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Review

Statistical and Ideal Convergences in Topology

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Abstract: The notion of convergence wins its own important part in the branch of Topology. Convergences in metric spaces, topological spaces, fuzzy topological spaces, fuzzy metric spaces, partially ordered sets (in short, posets), and fuzzy ordered sets (in short, fosets) develop significant chapters that attract the interest of many studies. In particular, statistical and ideal convergences play their own important role in all these areas. A lot of studies have been devoted to these special convergences, and many results have been proven. As a consequence, the necessity to produce and extend new results arises. Since there are many results on different kinds of convergences in different areas, we present a review paper on this research topic in order to collect the most essential results, which leads us to provide open questions for further investigation. More precisely, we present and gather definitions and results which have been proven for different kinds of convergences, mainly on statistical/ideal convergences, in metric spaces, topological spaces, fuzzy topological spaces, fuzzy metric spaces, posets, and fosets. Based on this presentation, we provide new open problems for further investigation on related topics. The study of these problems will create new "roads", enriching the branch of convergences in the field of Topology.

Keywords: convergence; statistical convergence; ideal convergence; metric space; topological space; fuzzy topological space; fuzzy metric space; partially ordered set (poset); fuzzy ordered set (foset)

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1. Introduction

Undoubtedly, the branches of Topology and Mathematical Analysis deal with many research topics. The notion of convergence is one of the most known and essential meanings for many studies. In particular, the classical notions of convergence in topological and metric spaces are studied in detail in order to get new results and solve topological problems in the view of convergence. There are many definitions, examples, and propositions/theorems that are related to these notions. Many properties and characterizations for topological/metric spaces are given through the notions of convergences. It is believed that the study of convergences gives advantages to the field of Topology as we can get various characterizations of topological properties (like the meanings of topology, closed sets, and compactness).

Simultaneously, the interest in finding new results tends to new research articles and new kinds of convergences, like the statistical and ideal convergence, enriching the Convergence Theory in Topology. The last proves that the field of convergence can be considered as a "source" that always gives the motivation to "produce" new results. Statistical and ideal convergences can provide new "tools" for future investigation in metric, topological, fuzzy spaces, and partially ordered sets. Among the many motivations for this study, the role of the ideal in different areas, like Dimension Theory and Rough Set Theory, shows the "power" of related investigations. Nowadays, various research papers are based on the notion of the ideal, investigating its applications to Mathematics and other scientific areas. For example, we refer to some recent research articles [1–3].

Therefore, with this paper, we mention the importance of the notion of convergence. We collect related definitions and results, which lead us to express new open questions and proposals for future study. These studies will enrich the branches of Topology, Mathematical Analysis, Fuzzy and Poset Theory.

2. Kinds of Convergences and Metric Spaces

We begin the article with the classical notion of the convergence of sequences in metric spaces, and then we focus on statistical and ideal kinds of convergences.

Definition 1 (see, for example [4]). *A sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space (X,d) converges to a point $x \in X$ if for every $\varepsilon > 0$ there exists a positive integer n_0 such that $x_n \in B(x,\varepsilon)$ for every $n \geqslant n_0$, where $B(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$. In this case we write $\lim\limits_{n\to\infty} x_n = x$ and the point x is called the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. Any sequence $(x_n)_{n\in\mathbb{N}}$ that converges to a point x is called a convergent sequence.*

For example, if $X = \mathbb{R}$ is the set of real numbers and d is the Euclidean metric, then the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to zero. However, the sequence $((-1)^n)_{n=1}^{\infty}$ does not converge to a point $x \in \mathbb{R}$.

Many results for the classical notion of convergence are proven, and a lot of them are used to get new facts. Among them, we state the following:

(1) The limit point of a sequence $(x_n)_{n\in\mathbb{N}}$ is unique.
(2) A sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space X converges to a point $x \in X$ if and only if each subsequence of it converges to x.
(3) Each convergent sequence is a Cauchy sequence.
(4) A subset M of a metric space X is closed if and only if each sequence $(x_n)_{n\in\mathbb{N}}$ of M converges to a point of M.
(5) The meanings of continuous functions, complete metric spaces, and sequentially compact metric spaces are studied through the notion of convergence.

However, in order to enrich the Convergence Theory in Metric Spaces, nowadays, a lot of papers that study the notions of statistical and ideal convergence develop new chapters (see, for instance [5,6]). In particular, the notion of statistical convergence enriches related studies [7–10] and is based on the notion of the asymptotic (natural) density of a set $A \subseteq \mathbb{N}$ as follows:

Let K be a subset of the set of positive integers $\mathbb{N}$ and $K(n) = \{k \leqslant n : k \in K\}$, for all $n \in \mathbb{N}$. The *asymptotic (natural) density* of K is given by the following limit (whenever this limit exists):

$$\delta(K) = \lim_{n\to\infty} \frac{|K(n)|}{n},$$

where $|K(n)|$ denotes the number of elements in $K(n) \subseteq \mathbb{N}$. We will say that the asymptotic density is not zero if it is greater than zero or does not exist. For example, the sets of odd and even numbers have an asymptotic density equal to $\frac{1}{2}$, and the set of prime numbers has an asymptotic density equal to zero. Also, $\delta(\mathbb{N}) = 1$.

Definition 2 (see for example [8,11]). *Let (X,d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. We say that $(x_n)_{n\in\mathbb{N}}$ statistically converges to x if for every $\varepsilon > 0$ we have*

$$\delta(\{n \in \mathbb{N} : d(x_n,x) \geqslant \varepsilon\}) = 0.$$

In this case we write st-lim $x_n = x$.

For example, if we consider the sequence $(x_n)_{n\in\mathbb{N}}$, which is defined as follows:

$$x_n = \begin{cases} 1, \text{ if } 1+2+\ldots+m = n \text{ and } m \in \mathbb{N}, \\ 0, \text{ otherwise,} \end{cases}$$

that is $x_n = (1,0,1,0,0,1,0,0,0,1,\ldots)$, in the metric space $X = \mathbb{R}$ with the Euclidean metric, then the sequence $(x_n)_{n\in\mathbb{N}}$ statistically converges to zero.

We mention the following facts:

(1) The statistic limit point of a sequence $(x_n)_{n\in\mathbb{N}}$ is unique.
(2) Each convergent sequence (under the classical point of view) statistically converges to the same point.
(3) A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space X is statistically convergent to $x \in X$, if and only if there exists a subset A of $\mathbb{N}$ with $\delta(A) = 1$ such that the sequence $(x_n)_{n\in A}$ converges to x [11,12].
(4) In this kind of convergence, we can observe that if a sequence statistically converges, it may not be either convergent or bounded. For example, we consider the sequence $(y_n)_{n\in\mathbb{N}}$, which is defined as follows:

$$y_n = \begin{cases} n, \text{ if } n = k^2 \text{ and } k \in \mathbb{N}, \\ L, \text{ otherwise,} \end{cases}$$

that is $y_n = (1,L,L,4,L,L,L,L,9,\ldots)$, in the metric space $X = \mathbb{R}$ with the Euclidean metric, then the sequence $(y_n)_{n\in\mathbb{N}}$ statistically converges to L, but it is not convergent and not bounded.
(5) Additionally, we can see the statistical convergence as a natural extension of the classical convergence. However, this extension has its own characteristics. For example, it is known that each subsequence of a convergent sequence is convergent, but this fact does not hold in the statistical convergence. We may have a sequence that statistically converges, but it has a subsequence that does not statistically converge. For example, the subsequence $(y_{k^2})_{k\in\mathbb{N}}$ of the sequence $(y_n)_{n\in\mathbb{N}}$, mentioned above, does not statistically converge.

At the same time, the ideal convergence comes to give us a different approach to the notion of convergence. The ideal convergence of sequences of points is based on the notion of the ideal of a set $A \subseteq \mathbb{N}$ [13,14]:

Generally, let D be a non-empty set. A family $\mathcal{I}$ of subsets of D is called *ideal* if $\mathcal{I}$ has the following properties:

(1) $\emptyset \in \mathcal{I}$.
(2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(3) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called *proper* if $D \notin \mathcal{I}$. Moreover, for all $n \in \mathbb{N}$ we set $M_n = \{n' \in \mathbb{N} : n' \geqslant n\}$ [15]. A proper ideal $\mathcal{I}$ of $\mathbb{N}$ is called *admissible*, if $\mathbb{N} \setminus M_n \in \mathcal{I}$, for all $n \in \mathbb{N}$ [16].

Definition 3 ([13,17])**.** *Let $\mathcal{I}$ be a non-trivial ideal on $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space (X,d) is said to be $\mathcal{I}$-convergent to $x \in X$ if for every $\varepsilon > 0$, $A(\varepsilon) = \{n \in \mathbb{N} : d(x_n, x) \geqslant \varepsilon\} \in \mathcal{I}$. In this case, we write $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$ and the point x is said to be the $\mathcal{I}$-limit of the sequence $(x_n)_{n\in\mathbb{N}}$. Any sequence $(x_n)_{n\in\mathbb{N}}$ that $\mathcal{I}$-converges to x is said to be $\mathcal{I}$-convergent.*

For example, let $\mathcal{I}_0 = \{\emptyset\}$. If a sequence $(x_n)_{n\in\mathbb{N}}$, in the space $X = \mathbb{R}$ with the usual topology, $\mathcal{I}_0$-converges, then this sequence is constant. The converse also holds. Another example states that if $\mathcal{I}_f$ denotes the class of all finite subsets of $\mathbb{N}$, then the $\mathcal{I}_f$-convergence coincides with the classical convergence in any metric space (X,d).

It has been proven that the notion of $\mathcal{I}$-convergence satisfies some classical axioms of convergence, some of which are summarized as follows:

(1) Every sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = x$ for each $n \in \mathbb{N}$, $\mathcal{I}$-converges to x.
(2) The $\mathcal{I}$-limit of a sequence $(x_n)_{n\in\mathbb{N}}$ is unique.
(3) If each subsequence y of an $\mathcal{I}$-convergent sequence x has a subsequence z which $\mathcal{I}$-converges to s, then x $\mathcal{I}$-converges to s.
(4) If the ideal $\mathcal{I}$ contains an infinite subset of $\mathbb{N}$, then the assumption that a sequence x $\mathcal{I}$-converges to s does not always imply that each subsequence of x $\mathcal{I}$-converges to s. For example, we suppose that the ideal $\mathcal{I}$ contains an infinite subset A of $\mathbb{N}$. Then $\mathbb{N}\backslash A$ is also an infinite set (in contrast, it is contained in $\mathcal{I}$ and thus, $A \cup (\mathbb{N}\backslash A) = \mathbb{N} \in \mathcal{I}$, which is a contradiction). Let s, t be distinct elements of a metric space (X, d). We consider the following sequence $x = (x_n)_{n\in\mathbb{N}}$:

$$x_n = \begin{cases} s, \text{ if } n \in A \\ t, \text{ if } n \in \mathbb{N} \setminus A. \end{cases}$$

The sequence x $\mathcal{I}$-converges to t but the subsequence y with $y_n = s$ for every $n \in A$ $\mathcal{I}$-converges to s.
(5) $\mathcal{I}$-convergence of real sequences coincides with the classical convergence if $\mathcal{I}$ is the ideal $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ (as mentioned above) and with the statistical convergence if $\mathcal{I}$ is the ideal of subsets of $\mathbb{N}$ of natural density zero.

The notion of $\mathcal{I}^*$-convergence of real sequences is based on the following remark [18]: A real sequence $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to ξ if and only if there exists a set

$$M = \{m_1 < m_2 < m_3 < \ldots < m_k < \ldots\} \subset \mathbb{N}$$

such that $\delta(M) = 1$ and $\lim_k x_{m_k} = \xi$. Extensive work on this concept has been done in [19]. In [13], it was proven that if $\mathcal{I}$ is an admissible ideal and $\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = x$, then $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = x$.

Definition 4 ([14])**.** *Let $\mathcal{I}$ be am admissible ideal of $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ of a metric space (X, d) is said to be $\mathcal{I}^*$-convergent to $x \in X$ if there is a set $H \in \mathcal{I}$ such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \ldots\}$ we have $\lim_{k\to\infty} x_{m_k} = x$. In this case, we write $\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = x$ and the point x is said to be the $\mathcal{I}^*$-limit of the sequence $(x_n)_{n\in\mathbb{N}}$.*

For this kind of convergence, we mention the following facts that shows its relation with the $\mathcal{I}$-convergence:
(1) If (X, d) is a metric space, $\mathcal{I}$ an admissible ideal of $\mathbb{N}$ and $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}^*$-converges to s, then $(x_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to s.
(2) Let (X, d) is a metric space.
   (i) If X has not an accumulation point, then $\mathcal{I}$-convergence coincides with $\mathcal{I}^*$-convergence, for any admissible ideal $\mathcal{I}$.
   (ii) If X has an accumulation point s, then there exists an admissible ideal $\mathcal{I}$ of $\mathbb{N}$ and a sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $(y_n)_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to s but the $\mathcal{I}^*$-limit point of $(y_n)_{n\in\mathbb{N}}$ does not exist.

Continuing the investigation of convergence in metric spaces, we see that in [20], the authors introduced and studied statistical analogs for convergences of sequences of sets followed by [21]. We use the same notations $\mathcal{S} = \{N \subseteq \mathbb{N} : \delta(N) = 1\}$ and $\mathcal{S}^\# = \{N \subseteq \mathbb{N} : \delta(N) \neq 0\}$.

If (X, d) is a metric space and $(C_n)_{n\in\mathbb{N}}$ is a sequence of closed subsets of X, then the *statistical inner limit* and the *statistical outer limit* of $(C_n)_{n\in\mathbb{N}}$ are defined, respectively, as follows:
(1) $\text{st-Li}C_n = \{x \in X : \forall U \in \mathcal{N}(x), \exists N \in \mathcal{S}^\#, \forall n \in N, C_n \cap U \neq \emptyset\}$ and

(2) st-Ls$C_n = \{x \in X : \forall U \in \mathcal{N}(x), \exists N \in \mathcal{S}, \forall n \in N, C_n \cap U \neq \emptyset\}$, where $\mathcal{N}(x)$ denotes the set of all neighborhoods of x [11].

Definition 5 ([11,20])**.** *In case that* st-LiC_n = st-Ls$C_n = C$*, we say that* $(C_n)_{n\in\mathbb{N}}$ *is Kuratowski statistical convergent to* C *and we write* st-lim $C_n = C$.

Definition 6 ([11,20])**.** *Let* (X, d) *be a metric space,* $(C_n)_{n\in\mathbb{N}}$ *be a sequence of nonempty closed subsets of* X *and* C *be a nonempty closed subset of* X*. We say that the sequence* $(C_n)_{n\in\mathbb{N}}$ *is Wijsman statistically convergent to* C *if the sequence* $(d(x, C_n))_{n\in\mathbb{N}}$ *is statistically convergent to* $d(x, C)$*, for all* $x \in X$*; i.e., for each* $\varepsilon > 0$ *and for each* $x \in X$*,*

$$\delta(\{n \in \mathbb{N} : |d(x, C_n) - d(x, C)| \geqslant \varepsilon\}) = 0.$$

In this case we write $\text{st}_W\text{-lim}\, C_n = C$.

Moreover, for a metric space (X, d) the Hausdorff distance $h(E, F)$ between the subsets E and F of X is given by:

$$h(E, F) = \max\{\sup_{e\in E} d(e, F), \sup_{f\in F} d(f, E)\}.$$

We have $h(\emptyset, \emptyset) = 0$ and $h(E, \emptyset) = 1$, whenever $E \neq \emptyset$ [22]. When E and F are closed, then $h(E, F) = 0$ if and only if $E = F$. Therefore, h can be considered as an "extended" metric on the collection of closed subsets of X. In general, h is not a metric because it may be infinite [11].

Definition 7 ([11,20])**.** *Let* (X, d) *be a metric space,* $(C_n)_{n\in\mathbb{N}}$ *be a sequence of nonempty closed subsets of* X *and* C *be a nonempty closed subset of* X*. We say that the sequence* $(C_n)_{n\in\mathbb{N}}$ *is Hausdorff statistically convergent to* C *if for every* $\varepsilon > 0$ *there exists* $N \in \mathcal{S}$ *such that* $h(C_n, C) < \varepsilon$*, for all* $n \in \mathbb{N}$*. In this case, we write* $\text{st}_H\text{-lim}\, C_n = C$.

Hausdorff statistical convergence implies Wijsman statistical convergence to the same limit [20]. Moreover, for a metric space (X, d), if $(C_n)_{n\in\mathbb{N}}$ is a sequence of closed subsets of X and C is a nonempty closed subset of X, then $\text{st}_H\text{-lim}\, C_n = C$ implies st-lim $C_n = C$. In [11], criteria for Kuratowski, Wijsman, and Hausdorff statistical convergences are given.

Proposition 1 ([11])**.** *Let* (X, d) *be a metric space,* $(C_n)_{n\in\mathbb{N}}$ *be a sequence of nonempty closed subsets of* X *and* C *be a nonempty closed subset of* X*. Then* C = st-lim C_n *if and only if both of the following conditions are satisfied:*

(1) For each $x \in C$ *there exist points* $x_n \in C_n$*, for each* $n \in \mathbb{N}$*, such that the sequence* $(x_n)_{n\in\mathbb{N}}$ *statistically converges to* x*;*

(2) Whenever there exist points $x_n \in C_n$*, for each* $n \in \mathbb{N}$*, such that the sequence* $(x_n)_{n\in\mathbb{N}}$ *has* x *a statistical cluster point, then* $x \in C$.

Definition 8 ([11])**.** *Let* (X, d) *be a metric space,* $(C_n)_{n\in\mathbb{N}}$ *be a sequence of nonempty closed subsets of* X *and* C *be a nonempty closed subset of* X*. We will say that* $(C_n)_{n\in\mathbb{N}}$ *is*

(1) lower Wijsman statistically convergent to C *if* $d(x, C) \geqslant \text{st-lim sup}\, d(x, C_n)$*, for all* $x \in X$ *and*

(2) upper Wijsman statistically convergent to C *if* $d(x, C) \geqslant \text{st-lim inf}\, d(x, C_n)$*, for all* $x \in X$.

Proposition 2 ([11])**.** *Let* (X, d) *be a metric space,* $(C_n)_{n\in\mathbb{N}}$ *be a sequence of nonempty closed subsets of* X *and* C *be a nonempty closed subset of* X*. Consider the following conditions:*

(i) For every open ball $B(x, \varepsilon)$ *with* $C \cap B(x, \varepsilon) \neq \emptyset$*, there exists* $N \in \mathcal{S}$ *such that* $C_n \cap B(x, \varepsilon) \neq \emptyset$*, for all* $n \in N$*;*

(ii) *If $0 < \eta < \varepsilon$, then for every open ball $B(x,\varepsilon)$ with $C \cap B(x,\varepsilon) = \emptyset$, there exists $N \in \mathcal{S}$ such that $C_n \cap B(x,\varepsilon) = \emptyset$, for all $n \in N$.*
Then $(C_n)_{n\in\mathbb{N}}$ is lower (resp. upper) Wijsman statistically convergent to C if and only if condition (i) (resp. (ii)) is satisfied.

Definition 9 ([11])**.** *Let (X,d) be a metric space, $(C_n)_{n\in\mathbb{N}}$ be a sequence of nonempty closed subsets of X and C be a nonempty closed subset of X. We will say that $(C_n)_{n\in\mathbb{N}}$ is*

(1) *lower Hausdorff statistically convergent to C if for every $\varepsilon > 0$ there exists $L \in \mathcal{S}$ such that for all $l \in L$, $C \subseteq B(C_l,\varepsilon)$ and*
(2) *upper Hausdorff statistically convergent to C if for every $\varepsilon > 0$, there exists $M \in \mathcal{S}$ such that for all $m \in M$, $C_m \subseteq B(C,\varepsilon)$.*
Consequently, $(C_n)_{n\in\mathbb{N}}$ is Hausdorff statistically convergent to C if and only if it is both lower and upper Hausdorff statistically convergent to C.

Proposition 3 ([11])**.** *Let (X,d) be a metric space, $(C_n)_{n\in\mathbb{N}}$ be a sequence of nonempty closed subsets of X and C be a nonempty closed subset of X. Consider the following conditions:*

(i) *For every open ball $B(x,\varepsilon)$ with $C \cap B(x,\varepsilon) \neq \emptyset$, there exists $N \in \mathcal{S}$ such that $C_n \cap B(x,\varepsilon) \neq \emptyset$, for all $n \in N$;*
(ii) *For every open ball U with $C \subseteq U$, there exists $N \in \mathcal{S}$ such that $C_n \subseteq U$, for all $n \in N$.*
Then $(C_n)_{n\in\mathbb{N}}$ is lower (resp. upper) Hausdorff statistically convergent to C if and only if condition (i) (resp. (ii)) is satisfied.

In the same paper, the relations between Kuratowski, Wijsman, and Hausdorff statistical convergences are given.

Proposition 4 ([11])**.** *Let (X,d) be a metric space. For a sequence of nonempty closed subsets $(C_n)_{n\in\mathbb{N}}$ of X and a nonempty closed subset C of X, whenever $(C_n)_{n\in\mathbb{N}}$ Wijsman statistically converges to C, then $(C_n)_{n\in\mathbb{N}}$ Kuratowski statistically converges to C. If we additionally suppose that X is a proper metric space, then the converse also holds.*

Proposition 5 ([11])**.** *Let (X,d) be a metric space. For a sequence of nonempty closed subsets $(C_n)_{n\in\mathbb{N}}$ of X and a nonempty closed subset C of X, if $(C_n)_{n\in\mathbb{N}}$ is lower (resp. upper) Hausdorff statistically convergent to C, then $(C_n)_{n\in\mathbb{N}}$ is lower (resp. upper) Wijsman statistically convergent to C.*

Further, in [11], the notion of statistical Cauchyness, with respect to the Hausdorff "extended" metric h, is used in order to study the relation between lower and upper Hausdorff statistical convergences with this statistical Cauchyness.

Definition 10 ([11])**.** *Let (X,d) be a metric space, $(C_n)_{n\in\mathbb{N}}$ be a sequence of nonempty closed subsets of X and C be a nonempty closed subset of X. We will say that $(C_n)_{n\in\mathbb{N}}$ is statistically Cauchy, with respect to the Hausdorff "extended" metric h if for arbitrary $\varepsilon > 0$, there exists $M \in \mathcal{S}$ such that for all $m,n \in M$, $h(C_m,C_n) < \varepsilon$.*

Proposition 6 ([11])**.** *Let (X,d) be a metric space and $(C_n)_{n\in\mathbb{N}}$ be a statistically Cauchy sequence, with respect to the Hausdorff "extended" metric h, of nonempty closed subsets of X. Then, the following hold:*
(1) There exists a Cauchy sequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \in C_{n_k}$, for all $k \in \mathbb{N}$.
(2) If every such sequence has a cluster point in X (i.e., converges in X), then $(C_n)_{n\in\mathbb{N}}$ is upper Hausdorff statistically convergent to LsC_n.

We consider the space of nonempty closed bounded subsets of X with respect to the Hausdorff metric h. Hausdorff statistical convergence can be considered statistical convergence with respect to the Hausdorff metric h, which we will call *statistical h-convergence*. Then:

(1) every statistically h-convergent sequence in this space is Kuratowski statistically convergent to the same limit [21];
(2) the set of all Hausdorff statistically convergent (i.e., statistically h-convergent) sequences coincides with the set of all statistically Cauchy sequences in this space if X is complete [23];
(3) every statistically Cauchy sequence in this space statistically h-converges and, therefore, Kuratowski statistically converges to its outer limit [11].

In [24], two kinds of $\mathcal{I}$-convergence for sequences of closed sets, which are called Kuratowski $\mathcal{I}$-convergence and Hausdorff $\mathcal{I}$-convergence, are introduced. For that, the authors of [24] consider the sets $\mathcal{N}_{\mathcal{I}} = \{N \subseteq \mathbb{N} : \mathbb{N} \setminus N \in \mathcal{I}\}$ and $\mathcal{N}_{\mathcal{I}}^{\#} = \{N \subseteq \mathbb{N} : N \notin \mathcal{I}\}$.

Definition 11 ([24])**.** *The $\mathcal{I}$-outer limit and $\mathcal{I}$-inner limit of a sequence $(C_n)_{n\in\mathbb{N}}$ of closed subsets of a metric space X are defined as follows:*

$$\mathcal{I} - \limsup_n C_n = \{x : \forall \varepsilon > 0, \exists N \in \mathcal{N}_{\mathcal{I}}^{\#}, \forall n \in N, C_n \cap B(x,\varepsilon) \neq \emptyset\}$$

and

$$\mathcal{I} - \liminf_n C_n = \{x : \forall \varepsilon > 0, \exists N \in \mathcal{N}_{\mathcal{I}}, \forall n \in N, C_n \cap B(x,\varepsilon) \neq \emptyset\}.$$

The $\mathcal{I}$-limit of $(C_n)_{n\in\mathbb{N}}$ exists if its $\mathcal{I}$-outer and $\mathcal{I}$-inner limits coincide. In this situation, we say that the sequence $(C_n)_{n\in\mathbb{N}}$ is Kuratowski $\mathcal{I}$-convergent and we write $\mathcal{I} - \limsup_n C_n = \mathcal{I} - \liminf_n C_n = \mathcal{I} - \lim_n C_n$.

Definition 12 ([24])**.** *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of a metric space X. We say that the sequence $(C_n)_{n\in\mathbb{N}}$ is Hausdorff $\mathcal{I}$-convergent to a closed subset C of X if $\mathcal{I} - \lim_n h(C_n, C) = 0$.*

In the same paper, [24], the properties of these notions of convergences are studied. Among them, we mention that every Kuratowski convergent sequence is Kuratowski $\mathcal{I}$-convergent and the following result.

Proposition 7 ([24])**.** *If $\{C, C_n : n \in \mathbb{N}\}$ is a family of closed subsets of a metric space X with $C \neq \emptyset$, then Hausdorff $\mathcal{I}$-convergence implies Kuratowski $\mathcal{I}$-convergence.*

Question 1. *Can we develop a study of Kuratowski $\mathcal{I}$-convergence for sequences of closed sets in topological spaces, where $\mathcal{I}$ is an ideal of the set of natural numbers?*

Question 2. *Can we define the Wijsman $\mathcal{I}$-convergence for sequences of closed sets in metric spaces? Which are its basic properties?*

3. Kinds of Convergences in Topological Spaces

It is known that from a classical topological point of view, the convergence of sequences is not able to describe essential topological properties because the convergence of sequences in a space does not uniquely determine its topology [4,25,26]. However, the convergence of nets and filters managed to successfully address the above problem. More precisely, the nets give a different road in order to study convergences in topological spaces. We have the so-called Moore–Smith sequences by Moore–Smith [27] and their study, in the view of convergence, in topological spaces by Birkhoff [19]. We state that J. Kelley [4,28] provides the right description of convergence in topological spaces under the prism of nets.

In order to present results that have been studied for the notion of convergence in topological spaces, we recall some of the most basic meanings (see for example [29]).

A net in a set X is an arbitrary function s from a non-empty directed set D to X. If $s(d) = s_d$, for all $d \in D$, then the net s will be denoted by $(s_d)_{d\in D}$.

A net $(t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *semisubnet* of the net $(s_d)_{d\in D}$ in X if there exists a function $\varphi : \Lambda \to D$ such that $t = s \circ \varphi$. We write $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ to indicate the fact that φ is the function mentioned above.

A net $(t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *subnet* of the net $(s_d)_{d\in D}$ in X if there exists a function $\varphi : \Lambda \to D$ with the following properties:

(1) $t = s \circ \varphi$, or equivalently, $t_\lambda = s_{\varphi(\lambda)}$ for every $\lambda \in \Lambda$.
(2) For every $d \in D$ there exists $\lambda_0 \in \Lambda$ such that $\varphi(\lambda) \geqslant d$ whenever $\lambda \geqslant \lambda_0$.

Definition 13 (see for example [4,25,30])**.** *We say that a net $(s_d)_{d\in D}$ converges to a point $x \in X$ if for every open neighborhood U of x there exists $d_0 \in D$ such that $x \in U$ for all $d \geqslant d_0$. In this case, we write $\lim_{d\in D} s_d = x$. The point x is called the limit of the net $(s_d)_{d\in D}$.*

For example, we consider a topological space X, $x \in X$ and $\beta(x)$ a base of X at the point x with the partial order given as: $U_1 \leqslant U_2$ if and only if $U_2 \subseteq U_1$, for $U_1, U_2 \in \beta(x)$. In each $U \in \beta(x)$ we assign an element x_U of X such that $x_U \in U$. Then we have a net $\{x_U : U \in \beta(x)\}$ which converges to x.

It is important to mention that convergence in topological spaces is one of the most basic notions in Topology. In contrast to metric spaces, the sequences of points of general topological spaces do not determine essential topological characteristics. For example, it is known that for a function $f : X \to Y$ of metric spaces X and Y, the following are equivalent:

(i) f is continuous,
(ii) For each $x \in X$, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in the metric space X which converges to x, then the sequence $(f(x_n))_{n\in\mathbb{N}}$ in the metric space Y converges to $f(x)$.

However, in topological spaces, the first condition implies the second condition, but in general, the converse does not hold. Many properties of metric spaces can not be extended to topological spaces. A different approach is to consider the convergence of nets and not sequences of points. As we have mentioned, the nets are defined in order to solve problems of sequences.

Inserting the meaning of net convergence, the classical meaning of convergence is extended. Then a natural question arises: *"Can we characterize the class of topological spaces that can be determined by their convergent sequences?"* V. Ponpmarev and A. Archangel'skii studied this problem. Then many mathematicians also studied this question [31–33]. Without a doubt, the contribution of J. Kelley to the convergence theory of nets is very important. He gave a standard characterization for a class of "convergent" nets in order to be topological [4].

Definition 14 ([4,29])**.** *Let X be a non-empty set and let $\mathcal{C}$ be a class consisting of pairs (s, x), where $s = (s_d)_{d\in D}$ is a net in X and $x \in X$. We say that $\mathcal{C}$ is a convergence class for X if it satisfies the conditions listed below.*

For convenience, we say that s converges $(\mathcal{C})$ to x or that $\lim_{d\in D} s_d \equiv x(\mathcal{C})$ if and only if $(s, x) \in \mathcal{C}$.

(C1) *If $(s_d)_{d\in D}$ is a net such that $s_d = x$, for every $d \in D$, then s converges $(\mathcal{C})$ to x.*
(C2) *If s converges $(\mathcal{C})$ to x, then so does each subnet of s.*
(C3) *If s does not converge $(\mathcal{C})$ to x, then there exists a subnet of s, no subnet of which converges $(\mathcal{C})$ to x.*
(C4) *We consider the following:*
  (1) *D is a directed set.*
  (2) *E_d is a directed set, for each $d \in D$.*
  (3) *$(s(d, e))_{e\in E_d}$ is a net from E_d to X, for each $d \in D$.*
  (4) *$\lim_{d\in D} t_d \equiv x(\mathcal{C})$, where $\lim_{e\in E_d} s(d, e) \equiv t_d(\mathcal{C})$, for every $d \in D$.*

Then, the net $r : D \times \prod_{d \in D} E_d \to X$*, where* $r(d, f) = s(d, f(d))$*, for every* $(d, f) \in D \times \prod_{d \in D} E_d$*, converges* $(\mathcal{C})$ *to* x*.*

The class of all convergence classes for X *is denoted by* $Con(X)$*.*

The class of convergent nets in X "forms" a convergence class. In the opposite direction, a convergence class $\mathcal{C}$ determines a topology on X as follows:

Theorem 1 ([4,29])**.** *(Convergence classes theorem) Let* $\mathcal{C}$ *be a convergence class for a set* X *and for each subset* A *of* X *let* $\mathrm{cl}(A)$ *be the set of all points* x *such that, for some net* s *in* A*,* s *converges* $(\mathcal{C})$ *to* x*. Then* cl *is a closure operator for a topology* $\tau(\mathcal{C})$ *on* X*, and* $(s, x) \in \mathcal{C}$ *if and only if* s *converges to* x *with respect to* $\tau(\mathcal{C})$*.*

In the study, [29], a modification of the above Kelley's theorem is introduced, giving an alternative approach to the meaning of convergent nets.

Definition 15 ([29])**.** *Let* X *be a non-empty set and let* $\mathcal{C}'$ *be a class consisting of pairs* (s, x)*, where* $s = (s_d)_{d \in D}$ *is a net in* X *and* $x \in X$*. We say that* $\mathcal{C}'$ *is a semi-convergence class for* X *if it satisfies the conditions listed below.*
For convenience, we say that s *semi-converges* $(\mathcal{C}')$ *to* x *or that* $\lim_{d \in D} s_d \equiv_s x(\mathcal{C}')$ *if and only if* $(s, x) \in \mathcal{C}'$*.*

- $(C'1)$ *If* $(s_d)_{d \in D}$ *is a net such that* $s_d = x$*, for every* $d \in D$*, then* s *semi-converges* $(\mathcal{C}')$ *to* x*.*
- $(C'2)$ *If* s *semi-converges* $(\mathcal{C}')$ *to* x*, then so does each subnet of* s*.*
- $(C'3)$ *If* s *does not semi-converge* $(\mathcal{C}')$ *to* x*, then there exists a subnet of* s*, no subnet of which semi-converges* $(\mathcal{C}')$ *to* x*.*
- $(C'4)$ *We consider the following:*
  - *(1)* D *is a directed set.*
  - *(2)* E_d *is a directed set, for each* $d \in D$*.*
  - *(3)* $(s(d, e))_{e \in E_d}$ *is a net from* E_d *to* X*, for each* $d \in D$*.*
  - *(4)* $\lim_{d \in D} t_d \equiv_s x(\mathcal{C}')$*, where* $\lim_{e \in E_d} s(d, e) \equiv_s t_d(\mathcal{C}')$*, for every* $d \in D$*.*

Then, the net $r : D \times \prod_{d \in D} E_d \to X$*, where* $r(d, f) = s(d, f(d))$*, for every* $(d, f) \in D \times \prod_{d \in D} E_d$*, semi-converges* $(\mathcal{C}')$ *to* x*.*

The class of all semi-convergence classes for X *is denoted by* $Con^s(X)$*.*

The following theorem sets up a one-to-one correspondence between the topologies for a set X and the semi-convergence classes on it.

Theorem 2 ([29])**.** *Let* $\mathcal{C}'$ *be a semi-convergence class for a set* X *and for each subset* A *of* X *let* $\mathrm{cl}(A)$ *be the set of all points* x *such that, for some net* s *in* A*,* s *semi-converges* $(\mathcal{C}')$ *to* x*. Then* cl *is a closure operator for a topology* $\tau^s(\mathcal{C}')$ *on* X*, and* $(s, x) \in \mathcal{C}'$ *if and only if* s *semi-converges to* x *with respect to* $\tau^s(\mathcal{C}')$*.*

Proposition 8 ([29])**.** *Let* X *be a non-empty set. Then*

(1) $Con(X) = Con^s(X)$ *and*
(2) $\{\tau(\mathcal{C}) : \mathcal{C} \in Con(X)\} = \{\tau^s(\mathcal{C}') : \mathcal{C}' \in Con^s(X)\}$*.*

At the same time, the meaning of the ideal convergence in topological spaces also attracts the interest of its investigation. This kind of convergence can be considered a natural extension of the corresponding ideal convergence in metric spaces.

In order to present the related studies, we recall some basic notations. Firstly, we suppose that $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ is a subnet of the net $(s_d)_{d \in D}$ in X. For every ideal $\mathcal{I}$ of the directed set D, the family $\{A \subseteq \Lambda : \varphi(A) \in \mathcal{I}\}$ is an ideal of the directed set Λ, which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

Let D be a directed set. For all $d \in D$ we set $M_d = \{d' \in D : d' \geqslant d\}$ [15]. A proper ideal $\mathcal{I}$ of a directed set D is called *admissible*, if $D \setminus M_d \in \mathcal{I}$, for all $d \in D$ [16]. Further, the set $\mathcal{I}_0(D) = \{A \subseteq D : A \subseteq D \setminus M_d \text{ for some } d \in D\}$ is a proper ideal of D [15].

Definition 16 ([15,34])**.** *Let X be a topological space. A net $(s_d)_{d\in D}$ in X is said to $\mathcal{I}$-(topologically)-converge to a point $x \in X$, where $\mathcal{I}$ is an ideal of D, if for every open neighborhood U of x, $\{d \in D : s_d \notin U\} \in \mathcal{I}$. In this case the point x is called the $\mathcal{I}$-limit of the net $(s_d)_{d\in D}$. In this case we write $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ (or simply, $(s_d)_{d\in D} \xrightarrow{\mathcal{I}} x$ either $\mathcal{I}\text{-}\lim_{d\in D} s_d = x$).*

For example, we consider the topological space (X, τ), where $X = \{a, b, c, d\}$ and

$$\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}.$$

If D is the set of all neighborhoods of $a \in X$, then D is directed by the order $\geqslant$ defined as follows: $U \geqslant V$ if and only if $U \subseteq V$, for $U, V \in D$. If we consider the ideal $\mathcal{I} = \{\emptyset, \{\{a\}\}, \{\{a, c\}\}, \{\{a\}, \{a, c\}\}$, then we define the net $s : D \to X$ as follows:

$$s_U = \begin{cases} d, & \text{if } U = \{a\}, \text{ or } \{a, c\}, \\ a, & \text{otherwise.} \end{cases}$$

If we consider any neighborhood U of a, then the set $\{d \in D : s_d \notin U\}$ is either $\emptyset \in \mathcal{I}$ or $\{\{a\}, \{a, c\}\} \in \mathcal{I}$. Thus, the net $(s_d)_{d\in D}$ $\mathcal{I}$-converges to a.

If $\mathcal{I}$ is D-admissible, then the net convergence in the topology τ implies its $\mathcal{I}$-convergence and the converse holds if $\mathcal{I} = \mathcal{I}_0(D)$, that is, $\mathcal{I}_0(D)$-convergence implies the net convergence. Also, some basic known results for the $\mathcal{I}$-convergence of nets are given as follows:

(1) If X is Hausdorff, then an $\mathcal{I}$-convergent net has a unique $\mathcal{I}$-limit.
(2) If every $\mathcal{I}$-convergent net in X has a unique $\mathcal{I}$-limit for every D-admissible ideal $\mathcal{I}$, then X is Hausdorff.

Definition 17 ([15,34])**.** *Let X be a topological space. A net $(s_d)_{d\in D}$ in X is said to $\mathcal{I}^*$-convergent to $x \in X$ if there exists a directed set M such that $D \setminus M \in \mathcal{I}$ and the net $(s_d)_{d\in M}$ is convergent to x. In this case, we write $\mathcal{I}^*\text{-}\lim_{d\in D} s_d = x$ and the point x is called the $\mathcal{I}^*$-limit of $(s_d)_{d\in D}$.*

If $\mathcal{I}$ is D-admissible, then $\mathcal{I}^*\text{-}\lim_{d\in D} s_d = x$ implies $\mathcal{I}\text{-}\lim_{d\in D} s_d = x$. In addition, if X has no limit point, then $\mathcal{I}$ and $\mathcal{I}^*$ convergence coincides for every D-admissible ideal $\mathcal{I}$. In the same paper [15] the authors investigated whenever $\mathcal{I}$-convergence of a net may imply its $\mathcal{I}^*$-convergence, introducing the so-called DP-condition.

Proposition 9 ([15])**.** *Let $\mathcal{I}$ be a D-admissible ideal of a directed set $(D, \geqslant)$.*

(i) If $\mathcal{I}$ satisfies the condition (DP) and X is a first axiom space, then for any net $(s_d)_{d\in D}$ in X, $\mathcal{I}\text{-}\lim_{d\in D} s_d = x$ implies $\mathcal{I}^*\text{-}\lim_{d\in D} s_d = x$.
(ii) Conversely if (X, τ) is a first axiom Hausdorff space containing at least one limit point and for each $x \in X$ and any net $(s_d)_{d\in D}$, $\mathcal{I}\text{-}\lim_{d\in D} s_d = x$ implies $\mathcal{I}^*\text{-}\lim_{d\in D} s_d = x$, then $\mathcal{I}$ satisfies the condition (DP).

In the study, [30], based on the concept of ideal convergence of nets in topological spaces, a modification of Kelley's theorem (Theorem 1) is presented.

Definition 18 ([30]). *Let X be a non-empty set and let $\mathcal{C}$ be a class consisting of triads $(s, x, \mathcal{I})$, where $s = (s_d)_{d \in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is an ideal of D. We say that the net s $\mathcal{I}$-converges $(\mathcal{C})$ to x if $(s, x, \mathcal{I}) \in \mathcal{C}$. We write $\mathcal{I}\text{-}\lim_{d \in D} s_d \equiv x(\mathcal{C})$.*

Definition 19 ([30]). *Let X be a non-empty set and let $\mathcal{C}$ be a class consisting of triads $(s, x, \mathcal{I})$, where $s = (s_d)_{d \in D}$ is a net in X, $x \in X$ and $\mathcal{I}$ is an ideal of D. We say that $\mathcal{C}$ is an ideal-convergence class for X if it satisfies the conditions listed below:*

(C1) *If $(s_d)_{d \in D}$ is a net such that $s_d = x$, for every $d \in D$ and $\mathcal{I}$ is an ideal of D, then $\mathcal{I}$-$\lim_{d \in D} s_d \equiv x(\mathcal{C})$.*

(C2) *If $\mathcal{I}_0(D)$-$\lim_{d \in D} s_d \equiv x(\mathcal{C})$, where $\mathcal{I}$ is an ideal of D, then for every subnet $(t_\lambda)_{\lambda \in \Lambda}$ of the net $(s_d)_{d \in D}$ we have $\mathcal{I}_0(\Lambda)$-$\lim_{\lambda \in \Lambda} t_\lambda \equiv x(\mathcal{C})$.*

(C3) *If $\mathcal{I} - \lim_{d \in D} s_d \equiv x(\mathcal{C})$, where $\mathcal{I}$ is an ideal of D, then for every semisubnet $(t_\lambda)_{\lambda \in \Lambda}$ of the net $(s_d)_{d \in D}$ we have $\mathcal{I}_\Lambda(\varphi)$-$\lim_{\lambda \in \Lambda} \equiv x(\mathcal{C})$.*

(C4) *If $\mathcal{I} - \lim_{d \in D} s_d \equiv x(\mathcal{C})$, where $\mathcal{I}$ is a proper ideal of D, then there exists a semisubnet $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the net $(s_d)_{d \in D}$ such that $\mathcal{I}_0(\Lambda)$-$\lim_{\lambda \in \Lambda} t_\lambda \equiv x(\mathcal{C})$.*

(C5) *Let D be a directed set and $\mathcal{I}$ a D-admissible ideal of D. If $(s_d)_{d \in D}$ does not $\mathcal{I}$-converge $(\mathcal{C})$ to x, then there exists a semisubnet $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ of the net $(s_d)_{d \in D}$ such that:*

   (1) $\Lambda \subseteq D$.
   (2) $\varphi(\lambda) = \lambda$, for every $\lambda \in \Lambda$.
   (3) No semisubnet $(r_k)^{f}_{k \in K}$ of $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ $\mathcal{I}_K$-converges to x, for every proper ideal $\mathcal{I}_K$ of K.
   (4) $\mathcal{I}_\Lambda(\varphi)$ is a proper and Λ-admissible ideal of Λ.

(C6) *We suppose the following:*

   (1) D is a directed set.
   (2) $\mathcal{I}_D$ is a proper ideal of D.
   (3) E_d is a directed set, for each $d \in D$.
   (4) $\mathcal{I}_{E_d}$ is a proper ideal of E_d, for each $d \in D$.
   (5) $(s(d,e))_{e \in E_d}$ is a net from E_d to X, for each $d \in D$.
   (6) $\mathcal{I}_D$-$\lim_{d \in D}(\mathcal{I}_{E_d}$-$\lim_{d \in E_d} s(d,e)) = x$.

   Then, the net $r : D \times \prod_{d \in D} E_d \to X$, where $r(d, f) = s(d, f(d))$, for every $(d, f) \in D \times \prod_{d \in D} E_d$, $\mathcal{I}_D \times \mathcal{I}_F$-converges to x, where $F = \prod_{d \in D} E_d$.

   The class of all ideal convergence classes for X is denoted by $Con_I(X)$.

Theorem 3 ([30]). *Let $\mathcal{C}$ be an ideal-convergence class for a set X. We consider the function $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathrm{cl}(A)$ is the set of all $x \in X$ such that, for some net $(s_d)_{d \in D}$ in A and a proper ideal $\mathcal{I}$ of the directed set D, $(s_d)_{d \in D}$ $\mathcal{I}$-converges $(\mathcal{C})$ to x. Then cl is a closure operator for a topology $\tau_I(\mathcal{C})$ on X and $((s_d)_{d \in D}, x, \mathcal{I}) \in \mathcal{C}$, where $\mathcal{I}$ is a proper D-admissible ideal, if and only if $(s_d)_{d \in D}$ $\mathcal{I}$-converges to x with respect to $\tau_I(\mathcal{C})$.*

In Theorem 3 it was proven that ideal convergence $(\mathcal{C})$, over proper D-admissible ideals, coincides with the ideal convergence relative to the topology $\tau_I(\mathcal{C})$. As it was observed in [29], the class $\mathcal{C}_\tau$ consisting of triads $((s_d)_{d \in D}, x, \mathcal{I})$, where $(s_d)_{d \in D}$ is a net in X, $x \in X$, $\mathcal{I}$ is an ideal of D and $(s_d)_{d \in D}$ $\mathcal{I}$-converges to x with respect to τ, is an ideal convergence class since it satisfies all the conditions of Definition 19. We say that the topology τ generates the ideal convergence class $\mathcal{C}_\tau$.

Proposition 10 ([29]). *Let $\mathcal{C}$ be an ideal convergence class for a set X and let $\mathcal{C}_{\tau_I(\mathcal{C})}$ be the ideal convergence class generated from the topology $\tau_I(\mathcal{C})$. Then, $\mathcal{C} \subseteq \mathcal{C}_{\tau_I(\mathcal{C})}$.*

Proposition 11 ([29]). *Let X be a non-empty set. There exists a map m of $Con_I(X)$ onto $Con(X)$, such that for every $\mathcal{C} \in Con_I(X)$ the following properties hold:*

(1) $\tau_I(\mathcal{C}) = \tau(m(\mathcal{C}))$,

(2) *$m(\mathcal{C})$ can be considered as a subclass of the class $\mathcal{C}$ in the sense that there exists a one-to-one map $e : m(\mathcal{C}) \to \mathcal{C}$ and each $((s_d)_{d\in D}, x) \in m(\mathcal{C})$ is identified with $e((s_d)_{d\in D}, x) \in \mathcal{C}$.*

In the study [29], the concept of ideal-semi-convergence classes on a non-empty set X is introduced in order to obtain a different modification of classical Kelley's theorem (Theorem 1).

Definition 20 ([29]). *Let X be a non-empty set and let $\mathcal{C}'$ be a class consisting of triads $(s, x, \mathcal{I})$, where $s = (s_d)_{d\in D}$ is a net in X, $x \in X$ and $\mathcal{I}$ is an ideal of D. We say that $\mathcal{C}'$ is an ideal semi-convergence class for X if it satisfies the conditions listed below.*

For convenience, we say that s $\mathcal{I}$-semi-converges $(\mathcal{C}')$ to x or that $\mathcal{I} - \lim\limits_{d\in D} s_d \equiv_s x(\mathcal{C}')$ if and only if $(s, x, \mathcal{I}) \in \mathcal{C}'$.

$(C'1)$ *If $(s_d)_{d\in D}$ is a net such that $s_d = x$, for every $d \in D$ and $\mathcal{I}$ is an ideal of D, then $\mathcal{I}\text{-}\lim\limits_{d\in D} s_d \equiv_s x(\mathcal{C}')$.*

$(C'2)$ *If $\mathcal{I}\text{-}\lim\limits_{d\in D} s_d \equiv_s x(\mathcal{C}')$, where $\mathcal{I}$ is an ideal of D, then for every semisubnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the net $(s_d)_{d\in D}$ we have $\mathcal{I}_\Lambda(\varphi)\text{-}\lim\limits_{\lambda\in\Lambda} t_\lambda \equiv_s x(\mathcal{C}')$.*

$(C'3)$ *If $\mathcal{I} - \lim\limits_{d\in D} s_d \equiv_s x(\mathcal{C}')$, where $\mathcal{I}$ is a proper ideal of of D, then there exists a semisubnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the net $(s_d)_{d\in D}$ such that $\mathcal{I}_0(\Lambda)\text{-}\lim\limits_{\lambda\in\Lambda} t_\lambda \equiv_s x(\mathcal{C}')$.*

$(C'4)$ *Let D be a directed set and $\mathcal{I}$ a proper ideal of D. If $(s_d)_{d\in D}$ does not $\mathcal{I}$-semi-converge $(\mathcal{C}')$ to x, then there exists a semisubnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the net $(s_d)_{d\in D}$ such that:*

(1) *No semisubnet $(r_k)^{f}_{k\in K}$ of $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ $\mathcal{I}_K$-semi-converges $(\mathcal{C}')$ to x, for every proper ideal $\mathcal{I}_K$ of K.*

(2) *$\mathcal{I}_\Lambda(\varphi)$ is a proper ideal of Λ.*

$(C'5)$ *We consider the following:*

(1) *D is a directed set.*

(2) *$\mathcal{I}_0(D)$ is a proper ideal of D.*

(3) *E_d is a directed set, for each $d \in D$.*

(4) *$\mathcal{I}_0(E_d)$ is a proper ideal of E_d, for each $d \in D$.*

(5) *$(s(d,e))_{e\in E_d}$ is a net from E_d to X, for each $d \in D$.*

(6) *$\mathcal{I}_0(D)\text{-}\lim\limits_{d\in D} t_d \equiv_s x(\mathcal{C}')$, where $\mathcal{I}_0(E_d)\text{-}\lim\limits_{e\in E_d} s(d,e) \equiv_s t_d(\mathcal{C}')$, for every $d \in D$.*

Then, the net $r : D \times \prod\limits_{d\in D} E_d \to X$, where $r(d, f) = s(d, f(d))$, for every $(d, f) \in D \times \prod\limits_{d\in D} E_d$, $\mathcal{I}_0(D \times \prod\limits_{d\in D} E_d)$-semi-converges $(\mathcal{C}')$ to x.

$(C'6)$ *If $(s_d)_{d\in D}$ is a net in X, then $\mathcal{P}(D)\text{-}\lim\limits_{d\in D} s_d \equiv_s x(\mathcal{C}')$, for every $\in X$, where $\mathcal{P}(D)$ denotes the powerset of D.*

The class of all ideal semi-convergence classes for X is denoted by $Con^s_I(X)$.

The following theorem sets up a one-to-one correspondence between the topologies for a set X and the ideal semi-convergence classes $\mathcal{C}'$ on it.

Theorem 4 ([29]). *(General ideal convergence classes theorem) Let $\mathcal{C}'$ be an ideal semi-convergence class for a non-empty set X. We consider the function $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$, where $\mathrm{cl}(A)$ is the set of all $x \in X$ such that for some net $(s_d)_{d\in D}$ in A and a proper ideal $\mathcal{I}$ of the directed set D, $(s_d)_{d\in D}$ $\mathcal{I}$-semi-converges $(\mathcal{C}')$ to x. Then cl is a closure operator for a topology $\tau^s_I(\mathcal{C}')$ on X and*

$((s_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}'$, where $\mathcal{I}$ is an ideal of D, if and only if $(s_d)_{d\in D}$ $\mathcal{I}$-converges to x with respect to $\tau_I^s(\mathcal{C}')$.

Proposition 12 ([29])**.** *Let X be a non-empty set. Then the following are satisfied:*

(1) The class $Con_I(X)$ is a refinement of the class $Con_I^s(X)$, that is for every $\mathcal{C} \in Con_I(X)$ there exists $\mathcal{C}' \in Con_I^s(X)$ such that $\mathcal{C} \subseteq \mathcal{C}'$.

(2) $Con_I^s(X) \subseteq Con_I(X)$.

Proposition 13 ([29])**.** *Let X be a non-empty set. There exists a one-to-one map m_s of $Con_I^s(X)$ onto $Con(X)$ such that for every $\mathcal{C}' \in Con_I^s(X)$ the following properties hold:*

(1) $\tau_I^s(\mathcal{C}') = \tau(m_s(\mathcal{C}'))$,
(2) $m_s(\mathcal{C}')$ can be considered as a subclass of the class $\mathcal{C}'$ in the sense that there exists an one-to-one map $e_s : m_s(\mathcal{C}') \to \mathcal{C}'$ and each $((s_d)_{d\in D}, x) \in m_s(\mathcal{C}')$ is identified with $e_s((s_d)_{d\in D}, x) \in \mathcal{C}'$.

Question 3. *Is it possible to achieve a similar characterization of the topology, through the ideal convergence of nets, with a smaller number of axioms for the ideal convergence classes?*

Question 4. *Is it possible to achieve an analogous to Theorem 4 characterization of the topology, for the case of $\mathcal{I}^*$-convergence of nets?*

4. Kinds of Convergences in Fuzzy Topological Spaces

The meaning of a fuzzy set is introduced by Zadeh in [35], who gives a different study of classical topological notions to a fuzzy environment. In [36], the meaning of the fuzzy topological space is given, and in [37–40], properties of such spaces are investigated. Moreover, in [37,38,41], the notion of fuzzy topology is characterized through fuzzy convergence classes, and in [38], a generalization of a Moore–Smith convergence of nets to fuzzy topological spaces was given. In [37], a characterization theorem between fuzzy topologies and fuzzy convergence classes was introduced. We refer to [35–38] for notions of Fuzzy Theory. Here, we recall some basic notions and notations.

A family δ of fuzzy sets in X is called a *fuzzy topology* for X (due to Chang [36]) if the following are satisfied:

(1) $\emptyset_{fuzzy}, X_{fuzzy} \in \delta$
(2) $A \wedge B \in \delta$, whenever $A, B \in \delta$, and
(3) $\vee\{A_\lambda : \lambda \in \Lambda\} \in \delta$, whenever $A_\lambda \in \delta$, for each $\lambda \in \Lambda$.

The pair (X, δ) is called a *fuzzy topological space*. Every member of δ is called a *δ-open* (or simply *open*) fuzzy set. The complement of a δ-open fuzzy set is called *δ-closed* (or simply *closed*) fuzzy set.

A fuzzy set in X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\lambda \in (0,1]$, we denote the fuzzy point by x_λ, where the point x is called its *support*. The set of all the fuzzy points in X is denoted by $FP(X)$.

A *fuzzy net* in X is an arbitrary function $s : D \to FP(X)$ where D is directed. If we set $s(d) = s_d$, for all $d \in D$, then the fuzzy net s will be denoted by $(s_d)_{d\in D}$.

A fuzzy net $t = (t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *fuzzy semisubnet* of the fuzzy net $s = (s_d)_{d\in D}$ in X if there exists a function $\varphi : \Lambda \to D$ such that $t = s \circ \varphi$, i.e., $t_\lambda = s_{\varphi(\lambda)}$ for every $\lambda \in \Lambda$. Similar to the classical topological case, we write $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ to indicate the fact that φ is the function mentioned above.

A fuzzy net $t = (t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *fuzzy subnet* of the fuzzy net $s = (s_d)_{d\in D}$ in X if t is a fuzzy semisubnet of the fuzzy net s and for every $d \in D$ there exists $\lambda_0 \in \Lambda$ such that $\varphi(\lambda) \geqslant d$ whenever $\lambda \in \Lambda$ with $\lambda \geqslant \lambda_0$.

Definition 21 ([38]). *We say that a fuzzy net $s = (s_d)_{d \in D}$ in a fuzzy topological space (X, δ) converges to a fuzzy point e in X, relative to δ, if s is eventually quasi-coincident with each Q-neighborhood of e. In this case, we write $\lim_{d \in D} s_d = e$.*

In [38], a survey on fuzzy topology, including closure, accumulation points, connectedness, separation axioms, and so on, is presented in detail. In this paper, we can find the properties of the above fuzzy convergence. We state the theorems ([38], Theorem 11.1, Theorem 11.2, Theorem 11.3), respectively, which state that:

(1) In a fuzzy topological space (X, δ), a fuzzy point $e \in_{fuzzy} \overline{A}$ if and only if there is a fuzzy net $s = (s_d)_{d \in D}$ in A such that s converges to e.
(2) A fuzzy subset A in a fuzzy topological space (X, δ) is closed if and only if every fuzzy net s in A cannot converge to a fuzzy point not belonging to A.
(3) In a fuzzy topological space (X, δ), x_λ is an accumulation point of A if and only if there is a fuzzy net in $A - x_\lambda$ converging to x_λ.
The following result of [38] shows a fact for the uniqueness of fuzzy limits.
(4) In a fuzzy topological space (X, δ), if a fuzzy net s converges to a fuzzy point x_λ, then for every $\mu \in (0, \lambda]$, s converges also to x_μ.

A study of fuzzy topology through fuzzy convergence classes has been done following the same thinking as in Kelley's theorem. In what follows, X is a non-empty set and $\mathcal{G}$ is a class consisting of pairs (s, e), where $s = (s_d)_{d \in D}$ is a fuzzy net in X and e is a fuzzy point in X.

Definition 22 ([37]). *We say that $\mathcal{G}$ is a fuzzy convergence class for X if it satisfies the conditions listed below.*

For convenience, we say that s converges $(\mathcal{G})$ to e or that $\lim_{d \in D} s_d \equiv e(\mathcal{G})$ if $(s, e) \in \mathcal{G}$.

(G1) If s is such that $s_d = e$, for each $d \in D$, then s converges $(\mathcal{G})$ to e.
(G2) If s converges $(\mathcal{G})$ to e, then so does each fuzzy subnet of s.
(G3) If s does not converge $(\mathcal{G})$ to e, then there exists a fuzzy subnet t of s, no fuzzy subnet of which converges $(\mathcal{G})$ to e.
(G4) We consider the following:
  (1) D is a directed set.
  (2) E_d is a directed set, for each $d \in D$.
  (3) $s^d = (s^d(n))_{n \in E_d}$ is a fuzzy net in X, converging $(\mathcal{G})$ to s_d, for each $d \in D$ and the fuzzy net $(s_d)_{d \in D}$, thus obtained, converges $(\mathcal{G})$ to e.

  Then, the induced net (associated with D and each s^d), converges $(\mathcal{G})$ to e.
(G5) For each point $x \in X$ and real directed set $D \subseteq (0, 1]$, if $r \leqslant \sup D$, then the fuzzy net $(x_d)_{d \in D}$ converges $(\mathcal{G})$ to x_r.

The class of all fuzzy convergence classes for X is denoted by $Con(X)$.

Proposition 14 ([37]). *Let (X, δ) be a fuzzy topological space. Then the class of pairs $\{(s, e) :$ the fuzzy net s converges to $e\}$ is a fuzzy convergence class, denoted by $\phi(\delta)$.*

Theorem 5 ([37]). *(fuzzy convergence classes theorem) We consider a map $c : I^X \to I^X$ induced as follows: for each $A \in I^X$, we define*

$$\mathcal{G}(A) = \{e : \text{for some fuzzy net } s \text{ in } A,\ (s, e) \in \mathcal{G}\}$$

and

$$c(A) = \vee \mathcal{G}(A).$$

Now if $\mathcal{G}$ is a fuzzy convergence class for X, then the following hold:

(1) The correspondence $A \mapsto c(A)$ is a fuzzy closure operator and the fuzzy topology thus obtained will be denoted by $\psi(\mathcal{G})$,

(2) $\phi(\psi(\mathcal{G})) = \mathcal{G}$ *and*
(3) $\psi(\phi(\delta)) = \delta$*, for a fuzzy topology* δ *on* X*.*

Therefore, there exists a bijective map between the set of all fuzzy topologies δ *for* X *and the set of all fuzzy convergence classes* $\mathcal{G}$ *for* X*. Moreover, this map is order-reversing, i.e., if* $\delta_1 \supseteq \delta_2$*, then* $\phi(\delta_1) \subseteq \phi(\delta_2)$*.*

In [41], the concept of a fuzzy semi-convergence class is presented, giving an alternative characterization of fuzzy topology in relation to the ordinary convergence of fuzzy nets.

Definition 23 ([41])**.** *We say that* $\mathcal{C}$ *is a fuzzy semi-convergence class for* X *if it satisfies the conditions listed below.*
For convenience, we say that s *semi-converges* $(\mathcal{C})$ *to* e *or that* $\lim_{d \in D} s_d \equiv_s e(\mathcal{C})$ *if* $(s, e) \in \mathcal{C}$*.*

$(G'1)$ *If* s *is such that* $s_d = e$*, for each* $d \in D$*, then* s *semi-converges* $(\mathcal{C})$ *to* e*.*
$(G'2)$ *If* s *semi-converges* $(\mathcal{C})$ *to* e*, then so does each fuzzy subnet of* s*.*
$(G'3)$ *If* s *does not semi-converge* $(\mathcal{C})$ *to* e*, then there exists a fuzzy subnet* t *of* s*, no fuzzy subnet of which semi-converges* $(\mathcal{C})$ *to* e*.*
$(G'4)$ *We consider the following:*

(1) D *is a directed set.*
(2) E_d *is a directed set, for each* $d \in D$*.*
(3) $s^d = (s^d(n))_{n \in E_d}$ *is a fuzzy net in* X*, semi-converging* $(\mathcal{C})$ *to* s_d*, for each* $d \in D$ *and the fuzzy net* $(s_d)_{d \in D}$*, thus obtained, semi-converges* $(\mathcal{C})$ *to* e*.*

Then, the induced net (associated with D *and each* s^d*), semi-converges* $(\mathcal{C})$ *to* e*.*
$(G'5)$ *For each point* $x \in X$ *and real directed set* $D \subseteq (0, 1]$*, if* $r \leqslant \sup D$*, then the fuzzy net* $(x_d)_{d \in D}$ *semi-converges* $(\mathcal{C})$ *to* x_r*.*

The class of all semi-convergence classes for X *is denoted by* $Con^s(X)$*.*

We hold that $Con(X) \subseteq Con^s(X)$. Moreover, $\phi(\delta) = \phi'(\delta)$, where $\phi'(\delta)$ is described in the following result.

Proposition 15 ([41])**.** *Let* (X, δ) *be a fuzzy topological space. Then the class of pairs* $\{(s, e) :$ *the fuzzy net* s *converges to* $e\}$ *is a fuzzy semi-convergence class, denoted by* $\phi'(\delta)$*.*

The following theorem sets up a one-to-one correspondence between the fuzzy topologies for a non-empty set X and the fuzzy semi-convergence classes on it.

Theorem 6 ([41])**.** *(fuzzy semi-convergence classes theorem) Let* $\mathcal{C}$ *be a fuzzy semi-convergence class for a non-empty set* X*. We consider a map* $\mathrm{cl} : I^X \to I^X$ *induced as follows: for each* $A \in I^X$*, we define* $\mathrm{cl}(A) \in I^X$ *to be such that a fuzzy point* $e \in_{fuzzy} \mathrm{cl}(A)$ *if and only if there exists a fuzzy net* s *in* A *such that* s *semi-converges* $(\mathcal{C})$ *to* e *i.e.,* $(s, e) \in \mathcal{C}$*.*

(1) *The correspondence* $A \mapsto \mathrm{cl}(A)$ *is a fuzzy closure operator and the fuzzy topology thus obtained will be denoted by* $\psi'(\mathcal{C})$*,*
(2) $\phi'(\psi'(\mathcal{C})) = \mathcal{C}$ *and*
(3) $\psi'(\phi'(\delta)) = \delta$*, for a fuzzy topology* δ *on* X*.*

Therefore, there exists a bijective map between the set of all fuzzy topologies δ *for* X *and the set of all fuzzy semi-convergence classes* $\mathcal{C}$ *for* X*. Moreover, this map is order-reversing, i.e., if* $\delta_1 \supseteq \delta_2$*, then* $\phi'(\delta_1) \subseteq \phi'(\delta_2)$*.*

Proposition 16 ([41])**.** *Let* X *be a non-empty set. Then the following are satisfied:*

(1) $Con(X) = Con^s(X)$*,*
(2) $\phi' = \phi$ *and* $\psi' = \psi$*.*

Question 5. *Is it possible to achieve a similar characterization of the fuzzy topology through fuzzy convergence of nets, with a smaller number of axioms for the fuzzy convergence classes?*

Further, in [41], the concept of fuzzy ideal convergence class is introduced in order to obtain analogous results in relation to the ideal convergence of fuzzy nets. (We state in [42] that we can find a different investigation of convergence in fuzzy topological spaces via prime filters. However, in our study, we pay attention to ideals.)

Definition 24 ([16])**.** *Let (X,δ) be a fuzzy topological space and $\mathcal{I}$ an ideal of a directed set D. We say that a fuzzy net $(s_d)_{d\in D}$ $\mathcal{I}$-converges to a fuzzy point e in X, relative to δ, if for every open Q-neighborhood U of e, we have $\{d \in D : s_d \bar{q} U\} \in \mathcal{I}$. In this case, we write $\mathcal{I}\text{-}\lim\limits_{d\in D} s_d = e$ and we say that e is the $\mathcal{I}$-limit of the fuzzy net $(s_d)_{d\in D}$.*

Definition 25 ([41])**.** *Let X be a non-empty set and let $\mathcal{H}$ be a class consisting of triads $(s,e,\mathcal{I})$, where $s = (s_d)_{d\in D}$ is a fuzzy net in X, e is a fuzzy point in X and $\mathcal{I}$ is an ideal of D. We say that $\mathcal{H}$ is a fuzzy ideal convergence class for X if it satisfies the conditions listed below.*

For convenience, we say that s $\mathcal{I}$-converges $(\mathcal{H})$ to e or that $\mathcal{I}\text{-}\lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$ if $(s,e,\mathcal{I}) \in \mathcal{H}$.

$(C'1)$ If $(s_d)_{d\in D}$ is a fuzzy net such that $s_d = e$ for every $d \in D$ and $\mathcal{I}$ is an ideal of D, then $\mathcal{I}\text{-}\lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$.

$(C'2)$ If $\mathcal{I}\text{-}\lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$, where $\mathcal{I}$ is an ideal of D, then for every fuzzy semisubnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the fuzzy net $(s_d)_{d\in D}$ we have $\mathcal{I}_\Lambda(\varphi)\text{-}\lim\limits_{\lambda\in\Lambda} t_\lambda \equiv e(\mathcal{H})$.

$(C'3)$ If $\mathcal{I}\text{-}\lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$, where $\mathcal{I}$ is a proper ideal of D, then there exists a fuzzy semisubnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the fuzzy net $(s_d)_{d\in D}$ such that $\mathcal{I}_0(\Lambda)\text{-}\lim\limits_{\lambda\in\Lambda} t_\lambda \equiv e(\mathcal{H})$.

$(C'4)$ Let D be a directed set and $\mathcal{I}_D$ a proper ideal of D. If the fuzzy net $(s_d)_{d\in D}$ does not $\mathcal{I}_D$-converge $(\mathcal{H})$ to e, then there exists a fuzzy semisubnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of the fuzzy net $(s_d)_{d\in D}$ such that:

(1) No fuzzy semisubnet $(r_k)^{f}_{k\in K}$ of $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ $\mathcal{I}_K$-converges $(\mathcal{H})$ to e, for every proper ideal $\mathcal{I}_K$ of K.

(2) $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal of Λ.

$(C'5)$ We consider the following:

(1) D is a directed set.

(2) $\mathcal{I}_0(D)$ is a proper ideal of D.

(3) E_d is a directed set, for each $d \in D$.

(4) $\mathcal{I}_0(E_d)$ is a proper ideal of E_d, for each $d \in D$.

(5) $(s(d,e))_{e\in E_d}$ is a fuzzy net in X for each $d \in D$.

(6) $\mathcal{I}_0(D)\text{-}\lim\limits_{d\in D} t_d \equiv e(\mathcal{H})$, where e is a fuzzy point in X, $\mathcal{I}_0(E_d)\text{-}\lim\limits_{e\in E_d} s(d,e) \equiv t_d(\mathcal{H})$ and t_d is a fuzzy point in X, for every $d \in D$.

Then, the fuzzy net $r : D \times \prod\limits_{d\in D} E_d \to X$, where $r(d,f) = s(d,f(d))$, for every $(d,f) \in D \times \prod\limits_{d\in D} E_d$, $\mathcal{I}_0(D \times \prod\limits_{d\in D} E_d)$-converges to e.

$(C'6)$ For each point $x \in X$ and real directed set $D \subseteq (0;1]$, if $r \leqslant \sup D$, then the fuzzy net $(x_d)_{d\in D}$ $\mathcal{I}_0(D)$-converges $(\mathcal{H})$ to x_r.

$(C'7)$ If $(s_d)_{d\in D}$ is a fuzzy net in X, then $\mathcal{P}(D)\text{-}\lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$, for every fuzzy point $e \in X$.

The class of all fuzzy ideal convergence classes for X is denoted by $Con_I(X)$.

If (X,δ) is a fuzzy topological space, then the class which consists of triads $((s_d)_{d\in D}, e, \mathcal{I})$, where $(s_d)_{d\in D}$ is a fuzzy net in X, e is a fuzzy point in X, $\mathcal{I}$ is an ideal of D and $(s_d)_{d\in D}$

$\mathcal{I}$-converges to x, relative to δ, is a fuzzy ideal convergence class, denoted by $\Phi(\delta)$. We say that the fuzzy topology δ generates the fuzzy ideal convergence class $\Phi(\delta)$.

The following theorem sets up a one-to-one correspondence between the fuzzy topologies for a non-empty set X and the fuzzy ideal convergence classes on it.

Theorem 7 ([41] (fuzzy ideal convergence classes theorem))**.** *Let $\mathcal{H}$ be a fuzzy ideal convergence class for a non-empty set X. We consider a map $\text{cl} : I^X \to I^X$ induced as follows: for each $A \in I^X$, we define $\text{cl}(A) \in I^X$ to be such that a fuzzy point $e \in_{fuzzy} \text{cl}(A)$ if and only if for some fuzzy net $(s_d)_{d\in D}$ in A and a proper ideal $\mathcal{I}$ of the directed set D, $(s_d)_{d\in D}$ $\mathcal{I}$-converges ($\mathcal{H}$) to e i.e., $(s, e, \mathcal{I}) \in \mathcal{H}$. Then cl is a fuzzy closure operator for a fuzzy topology denoted by $\Psi(\mathcal{H})$ on X and $((s_d)_{d\in D}, e, \mathcal{I}) \in \mathcal{H}$ if and only if $(s_d)_{d\in D}$ $\mathcal{I}$-converges to e with respect to $\Psi(\mathcal{H})$.*

Proposition 17 ([41])**.** *Let $\mathcal{H}$ be a fuzzy ideal convergence class and δ be a fuzzy topology for a non-empty set X. We have the following:*

(1) $\Phi(\Psi(\mathcal{H})) = \mathcal{H}$ and
(2) $\Psi(\Phi(\delta)) = \delta$.

Therefore, there exists a bijective map between the set of all fuzzy topologies δ for X and the set of all fuzzy ideal convergence classes $\mathcal{H}$ for X. Moreover, this map is order-reversing, i.e., if $\delta_1 \supseteq \delta_2$, then $\Phi(\delta_1) \subseteq \Phi(\delta_2)$.

Proposition 18 ([41])**.** *Let X be a non-empty set. Then there exists a one-to-one map m of $Con_I(X)$ onto $Con(X)$ such that for every $\mathcal{H} \in Con_I(X)$ the following properties hold:*

(1) $\Psi(\mathcal{H}) = \psi(m(\mathcal{H}))$,
(2) $m(\mathcal{H})$ can be considered as a subclass of the class $\mathcal{H}$ in the sense that there exists a one-to-one map $\varepsilon : m(\mathcal{H}) \to \mathcal{H}$ and each $((s_d)_{d\in D}, e) \in m(\mathcal{H})$ is identified with $\varepsilon((s_d)_{d\in D}, e) \in \mathcal{H}$.

Question 6. *Can we define new meanings of ideal convergences in fuzzy topological spaces? Which are their basic properties?*

L-fuzzy topological spaces have come to enrich the theory of fuzzy topological spaces (see for example [43]). In what follows, X denotes a nonempty ordinary set, L is a completely distributive lattice with an ordering-reversing involution $' : L \to L$, usually called *F-lattice*, and L^X is the set of all L-fuzzy sets on X.

For every $x \in X$ and $a \in L$, we denote the L-fuzzy subset taking value a at x and value 0 at other points of X by x_a, call it an *L-fuzzy point a* on X. For every $\mathcal{A} \subset L^X$, we denote the set of all L-fuzzy points on X, contained in $\mathcal{A}$ by $Pt(\mathcal{A})$; especially for every $A \in L^X$, $Pt(\downarrow A)$ means the set of all L-fuzzy points contained in $\downarrow A$ [44].

$\delta \subset L^X$ is called an *L-fuzzy topology* on X, if δ is closed under arbitrary joins and finite meets; especially $\mathbf{0}, \mathbf{1} \in \delta$, where $\mathbf{0}$ and $\mathbf{1}$ stands for the least and the greatest element of L^X, respectively. The pair (L^X, δ) is called an *L-fuzzy topological space*. Every $U \in \delta$ is called an *open subset* in L^X and every $P \in L^X$ such that $P' \in \delta$ is called a *closed subset* in L^X [44].

Let D be a directed set. A net s in L^X with index set D is the mapping $s : D \to L^X$, denoted by $s = (s_d)_{d\in D}$. If $s : D \to L^X$ and $t : E \to L^X$ are two nets in L^X, then t is said to be a *subnet* of s, if there exists a mapping $N : E \to D$, called a *cofinal selection* on s, such that

(1) $t = s \circ N$;
(2) For every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geqslant n_0$ for $m \geqslant m_0$ [44].

Definition 26 ([44])**.** *If $e \in Pt(L^X)$, then e is called a limit of the net s if for every Q-neighborhood U of e, s eventually quasi-coincides with U; in this case we also say s converges to e.*

Moreover, $e \in Pt(L^X)$ is called a *cluster point* of a net s, denoted by $s\infty e$, if for every Q-neighborhood U of e, s frequently quasi-coincides with U. We denote the join of all cluster points of s using $\text{clu}s$ and the join of all limit points of s with $\lim s$ [44].

In [44] properties of this L-fuzzy convergence are studied (see for example ([44], Theorem 2.3, Theorem 2.5, Theorem 2.6, Theorem 2.7)) some of which are presented as follows:

(1) If a net s in an L-fuzzy topological space converges to e, where $e \in Pt(L^X)$, then $s \infty e$.
(2) $\lim s \leqslant \text{clu} s$, for any net s in an L-fuzzy topological space.
(3) If a net s in an L-fuzzy topological space converges to e, where $e \in Pt(L^X)$, and $e \geqslant d$, then s also converges to d.
(4) If s is a net in an L-fuzzy topological space that converges to e, where $e \in Pt(L^X)$, and t is a subnet of s, then t also converges to e. Further, $\lim s \leqslant \lim t$ and $\text{clu} t \leqslant \text{clu} s$.
(5) A net s in an L-fuzzy topological space converges to e, where $e \in Pt(L^X)$, if and only if $t \infty e$, for every subnet t of s.

Moreover, in this paper, we can also find a study of convergence classes in the environment of L-fuzzy topological spaces.

Let L^X be an L-fuzzy topological space, D a directed set, $\{E^n : n \in D\}$ a family of directed sets and $s^n = \{s^n(m) : m \in E^n\}$ a net in $Pt(L^X)$, for every $n \in D$. Then for the product directed set $\mathbf{D} = D \times \prod_{n \in D} E_n$, the net $\mathbf{s} : D \to L^X$ defined as: for every $(n, f) \in \mathbf{D}$, $\mathbf{s}(n, f) = s^n(f(n))$ is called the *induced net* of the net family $\{s^n : n \in D\}$.

Definition 27 ([44])**.** *Let $\mathcal{A} \subset L^X$. Let also $\mathcal{S}_M(\mathcal{A})$ denote the class of all nets $s = (s_d)_{d \in D}$ such that $s(d) \in M(\mathcal{A})$ for every $d \in D$. Let $\mathcal{C} \subset \mathcal{S}_M(L^X) \times M(L^X)$ and $(s, e) \in \mathcal{S}_M(L^X) \times M(L^X)$. We say that s $\mathcal{C}$-converges to e, denoted by $\lim_{d \in D} s_d \equiv_{\mathcal{C}} e$, if $(s, e) \in \mathcal{C}$. $\mathcal{C}$ is called an L-fuzzy convergence class on L^X if it fulfills the following conditions:*

(L1) If s is such that $s_d = e$, for each $d \in D$, then s converges ($\mathcal{C}$) to e.
(L2) If s converges ($\mathcal{C}$) to e, then so does each subnet of s.
(L3) If s does not converge ($\mathcal{C}$) to e, then there exists a subnet t of s, no subnet of which converges ($\mathcal{C}$) to e.
(L4) For every directed set D and every $\{s^n : n \in D\} \subset \mathcal{S}_M(L^X)$, where $s^n = \{s^n(m) : m \in E^n\}$ and $\lim s^n \equiv_{\mathcal{C}} s(n)$, for every $n \in D$, if $\lim s \equiv_{\mathcal{C}} e$ for the obtained molecule net $s = \{s(n) : n \in D\}$, then for the induced net $\mathbf{s}$ of $\{s^n : n \in D\}$, $\lim \mathbf{s} \equiv_{\mathcal{C}} e$.
(L5) For every $x \in X$, every $A \subset M(L)$ and every molecule $\lambda \leqslant \bigvee A$, there exists $s \in \mathcal{S}_M(\{x_\xi : \xi \in A\})$ such that $\lim \mathbf{s} \equiv_{\mathcal{C}} x_\lambda$.

Let (L^X, δ) be an L-fuzzy topological space. We denote this with

$$\varphi(\delta) = \{(s, e) \in \mathcal{S}_M(L^X) \times M(L^X) : \lim s \equiv_{\mathcal{C}} e\}$$

the so-called *L-fuzzy convergence class* on L^X generated by δ. Then $\varphi(\delta)$ is an L-fuzzy convergence class on L^X [44].

Let $\mathcal{C} \subset \mathcal{S}_M(L^X) \times M(L)$. For every $A \in L^X$, we denote using

$$\text{clu}_{\mathcal{C}}(A) = \{e \in M(L^X) : \exists s \in Pt(\downarrow A), \lim s \equiv_{\mathcal{C}} e\}.$$

We define an operator c on L^X, called the *closure operator generated by $\mathcal{C}$*, as: $c(A) = \bigvee \text{clu}_{\mathcal{C}} A,\ A \in L^X$.

Proposition 19 ([44])**.** *Let X be a nonempty ordinary set, L an F-lattice and $\mathcal{C}$ an L-fuzzy convergence class on L^X. Then the closure operator generated by $\mathcal{C}$ is a closure operator on L^X.*

Let X be a nonempty ordinary set, L an F-lattice and $\mathcal{C}$ an L-fuzzy convergence class on L^X. Denote the L-fuzzy topology on X generated by the closure operator on L^X generated by $\mathcal{C}$ as $\psi(\mathcal{C})$. Then the following results prove the correspondence between L-fuzzy topologies and L-fuzzy convergence classes.

Proposition 20 ([44]). *Let X be a nonempty ordinary and L an F-lattice. Then:*

(1) *For every L-fuzzy topology δ on X, $\psi(\varphi(\delta)) = \delta$.*
(2) *For every L-fuzzy convergence class $\mathcal{C}$ on L^X, $\varphi(\psi(\mathcal{C})) = \mathcal{C}$.*
(3) *For every pair δ, μ of L-fuzzy topologies on L^X such that $\delta \subset \mu$, $\varphi(\delta) \supset \varphi(\mu)$.*
(4) *For every pair $\mathcal{C}, \mathcal{D}$ of L-fuzzy convergence classes on L^X such that $\mathcal{C} \subset \mathcal{D}$, $\psi(\mathcal{C}) \supset \psi(\mathcal{D})$.*

Question 7. *Is it possible to achieve a similar characterization of the topology through L-fuzzy convergence of nets, with a smaller number of axioms, for the L-fuzzy convergence classes?*

Question 8. *Does there exist a relative theory of axioms for the L-fuzzy semi-convergence in L-fuzzy topological spaces?*

Question 9. *Does there exist a theory of L-fuzzy ideal convergence in L-fuzzy topological spaces? Which is the precise notion of such kind of ideal convergence? Which are its properties?*

Question 10. *Is it possible to achieve a similar characterization of the topology through L-fuzzy ideal convergence of nets, with some axioms (as in work [44]) for the L-fuzzy ideal convergence classes?*

5. Kinds of Convergences in Fuzzy Metric Spaces

In [45,46], the notion of fuzzy metric space is introduced and studied, based on continuous t-norms, and many other articles have contributed to the development of this theory (for example, [47–50]). In this section, we pay attention to the notion of convergence in such spaces, introducing the most basic results.

A *fuzzy metric space* is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions for every $x, y, z, \in X$ and $s, t > 0$:

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, then we say that $(M, *)$ or simply M is a *fuzzy metric* on X. Further, we say that (X, M) or, simply, X is a fuzzy metric space (see also [51]).

Every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form

$$\{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\},$$

where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$.

Definition 28 ([45]). *Let $(X, M, *)$ be a fuzzy metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to converge to x if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.*

Definition 29 ([50]). *Let $(X, M, *)$ be a fuzzy metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to point-converge to x_0 if $\lim_{n\to\infty} M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$. In such case, we say that $(x_n)_{n\in\mathbb{N}}$ is p-convergent to x_0 for $t_0 > 0$, or, simply, $(x_n)_{n\in\mathbb{N}}$ is p-convergent.*

Some of the most basic facts for the above types of convergences are given as follows:

(1) A sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x_0 if and only if $(x_n)_{n\in\mathbb{N}}$ is p-convergent to x_0 for all $t > 0$.
(2) If $\lim_{n\to\infty} M(x_n, x, t_1) = 1$ and $\lim_{n\to\infty} M(x_n, y, t_2) = 1$, then $x = y$.

(3) If $\lim_{n\to\infty} M(x_n, x_0, t_0) = 1$, then $\lim_{k\to\infty} M(x_{n_k}, x_0, t_0) = 1$ for each subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$.
(4) If $(x_n)_{n\in\mathbb{N}}$ p-converges to x_0 and it is convergent, then $(x_n)_{n\in\mathbb{N}}$ converges to x_0.

However, in [50], we can find an example for this property (4). Let $(x_n)_{n\in\mathbb{N}} \subset (0,1)$ be a strictly increasing sequence convergent to 1 with respect to the usual topology of $\mathbb{R}$ and $X = (x_n)_{n\in\mathbb{N}} \cup \{1\}$. We define on $X \times X \times (0, +\infty)$ the function M given by:

(i) $M(x, x, t) = 1$ for each $x \in X$, $t > 0$,
(ii) $M(x_n, x_m, t) = \min\{x_n, x_m\}$, for all $m, n \in \mathbb{N}$, $t > 0$, and
(iii) $M(x_n, 1, t) = M(1, x_n, t) = \min\{x_n, t\}$ for all $n \in \mathbb{N}$, $t > 0$.

Then the sequence $(x_n)_{n\in\mathbb{N}}$ is not convergent since $\lim_{n\to\infty} M(x_n, 1, \frac{1}{2}) = \frac{1}{2}$, but it is p-convergent to 1, as $\lim_{n\to\infty} M(x_n, 1, 1) = 1$.

Definition 30 ([52])**.** *Let $(X, M, *)$ be a fuzzy metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to s-converge to x_0 if $\lim_{n\to\infty} M(x_n, x_0, \frac{1}{n}) = 1$.*

Some of the most basic facts for this type of convergence are given as follows:

(1) If τ_M is the discrete topology, then convergent sequences are s-convergent.
(2) Each s-convergent sequence in a fuzzy metric space X is convergent.
However, in [51], we can find an example that shows that the converse of (2) does not always hold. On $(0, +\infty)$ we consider the principal fuzzy metric $(M, *)$, where M is defined by

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}, \ x, y \in (0, +\infty), \ t > 0.$$

Since $\lim_{n\to\infty} M\left(\frac{1}{n}, 0, t\right) = 1$, for each $t > 0$, $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to zero, but it is not s-convergent to zero, as $\lim_{n\to\infty} M\left(\frac{1}{n}, 0, \frac{1}{n}\right) = \frac{1}{2}$.
(3) Each subsequence of an s-convergent sequence in a fuzzy metric space X is s-convergent.
(4) Each convergent sequence in a fuzzy metric space X admits an s-convergent subsequence.

Definition 31 ([53])**.** *Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $(x_n)_{n\in\mathbb{N}}$ is said to strong converge to x_0 if $\lim_{n,m} M(x_n, x_0, \frac{1}{m}) = 1$. In this case, we say that $(x_n)_{n\in\mathbb{N}}$ st-converges to x_0.*

The above strong convergence is a stronger concept than convergence, in support of which it is proven that:

(1) Each st-convergent sequence is s-convergent, but the converse does not always hold [53].
Let $(X, M_d, *)$ be the standard fuzzy metric, where $X = \mathbb{R}$, d is the usual metric on $\mathbb{R}$ and M_d is a function on $X \times X \times (0, +\infty)$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \ x, y \in X \ t > 0.$$

We consider the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ is s-convergent to zero, but it is not st-convergent to zero.
(2) Each subsequence of an st-convergent sequence in a fuzzy metric space X is st-convergent.

(3) Every convergent sequence in a fuzzy metric space X is st-convergent if and only if every convergent sequence in X is s-convergent.

Definition 32 ([54])**.** *Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $(x_n)_{n\in\mathbb{N}}$ is said to s_p-converge to x_0 for $p \in \mathbb{N}$ if $\lim_{n\to\infty} M(x_n, x_0, \frac{1}{n^p}) = 1$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be s_∞-convergent to x_0 if $(x_n)_{n\in\mathbb{N}}$ is s_p-convergent to x_0, for all $p \in \mathbb{N}$.*

For these kinds of convergences, we state the following:

(1) s_1-convergence coincides with the s-convergence in any fuzzy metric space.
(2) s_{k+1}-convergence implies s_k-convergence in any fuzzy metric space for $k \in \mathbb{N}$.
(3) st-convergence implies s_∞-convergence in any fuzzy metric space.
Various examples that show that the converse of the above statements do not hold can be found in the paper [54].
(4) Each subsequence of an s_p-convergent sequence in a fuzzy metric space X is s_p-convergent for all $p \in \mathbb{N}$ and each subsequence of an s_∞-convergent sequence in a fuzzy metric space X is s_∞-convergent.
(5) Each convergent sequence in a fuzzy metric space X admits an s_p-convergent subsequence for all $p \in \mathbb{N}$. Further, each convergent sequence in a fuzzy metric space X admits an s_∞-convergent subsequence.

Question 11. *Can we define new meanings of convergences and statistical/ideal convergences in fuzzy metric spaces? Which are their basic properties?*

6. Kinds of Convergences in Partially Ordered Sets

As the theory of convergence produces new motivations for more results, partially ordered sets and the convergence of nets in these objects have come to construct new chapters. Therefore, the meanings of order-convergence, o_2-convergence (as a generalization of order-convergence), lim-inf-convergence, and similar kinds are defined, and basic properties are investigated. The main problem in the theory of convergence in partially ordered sets focuses on finding partially ordered sets in which each of the above convergence is topological [55–63]. Thus, in this section, we present notions of convergence in partially ordered sets (in short, posets) and basic results. Mainly, we present the main research problem of being notions of convergences in posets topological. We shall denote partially ordered sets by their underlying sets, and we will write X for $(X, \leqslant)$.

Definition 33 ([64–66])**.** *A net $(s_d)_{d\in D}$ in a poset X is said to order-converge (in short o-converge) to $y \in X$ if there exist subsets A and B of X such that*

(1) A is up-directed and B is down-directed,
(2) $y = \sup A = \inf B$, and
(3) for each $a \in A$ and $b \in B$, there exists $k \in D$ such that $a \leqslant s_d \leqslant b$ holds for all $d \geqslant k$.

In this case we write $(s_d)_{d\in D} \xrightarrow{o} x$.

If X is a complete lattice, then a net $(s_d)_{d\in D}$ in X is said to *o–converge* to $y \in X$ if and only if $y = \liminf s_d = \limsup s_d$, where $\liminf s_d = \sup_{d\in D} \inf_{j\geqslant d} s_j$ and $\limsup s_d = \inf_{d\in D} \sup_{j\geqslant d} s_j$ [56,67].

Some of the main facts for the o-convergence are summarized as follows:

(1) In a complete lattice, the alternative definitions of o-convergence in the cases of a complete lattice and a poset agree.
(2) The o-convergent point of a net $(s_d)_{d\in D}$ in a poset, if it exists, is unique.
(3) Any constant net $(s_d)_{d\in D}$ in a poset X with value x o-converges to x.

(4) Further, given a poset X, by $\mathcal{T}_X^o$ we denote the set consisting of all subsets U of X satisfying the following property: *If* $(s_d)_{d\in D} \xrightarrow{o} x \in U$, *then there exists* $d_0 \in D$ *such that* $s_d \in U$ *for every* $d \geqslant d_0$. The set $\mathcal{T}_X^o$ forms a topology on X, which is called the *order topology* on X [67–69].

Definition 34 (see for example [67])**.** *Let $\mathcal{L}$ be the class consisting of all pairs $((s_d)_{d\in D}, x)$ of a net $(s_d)_{d\in D}$ and an element x in a poset X with $(s_d)_{d\in D}$ o-converging to x. The class $\mathcal{L}$ is called topological if there is a topology τ on X such that $((s_d)_{d\in D}, x) \in \mathcal{L}$ if and only if the net $(s_d)_{d\in D}$ converges to x with respect to the topology τ.*

In [63], it is observed that, in general, o-convergence is not topological. In [58,68], some examples show that o-convergence is not topological even though X is a complete lattice. Further, for a completely distributive lattice and any antichain, the o-convergence is topological [67]. Thus, a natural problem arises: *"For what posets is the o-convergence topological?"* In [59,60,67,68,70], conditions under of which o-convergence is topological are given. For that, new classes of posets are introduced, such as the notions of a doubly-continuous poset, $\mathcal{S}$-doubly continuous posets, and $\mathcal{O}$-doubly continuous poset (see for example [56,71]). For these posets, under some restricted conditions, the o-convergence is topological.

Definition 35 ([61,72])**.** *A net $(s_d)_{d\in D}$ in a poset X is said to o_2-converge to $y \in X$ if there exist subsets M and N of X such that*

(1) $y = \sup M = \inf N$, and
(2) for each $a \in M$ and $b \in N$, there exists $k \in D$ such that $a \leqslant s_d \leqslant b$ holds for all $d \geqslant k$.
In this case we write $(s_d)_{d\in D} \xrightarrow{o_2} x$.

If X is a complete lattice, then a net $(s_d)_{d\in D}$ in X is said to *o_2-converge* to $y \in X$ if $y = \liminf s_d = \limsup s_d$, where $\liminf s_d = \sup\limits_{d\in D} \inf\limits_{j\geqslant d} s_j$ and $\limsup s_d = \inf\limits_{d\in D} \sup\limits_{j\geqslant D} s_j$ [62].

Some of the main properties of o_2-convergence are summarized as follows:

(1) In a complete lattice, the alternative definitions of o_2-convergence in the cases of a complete lattice and a poset agree.
(2) The o_2-convergent point of a net $(s_d)_{d\in D}$ in a poset, if it exists, is unique.
(3) Any constant net $(s_d)_{d\in D}$ in a poset with value x o_2-converges to x.
(4) If $(s_d)_{d\in D}$ o-converges to x, then it o_2-converges to x [61], that is the o_2-convergence is a generalization of o-convergence.
However, in [62], we can find an example that shows that the converse of (4) does not always hold. Let $P = \{d_1, d_2, \ldots\} \cup \{c\} \cup \{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\}$. The order $\leqslant$ on P is defined as follows:
    (i) $a_i \leqslant c$, $b_i \leqslant c$, $c \leqslant d_i$, for all $i = 1, 2, 3, \ldots$
    (ii) if $k \geqslant i$, then $a_k \geqslant b_i$.
The net $(a_i)_{i\in\mathbb{N}}$ o_2-converges to c but it does not o-converge to c.
(5) In a lattice o_2-convergence is equivalent to o-convergence.
(6) Moreover, given a poset X, we denote by $\mathcal{T}_X^{o_2}$ the set consisting of all subsets U of X satisfying the following property: *If* $(s_d)_{d\in D} \xrightarrow{o_2} x \in U$, *then there exists* $d_0 \in D$ *such that* $x_d \in U$ *for every* $d \geqslant d_0$. The set $\mathcal{T}_X^{o_2}$ forms a topology on X, which is called the *o_2-topology* on X [69].

Definition 36 (see for example [67])**.** *Let $\mathcal{L}$ be the class consisting of all pairs $((s_d)_{d\in D}, x)$ of a net $(s_d)_{d\in D}$ and an element x in a poset X with $(s_d)_{d\in D}$ o_2-converging to x. The class $\mathcal{L}$ is called topological if there is a topology τ on X such that $((s_d)_{d\in D}, x) \in \mathcal{L}$ if and only if the net $(s_d)_{d\in D}$ converges to x with respect to the topology τ.*

The o_2-convergence is also not topological generally. However, in [34,57,62], the authors give sufficient and necessary conditions for o_2-convergence to be topological. For that, they study the notions of α-double continuous poset and α^*-double continuous poset. For these posets, under some restricted conditions, the o_2-convergence is topological.

Definition 37 ([63])**.** *A net $(s_d)_{d\in D}$ in a poset X is said to* lim-inf*-converge to an element $y \in X$ if there exists an up-directed subset M of X such that*

(1) $\sup M$ *exists with* $\sup M \geqslant y$ *and*
(2) *for any $m \in M$, $s_d \geqslant m$ holds eventually that is, there exists $k \in D$ such that $s_d \geqslant m$ for all $d \geqslant k$.*

In particular, a net $(s_d)_{d\in D}$ in a complete lattice lim-inf-*converges* to x if $x \leqslant \inf\{\sup_{d\geqslant k}\{x_d\} : k \in D\}$ [56].

Some of the main results for the lim-inf-convergence are given as follows:

(1) Let $(s_d)_{d\in D}$ be a net in a poset X such that $x = \inf\{x_d : d \in D\}$ exists. Then $(s_d)_{d\in D}$ lim-inf-converges to x.
(2) If $(s_d)_{d\in D}$ lim-inf-converges to x, then it lim-inf-converges to every y with $y \leqslant x$. Thus, the lim-inf-limits of a net are generally not unique.

Definition 38 (see for example [67])**.** *Let $\mathcal{L}$ be the class consisting of all pairs $((s_d)_{d\in D}, x)$ of a net $(s_d)_{d\in D}$ and an element x in a poset X with $(s_d)_{d\in D}$* lim-inf*-converging to x. The class $\mathcal{L}$ is called topological if there is a topology τ on X such that $((s_d)_{d\in D}, x) \in \mathcal{L}$ if and only if the net $(s_d)_{d\in D}$ converges to x with respect to the topology τ.*

It was proven that for a complete lattice L, the lim-inf-convergence was topological if and only if L was a continuous lattice [56]. Further, for any poset X the lim-inf-convergence was topological if and only if X was a continuous poset [63].

In [63], the authors also consider the lim-inf$_2$-convergence, a part of generalized o_2-convergence for lim-inf-convergence. They also establish a characterization for this convergence to be topological, proving that for a poset X the lim-inf$_2$-convergence is topological if and only if X is α-continuous.

Definition 39 ([63])**.** *A net $(s_d)_{d\in D}$ in a poset X is said to* lim-inf$_2$*-converge to $x \in X$ if there exists a subset $M \subseteq X$, such that*

(1) $\sup M$ *exists and* $x \leqslant \sup M$ *and*
(2) *for each $m \in M$, $x_d \geqslant m$ holds eventually.*

Definition 40 (see for example [67])**.** *Let $\mathcal{L}$ be the class consisting of all pairs $((s_d)_{d\in D}, x)$ of a net $(s_d)_{d\in D}$ and an element x in a poset X with $(s_d)_{d\in D}$* lim-inf$_2$*-converging to x. The class $\mathcal{L}$ is called topological if there is a topology τ on X such that $((s_d)_{d\in D}, x) \in \mathcal{L}$ if and only if the net $(s_d)_{d\in D}$ converges to x with respect to the topology τ.*

Definition 41 ([67])**.** *Let X be a poset and $\mathcal{M} \subseteq \mathcal{P}(X)$. A net $(s_d)_{d\in D}$ in a poset X is said to* lim-inf$_{\mathcal{M}}$*-converge to $x \in X$ if there exists a subset $M \in \mathcal{M}$ such that*

(1) $\sup M$ *exists and* $x \leqslant \sup M$ *and*
(2) *for each $m \in M$, $x_d \geqslant m$ holds eventually.*

As it is observed in [67], lim-inf-convergence and lim-inf$_2$-convergence are particular cases of lim-inf$_{\mathcal{M}}$-convergence.

Definition 42 (see for example [67])**.** *Let $\mathcal{L}$ be the class consisting of all pairs $((s_d)_{d\in D}, x)$ of a net $(s_d)_{d\in D}$ and an element x in a poset X with $(s_d)_{d\in D}$* lim-inf$_{\mathcal{M}}$*-converging to x. The class $\mathcal{L}$*

is called topological if there is a topology τ on X such that $((s_d)_{d\in D}, x) \in \mathcal{L}$ if and only if the net $(s_d)_{d\in D}$ converges to x with respect to the topology τ.

Similar to the above discussions, in [67], continuous posets and α-continuous posets are classes of posets in which the lim-inf$_{\mathcal{M}}$-convergence is topological.

Question 12. *Define new notions of convergences in posets. Which are their properties? In which posets are such kinds of convergences topological?*

Recently, a new research issue has been investigated, combining the notions of convergence in posets with ideals. Therefore, notions of ideal convergence in posets are introduced and studied.

Definition 43 ([34,69])**.** *Let X be a poset. A net $(s_d)_{d\in D}$ in X is said to $\mathcal{I}$-order-converge to a point $x \in X$, where $\mathcal{I}$ is an ideal on D, if there exist subsets A and B of P such that:*

(1) A is directed and B is filtered.
(2) $x = \sup A = \inf B$.
(3) For every $a \in A$ and $b \in B$, $\{d \in D : s_d \notin [a, b]\} \in \mathcal{I}$.

In this case the point x is called the $\mathcal{I}$-o-limit of the net $(s_d)_{d\in D}$ and we write $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$.

Some of the basic properties of the ideal-order-convergence are given as follows:

(1) The $\mathcal{I}$-o-limit is unique.
(2) If $(s_d)_{d\in D}$ is a net with $s_d = x$ for every $d \in D$, then $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ holds for every ideal $\mathcal{I}$ of D.
(3) If $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$, then for every semi-subnet $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ of $(s_d)_{d\in D}$ we have $(t_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{\mathcal{I}_\Lambda(\varphi)-o} x$.

Proposition 21 ([69])**.** *Let $(s_d)_{d\in D}$ be a net in a poset X and $\mathcal{I}$ a non-trivial ideal on D. Then there exists a semi-subnet $(t_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda\in\Lambda_{\mathcal{I}}}$ of $(s_d)_{d\in D}$ such that for every $A \subseteq X$, $\{d \in D : s_d \notin A\} \in \mathcal{I}$ if and only if there exists $\lambda_0 \in \Lambda_{\mathcal{I}}$ such that $t_\lambda \in A$ for all $\lambda \geqslant \lambda_0$. In particular, for $x \in X$ and a topology τ on X,*

(1) $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ if and only if $(t_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda\in\Lambda_{\mathcal{I}}} \xrightarrow{\mathcal{I}-t} x$ with respect to τ.
(2) $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$ if and only if $(t_\lambda)^{\varphi_{\mathcal{I}}}_{\lambda\in\Lambda_{\mathcal{I}}} \xrightarrow{o} x$.

In [69], in order to succeed, topologies are inserted in posets under conditions in which the ideal-order convergence in posets is topological.

Proposition 22 ([69])**.** *Let X be a set and let $\mathcal{C}_X$ be a class consisting of triads $((s_d)_{d\in D}, x, \mathcal{I})$, where $(s_d)_{d\in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is a non-trivial ideal on D. The family*

$$\{U \subseteq X : \{d \in D : s_d \notin U\} \in \mathcal{I} \text{ for every } ((s_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X,\ x \in U\}$$

is a topology $\tau(\mathcal{C}_X)$ on X.

Proposition 23 ([69])**.** *Let X be a poset. Then the following are satisfied:*

(1) If $((s_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X$, then $(s_d)_{d\in D} \xrightarrow{I-t} x$ with respect to $\tau(\mathcal{C}_X)$;
(2) $\tau(\mathcal{C}_X) = \mathcal{T}^o_X$;
(3) If $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o} x$, then $(s_d)_{d\in D} \xrightarrow{I-t} x$ with respect to $\mathcal{T}^o_X$.

Proposition 24 ([69])**.** *Let X be a poset. The topology $\mathcal{T}^o_X$ is the finest topology τ on X such that ideal-order-convergence implies ideal-topology-convergence with respect to τ.*

Definition 44 ([69]). *The ideal-order-convergence in a poset X is called topological if there exists a topology τ on X such that for every net $(s_d)_{d\in D}$ in X, $x \in X$ and for every non-trivial ideal $\mathcal{I}$ of D, $(s_d)_{d\in D} \xrightarrow{I-o} x$ if and only if $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ.*

Proposition 25 ([69]). *Let X be a poset such that the ideal-order-convergence is topological and let τ be the corresponding topology on X. Then, $\tau \subseteq \mathcal{T}_X^o$.*

Proposition 26 ([69]). *The ideal-order-convergence in a poset X is topological if and only if the order-convergence in X is topological.*

Based on the above results, it is proven that for a poset X, the ideal-order convergence is topological for the $\mathcal{T}_X^o$ topology if and only if X is an S^*-doubly continuous poset [69].

Definition 45 ([34,69]). *Let X be a poset. A net $(s_d)_{d\in D}$ in X is said to $\mathcal{I}$-o_2-converge to a point $x \in X$, where $\mathcal{I}$ is an ideal of D, if there exist subsets M and N of X such that:*

(1) $x = \sup M = \inf N$.
(2) For each $m \in M$ and $n \in N$, $\{d \in D : s_d \notin [m, n]\} \in \mathcal{I}$.

In this case the point x is called the $\mathcal{I}$-o_2-limit of the net $(s_d)_{d\in D}$ and we write $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$.

Some of the basic properties of the ideal-o_2-convergence are given as follows:

(1) The $\mathcal{I}$-o_2-limit is unique.
(2) If $\mathcal{I}$ is a non-trivial ideal on D, then $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$ if and only if $(t_\lambda)^{\varphi}_{\lambda\in\Lambda} \xrightarrow{o_2} x$.
(3) If $\mathcal{I}$ is a non-trivial ideal on D, then the $\mathcal{I}$-order-convergence implies the $\mathcal{I}$-o_2-convergence.

In [69], it is observed that the converse of (3) does not always hold. We consider the poset $(\mathbb{Z}, \leqslant)$, which is represented in Figure 1.

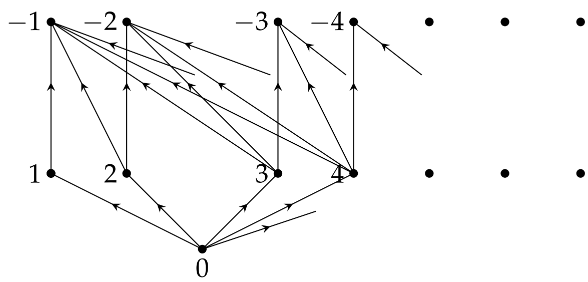

Figure 1. The poset $(\mathbb{Z}, \leqslant)$.

Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. Then the net $(a_n)_{n\in\mathbb{N}}$, where $a_n = n$, $n \in \mathbb{N}$, $\mathcal{I}$-o_2-converges to zero but it does not $\mathcal{I}$-o-converge to zero.

For an arbitrary poset X, we denote by $\mathcal{C}_X^{o_2}$ the class consisting of triads $((s_d)_{d\in D}, x, \mathcal{I})$, where $(s_d)_{d\in D}$ is a net in X, $x \in X$ and $\mathcal{I}$ is a non-trivial ideal on D such that $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$. The corresponding topology $\tau(\mathcal{C}_X^{o_2})$ on X is called the *ideal-o_2-topology* on X [69].

Proposition 27 ([69]). *For any poset X, the following are satisfied:*

(1) $\tau(\mathcal{C}_X^{o_2}) = \mathcal{T}_X^{o_2} \subseteq \mathcal{T}_X^o$;
(2) If $((s_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X^{o_2}$, then $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\tau(\mathcal{C}_X^{o_2})$.

Definition 46 ([34,69]). *The ideal-o_2-convergence in a poset X is called topological, if there exists a topology τ on X such that for every net $(s_d)_{d\in D}$ in X, $x \in X$, and for every ideal $\mathcal{I}$ of D, $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-o_2} x$ if and only if $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ.*

Proposition 28 ([69])**.** *The ideal-o_2-convergence in a poset X is topological if and only if the o_2-convergence in X is topological.*

Based on the above facts, it is proven that the ideal-o_2-convergence in a poset X is topological if and only if X is an $\mathcal{O}_2$-doubly continuous poset. In addition, the ideal-o_2-convergence in every finite lattice, every chain or antichain is topological [69]. In [34], it was proven that the ideal-o_2-convergence on a poset X was topological if and only if X was $\mathcal{O}_2$-doubly continuous, presenting a modification of Kelley's theorem for convergence classes of nets (see Theorem 4). Further, by studying some open subsets of the o_2-topology $\mathcal{T}_X^{o_2}$, the same result followed.Similar to this task, in [34], we have that the ideal–order convergence on a poset P is topological if and only if X is S^*-doubly continuous.

In [34], the ideal-lim-inf-convergence is also introduced as a generalization of lim-inf-convergence on an arbitrary poset.

Definition 47 ([34,69])**.** *A net $(s_d)_{d\in D}$ in a poset X is said to $\mathcal{I}$-*lim-inf*-converge to an element $x \in X$, where $\mathcal{I}$ is an ideal of D, if there exists a directed subset M of X such that the following conditions hold:*

(1) sup *M*exists with* sup *$M \geqslant x$, and*
(2) *for each $m \in M$, $\{d \in D : s_d \ngeqslant m\} \in \mathcal{I}$.*

*In this case the point x is called the $\mathcal{I}$-*lim-inf*-limit of the net $(s_d)_{d\in D}$ and we write $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-\lim-\inf} x$.*

Since a net $(s_d)_{d\in D} \xrightarrow{\lim-\inf} x$ if and only if $(s_d)_{d\in D} \xrightarrow{\mathcal{I}_0(D)-\lim-\inf} x$, the lim-inf-convergence on posets can be considered as a special case of ideal-lim-inf-convergence [34]. On the other side, ideal-lim-inf-convergence can be reduced to lim-inf-convergence, since a net $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-\lim-\inf} x$ if and only if $(t_\lambda)_{\lambda\in\Lambda_I}^{\varphi_I} \xrightarrow{\lim-\inf} x$. [34]

Moreover, in [34], two new topologies on a poset X are introduced as follows:

(1) We denote by $\mathcal{T}_X^{\lim-\inf}$ the family of all subsets U of X which satisfy the property: $(s_d)_{d\in D} \xrightarrow{\lim-\inf} x \in U$ *implies* $(s_d)_{d\in D}$ *is eventually in* U. Then, $\mathcal{T}_X^{\lim-\inf}$ is a topology on X called the lim-inf-*topology* on X.
(2) We denote by $\mathcal{C}_X^{\lim-\inf}$ the class consisting of triads $((s_d)_{d\in D}, x, \mathcal{I})$, where $(s_d)_{d\in D}$ is a net in X, $x \in X$ and $\mathcal{I}$ is an ideal of D such that $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-\lim-\inf} x$. The corresponding topology $\tau(\mathcal{C}_X^{\lim-\inf})$, which consists of all subsets U of X which satisfy the property: $((s_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}_X^{\lim-\inf}$, *where $\mathcal{I}$ is a non-trivial ideal of D and $x \in U$ implies* $\{d \in D : s_d \notin U\} \in \mathcal{I}$, is called the *ideal*-lim-inf-*topology* on X.

Proposition 29 ([34])**.** *Let X be a poset.*

(1) *If $(s_d)_{d\in D} \xrightarrow{\lim-\inf} x \in X$, then $(s_d)_{d\in D}$ converges to x with respect to $\mathcal{T}_X^{\lim-\inf}$.*
(2) *If $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-\lim-\inf} x \in X$, where $\mathcal{I}$ is an ideal of D, then $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to $\tau(\mathcal{C}_X^{\lim-\inf})$.*

Proposition 30 ([34])**.** *Let X be a poset.*

(1) *$\mathcal{T}_X^{\lim-\inf}$ is the finest topology τ on X such that* lim-inf-*convergence implies convergence with respect to τ.*
(2) *$\tau(\mathcal{C}_X^{\lim-\inf})$ is the finest topology τ on X such that ideal-*lim-inf-*convergence implies ideal convergence with respect to τ.*
(3) *$\tau(\mathcal{C}_X^{\lim-\inf}) = \mathcal{T}_X^{\lim-\inf}$.*

Question 13. *Which are the main properties of the topologies $\mathcal{T}_X^{o}$, $\mathcal{T}_X^{o_2}$ and $\mathcal{T}_X^{\lim-\inf}$?*

In [34] the ideal-lim-inf-convergence in a poset X has been studied under the prism of being topological, investigating posets in which this convergence is topological.

Definition 48 ([34,69])**.** *The ideal-*lim-inf*-convergence in a poset X is called topological if there exists a topology τ on X such that for every net $(s_d)_{d\in D}$ in X, $x \in X$ and for every ideal $\mathcal{I}$ of D, $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-\lim-\inf} x$ if and only if $(s_d)_{d\in D} \xrightarrow{\mathcal{I}-t} x$ with respect to τ.*

It is proven that for a poset X, the ideal-lim-inf-convergence is topological if and only if P is a continuous poset [34]. In addition, as observed in [34], we can get similar results if we consider the ideal-lim-inf$_2$-convergence in posets. Particularly, for a poset X, the ideal-lim-inf$_2$-convergence is topological if and only if X is an α-continuous poset.

Question 14. *Define other notions of ideal-convergences. Which are their basic properties?*

Question 15. *Let $\mathcal{P}$ denote any of the new notions of ideal convergences in a poset (by Question 14). Does there exist classes of posets in which this convergence $\mathcal{P}$ is topological?*

Question 16. *What are the main properties of the induced topologies of the new notions of convergences relative to Question 14?*

Question 17. *Let $\mathcal{P}$ denote any of the new notions of ideal convergences in a poset (by Question 14). Does there exist a modification of Kelley's theorem for ideal-convergence classes of nets that gives an alternative approach to proving that this convergence is topological?*

7. Kinds of Convergences in Fuzzy Ordered Sets

A particular interest had been shown in the convergence in fuzzy ordered sets (in short, fosets). The sequential convergence in fuzzy partially ordered sets, under the name o_F-convergence, is known. In particular, using a notion of fuzzy order, the authors in [73] defined and studied a notion of convergence for sequences in the sense of Birkhoff [64]. In this section, we recall notions of convergence in fosets and present some open problems. We recall some basic notions, and we refer to [74–76] for more information on related meanings.

Let X be a nonempty set. A *fuzzy order* on X is a fuzzy set on $X \times X$ whose membership function μ satisfies the following properties:

(1) (reflexivity) for all $x \in X$, $\mu(x,x) = 1$;
(2) (antisymmetry) for all $x, y \in X$, $\mu(x,y) + \mu(y,x) > 1$ implies $x = y$; and,
(3) (transitivity) for all $x, z \in X$, $\mu(x,z) \geqslant \vee_{y\in X}[\mu(x,y) \wedge \mu(y,z)]$.

A set with a fuzzy order defined on it is called a *fuzzy ordered set* (or foset for short.)

Definition 49 ([73,77])**.** *Let X be a foset. We say that a net $(s_d)_{d\in D}$ in X is order-converging or (o_F)-converging to a point $x \in X$ and we write $(s_d)_{d\in D} \xrightarrow{o_F} x$ if there exists a pair of nets $(u_d)_{d\in D}$ and $(v_d)_{d\in D}$, in X, such that:*

(1) $(u_d)_{d\in D}\uparrow x$, $(v_d)_{d\in D}\downarrow x$ and
(2) $\mu(u_d, s_d) > 1/2$ and $\mu(s_d, v_d) > 1/2$, for all $d \in D$.

Some of the basic properties of this convergence are given as follows:

(1) If $(s_n)_{n\in\mathbb{N}}\uparrow$, then $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$ if and only if $(s_n)_{n\in\mathbb{N}}\uparrow x$.
(2) If $(s_n)_{n\in\mathbb{N}}\downarrow$, then $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$ if and only if $(s_n)_{n\in\mathbb{N}}\downarrow x$.
(3) If $\mu(s_n, t_n) > 1/2$, for all $n \in \mathbb{N}$, and $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$, $(t_n)_{n\in\mathbb{N}} \xrightarrow{o_F} y$, then $\mu(x,y) > 1/2$.
(4) If $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$ and $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} y$, then $x = y$.
(5) If $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$, then any subsequence of $(s_n)_{n\in\mathbb{N}}$ o_F-converges to the same limit.

(6) If $\mu(t_n, s_n) > 1/2$ and $\mu(s_n, r_n) > 1/2$, for all $n \in \mathbb{N}$ and $(t_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$, $(r_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$, then $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$.

In [77], a notion of net convergence, with respect to the fuzzy order relation, named o-convergence, is introduced and studied, in the sense of McShane [66].

Definition 50 ([77]). *Let X be a foset. We say that a net $(s_d)_{d\in D}$ in X is o-converging to a point $x \in X$ and we write $(s_d)_{d\in D} \xrightarrow{o} x$ if there exist a directed to the right subset A of X and a directed to the left subset B of X, such that*

(1) $\vee A = \wedge B = x$ and
(2) for every $a \in A$ and every $b \in B$, $\mu(a, s_d) > 1/2$ and $\mu(s_d, b) > 1/2$, eventually.

Proposition 31 ([77]). *Let X be a foset, $(s_d)_{d\in D}$ be a net in X and $x \in X$. If $(s_d)_{d\in D} \xrightarrow{o_F} x$, then $(s_d)_{d\in D} \xrightarrow{o} x$.*

In [77], we can find various examples that show that the converse of the above proposition does not always hold. Further, the o-convergence has similar properties to o_F-convergence.

(1) $(s_d)_{d\in D}\uparrow x$ if and only if $(s_d)_{d\in D}\uparrow x$ is increasing and $(s_d)_{d\in D} \xrightarrow{o} x$.
(2) $(s_d)_{d\in D}\downarrow x$ if and only if $(s_d)_{d\in D}\downarrow x$ is decreasing and $(s_d)_{d\in D} \xrightarrow{o} x$.
(3) If $\mu(s_d, t_d) > 1/2$, for all $d \in D$, and $(s_d)_{d\in D} \xrightarrow{o} x$, $(t_d)_{d\in D} \xrightarrow{o} y$, then $\mu(x, y) > 1/2$.
(4) If $(s_d)_{d\in D} \xrightarrow{o} x$ and $(s_d)_{d\in D} \xrightarrow{o} y$, then $x = y$.
(5) If $(s_d)_{d\in D} \xrightarrow{o} x$, then any subnet of $(s_d)_{d\in D}$ o-converges to the same limit.
(6) If $\mu(t_d, s_d) > 1/2$ and $\mu(s_d, r_d) > 1/2$, for all $d \in D$ and $(t_d)_{d\in D} \xrightarrow{o} x$, $(r_d)_{d\in D} \xrightarrow{o} x$, then $(s_d)_{d\in D} \xrightarrow{o} x$.

The main result of the paper [77] is that the two notions of convergence are identical in the area of complete F-lattices.

Proposition 32 ([77]). *Let $(s_d)_{d\in D}$ be a net in a complete F-lattice X and $x \in X$. Then $(s_d)_{d\in D} \xrightarrow{o} x$ if and only if $(s_d)_{d\in D} \xrightarrow{o_F} x$.*

Question 18. *Can we develop studies of new notions of convergences/statistical or ideal convergences in fosets?*

Question 19. *In which fosets any of the new notions of convergences/statistical or ideal convergences (by Question 18) is topological?*

Question 20. *Can we develop studies of convergence classes/ideal-convergence classes in fosets as a modification of Kelley's theorem? When will such classes be topological?*

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