{y_{33i}}{\pi_{33i}} \right) \mu_{22i} + \left(\frac{y_{32i}}{\pi_{32i}} - \frac{y_{33i}}{\pi_{33i}} \right) \right] \xi_{\bullet 2i} \quad (21)$$

- The first partial derivative of the ln-likelihood function to the parameter $\beta_{2j}; j = 1, 2, \dots, k$

$$\frac{\partial Q}{\partial \beta_{2j}} = \sum_{i=1}^n \left[\left\{ \left(\frac{y_{11i}}{\pi_{11i}} - \frac{y_{12i}}{\pi_{12i}} - \frac{y_{21i}}{\pi_{21i}} + \frac{y_{22i}}{\pi_{22i}} \right) \mu_{11i} + \left(\frac{y_{21i}}{\pi_{21i}} - \frac{y_{22i}}{\pi_{22i}} - \frac{y_{31i}}{\pi_{31i}} + \frac{y_{32i}}{\pi_{32i}} \right) \mu_{21i} + \left(\frac{y_{31i}}{\pi_{31i}} - \frac{y_{32i}}{\pi_{32i}} \right) \right\} \xi_{\bullet 1i} \mathbf{x}_{j(r_j)}^* + \left\{ \left(\frac{y_{12i}}{\pi_{12i}} - \frac{y_{13i}}{\pi_{13i}} - \frac{y_{22i}}{\pi_{22i}} + \frac{y_{23i}}{\pi_{23i}} \right) \mu_{12i} + \left(\frac{y_{22i}}{\pi_{22i}} - \frac{y_{23i}}{\pi_{23i}} - \frac{y_{32i}}{\pi_{32i}} + \frac{y_{33i}}{\pi_{33i}} \right) \mu_{22i} + \left(\frac{y_{32i}}{\pi_{32i}} - \frac{y_{33i}}{\pi_{33i}} \right) \right\} \xi_{\bullet 2i} \mathbf{x}_{j(r_j)}^* \right] \quad (22)$$

- The first partial derivative of the ln-likelihood function to the parameter Δ_{11}

$$\frac{\partial Q}{\partial \Delta_{11}} = \sum_{i=1}^n \left[\left(\frac{y_{11i}}{\pi_{11i}} - \frac{y_{12i}}{\pi_{12i}} - \frac{y_{21i}}{\pi_{21i}} + \frac{y_{22i}}{\pi_{22i}} \right) z_{11i} \psi_{11i} \right] \quad (23)$$

- The first partial derivative of the ln-likelihood function to the parameter Δ_{12}

$$\frac{\partial Q}{\partial \Delta_{12}} = \sum_{i=1}^n \left[\left(\frac{y_{12i}}{\pi_{12i}} - \frac{y_{13i}}{\pi_{13i}} - \frac{y_{22i}}{\pi_{22i}} + \frac{y_{23i}}{\pi_{23i}} \right) z_{12i} \psi_{12i} \right] \quad (24)$$

- The first partial derivative of the ln-likelihood function to the parameter Δ_{21}

$$\frac{\partial Q}{\partial \Delta_{21}} = \sum_{i=1}^n \left[\left(\frac{y_{21i}}{\pi_{21i}} - \frac{y_{22i}}{\pi_{22i}} - \frac{y_{31i}}{\pi_{31i}} + \frac{y_{32i}}{\pi_{32i}} \right) z_{21i} \psi_{21i} \right] \quad (25)$$

- The first partial derivative of the ln-likelihood function to the parameter Δ_{22}

$$\frac{\partial Q}{\partial \Delta_{22}} = \sum_{i=1}^n \left[\left(\frac{y_{22i}}{\pi_{22i}} - \frac{y_{23i}}{\pi_{23i}} - \frac{y_{32i}}{\pi_{32i}} + \frac{y_{33i}}{\pi_{33i}} \right) z_{22i} \psi_{22i} \right] \tag{26}$$

- The first partial derivative of the ln-likelihood function to the parameter γ_j ; $j = 1, 2, \dots, k$

$$\frac{\partial Q}{\partial \gamma_j} = \sum_{i=1}^n \left[\left\{ \left(\frac{y_{11i}}{\pi_{11i}} - \frac{y_{12i}}{\pi_{12i}} - \frac{y_{21i}}{\pi_{21i}} + \frac{y_{22i}}{\pi_{22i}} \right) z_{11i} \psi_{11i} + \left(\frac{y_{12i}}{\pi_{12i}} - \frac{y_{13i}}{\pi_{13i}} - \frac{y_{22i}}{\pi_{22i}} + \frac{y_{23i}}{\pi_{23i}} \right) z_{12i} \psi_{12i} + \left(\frac{y_{21i}}{\pi_{21i}} - \frac{y_{22i}}{\pi_{22i}} - \frac{y_{31i}}{\pi_{31i}} + \frac{y_{32i}}{\pi_{32i}} \right) z_{21i} \psi_{21i} + \left(\frac{y_{22i}}{\pi_{22i}} - \frac{y_{23i}}{\pi_{23i}} - \frac{y_{32i}}{\pi_{32i}} + \frac{y_{33i}}{\pi_{33i}} \right) z_{22i} \psi_{22i} \right\} \mathbf{x}_{j(r_j)}^* \right] \tag{27}$$

where $\delta_{ab} = \frac{1}{2} \{1 - S_{ab}^{-1} [1 + \psi_{ab}(F_{a\bullet} - F_{\bullet b}) - F_{a\bullet} - F_{\bullet b}]\}$; $\mu_{ab} = \frac{1}{2} \{1 - S_{ab}^{-1} [1 + \psi_{ab}(F_{\bullet b} - F_{a\bullet}) - F_{a\bullet} - F_{\bullet b}]\}$

$S_{ab} = \left[1 + (\psi_{ab} - 1)(F_{a\bullet} + F_{\bullet b}) \right]^2 - 4\psi_{ab}(\psi_{ab} - 1)F_{a\bullet}F_{\bullet b}$; $\zeta_{a\bullet} = F_{a\bullet}(1 - F_{a\bullet})$; $\zeta_{\bullet b} = F_{\bullet b}(1 - F_{\bullet b})$

$z_{ab} = \left[2(\psi_{ab} - 1)^2 S_{ab} \right]^{-1} \{1 - S_{ab} + (\psi_{ab} - 1)[F_{a\bullet} + F_{\bullet b} - 2F_{a\bullet}F_{\bullet b}]\}$.

The result of the first partial derivatives are not closed-form, thus, a numerical optimization such as the Berndt–Hall–Hall–Hausman (BHHH) method is needed to obtain the maximum likelihood estimators and the BHHH algorithm is as follows:

- Step 1. Determine the initial value for

$$\theta^{(0)} = \left[\alpha_{11}^{(0)} \quad \alpha_{12}^{(0)} \quad \beta_{011}^{(0)} \quad \beta_{111}^{(0)} \quad \dots \quad \beta_{r1k}^{(0)} \quad \alpha_{21}^{(0)} \quad \alpha_{22}^{(0)} \quad \beta_{021}^{(0)} \quad \beta_{121}^{(0)} \quad \dots \quad \beta_{r2k}^{(0)} \right. \\ \left. \Delta_{11}^{(0)} \quad \Delta_{12}^{(0)} \quad \Delta_{21}^{(0)} \quad \Delta_{22}^{(0)} \quad \gamma_{01}^{(0)} \quad \gamma_{11}^{(0)} \quad \dots \quad \gamma_{rk}^{(0)} \right]^T$$

obtained from the parameter estimator of the ordinal logistic regression model on each response variable.

- Step 2. Calculate the gradient vector elements obtained from the first partial derivative of the ln-likelihood function for each parameter

$$\mathbf{q}(\theta) = \left[\frac{\partial Q}{\partial \alpha_{11}} \quad \frac{\partial Q}{\partial \alpha_{12}} \quad \frac{\partial Q}{\partial \beta_{011}} \quad \frac{\partial Q}{\partial \beta_{111}} \quad \dots \quad \frac{\partial Q}{\partial \beta_{r1k}} \quad \frac{\partial Q}{\partial \alpha_{21}} \quad \frac{\partial Q}{\partial \alpha_{22}} \quad \frac{\partial Q}{\partial \beta_{021}} \quad \frac{\partial Q}{\partial \beta_{121}} \quad \dots \quad \frac{\partial Q}{\partial \beta_{r2k}} \right. \\ \left. \frac{\partial Q}{\partial \Delta_{11}} \quad \frac{\partial Q}{\partial \Delta_{12}} \quad \frac{\partial Q}{\partial \Delta_{21}} \quad \frac{\partial Q}{\partial \Delta_{22}} \quad \frac{\partial Q}{\partial \gamma_{01}} \quad \frac{\partial Q}{\partial \gamma_{11}} \quad \dots \quad \frac{\partial Q}{\partial \gamma_{rk}} \right]^T$$

- Step 3. Calculate the Hessian matrix $\mathbf{H}(\theta)$ that can be obtained from the following formula

$$\mathbf{H}(\theta) = - \left[\mathbf{q}^T(\theta) \mathbf{q}(\theta) \right]$$

- Step 4. Start the BHHH iteration process with the following formula

$$\hat{\theta}^{(t+1)} = \hat{\theta}^{(t)} - \mathbf{H}^{-1}(\hat{\theta}^{(t)}) \mathbf{q}(\hat{\theta}^{(t)}), \text{ for } t = 0, 1, 2, \dots$$

- Step 5. The iteration will stop if $\|\hat{\theta}^{(t+1)} - \hat{\theta}^{(t)}\| < \epsilon$, where ϵ is a very small positive number. The last iteration produces an estimator value for each parameter.

Furthermore, the best model in this study is determined using the criteria of Akaike information criterion corrected (AICc), Bayesian information criterion (BIC), and deviance which are defined as follows:

$$AICc = -2 \ln L(\hat{\theta}) + 2K + \frac{2K(K+1)}{n-K-1} \tag{28}$$

$$BIC = -2 \ln L(\hat{\theta}) + K \ln n \tag{29}$$

$$Deviance = -2 \ln L(\hat{\theta}) \tag{30}$$

where $L(\hat{\theta})$ is the likelihood value of the parameter's estimate, K is the number of covariates, and n is the sample size. The best model is the BPOLR model, which has the smallest values of AICc, BIC, and deviance.

4. Hypothesis Testing of The BPOLR Model

The simultaneous hypothesis testing of the BPOLR model aims to determine the influence of the predictor variable on the response variable simultaneously or to discover at least one predictor variable which has a significant effect against the response variable. The hypothesis used is as follows:

$$H_0 : \beta_{p1q} = \beta_{p2q} = \gamma_{pq} = 0 ; p = 1, 2, \dots, r_q ; q = 1, 2, \dots, k \tag{31}$$

$$H_1 : \text{at least one } \beta_{p1q} \neq 0 \text{ or } \beta_{p2q} \neq 0 \text{ or } \gamma_{pq} \neq 0$$

In this study, we used the MLRT method to obtain the test statistics by calculating the ratio between the maximum value of the likelihood function under H_0 , $L(\omega)$, with the maximum value of the likelihood function under population, $L(\Omega)$.

The first step to obtaining the maximum value of $L(\omega)$ is to define the parameter space under H_0 , $\omega = \{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22}, \beta_{011}, \dots, \beta_{01k}, \beta_{021}, \dots, \beta_{02k}, \gamma_{01}, \dots, \gamma_{0k}\}$. The likelihood function under H_0 is:

$$L(\omega) = \prod_{i=1}^n \prod_{a=1}^3 \prod_{b=1}^3 (\pi_{abi}^*)^{y_{abi}} \tag{32}$$

Let $Q_\omega = \ln L(\omega)$, then the log-likelihood function under H_0 is

$$Q_\omega = \sum_{i=1}^n (y_{11i} \ln \pi_{11i}^* + y_{12i} \ln \pi_{12i}^* + \dots + y_{33i} \ln \pi_{33i}^*) \tag{33}$$

Furthermore, we can obtain the estimated value of the maximum log-likelihood function under H_0 :

$$Q_{\hat{\omega}} = \sum_{i=1}^n (y_{11i} \ln \hat{\pi}_{11i}^* + y_{12i} \ln \hat{\pi}_{12i}^* + \dots + y_{33i} \ln \hat{\pi}_{33i}^*) \tag{34}$$

where

$$\begin{aligned} \hat{\pi}_{11i}^* &= \hat{F}_{11i}^* ; \hat{\pi}_{12i}^* = \hat{F}_{12i}^* - \hat{F}_{11i}^* ; \hat{\pi}_{13i}^* = \hat{F}_{1\bullet i}^* - \hat{F}_{12i}^* ; \hat{\pi}_{21i}^* = \hat{F}_{21i}^* - \hat{F}_{11i}^* ; \\ \hat{\pi}_{22i}^* &= \hat{F}_{22i}^* - \hat{F}_{12i}^* - \hat{F}_{21i}^* + \hat{F}_{11i}^* ; \hat{\pi}_{23i}^* = \hat{F}_{2\bullet i}^* - \hat{F}_{1\bullet i}^* - \hat{F}_{22i}^* + \hat{F}_{12i}^* ; \hat{\pi}_{31i}^* = \hat{F}_{\bullet 1i}^* - \hat{F}_{21i}^* ; \\ \hat{\pi}_{32i}^* &= \hat{F}_{\bullet 2i}^* - \hat{F}_{22i}^* - \hat{F}_{\bullet 1i}^* + \hat{F}_{21i}^* ; \hat{\pi}_{33i}^* = 1 - \hat{F}_{2\bullet i}^* - \hat{F}_{\bullet 2i}^* + \hat{F}_{22i}^* \end{aligned}$$

with

$$\begin{aligned} \hat{F}_{abi}^* &= \frac{p_3 - \sqrt{p_3^2 + q_3}}{2(\hat{\psi}_{ab} - 1)}, \hat{\psi}_{ab} \neq 1; p_3 = 1 + (\hat{\psi}_{ab} - 1)(\hat{F}_{a\bullet i}^* + \hat{F}_{\bullet bi}^*); q_3 = -4\hat{\psi}_{ab}(\hat{\psi}_{ab} - 1)\hat{F}_{a\bullet i}^*\hat{F}_{\bullet bi}^* \\ \hat{F}_{a\bullet i}^* &= \frac{\exp\left(\hat{\alpha}_{1a} + \sum_{j=1}^k \hat{\beta}_{01j}\right)}{\exp\left(\hat{\alpha}_{1a} + \sum_{j=1}^k \hat{\beta}_{01j}\right)}; \hat{F}_{\bullet bi}^* = \frac{\exp\left(\hat{\alpha}_{2b} + \sum_{j=1}^k \hat{\beta}_{02j}\right)}{\exp\left(\hat{\alpha}_{2b} + \sum_{j=1}^k \hat{\beta}_{02j}\right)}; \hat{\psi}_{11} = \frac{n_{11i}(n_{22i} + n_{23i} + n_{32i} + n_{33i})}{(n_{12i} + n_{13i})(n_{21i} + n_{31i})}; \\ \hat{\psi}_{12} &= \frac{(n_{11i} + n_{12i})(n_{23i} + n_{33i})}{n_{13i}(n_{21i} + n_{22i} + n_{31i} + n_{32i})}; \hat{\psi}_{21} = \frac{(n_{11i} + n_{21i})(n_{32i} + n_{33i})}{(n_{12i} + n_{13i} + n_{22i} + n_{23i})n_{31i}}; \hat{\psi}_{22} = \frac{(n_{11i} + n_{12i} + n_{21i} + n_{22i})n_{33i}}{(n_{13i} + n_{23i})(n_{31i} + n_{32i})}. \end{aligned}$$

Furthermore, to obtain the maximum value of $L(\Omega)$ is to define the parameter space under population, $\Omega = \{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22}, \beta_{p1q}, \beta_{p2q}, \gamma_{pq}; p = 0, 1, 2, \dots, r; q; q = 1, 2, \dots, k\}$. Then, the likelihood function under population is

$$L(\Omega) = \prod_{i=1}^n \prod_{a=1}^3 \prod_{b=1}^3 (\pi_{abi})^{y_{abi}} \tag{35}$$

Let $Q_{\Omega} = \ln L(\Omega)$, then the log-likelihood function under population is

$$Q_{\Omega} = \sum_{i=1}^n (y_{11i} \ln \pi_{11i} + y_{12i} \ln \pi_{12i} + \dots + y_{33i} \ln \pi_{33i}) \tag{36}$$

Furthermore, we can obtain the estimated value of the maximum log-likelihood function under population, that is:

$$Q_{\hat{\Omega}} = \sum_{i=1}^n (y_{11i} \ln \hat{\pi}_{11i} + y_{12i} \ln \hat{\pi}_{12i} + \dots + y_{33i} \ln \hat{\pi}_{33i}) \tag{37}$$

The test statistic using MLRT method is obtained by

$$G^2 = -2 \ln \left(\frac{L(\hat{\omega})}{L(\hat{\Omega})} \right) = 2(Q_{\hat{\Omega}} - Q_{\hat{\omega}}) \tag{38}$$

where $Q_{\hat{\omega}}$ and $Q_{\hat{\Omega}}$ are given in Equations (36) and (39), respectively. Furthermore, the test statistic in Equation (38) can be expressed as follows

$$G^2 = 2 \left[\sum_{i=1}^n \left(y_{11i} \ln \left(\frac{\hat{\pi}_{11i}}{\hat{\pi}_{11i}^*} \right) + y_{12i} \ln \left(\frac{\hat{\pi}_{12i}}{\hat{\pi}_{12i}^*} \right) + \dots + y_{33i} \ln \left(\frac{\hat{\pi}_{33i}}{\hat{\pi}_{33i}^*} \right) \right) \right] \\ = 2 \left[\sum_{i=1}^n \left(n \hat{\pi}_{11i} \ln \left(1 + \frac{\hat{\pi}_{11i} - \hat{\pi}_{11i}^*}{\hat{\pi}_{11i}^*} \right) \right) + \sum_{i=1}^n \left(n \hat{\pi}_{12i} \ln \left(1 + \frac{\hat{\pi}_{12i} - \hat{\pi}_{12i}^*}{\hat{\pi}_{12i}^*} \right) \right) + \dots + \sum_{i=1}^n \left(n \hat{\pi}_{33i} \ln \left(1 + \frac{\hat{\pi}_{33i} - \hat{\pi}_{33i}^*}{\hat{\pi}_{33i}^*} \right) \right) \right]$$

Based on the Taylor expansion of $\ln \left(1 + \frac{\hat{\pi}_{abi} - \hat{\pi}_{abi}^*}{\hat{\pi}_{abi}^*} \right)$ with $a, b = 1, 2, 3$ and $i = 1, 2, \dots, n$ it will be proved that $\lim_{n \rightarrow \infty} G^2 = G^*$.

$$G^* = \sum_{i=1}^n \frac{(n \hat{\pi}_{11i} - n \hat{\pi}_{11i}^*)^2}{n \hat{\pi}_{11i}^*} + \sum_{i=1}^n \frac{(n \hat{\pi}_{12i} - n \hat{\pi}_{12i}^*)^2}{n \hat{\pi}_{12i}^*} + \dots + \sum_{i=1}^n \frac{(n \hat{\pi}_{33i} - n \hat{\pi}_{33i}^*)^2}{n \hat{\pi}_{33i}^*} \\ = \sum_{i=1}^n \sum_{a=1}^3 \sum_{b=1}^3 \frac{(n \hat{\pi}_{abi} - n \hat{\pi}_{abi}^*)^2}{n \hat{\pi}_{abi}^*} \tag{39} \\ = \sum_{i=1}^n \sum_{a=1}^3 \sum_{b=1}^3 \frac{(y_{abi} - n \hat{\pi}_{abi}^*)^2}{n \hat{\pi}_{abi}^*}$$

The test statistic in Equation (39) has an asymptotic Chi-square limit distribution with v degree of freedom, where v is the difference between the number of parameters in the BPOLR model under population and H_0 , $v = \left(8 + 3 \left(\sum_{j=1}^k r_j + k \right) \right) - (8 + 3k) = 3 \sum_{j=1}^k r_j$. Therefore, reject H_0 when $G^2 > \chi_{(v, 1-\alpha)}^2$, where $\chi_{(v, 1-\alpha)}^2$ is the $1 - \alpha$ quantile from a Chi-square distribution (χ^2) with v degree of freedom.

5. Simulation Study

In this section, we outline a simulation study for the present performance of parameter estimation of the BPOLR model based on the BHHH algorithm. The simulation was constructed by two ordinal response variables that each have three categories and three predictor variables. In this paper, the simulation is carried out for one example of the BPOLR model, i.e., the BPOLR model with degree polynomial (1,2,1). The simulation

was carried out for 100 replications with different sample sizes, i.e., $n = 100$, $n = 200$, and $n = 300$. The simulation scenarios are decided as follows:

- Generate three predictor variables (X_1 , X_2 and X_3) that are constructed from a standard uniform distribution
- Set the initial coefficients of the BPOLR model as follows:

$$\alpha_{11} = 0.05 ; \alpha_{12} = 0.5 ; \beta_{111} = -1 ; \beta_{112} = 0.5 ; \beta_{212} = 2 ; \beta_{113} = 1.5 ; \alpha_{21} = 0.25 ; \alpha_{22} = 1.25 ;$$

$$\beta_{121} = -2 ; \beta_{122} = 2 ; \beta_{222} = 2 ; \beta_{123} = 2 ; \Delta_{11} = 0.1 ; \Delta_{12} = 0.15 ; \Delta_{21} = 0.25 ; \Delta_{22} = 0.3 ;$$

$$\gamma_{11} = -1.5 ; \gamma_{12} = 2 ; \gamma_{22} = 3 ; \gamma_{13} = 2$$

- Generate two ordinal response variables (Y_1 and Y_2) with the following steps:
 - Determine the cumulative logit model for Y_1 and Y_2 as in Equations (2)–(5) and the odds ratio transformation model, as in Equations (6)–(9)
 - Determine the marginal cumulative probability for Y_1 and Y_2 as in Equations (10) and (11) and the joint cumulative probability as in Equation (13).
 - Determine the joint probability of Y_1 and Y_2
 - Generate two ordinal response variables based on the joint probability obtained
- Examine the independence of the response variables using the Mantel–Haenszel test to fulfill the assumption of dependence between the response variables in the bivariate model. If the response variable is independent, then the data generation process is repeated until the dependent response variable is obtained.
- Estimate the parameters of the BPOLR model based on the BHHH algorithm
- Repeat the process for up to 100 replications for each sample size
- Calculate the mean of parameter estimated and its standard error (SE)

The results and comments regarding the simulation study are given in the following tables and figures below. The mean of the estimated parameters and their corresponding standard error (SE) for different sample sizes are presented in Table 2. The results show that the estimated parameters approach their true coefficient on average and the corresponding standard errors also decrease as the sample size increases.

Furthermore, the suitability of the estimated parameter with the true coefficients for each sample size can be presented in Figure 1 below:

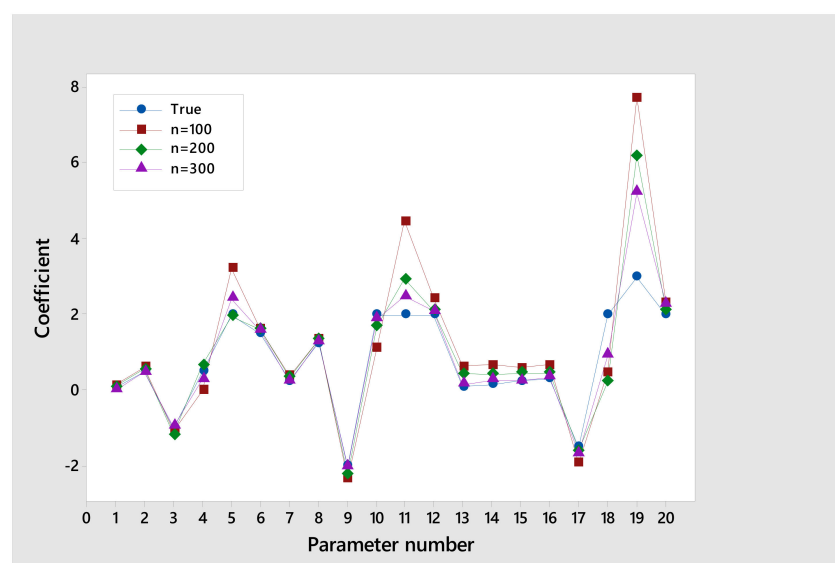


Figure 1. Mean of the true and estimated parameter coefficients for different sample sizes, i.e $n = 100$, $n = 200$ and $n = 300$.

Table 2. Mean of the estimated parameter and the corresponding standard errors of the BPOLR model for different sample sizes.

No.	Parameter	True Coeff.	Mean of the est. Parameter			Standard Error		
			<i>n</i> = 100	<i>n</i> = 200	<i>n</i> = 300	<i>n</i> = 100	<i>n</i> = 200	<i>n</i> = 300
1	α_{11}	0.05	0.138	0.097	0.005	1.193	0.662	0.519
2	α_{12}	0.5	0.615	0.549	0.474	1.195	0.666	0.521
3	β_{111}	−1	−1.069	−1.176	−0.96	1.144	0.632	0.501
4	β_{112}	0.5	−0.005	0.679	0.283	4.856	2.566	2.004
5	β_{212}	2	3.213	1.949	2.418	5.492	2.762	2.169
6	β_{113}	1.5	1.607	1.604	1.582	1.159	0.652	0.516
7	α_{21}	0.25	0.382	0.337	0.236	1.556	0.777	0.604
8	α_{22}	1.25	1.35	1.349	1.282	1.573	0.789	0.614
9	β_{121}	−2	−2.34	−2.216	−2.032	1.546	0.826	0.625
10	β_{122}	2	1.119	1.686	1.873	6.949	3.417	2.513
11	β_{222}	2	4.466	2.928	2.47	9.294	4.291	3
12	β_{123}	2	2.442	2.115	2.094	1.565	0.823	0.638
13	Δ_{11}	0.1	0.636	0.44	0.155	5.038	1.941	1.354
14	Δ_{12}	0.15	0.677	0.414	0.269	5.101	1.98	1.387
15	Δ_{21}	0.25	0.587	0.461	0.25	5.047	1.952	1.354
16	Δ_{22}	0.3	0.671	0.455	0.351	5.145	1.978	1.383
17	γ_{11}	−1.5	−1.926	−1.585	−1.662	5.088	1.992	1.396
18	γ_{12}	2	0.475	0.226	0.929	26.556	9.533	6.397
19	γ_{22}	3	7.729	6.204	5.216	40.777	13.805	8.713
20	γ_{13}	2	2.33	2.134	2.273	5.22	2.008	1.447

Based on Figure 1, the results show that the greater of samples size, the closer the estimated parameter values get to the true coefficient values. This can be seen from the movement of the estimated parameter coefficients for *n* = 300, which are closest to the true coefficients, and for *n* = 100, the coefficients which are farthest for each parameter. In addition, the standard error range values from parameter estimation was also investigated for different samples in the form of boxplots. When the sample size increases, the standard error values of the model parameter estimation becomes less, as presented in Figure 2 as follows:

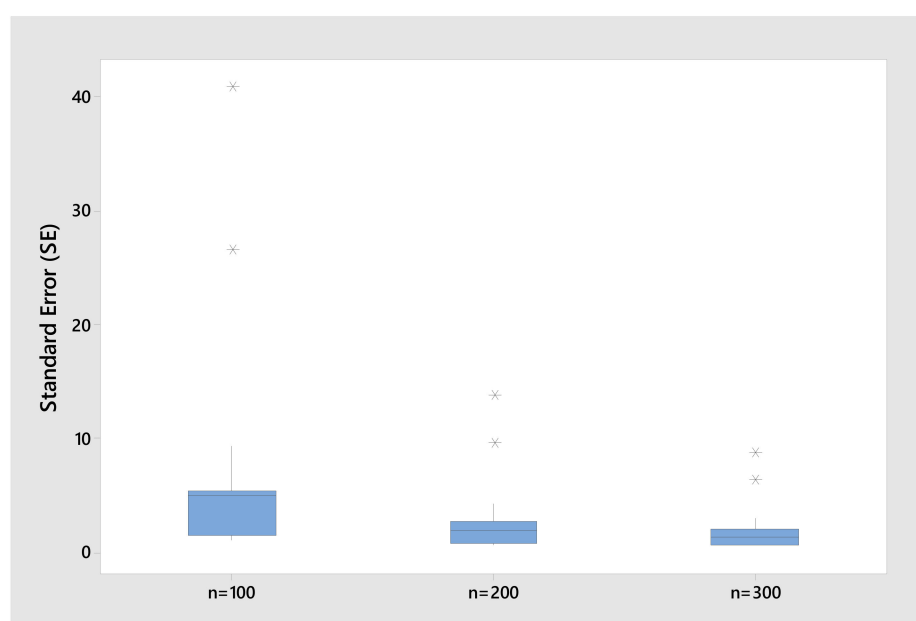


Figure 2. Boxplot of Standard Error (SE) for different sample sizes.

6. Conclusions

We developed ordinal logistic regression with two correlated response variables in which the relationship between the continuous predictor variable and its logit is modeled as a polynomial form, called the BPOLR model. The proposed BPOLR model contributes to the development of statistical science that can be used as an alternative solution in statistical modeling related to the bivariate ordinal logistic regression, where the relationship between continuous predictor variables and their logit links is assumed to be not always linear, so that it is more flexible. Therefore, the advantage of the BPOLR model can accommodate linear, quadratic, cubic or other relationships between continuous predictor variables and their logit links in a model. We used the MLE method to obtain parameter estimators of the BPOLR model. The first partial derivatives are not closed-form, thus, a numerical optimization such as the method of BHHH is needed to obtain the maximum likelihood estimators. We used the MLRT method to obtain test statistics of parameters. The simultaneous test was used to make the simultaneous of the parameters significantly. The statistical test distribution has a Chi-square limit that, asymptotically with the degree of freedom, is the difference between the parameter effective number in the reduced and full models. The optimal degree value for each predictor variable is obtained when the value of AICc, BIC, and deviance are minimum. Based on the simulation study results, the estimated parameters approach their true coefficient on average and the corresponding standard errors also decrease as the sample size increases.

Author Contributions: Conceptualization, V.R., M.R. and P.P.; methodology, V.R., M.R. and P.P.; writing original draft preparation, M.R.; writing—review and editing, V.R., M.R. and P.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Directorate of Research and Community Service (DRPM) Institut Teknologi Sepuluh Nopember with grant number 959/PKS/ITS/2022.

Data Availability Statement: Not applicable.

Acknowledgments: All authors thank the editor and reviewers for the improvement of this paper through criticism and suggestions provided.

Conflicts of Interest: The authors declare no conflict of interest.

Nomenclature

AICc	Akaike's Information Criterion Correction
BHHH	Berndt–Hall–Hall–Hausman
BIC	Bayesian Information Criterion
BPOLR	Bivariate Polynomial Ordinal Logistic Regression
MLE	Maximum Likelihood Estimation
MLRT	Maximum Likelihood Ratio Test.

References

1. Hosmer, D.W.; Lemeshow, S. Regression. In *Applied Logistic Regression*, 2nd ed.; John Wiley & Sons: New York, NY, USA, 2000; pp. 260–288.
2. Dale, J.R. Global Cross-Ratio Models for Bivariate, Discrete, Ordered Responses. *Biometrics* **1986**, *42*, 909–917. [[CrossRef](#)] [[PubMed](#)]
3. Williamson, J.M.; Kim, K.M.; Lipsitz, S.R. Analyzing Bivariate Ordinal Data using a Global Odds Ratio. *J. Am. Stat. Assoc.* **1995**, *90*, 1432–1437. [[CrossRef](#)]
4. Enea, M.; Lovison, G. A Penalized Approach for the Bivariate Ordered Logistic Model with Applications to Social and Medical Data. *Stat. Model.* **2018**, *19*, 467–500. [[CrossRef](#)]
5. Royston, P.; Altman, D.G. Regression Using Fractional Polynomials of Continuous Covariates: Parsimonious Parametric Modelling. *J. R. Stat. Soc. Ser. C Appl. Stat.* **1994**, *43*, 429–453. [[CrossRef](#)]
6. Sauerbrei, W.; Royston, P. Building Multivariable Prognostic and Diagnostic Models: Transformation of The Predictors by Using Fractional Polynomials. *J. R. Stat. Soc. Ser. A Stat. Soc.* **1999**, *162*, 71–94. [[CrossRef](#)]

7. Sauerbrei, W.; Royston, P.; Binder, H. Selection of Important Variables and Determination of Functional Form for Continuous Predictors in Multivariable Model Building. *Stat. Med.* **2007**, *26*, 5512–5528. [[CrossRef](#)] [[PubMed](#)]
8. Regier, M.; Parker, R.D. Smoothing Using Fractional Polynomials: An Alternative to Polynomials and Splines in Applied Research. *Wiley Interdiscip. Rev. Comput. Stat.* **2015**, *7*, 275–283. [[CrossRef](#)]
9. Zhang, Z. Multivariable Fractional Polynomial Method for Regression Model. *Ann. Transl. Med.* **2016**, *4*, 174. [[CrossRef](#)] [[PubMed](#)]
10. Silke, B.; Kellett, J.; Rooney, T.; Bennet, K.; O’Riordan, D. An Improved Medical Admissions Risk System using Multivariable Fractional Polynomial Logistic Regression Modelling. *Q. J. Med.* **2010**, *103*, 23–32. [[CrossRef](#)] [[PubMed](#)]
11. Omer, D.; Musa, A.B. Modelling Logistic Regression using Multivariable Fractional Polynomials. *Imp. J. Interdiscip. Res.* **2017**, *3*, 8–16.
12. Sohail, M.N.; Ren, J.; Muhammad, M.U.; Rizwan, T.; Iqbal, W.; Abir, S.I.; Irshad, M.; Bilal, M. Group Covariates Assessment on Real-Life Diabetes Patients by Fractional Polynomials: A Study based on Logistic Regression Modeling. *J. Biotech. Res.* **2019**, *10*, 116–125.
13. Ratnasari, V.; Purhadi; Aviantholib, I.C.; Dani, A.T.R. Parameter estimation and hypothesis testing the second order of bivariate binary logistic regression (S-BBLR) model with Berndt Hall-Hall-Hausman (BHHH) iterations. *Commun. Math. Biol. Neurosci.* **2022**, *2022*, 35.
14. Narendra, M.B.; Sularyo, T.S.; Soetjningsih, S.S.; Suyitno, H.; Ranuh, I.G.N.G.; Wiradisuria, S. *Tumbuh Kembang Anak dan Remaja*; Sagung Seto: Jakarta, Indonesia, 2002.
15. Chamidah, N.; Saifudin, T. Estimation of Children Growth Curve Based on Kernel Smoothing in Multi-Response Nonparametric Regression. *Appl. Math. Sci.* **2013**, *7*, 1839–1847. [[CrossRef](#)]
16. Tilling, K.; Wallis, C.M.; Lawlor, D.A.; Hughes, R.A.; Howe, L.D. Modelling Childhood Growth Using Fractional Polynomials and Linear Splines. *Ann. Nutr. Metab.* **2014**, *65*, 129–138. [[CrossRef](#)] [[PubMed](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.