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Mean-Field and Anticipated BSDEs with Time-Delayed Generator

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Abstract: In this paper, we discuss a new type of mean-field anticipated backward stochastic differential equation with a time-delayed generator (MF-DABSDEs) which extends the results of the anticipated backward stochastic differential equation to the case of mean-field limits, and in which the generator considers not only the present and future times but also the past time. By using the fixed point theorem, we shall demonstrate the existence and uniqueness of the solutions to these equations. Finally, we shall establish a comparison theorem for the solutions.

Keywords: anticipated backward stochastic differential equations; mean-field limits; time-delayed; comparison theorem

MSC: 60H10; 60H20

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1. Introduction

Since Pardoux and Peng [1] first proposed a general form of non-linear backward stochastic differential equations (BSDEs) in 1990, the theoretical research of BSDEs has developed rapidly. In our research, we are looking at the case where there exists a pair of adapted processes $(Y_\cdot, Z_\cdot)$ that satisfy the following type of BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s,\ \ 0 \le t \le T,$$

where ξ is the terminal value, f is the generator related to the present time, and B_s is a Brownian process. In the last three decades, research on BSDEs has seen significant advances in many fields: for example, various BSDE models and the uniqueness and existence of the solutions to these models (Bahlali et al. [2]; Al-Hussein [3]; Zhang et al. [4]), a new nonlinear expectation named g-expectation which is based on BSDEs (Peng [5]; Luo et al. [6]), the numerical solution of BSDEs (Ma et al. [7]; Gobet et al. [8]; Zhao et al. [9]; Han [10]), the relationship between BSDEs and partial differential equations (PDEs) (Ren and Xia [11]; Pardoux and Răşcanu [12]), and the numerous applications of BSDEs in various areas including optimal control, finance, biology, and physics (for examples, refer to [13–17]).

In numerous fields, including economics and finance, statistical mechanics, physics, and game theory, the use of mathematical mean-field approaches is crucial. Buckdahn et al. [18,19] introduced a new type of BSDE, called the mean-field BSDE, and then demonstrated the existence and uniqueness of the solution for that type of mean-field BSDE, which is given by

$$Y_t = \xi + \int_t^T E'\Big[f(s, Y'_s, Z'_s, Y_s, Z_s)\Big]\,\mathrm{d}t - \int_t^T Z_s\mathrm{d}B_s, \quad 0 \le t \le T.$$

Additionally, the authors also showed that in a Markovian setting, mean-field BSDEs generate the viscosity solution of a non-local PDE.

Meanwhile, Peng and Yang [20] initially discussed a fundamental class of BSDEs in 2009, namely, anticipated BSDEs, where

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)})\, \mathrm{d}t - \int_t^T Z_s \mathrm{d}B_s, \quad 0 \le t \le T; \\ Y_t = \xi_t, \quad T \le t \le T+K; \\ Z_t = \eta_t, \quad T \le t \le T+K. \end{cases}$$

The two deterministic $\mathbb{R}^+$-valued continuous functions $\delta(s), \zeta(s)$ defined on $[0, T]$ satisfy (i) $t \le t + \delta(t) \le T + K$, $t \le t + \zeta(t) \le T + K$, and (ii) $\int_t^T f(s + \delta(s))\, \mathrm{d}s \le L \int_t^{T+K} f(s)\mathrm{d}s$, $\int_t^T f(s + \zeta(s))\, \mathrm{d}s \le L \int_t^{T+K} f(s)\mathrm{d}s$; the authors also demonstrated the existence and uniqueness of the solution to the above equations. Feng [21] investigated the uniqueness and existence of the solution of an anticipated BSDE with a reflecting boundary. Wang and Cui [22] also proposed a new type of differential equation called the anticipated backward doubly stochastic differential equation; the authors solved certain stochastic control problems by utilizing the duality between anticipated BSDEs and stochastic differential delay equations. Later, Wang and Yu [23] extended this theory to generalized anticipated backward doubly stochastical differential equations. Henceforth, the amount of study carried out on the combination of mean-field and anticipated BSDEs is progressively growing; for example, Douissi et al. [24] showed a new kind of mean-field anticipated BSDE driven by fractional Brownian motion. Furthermore, Liu and Da [25] focused on mean-field anticipated BSDEs driven by time-changed Lévy noises, while Hao [26] discussed anticipated mean-field BSDEs with jumps.

On the other hand, Delong and Imkeller [27] addressed BSDEs with time-delayed generators as follows:

$$Y_t = \xi + \int_t^T f(s, Y_{s-u(s)}, Z_{s-v(s)})\, \mathrm{d}t - \int_t^T Z_s \mathrm{d}B_s, \quad 0 \le t \le T,$$

where f is a generator that depends on the past value of a solution and $0 \le u(s) \le T$, $0 \le v(s) \le T$.

As a generalisation of Delong and Imkeller [27] or Peng and Yang [20], He et al. [28] investigated a type of delay and anticipated BSDEs. Ma and Liu [29] provided results for the existence and uniqueness of the solution for a mean-field BSDE with an average delay and applied the theoretical results to the study of the infinite-horizon linear-quadratic control issue. Under partial information, Zhuang [30] studied non-zero and differential games for the anticipated forward-backward stochastic differential delay equation, which can be used to resolve a problem involving the management of time-delayed pension funds with non-linear expectations.

However, under the condition of mean-field, the case where the generator considers not only the current time and the future time but also the past time has not been studied yet. Therefore, our study will focus on studying the BSDEs of this case to enrich the theory of BSDEs. This study might then encourage researchers to investigate stochastic optimal control problems more realistically; furthermore, the theory will be useful to connect mean-field BSDEs of this type with non-local PDE.

Based on the motivations discussed above, an essential and meaningful question is that if we construct the mean-field and anticipated BSDEs with a time-delayed generator, how can we prove the existence and uniqueness of its solution? In addition, what about the relative comparison theorem? Firstly, the BSDE model considered in our study is given by

$$\begin{cases} -\mathrm{d}Y_t = E'\Big[f\big(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y'_{t-d_1(t)}, Z'_{t-d_2(t)}, Y_t, Z_t, Y'_t, Z'_t, Y_{t+d_3(t)}, \\ \qquad Z_{t+d_4(t)}, Y'_{t+d_3(t)}, Z'_{t+d_4(t)}\big)\Big]\, \mathrm{d}t - Z_t \mathrm{d}B_t, \quad 0 \le t \le T; \\ Y_t = \xi_t, \quad T \le t \le T+K; \\ Z_t = \eta_t, \quad T \le t \le T+K. \end{cases} \tag{1}$$

The rest of the framework for this study is organised as follows. Section 2 introduces some basic information on the new BSDE model that we are proposing, which is the mean-field anticipated backward stochastic differential equation with a time-delayed generator (MF-DABSDE for short). In Section 3, by using the fixed point theorem, we demonstrate the existence and uniqueness of the solutions for this type of BSDE. Section 4 focuses on studying the comparison theorem of the solutions for this kind of model.

2. Preliminaries

We assume a complete probability space $(\Omega, \mathcal{F}, P)$ with natural filtration $\mathcal{F}_t = \sigma\{W_s, s \leq t\} \vee \mathcal{N}_P, t \in [0,T]$, which is generated by a d-dimensional standard Brownian motion $\{B_t\}_{t\geq 0}$, where $T > 0$ is a fixed real-time horizon and $\mathcal{N}_P$ denotes the set of all P-null subsets and a real-time horizon. We denote the norm in $\mathbb{R}^m$ by $|\cdot|$. To simplify the presentation, we only discuss the one-dimensional case in this study. Consider the following sets:

$$
\begin{aligned}
L^2(\mathcal{F}_t;\mathbb{R}^m) &:= \left\{\varphi : \Omega \to \mathbb{R}^m |\ \varphi \text{ is } \mathcal{F}_t - \text{measurable}, E[|\varphi|^2] < \infty\right\}; \\
L^2_{\mathcal{F}}(0,T;\mathbb{R}^m) &:= \Bigg\{\varphi : [0,T]\times\Omega \to \mathbb{R}^m |\ \varphi \text{ is progressively measurable process,} \\
&\qquad E\left[\int_0^T |\varphi(t)|^2]\,\mathrm{d}t\right] < \infty\Bigg\}; \\
S^2_{\mathcal{F}}(0,T;\mathbb{R}^m) &:= \Bigg\{\phi : [0,T]\times\Omega \to \mathbb{R}^m |\ \varphi \text{ is continuous adapted process,} \\
&\qquad E\left[\sup_{0\leq t\leq T} |\varphi(t)|^2\right] < \infty\Bigg\}.
\end{aligned}
$$

If $m = 1$, we denote the above spaces, respectively, by $L^2(\mathcal{F}_t), L^2_{\mathcal{F}}(0,T)$ and $S^2_{\mathcal{F}}(0,T)$.

In addition, we introduce assumptions about d_i. Let $d_i(\cdot), i = 1,2,3,4$ represent four $\mathbb{R}^+$-valued continuous functions defined on $[0,T]$, and consider the following assumptions:

(D1) There exists a constant $K \geq 0$, such that for all $t \in [0,T]$, $0 \leq t - d_1(t) \leq t$, $0 \leq t - d_2(t) \leq t$, $t \leq t + d_3(t) \leq T + K$, $t \leq t + d_4(t) \leq T + K$;

(D2) There exists a constant $L \geq 0$, such that for all non-negative and integrable $f(\cdot)$, $\int_t^T f(s - d_1(s))\,\mathrm{d}s \leq L\int_t^{T+K} f(s)\mathrm{d}s$, $\int_t^T f(s - d_2(s))\,\mathrm{d}s \leq L\int_t^{T+K} f(s)\mathrm{d}s$, $\int_t^T f(s + d_3(s))\,\mathrm{d}s \leq L\int_t^{T+K} f(s)\mathrm{d}s$, $\int_t^T f(s + d_4(s))\,\mathrm{d}s \leq L\int_t^{T+K} f(s)\mathrm{d}s$.

Then, we introduce the space required by mean-field limits and the assumptions; we first let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) = (\Omega\times\Omega, \mathcal{F}\otimes\mathcal{F}, P\otimes P)$ be the (non-completed) product of $(\Omega, \mathcal{F}, P)$ with itself, and the product space has been filtered by $\bar{\mathbb{F}} = \left\{\bar{\mathcal{F}}_t = \mathcal{F}\otimes\mathcal{F}_t, \quad 0 \leq t \leq T\right\}$. As a random variable originally defined on Ω, $\xi \in L^0(\Omega, \mathcal{F}, P;\mathbb{R}^m)$ is canonically extended to $\bar{\Omega} : \xi'(\omega', \omega) = \xi(\omega'), (\omega', \omega) \in \bar{\Omega} = \Omega\times\Omega$. For any $\varphi \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}), \varphi(.,\omega) : \Omega \to \mathbb{R}$ belongs to $L^1(\Omega, \mathcal{F}, P)$, $P(\mathrm{d}\omega)$-a.s.; and then we denote the expectation of $\varphi(.,\omega)$ by

$$E'[\varphi(.,\omega)] = \int_\Omega \varphi(\omega', \omega)P(\mathrm{d}\omega').$$

Note that $E'[\varphi] = E'[\varphi(.,\omega)] \in L^1(\Omega, \mathcal{F}, P)$, and

$$
\begin{aligned}
\bar{E}[\varphi] &= \int_\Omega \varphi \mathrm{d}\bar{\mathrm{P}} \\
&= \int_\Omega E'[(.,\Omega)]P(\mathrm{d}\omega) \\
&= E[E'[\varphi]].
\end{aligned}
$$

Now we observe that the generator of model (1) is $E'[f(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y'_{t-d_1(t)}, Z'_{t-d_2(t)}, Y_t, Z_t, Y'_t, Z'_t, Y_{t+d_3(t)}, Z_{t+d_4(t)}, Y'_{t+d_3(t)}, Z'_{t+d_4(t)})]$, which includes not only the present and the past, but also the future solutions. Because of the preceding notation, we consider the following derivation:

$$
\begin{aligned}
&E'\Big[f(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y'_{t-d_1(t)}, Z'_{t-d_2(t)}, Y_t, Z_t, Y'_t, Z'_t, Y_{t+d_3(t)}, Z_{t+d_4(t)}, Y'_{t+d_3(t)}, Z'_{t+d_4(t)})\Big](\omega)\\
&= E'\Big[f(t, Y_{t-d_1(t)}(\omega), Z_{t-d_2(t)}(\omega), Y'_{t-d_1(t)}, Z'_{t-d_2(t)}, Y_t(\omega), Z_t(\omega), Y'_t, Z'_t, Y_{t+d_3(t)}(\omega),\\
&\quad Z_{t+d_4(t)}(\omega), Y'_{t+d_3(t)}, Z'_{t+d_4(t)})\Big]\\
&= \int_\Omega f(t, Y_{t-d_1(t)}(\omega), Z_{t-d_2(t)}(\omega), Y'_{t-d_1(t)}(\omega'), Z'_{t-d_2(t)}(\omega'), Y_t(\omega), Z_t(\omega), Y'_t(\omega'),\\
&\quad Z'_t(\omega'), Y_{t+d_3(t)}(\omega), Z_{t+d_4(t)}(\omega), Y'_{t+d_3(t)}(\omega'), Z'_{t+d_4(t)}(\omega'))P(\mathrm{d}\omega').
\end{aligned}
$$

Indeed, based on the definition of expectation given above, we can derive the following two special cases:

$$
\begin{aligned}
&E\Big[E'\Big[f(t, Y'_{t-d_1(t)}, Z'_{t-d_2(t)}, Y'_t, Z'_t, Y'_{t+d_3(t)}, Z'_{t+d_4(t)})\Big]\Big]\\
&= E\Big[E\Big[f(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y_t, Z_t, Y_{t+d_3(t)}, Z_{t+d_4(t)})\Big]\Big]\\
&= E\Big[f(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y_t, Z_t, Y_{t+d_3(t)}, Z_{t+d_4(t)})\Big];\\
&E\Big[E'\Big[f(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y_t, Z_t, Y_{t+d_3(t)}, Z_{t+d_4(t)})\Big]\Big]\\
&= E\Big[E\Big[f(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y_t, Z_t, Y_{t+d_3(t)}, Z_{t+d_4(t)})\Big]\Big]\\
&= E\Big[f(t, Y_{t-d_1(t)}, Z_{t-d_2(t)}, Y_t, Z_t, Y_{t+d_3(t)}, Z_{t+d_4(t)})\Big].
\end{aligned}
\tag{2}
$$

Next, we present assumptions about the generator f. Let the mapping $f(t, \omega, u', v', u, v, y', z', y, z, \phi', \psi', \phi, \psi) : [0, T] \times \Omega \times L^2(\mathcal{F}_{r'}, \mathbb{R}^m) \times L^2(\mathcal{F}_r, \mathbb{R}^m) \times L^2(\mathcal{F}_{r'}, \mathbb{R}^m) \times L^2(\mathcal{F}_r, \mathbb{R}^m) \times L^2(\mathcal{F}_{r'}, \mathbb{R}^m) \times L^2(\mathcal{F}_r, \mathbb{R}^m) \to L^2(\mathcal{F}_t, \mathbb{R}^m), s \leq r, r' \leq T + K$, satisfy the following two assumptions:

(H1) There exists a constant $C > 0$, such that for every $t \in [0, T]$, we have

$$
\begin{aligned}
&|f(t, u', v', u, v, y', z', y, z, \phi', \psi', \phi, \psi) - f(t, \bar{u}', \bar{v}', \bar{u}, \bar{v}, \bar{y}', \bar{z}', \bar{y}, \bar{z}, \bar{\phi}', \bar{\psi}', \bar{\phi}, \bar{\psi})|\\
&\leq C\big(|u' - \bar{u}'| + |v' - \bar{v}'| + |u - \bar{u}| + |v - \bar{v}| + |y' - \bar{y}'| + |z' - \bar{z}'| + |y - \bar{y}|\\
&\quad + |z - \bar{z}| + E'\Big[|\phi' - \bar{\phi}'| + |\psi' - \bar{\psi}'|\Big|\mathcal{F}_t\Big] + E\Big[|\phi - \bar{\phi}| + |\psi - \bar{\psi}|\Big|\mathcal{F}_t\Big]\big),
\end{aligned}
$$

where $u', u, \bar{u}', \bar{u} \in L^2_{\mathcal{F}}(0, t; \mathbb{R}^m)$; $v', v, \bar{v}', \bar{v} \in L^2_{\mathcal{F}}(0, t; \mathbb{R}^{m\times d})$; $y', y, \bar{y}', \bar{y} \in \mathbb{R}^m$; $z', z, \bar{z}', \bar{z} \in \mathbb{R}^{m\times d}$; $\phi', \phi, \bar{\phi}', \bar{\phi} \in L^2_{\mathcal{F}}(t, T + K; \mathbb{R}^m)$; and $\psi', \psi, \bar{\psi}', \bar{\psi} \in L^2_{\mathcal{F}}(t, T + K; \mathbb{R}^{m\times d})$;

(H2) $E\Big[\int_0^T |f(t, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)|^2 \mathrm{d_t}\Big] < \infty$, and $f(t, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \in L^2_{\mathcal{F}}(0, T + K; \mathbb{R}^m)$.

We shall now review some basic results of propositions that will be used throughout the paper: Itô's formula, the Burkeholder–Davis–Gundy inequality, and the fixed point theorem. Firstly, as we know, Itô's formula is the most famous formula in stochastic calculus; it was proposed by Kiyosi Itô [31] in 1951 and is frequently used in the field of stochastic differential equations. This formula points out the rules for differentiating the functions of a stochastic process, and it is given below.

Proposition 1 (Øksendal [32], Theorem 4.1.2)**.** *Let X_t be an Itô process given by $\mathrm{d}X_t = u\mathrm{d}t + v\mathrm{d}B_t$, where B_t is a Brownian process and the functions u, v are deterministic functions of time. For any twice differentiable scalar function $g(t,x)$ of two real variables t and x, we have*

$$\mathrm{d}g(t,X_t) = \frac{\partial g}{\partial t}(t,X_t)\mathrm{d}t + \frac{\partial g}{\partial x}(t,X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(\mathrm{d}X_t)^2.$$

Next, the Burkholder-Davis-Gundy inequality is given as follows:

Proposition 2 (Burkholder et al. [33], Theorem 2.3)**.** *For any $1 \le p < \infty$, there exist positive constants c_p and C_p such that, for all local martingales X with $X_0 = 0$ and stopping times τ, the following inequality holds:*

$$c_p\mathbb{E}\Big[[X]_\tau^{p/2}\Big] \le \mathbb{E}\Big[(\sup_{0\le t\le\tau} X_t)^p\Big] \le C_p\mathbb{E}\Big[[X]_\tau^{p/2}\Big].$$

This paper will use the special case of $p = 2$ for the Burkholder-Davis-Gundy inequality.

Lastly, the fixed point theorem is an important principle in mathematics, and there have been several theorems that fall under it, for example, the contraction mapping theorem or Banach theorem, the Brouwer fixed point theorem, the Kakutani fixed-point theorem, Tarski's theorem, and so on. These fixed point theorems often play a key role in proving the existence and uniqueness of fixed points for a self-mapping on complete metric spaces. Interested readers can refer to Granas and Dugundji [34] and Zhou et al. [35]. The contraction mapping theorem, which will be used in this paper, is briefly introduced below.

Proposition 3 (Granas and Dugundji [34], Theorem 1.1)**.** *Let (Y,d) be a complete metric space and $F: Y \to Y$ be contractive. Then F has a unique fixed point u, and $F^n(y) \to u$ for each $y \in Y$.*

3. An Existence and Uniqueness Result for MF-DABSDEs

In this section, our aim is to seek out a pair of processes $(Y_t, Z_t) \in S^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m \times L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^{m\times d})$ satisfying the mean-field BSDEs of model (1). Lemma 3.1 of Peng [5] can be extended to MF-DABSDEs by the following simple deduction.

Lemma 1. *Given a terminal condition $\xi \in L^2(\mathcal{F}_T;\mathbb{R}^m)$, i.e., ξ is a $\mathbb{R}^m$-value $\mathcal{F}_T$-measurable random variable that satisfies $E\big[|\xi|^2\big] < \infty$, and $f_0(t)$ is an $\mathcal{F}_t$-adapted process that satisfies $E\Big[\int_0^T |f_0(t)|^2\,\mathrm{d}t\Big] < \infty$. Therefore, $(y_t, z_t) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m \times \mathbb{R}^{m\times d})$ is a pair of processes that satisfy the following type of BSDEs:*

$$y_t = \xi + \int_t^T E'[f_0(s)]\mathrm{d}s - \int_t^T z_s\mathrm{d}B_t, t\in[0,T].$$

If $f_0(t) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, then $(y_t,z_t) \in S^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m\times d})$. Hence, for $\beta > 0$, which is an argitrary constant, the following estimate can be obtained:

$$|y_0|^2 + E\left[\int_0^T e^{\beta s}\left(\frac{\beta}{2}|y_s|^2 + |z_s|^2\right)\mathrm{d}s\right] \le E\left[e^{\beta T}|\xi|^2\right] + \frac{2}{\beta}E\left[\int_0^T e^{\beta s}\big|E'[f_0(s)]\big|^2\mathrm{d}s\right].$$

We also have

$$E\left[\sup_{0\le t\le T}|y_t|^2\right] \le CE\left[|\xi|^2 + \int_0^T \big|E'[f_0(s)]\big|^2\mathrm{d}s\right] < \infty.$$

Proof. Applying Itô's formula for $e^{\beta s}|Y_s|^2$ for $s \in [t,T]$, one has

$$e^{\beta t}|y_t|^2 + \int_t^T e^{\beta s}(\beta|y_s|^2 + |z_s|^2)\mathrm{d}s = e^{\beta T}|\xi|^2 + 2\int_t^T e^{\beta s}|y_s|E'[f_0(s)]\mathrm{d}s - 2\int_t^T e^{\beta s}|y_s||z_s|\mathrm{d}s.$$

Taking conditional expectation under $\mathcal{F}_t$ and multiplying $e^{-\beta t}$ on both sides of the above equation,

$$\begin{aligned}
&|y_t|^2 + E\left[\int_t^T e^{\beta(s-t)}(\beta|y_s|^2 + |z_s|^2)\mathrm{d}s\Big|\mathcal{F}_t\right] \\
&= E\left[e^{\beta(T-t)}|\xi|^2\Big|\mathcal{F}_t\right] + 2E\left[\int_t^T e^{\beta(s-t)}|y_s|E'[f_0(s)]\mathrm{d}s\Big|\mathcal{F}_t\right] \\
&\leq E\left[e^{\beta(T-t)}|\xi|^2\Big|\mathcal{F}_t\right] + E\left[\int_t^T e^{\beta(s-t)}\left(\frac{\beta}{2}|y_s|^2 + \frac{2}{\beta}\left(E'[f_0(s)]\right)^2\right)\mathrm{d}s\Big|\mathcal{F}_t\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&|y_t|^2 + E\left[\int_t^T e^{\beta(s-t)}\left(\frac{\beta}{2}|y_s|^2 + |z_s|^2\right)\mathrm{d}s\Big|\mathcal{F}_t\right] \\
&\leq E\left[e^{\beta(T-t)}|\xi|^2\Big|\mathcal{F}_t\right] + \frac{2}{\beta}E\left[\int_t^T e^{\beta(s-t)}\left(E'[f_0(s)]\right)^2\mathrm{d}s\Big|\mathcal{F}_t\right].
\end{aligned}$$

When $t = 0$, we have

$$|y_0|^2 + E\left[\int_0^T e^{\beta s}\left(\frac{\beta}{2}|y_s|^2 + |z_s|^2\right)\mathrm{d}s\right] \leq E\left[e^{\beta T}|\xi|^2\right] + \frac{2}{\beta}E\left[\int_0^T e^{\beta s}\left(E'[f_0(s)]\right)^2\mathrm{d}s\right].$$

By using Burkholder–Davis–Gundy inequality, we have

$$E\left[\sup_{0\leq t\leq T}|y_t|^2\right] \leq CE\left[|\xi|^2 + \int_0^T \left|E'[f_0(s)]\right|^2\mathrm{d}s\right] < \infty,$$

where C is a constant that varies with T. Therefore, $y_t \in S^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$. □

Lemma 2. *Suppose $f(s,\cdot)$ satisfies (H1) and (H2) for $s \in [0,T]$. Further, let $y'_{s-d_1(s)}, y_{s-d_1(s)}$, $\bar{y}'_{s-d_1(s)}, \bar{y}_{s-d_1(s)} \in L^2_{\mathcal{F}}(0,t;\mathbb{R}^m)$; $z'_{s-d_2(s)}, z_{s-d_2(s)}, \bar{z}'_{s-d_2(s)}, \bar{z}_{s-d_2(s)} \in L^2_{\mathcal{F}}(0,t;\mathbb{R}^{m\times d})$; $y_s{}', y_s$, $\bar{y}'_s, \bar{y}_s \in \mathbb{R}^m$; $z_s{}', z_s, \bar{z}'_s, \bar{z}_s \in \mathbb{R}^{m\times d}$; $y'_{s+d_3(s)}, y_{s+d_3(s)}, \bar{y}'_{s+d_3(s)}, \bar{y}_{s+d_3(s)} \in L^2_{\mathcal{F}}(t,T+K;\mathbb{R}^m)$; $z'_{s+d_4(s)}, z_{s+d_4(s)}, \bar{z}'_{s+d_4(s)}, \bar{z}_{s+d_4(s)} \in L^2_{\mathcal{F}}(t,T+K;\mathbb{R}^{m\times d})$ for $t \in [0,T]$; and let $d_i(t)$, i = 1,2,3,4, satisfy (D1) and (D2). Hence,*

$$\begin{aligned}
E\Bigg[\int_t^T e^{\beta s}\Bigg|&E'\left[f(s,y'_{s-d_1(s)},z'_{s-d_2(s)},y_{s-d_1(s)},z_{s-d_2(s)},y_s{}',z_s{}',y_s,z_s,y'_{s+d_3(s)},z'_{s+d_4(s)},y_{s+d_3(s)},z_{s+d_4(s)})\right] \\
&-E'\left[f(s,\bar{y}'_{s-d_1(s)},\bar{z}'_{s-d_2(s)},\bar{y}_{s-d_1(s)},\bar{z}_{s-d_2(s)},\bar{y}'_s,\bar{z}'_s,\bar{y}_s,\bar{z}_s,\bar{y}'_{s+d_3(s)},\bar{z}'_{s+d_4(s)},\bar{y}_{s+d_3(s)},\bar{z}_{s+d_4(s)})\right]\Bigg|^2\mathrm{d}s\Bigg] \\
&\leq 24C^2(2L+1)E\left[\int_t^{T+K} e^{\beta s}\left(|\hat{y}_{s-d_1(s)}|^2 + |\hat{z}_{s-d_2(s)}|^2 + |\hat{y}_s,\hat{z}_s|^2 + |\hat{y}_{s+d_3(s)}|^2 + |\hat{z}_{s+d_4(s)}|^2\right)\mathrm{d}s\right],
\end{aligned}$$

where we denote the differences by $\hat{y}_{s-d_1(s)} = y_{s-d_1(s)} - \bar{y}_{s-d_1(s)}$, $\hat{z}_{s-d_2(s)} = z_{s-d_2(s)} - \bar{z}_{s-d_2(s)}$, $\hat{y}_s = y_s - \bar{y}_s$, $\hat{z}_s = z_s - barz_s$, $\hat{y}_{s+d_3(s)} = y_{s+d_3(s)} - \bar{y}_{s+d_3(s)}$, $\hat{z}_{s+d_4(s)} = z_{s+d_4(s)} - \bar{z}_{s+d_4(s)}$.

Proof. From assumptions (H1), (D1), and (D2), Equation (2), and Jensen's inequality, we have

$$
\begin{aligned}
&E\bigg[\int_t^T e^{\beta s}\bigg|E'\Big[f(s,y'_{s-d_1(s)},z'_{s-d_2(s)},y_{s-d_1(s)},z_{s-d_2(s)},y_s{}',z_s{}',y_s,z_s,y'_{s+d_3(s)},z'_{s+d_4(s)},y_{s+d_3(s)},z_{s+d_4(s)})\Big]\\
&\quad-E'\Big[f(s,\bar{y}'_{s-d_1(s)},\bar{z}'_{s-d_2(s)},\bar{y}_{s-d_1(s)},\bar{z}_{s-d_2(s)},\bar{y}'_s,\bar{z}'_s,\bar{y}_s,\bar{z}_s,\bar{y}'_{s+d_3(s)},\bar{z}'_{s+d_4(s)},\bar{y}_{s+d_3(s)},\bar{z}_{s+d_4(s)})\Big]\bigg|^2\mathrm{d}s\bigg]\\
&\le E\bigg[\int_t^T e^{\beta s}E'\bigg[\Big|f(s,y'_{s-d_1(s)},z'_{s-d_2(s)},y_{s-d_1(s)},z_{s-d_2(s)},y_s{}',z_s{}',y_s,z_s,y'_{s+d_3(s)},z'_{s+d_4(s)},y_{s+d_3(s)},z_{s+d_4(s)})\\
&\quad-f(s,\bar{y}'_{s-d_1(s)},\bar{z}'_{s-d_2(s)},\bar{y}_{s-d_1(s)},\bar{z}_{s-d_2(s)},\bar{y}'_s,\bar{z}'_s,\bar{y}_s,\bar{z}_s,\bar{y}'_{s+d_3(s)},\bar{z}'_{s+d_4(s)},\bar{y}_{s+d_3(s)},\bar{z}_{s+d_4(s)})\Big|^2\bigg]\mathrm{d}s\bigg]\\
&\le 24C^2E\bigg[\int_t^T e^{\beta s}\Big(|\hat{y}_{s-d_1(s)}|^2+|\hat{z}_{s-d_2(s)}|^2+|\hat{y}_s|^2+|\hat{z}_s|^2+|\hat{y}_{s+d_3(s)}|^2+|\hat{z}_{s+d_4(s)}|^2\Big)\mathrm{d}s\bigg]\\
&\le 24C^2(2L+1)E\bigg[\int_t^{T+K} e^{\beta s}\Big(|\hat{y}_s|^2+|\hat{z}_s|^2\Big)\mathrm{d}s\bigg].
\end{aligned}
$$

□

Theorem 1. *Suppose that $\xi(t)\in S^2_{\mathcal{F}}(T,T+K;\mathbb{R}^m$ and $\eta(t)\in L^2_{\mathcal{F}}(T,T+K;\mathbb{R}^{m\times d})$ satisfy the conditions (H1) and (H2), and $d_i(t), i=1,2,3,4$, satisfy (D1) and (D2), then there will exist a unique solution $(Y_t,Z_t)_{t\in[0,T+K]}\in S^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^{m\times d})$ for the MF-DABSDEs.*

Proof. Firstly, we define a norm on $L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^{m\times d})$ which is equivalent to the following norm

$$
\|\varphi(\cdot)\|_\beta=\bigg\{E\bigg[\int_0^{T+K}e^{\beta t}|\varphi_t|^2\mathrm{d}t\bigg]\bigg\}^2.
$$

We rewrite the MF-DABSDE given in Equation (1) as

$$
\begin{cases}
Y_t=\xi_T+\displaystyle\int_t^T E'\Big[f(t,Y'_{t-d_1(t)},Z'_{t-d_2(t)},Y_{t-d_1(t)},Z_{t-d_2(t)},Y'_t,Z'_t,Y_t,Z_t,Y'_{t+d_3(t)},\\
\qquad\qquad Z'_{t+d_4(t)},Y_{t+d_3(t)},Z_{t+d_4(t)})\Big]\,\mathrm{d}t-\displaystyle\int_t^T Z_t\mathrm{d}B_t,\quad 0\le t\le T;\\
Y_t=\xi_t,\quad T\le t\le T+K;\\
Z_t=\eta_t,\quad T\le t\le T+K.
\end{cases}
\tag{3}
$$

Then we define the mapping I: $L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times\mathbb{R}^{m\times d})\to L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times\mathbb{R}^{m\times d})$ such that $(Y.,Z.)=\mathrm{I}(\mathrm{y.},\mathrm{z.})$. For an arbitrary pair $(y.,z.),(\bar{y}.,\bar{z}.)\in L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times\mathbb{R}^{m\times d})$, let $(Y.,Z.)=\mathrm{I}(\mathrm{y.},\mathrm{z.})$, $(\bar{Y}.,\bar{Z}.)=\mathrm{I}(\bar{\mathrm{y}}.,\bar{\mathrm{z}}.)$, and we put the differences as follows:

$$
(\hat{Y}.,\hat{Z}.)=(Y.-\bar{Y}.,Z.-\bar{Z}.),\qquad(\hat{y}.,\hat{z}.)=(y.-\bar{y}.,z.-\bar{z}.).
$$

Now we will prove that the pair $(Y_t)\in S^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m),(Z_t)\in L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^{m\times d})$ can solve Equation (3) if and only if it is a fixed point of I.

Applying Itô's formula for $e^{\beta t}|\hat{Y}_t|^2$ we have

$$
\mathrm{d}\Big(e^{\beta t}|\hat{Y}_t|^2\Big)=\beta e^{\beta t}|\hat{Y}_t|^2+2e^{\beta t}|\hat{Y}_t|\mathrm{d}|\hat{Y}_t|+e^{\beta t}\mathrm{d}|\hat{Y}_t|^2.
$$

Combining Equation (1) and the properties of the Itô's integral, and then taking the integral on $[0,T]$, we have

$$
\begin{aligned}
&e^{\beta T}|\hat{Y}_T|^2-|\hat{Y}_0|^2\\
&=\int_0^T e^{\beta t}\Big(\beta|\hat{Y}_t|^2+|\hat{Z}_t|^2\Big)\mathrm{d}t-2\int_0^T e^{\beta t}|\hat{Y}_t|\Big\{E'\Big[f(t,y'_{t-d_1(t)},z'_{t-d_2(t)},y_{t-d_1(t)},z_{t-d_2(t)},\\
&\quad y_t{}',z_t{}',y_t,z_t,y'_{t+d_3(t)},z'_{t+d_4(t)},y_{t+d_3(t)},z_{t+d_4(t)})\Big]-E'\Big[f(t,\bar{y}'_{t-d_1(t)},\bar{z}'_{t-d_2(t)},\bar{y}_{t-d_1(t)},\\
&\quad \bar{z}_{t-d_2(t)},\bar{y}'_t,\bar{z}'_t,\bar{y}_t,\bar{z}_t,\bar{y}'_{t+d_3(t)},\bar{z}'_{t+d_4(t)},\bar{y}_{t+d_3(t)},\bar{z}_{t+d_4(t)})\Big]\Big\}\mathrm{d}t+2\int_0^T e^{\beta t}|\hat{Y}_t||\hat{Z}_t|\mathrm{d}B_t.
\end{aligned}
$$

Rearranging the terms and taking expectations on both sides, we obtain

$$
\begin{aligned}
&|\hat{Y}_0|^2+\beta E\left[\int_0^T e^{\beta t}|\hat{Y}_t|^2\mathrm{d}t\right]+E\left[\int_0^T e^{\beta t}|\hat{Z}_t|^2\mathrm{d}t\right]\\
&=2E\bigg[\int_0^T e^{\beta t}|\hat{Y}_t|\Big\{E'\Big[f(t,y'_{t-d_1(t)},z'_{t-d_2(t)},y_{t-d_1(t)},z_{t-d_2(t)},y_t{}',z_t{}',y_t,z_t,y'_{t+d_3(t)},z'_{t+d_4(t)},\\
&\quad y_{t+d_3(t)},z_{t+d_4(t)})\Big]-E'\Big[f(t,\bar{y}'_{t-d_1(t)},\bar{z}'_{t-d_2(t)},\bar{y}_{t-d_1(t)},\bar{z}_{t-d_2(t)},\bar{y}'_t,\bar{z}'_t,\bar{y}_t,\bar{z}_t,\bar{y}'_{t+d_3(t)},\bar{z}'_{t+d_4(t)},\\
&\quad \bar{y}_{t+d_3(t)},\bar{z}_{t+d_4(t)})\Big]\Big\}\mathrm{d}t\bigg]\\
&\le\frac{\beta}{2}E\left[\int_0^T e^{\beta t}|\hat{Y}_t|^2\mathrm{d}t\right]+\frac{2}{\beta}E\bigg[\int_0^T e^{\beta t}\Big\{E'\Big[f(t,y'_{t-d_1(t)},z'_{t-d_2(t)},y_{t-d_1(t)},z_{t-d_2(t)},y_t{}',z_t{}',\\
&\quad y_t,z_t,y'_{t+d_3(t)},z'_{t+d_4(t)},y_{t+d_3(t)},z_{t+d_4(t)})\Big]-E'\Big[f(t,\bar{y}'_{t-d_1(t)},\bar{z}'_{t-d_2(t)},\bar{y}_{t-d_1(t)},\bar{z}_{t-d_2(t)},\\
&\quad \bar{y}'_t,\bar{z}'_t,\bar{y}_t,\bar{z}_t,\bar{y}'_{t+d_3(t)},\bar{z}'_{t+d_4(t)},\bar{y}_{t+d_3(t)},\bar{z}_{t+d_4(t)})\Big]\Big\}^2\mathrm{d}t\bigg].
\end{aligned}
$$

Rearranging the terms again and applying Lemma 2, we obtain the following estimate:

$$
\begin{aligned}
&E\left[\int_0^T e^{\beta t}\left(\frac{\beta}{2}|\hat{Y}_t|^2+|\hat{Z}_t|^2\right)\mathrm{d}t\right]\\
&\le\frac{2}{\beta}E\bigg[\int_0^T e^{\beta t}\Big\{E'\Big[f(t,y'_{t-d_1(t)},z'_{t-d_2(t)},y_{t-d_1(t)},z_{t-d_2(t)},y_t{}',z_t{}',y_t,z_t,y'_{t+d_3(t)},z'_{t+d_4(t)},y_{t+d_3(t)},z_{t+d_4(t)})\Big]\\
&\quad -E'\Big[f(t,\bar{y}'_{t-d_1(t)},\bar{z}'_{t-d_2(t)},\bar{y}_{t-d_1(t)},\bar{z}_{t-d_2(t)},\bar{y}'_t,\bar{z}'_t,\bar{y}_t,\bar{z}_t,\bar{y}'_{t+d_3(t)},\bar{z}'_{t+d_4(t)},\bar{y}_{t+d_3(t)},\bar{z}_{t+d_4(t)})\Big]\Big\}^2\mathrm{d}t\bigg]\\
&\le\frac{48C^2(2L+1)}{\beta}E\left[\int_0^{T+K}e^{\beta s}\Big(|\hat{y}_s|^2+|\hat{z}_s|^2\Big)\mathrm{d}s\right]. \qquad (4)
\end{aligned}
$$

Finally, by taking $\beta=96C^2(2L+1)+2$, we obtain

$$
E\left[\int_0^{T+K}e^{\beta t}\Big(|\hat{Y}_t|^2+|\hat{Z}_t|^2\Big)\mathrm{d}t\right]\le\frac{1}{2}E\left[\int_0^{T+K}e^{\beta t}\Big(|\hat{y}_t|^2+|\hat{z}_t|^2\Big)\mathrm{d}t\right].
$$

That is,

$$
\big\|(\hat{Y}_\cdot,\hat{Z}_\cdot)\big\|_\beta\le\frac{1}{\sqrt{2}}\|(\hat{y}_\cdot,\hat{z}_\cdot)\|_\beta.
$$

Thus, this mapping I is a contraction mapping on $L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times\mathbb{R}^{m\times d})$ that allows us to apply the fixed point theorem; the mapping I has a unique fixed point. That means Equation (3) has a unique solution $(Y_t,Z_t)\in L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times\mathbb{R}^{m\times d})$ on $[0,T+K]$ such that $\mathrm{I}(y_t,z_t)=(Y_t,Z_t)$. On the other hand, as f satisfies the assumptions (H1) and (H2), and $d_i(t), i=1,2,3,4$, satisfy (D1) and (D2), we have $f(t,y'_{t-d_1(t)},z'_{t-d_2(t)},y_{t-d_1(t)},z_{t-d_2(t)},y_t{}',z_t{}',$

$y_t, z_t, y'_{t+d_3(t)}, z'_{t+d_4(t)}, y_{t+d_3(t)}, z_{t+d_4(t)}) \in L^2_{\mathcal{F}}(0, T+K; \mathbb{R}^m)$. Then, by applying Lemma 1, we obtain $Y_t \in S^2_{\mathcal{F}}(0, T+K; \mathbb{R}^m)$. $\square$

4. Comparison Theorem

In this section, we investigate a comparison theorem of MF-DABSDEs of the one-dimensional kind shown below:

$$\begin{cases} Y_t = \xi_T + \int_t^T E'\Big[f(s, Y'_{s-d_1(s)}, Y_s, Z_s, Y'_{s+d_3(s)})\Big]\,\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \quad 0 \le t \le T; \\ Y_t = \xi_t, \quad T \le t \le T+K; \\ Z_t = \eta_t, \quad T \le t \le T+K. \end{cases}$$

Firstly, we introduce the classical case of the comparison theorem of BSDEs; Lemma 3 refers to Lemma 3.4 of Peng and Yang [20].

Lemma 3. *Let* $(Y^{(1)}_\cdot, Z^{(1)}_\cdot), (Y^{(2)}_\cdot, Z^{(2)}_\cdot)$ *be the solutions of the following classical type of BSDE:*

$$Y^{(j)}_t = \xi^{(j)}_T + \int_t^T f_j(s, Y^{(j)}_s, Z^{(j)}_s)\,\mathrm{d}s - \int_t^T Z^{(j)}_s\,\mathrm{d}B_s, \ \ 0 \le t \le T.$$

Here $j = 1, 2$, *and for* (y, z), $f_j(t, y, z) : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ *satisfies the Lipschitz condition, meaning that for any* $y, \bar{y} \in \mathbb{R}^m$ *and* $z, \bar{z} \in \mathbb{R}^{m\times d}$, *there exists* $C > 0$ *such that* $|f_j(t, y, z) - f_j(t, \bar{y}, \bar{z})| \le C(|y - \bar{y}| + |z - \bar{z}|)$ *and* $f_j(\cdot, 0, 0) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$. *If* $\xi^{(1)} \le \xi^{(2)}, f_1(t, y, z) \le f_2(t, y, z), t \in [0, T], y \in \mathbb{R}^m, z \in \mathbb{R}^{m\times d}$, *then*

$$Y^{(1)}_t \le Y^{(2)}_t, \quad a.e., a.s.$$

Next, let $(Y^{(1)}_\cdot, Z^{(1)}_\cdot), (Y^{(2)}_\cdot, Z^{(2)}_\cdot)$ be the solutions of the two one-dimensional MF-DABSDEs shown below,

$$\begin{cases} Y^{(j)}_t = \xi^{(j)}_T + \int_t^T E'\Big[f_j(s, Y'^{(j)}_{s-d_1(s)}, Y^{(j)}_s, Z^{(j)}_s, Y'^{(j)}_{s+d_3(s)})\Big]\,\mathrm{d}s - \int_t^T Z^{(j)}_s \mathrm{d}B_s, \quad 0 \le t \le T; \\ Y^{(j)}_t = \xi^{(j)}_t, \quad T \le t \le T+K; \\ Z^{(j)}_t = \eta^{(j)}_t, \quad T \le t \le T+K, \end{cases} \tag{5}$$

where $j = 1, 2$. The end outcome is as follows.

Theorem 2. *Suppose* $f_j(t, \cdot), j = 1, 2$ *satisfy the assumptions (H1) and (H2),* $\xi^{(j)}_t \in S^2_{\mathcal{F}}(T, T+K; \mathbb{R}^m)$, *and* $d_i(t), i = 1, 2, 3, 4$ *satisfy (D1) and (D2). Moreover, assume that*

(i) $f_1(t, u', y, z, \phi')$ *is increasing in* u' *and* ϕ';
(ii) $\xi^{(1)}_t \le \xi^{(2)}_t$;
(iii) $f_1(t, y'_{t-d_1(t)}, y_t, z_t, y'_{t+d_3(t)}) \le f_2(t, y'_{t-d_1(t)}, y_t, z_t, y'_{t+d_3(t)})$, $y'_{t-d_1(t)} \in L^2_{\mathcal{F}}(0, t)$, $y'_{t+d_3(t)} \in L^2_{\mathcal{F}}(t, T+K)$.

It is then true that $Y^{(1)}_t \le Y^{(2)}_t$ *almost surely.*

Proof. Since $(Y^{(1)}_\cdot, Z^{(1)}_\cdot)$ is the solution of the one-dimensional MF-DABSDE given in Equation (5), we have

$$\begin{cases} Y^{(1)}_t = \xi^{(1)}_T + \int_t^T E'\Big[f_1(s, Y'^{(1)}_{s-d_1(s)}, Y^{(1)}_s, Z^{(1)}_s, Y'^{(1)}_{s+d_3(s)})\Big]\,\mathrm{d}s - \int_t^T Z^{(1)}_s \mathrm{d}B_s, \quad 0 \le t \le T; \\ Y^{(1)}_t = \xi^{(1)}_t, \quad T \le t \le T+K. \end{cases} \tag{6}$$

Next, we consider the following BSDEs:

$$\begin{cases} Y_t^{(3)} = \xi_T^{(2)} + \int_t^T E'\Big[f_2(s, Y'^{(1)}_{s-d_1(s)}, Y_s^{(3)}, Z_s^{(3)}, Y'^{(1)}_{s+d_3(s)})\Big]\,\mathrm{d}s - \int_t^T Z_s^{(3)}\mathrm{d}B_s, \quad 0 \le t \le T; \\ Y_t^{(3)} = \xi_t^{(2)}, \quad T \le t \le T+K. \end{cases} \tag{7}$$

From the classical existence and uniqueness theorem of classical BSDEs (Peng 2004, Theorem 3.2), we know there exists a unique solution $(Y_t^{(3)}, Z_t^{(3)}) \in S^2_{\mathcal{F}}(0,T) \times L^2_{\mathcal{F}}(0,T)$. Considering Equations (6) and (7), as $\xi_t^{(1)} \le \xi_t^{(2)}$, $f_1(s, Y'^{(1)}_{s-d_1(s)}, Y_s^{(1)}, Z_s^{(1)}, Y'^{(1)}_{s+d_3(s)}) \le f_2(s, Y'^{(1)}_{s-d_1(s)}, Y_s^{(3)}, Z_s^{(3)}, Y'^{(1)}_{s+d_3(s)})$, by Lemma 3, we have

$$Y_t^{(1)} \le Y_t^{(3)} \ \ a.e., a.s.$$

Set

$$\begin{cases} Y_t^{(4)} = \xi_T^{(2)} + \int_t^T E'\Big[f_2(s, Y'^{(3)}_{s-d_1(s)}, Y_s^{(4)}, Z_s^{(4)}, Y'^{(3)}_{s+d_3(s)})\Big]\,\mathrm{d}s - \int_t^T Z_s^{(4)}\mathrm{d}B_s, \quad 0 \le t \le T; \\ Y_t^{(4)} = \xi_t^{(2)}, \quad T \le t \le T+K. \end{cases} \tag{8}$$

Consider Equations (7) and (8); $f_1(t, u', y, z, \phi')$ is increasing in u' and ϕ', and $Y_t^{(1)} \le Y_t^{(3)}$, which imply $f_2(s, Y'^{(1)}_{s-d_1(s)}, Y_s^{(3)}, Z_s^{(3)}, Y'^{(1)}_{s+d_3(s)}) \le f_2(s, Y'^{(3)}_{s-d_1(s)}, Y_s^{(4)}, Z_s^{(4)}, Y'^{(3)}_{s+d_3(s)})$. Similar to the above, we have

$$Y_t^{(3)} \le Y_t^{(4)} \ \ a.e., a.s.$$

For $n = 5, 6, \dots$, we consider the following BSDEs:

$$\begin{cases} Y_t^{(n)} = \xi_T^{(2)} + \int_t^T E'\Big[f_2(s, Y'^{(n-1)}_{s-d_1(s)}, Y_s^{(n)}, Z_s^{(n)}, Y'^{(n-1)}_{s+d_3(s)})\Big]\,\mathrm{d}s - \int_t^T Z_s^{(n)}\mathrm{d}B_s, \quad 0 \le t \le T; \\ Y_t^{(n)} = \xi_t^{(2)}, \quad T \le t \le T+K. \end{cases}$$

Similarly, we obtain

$$Y_t^{(4)} \le Y_t^{(5)} \le \dots \le Y_t^{(n-1)} \le Y_t^{(n)} \le \dots, \ \ a.s.$$

Next, we will show that $n \ge 4$, $Y_t^{(n)}$ and $Z_t^{(n)}$ are, respectively, Cauchy sequences. Denote $\hat{Y}_t^{(n)} := Y_t^{(n)} - Y_t^{(n-1)}$, $\hat{Z}_t^{(n)} := Z_t^{(n)} - Z_t^{(n-1)}$, $n \ge 4$, then from estimate (4), we obtain

$$\begin{aligned} E\Big[\int_0^T e^{\beta t}\Big(\frac{\beta}{2}|\hat{Y}_t^{(n)}|^2 + |\hat{Z}_t^{(n)}|^2\Big)\mathrm{d}t\Big] \le \frac{2}{\beta}E\Big[\int_0^T e^{\beta t}\Big\{E'\Big[f_2(t, Y'^{(n-1)}_{s-d_1(s)}, Y_t^{(n)}, Z_t^{(n)}, Y'^{(n-1)}_{s+d_3(s)}) \\ - f_2(t, Y'^{(n-2)}_{s-d_1(s)}, Y_t^{(n-1)}, Z_t^{(n-1)}, Y'^{(n-2)}_{s+d_3(s)})\Big]\Big\}^2 \mathrm{d}t\Big]. \end{aligned}$$

When we apply Jensen's inequality, assumptions (H1), (D1) and (D2), and the fact that $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$, one has

$$\begin{aligned} & E\Big[\int_0^T e^{\beta t}\Big(\frac{\beta}{2}|\hat{Y}_t^{(n)}|^2 + |\hat{Z}_t^{(n)}|^2\Big)\mathrm{d}t\Big] \\ & \le \frac{8C^2(2L+1)}{\beta}E\Big[\int_0^T e^{\beta t}\Big(|\hat{Y}_t^{(n-1)}|^2 + \hat{Y}_t^{(n)}|^2 + \hat{Z}_t^{(n)}|^2 + \hat{Y}_t^{(n-1)}|^2\Big)\mathrm{d}t. \end{aligned}$$

Let $\beta = 32C^2(2L+1) + 2$, then we obtain

$$E\left[\int_0^T e^{\beta t}\left(|\hat{Y}_t^{(n)}|^2+|\hat{Z}_t^{(n)}|^2\right)\mathrm{d}t\right]\leq\frac{1}{4}E\left[\int_0^T e^{\beta t}\left(|\hat{Y}_t^{(n)}|^2+\hat{Z}_t^{(n)}|^2\right)\mathrm{d}t\right]+\frac{1}{2}E\left[\int_0^T e^{\beta t}|\hat{Y}_t^{(n-1)}|^2\mathrm{d}t\right].$$

Hence,

$$\begin{aligned}E\left[\int_0^T e^{\beta t}\left(|\hat{Y}_t^{(n)}|^2+|\hat{Z}_t^{(n)}|^2\right)\mathrm{d}t\right]&\leq\frac{2}{3}E\left[\int_0^T e^{\beta t}|\hat{Y}_t^{(n-1)}|^2\mathrm{d}t\right]\\&\leq\frac{2}{3}E\left[\int_0^T e^{\beta t}\left(|\hat{Y}_t^{(n-1)}|^2+|\hat{Z}_t^{(n-1)}|^2\right)\mathrm{d}t\right].\end{aligned}$$

Therefore,

$$E\left[\int_0^T e^{\beta t}\left(|\hat{Y}_t^{(n)}|^2+|\hat{Z}_t^{(n)}|^2\right)\mathrm{d}t\right]\leq\left(\frac{2}{3}\right)^{n-4}E\left[\int_0^T e^{\beta t}\left(|\hat{Y}_t^{(4)}|^2+|\hat{Z}_t^{(4)}|^2\right)\mathrm{d}t\right].$$

This means $(\hat{Y}_t^{(n)},\hat{Z}_t^{(n)})_{n\geq 4}$ is Cauchy sequence in $L^2_{\mathcal{F}}(0,T+K)\times L^2_{\mathcal{F}}(0,T)$. Let the limit of $(\hat{Y}_t^{(n)},\hat{Z}_t^{(n)})$ be $(Y_\cdot,Z_\cdot)$ for all $0\leq t\leq T$, when $n\to\infty$, hence

$$\begin{aligned}&E\left[\int_t^T e^{\beta s}\left(E'\left[f_2(s,Y'^{(n-1)}_{s-d_1(s)},Y_s^{(n)},Z_s^{(n)},Y'^{(n-1)}_{s+d_3(s)})\right]-E'\left[f_2(s,Y'_{s-d_1(s)},Y_s,Z_s,Y'_{s+d_3(s)})\right]\right)\mathrm{d}s\right]\\&\quad\leq 4C^2E\left[\int_t^T e^{\beta s}\left(|Y_s^{(n)}-Y_s|^2+|Z_s^{(n)}-Z_s|^2+2L|Y_s^{(n-1)}-Y_s|^2\right)\right]\longrightarrow 0.\end{aligned}$$

Thus, (Y_t,Z_t) is a solution of the following MF-DABSDEs:

$$\begin{cases}Y_t=\xi_T^{(2)}+\int_t^T E'\left[f_2(s,Y'_{s-d_1(s)},Y_s,Z_s,Y'_{s+d_3(s)})\right]\mathrm{d}s-\int_t^T Z_s\mathrm{d}B_s, & 0\leq t\leq T;\\ Y_t=\xi_t^{(2)}, \quad T\leq t\leq T+K.\end{cases}$$

Then, by Theorem 1 on the uniqueness of the solution, we know that

$$Y_t=Y_t^{(2)},a.s.$$

Since

$$Y_t^{(1)}\leq Y_t^{(3)}\leq Y_t^{(4)}\leq Y_t,$$

then we obtain the desired result $Y_t^{(1)}\leq Y_t^{(2)},a.s.$ □

5. Conclusions

Our study contributes to the introduction of a new type of BSDE, the mean-field anticipated BSDE with a time-delayed generator, and uses the fixed point theorem, which is more convenient than another method (Picard's iterative method), to prove the existence and uniqueness of the solution to this class of equations. Moreover, a comparison theorem is also obtained. A potential limitation of this study, when compared with the core work of Peng and Yang [20], stems from the fact that it involves mean-field limits and a more general generator f, which necessitates more elaborate steps. Also, this paper is slightly more demanding in terms of assumptions because of the simpler fixed point theorem method. Therefore, as a follow-up study and as the application of this paper, we aim to establish the relationship between the MF-DABSDEs and a nonlocal partial differential equation. In addition, it should be pointed out that, similar to the study of mean-field anticipated BSDEs driven by fractional Brownian motion, theoretically, our equation can also be applied to stochastic optimal control problems. In the future, further research may be conducted on this topic utilising broader assumptions and simpler approaches.

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References

1. Pardoux, E.; Peng, S. Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.* **1990**, *14*, 55–61. [CrossRef]
2. Bahlali, K.; Essaky, E.H.; Oukine, Y. Reflected backward stochastic differential equation with jumps and locally Lipschitz coefficient. *Random Oper. Stoch. Equ.* **2002**, *10*, 481–486. [CrossRef]
3. Al-Hussein, A. Backward stochastic partial differential equations driven by infinite-dimensional martingales and applications. *Stochastics* **2009**, *81*, 601–626. [CrossRef]
4. Zhang, P.; Ibrahim, A.I.N.; Mohamed, N.A. Backward Stochastic Differential Equations (BSDEs) Using Infinite-Dimensional Martingales with Subdifferential Operator. *Axioms* **2022**, *11*, 536. [CrossRef]
5. Peng, S. Nonlinear Expectations, Nonlinear Evaluations and Risk Measures. In *Stochastic Methods in Finance*; Springer: Berlin, Germany, 2004; pp. 165–253. [CrossRef]
6. Luo, M.; Fečkan, M.; Wang, J.R.; O'Regan, D. g-Expectation for Conformable Backward Stochastic Differential Equations. *Axioms* **2022**, *11*, 75. [CrossRef]
7. Ma, J.; Protter, P.; Martín, J.S.; Torres, S. Numberical Method for Backward Stochastic Differential Equations. *Ann. Appl. Probab.* **2002**, *12*, 302–316. [CrossRef]
8. Gobet, E.; Lemor, J.P.; Warin, X. A regression-based Monte Carlo method to solve backward stochastic differential equations. *Ann. Appl. Probab.* **2005**, *15*, 2172–2202. [CrossRef]
9. Zhao, W.; Zhang, W.; Ju, L. A Numerical Method and its Error Estimates for the Decoupled Forward-Backward Stochastic Differential Equations. *Commun. Comput. Phys.* **2014**, *15*, 618–646. [CrossRef]
10. Han, Q. Variable Step Size Adams Methods for BSDEs. *J. Math.* **2021**, *2021*, 9799627. [CrossRef]
11. Ren, Y.; Xia, N. Generalized Reflected BSDE and an Obstacle Problem for PDEs with a Nonlinear Neumann Boundary Condition. *Stoch. Anal. Appl.* **2006**, *24*, 1013–1033. [CrossRef]
12. Pardoux, E.; Răşcanu, A. Backward Stochastic Differential Equations. In *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations*; Springer: New York, NY, USA, 2014; pp. 353–515. [CrossRef]
13. Karoui, N.E.; Peng, S.; Quenez, M.C. Backward stochastic differential equations in finance. *Math. Financ.* **1997**, *7*, 1–71. [CrossRef]
14. Peng, S.; Wu, Z. Fully Coupled Forward-Backward Stochastic Differential Equations and Applications to Optimal Control. *SIAM J. Control Optim.* **1999**, *37*, 825–843. [CrossRef]
15. El Asri, B.; Hamadene, S.; Oufdil, K. On the stochastic control-stopping problem. *J. Differ. Equ.* **2022**, *336*, 387–426. [CrossRef]
16. Perninge, M. Sequential Systems of Reflected Backward Stochastic Differential Equations with Application to Impulse Control. *Appl. Math. Optim.* **2022**, *86*, 19. [CrossRef]
17. Li, J.; Peng, S. Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton–Jacobi–Bellman equations. *Nonlinear Anal.* **2009**, *70*, 1776–1796. [CrossRef]
18. Buckdahn, R.; Djehiche, B.; Li, J.; Peng, S. Mean-field backward stochastic differential equations: A limit approach. *Ann. Appl. Probab.* **2009**, *37*, 1524–1565. [CrossRef]
19. Buckdahn, R.; Li, J.; Peng, S. Mean-field backward stochastic differential equations and related partial differential equations. *Stoch. Process. Appl.* **2009**, *119*, 3133–3154. [CrossRef]
20. Peng, S.; Yang, Z. Anticipated backward stochastic differential equations. *Ann. Appl. Probab.* **2009**, *37*, 877–902. [CrossRef]
21. Feng, X. Anticipated Backward Stochastic Differential Equation with Reflection. *Commun. Stat.-Simul. Comput.* **2016**, *45*, 1676–1688. [CrossRef]
22. Wang, T.; Cui, S. Anticipated Backward Doubly Stochastic Differential Equations with Non-Lipschitz Coefficients. *Mathematics* **2022**, *10*, 396. [CrossRef]
23. Wang, T.; Yu, J. Anticipated Generalized Backward Doubly Stochastic Differential Equations. *Symmetry* **2022**, *14*, 114. [CrossRef]
24. Douissi, S.; Wen, J.; Shi, Y. Mean-field anticipated BSDEs driven by fractional Brownian motion and related stochastic control problem. *Appl. Math. Comput.* **2019**, *355*, 282–298. [CrossRef]
25. Liu, Y.; Dai, Y. Mean-field anticipated BSDEs driven by time-changed Lévy noises. *Adv. Differ. Equ.* **2020**, *2020*, 621. [CrossRef]
26. Hao, T. Anticipated mean-field backward stochastic differential equations with jumps. *Lith. Math. J.* **2020**, *60*, 359–375. [CrossRef]

27. Delong, Ł.; Imkeller, P. Backward stochastic differential equations with time delayed generators—Results and counterexamples. *Ann. Appl. Probab.* **2010**, *20*, 1512–1536. [CrossRef]
28. He, P.; Ren, Y.; Zhang, D. A Study on a New Class of Backward Stochastic Differential Equation. *Math. Probl. Eng.* **2020**, *2020*, 1518723. [CrossRef]
29. Ma, H.; Liu, B. Infinite horizon optimal control problem of mean-field backward stochastic delay differential equation under partial information. *Eur. J. Control* **2017**, *36*, 43–50. [CrossRef]
30. Zhuang, Y. Non-zero sum differential games of anticipated forward-backward stochastic differential delayed equations under partial information and application. *Adv. Differ. Equ.* **2017**, *2017*, 383. [CrossRef]
31. Itô, K. On stochastic differential equations. *Mem. Am. Math. Soc.* **1951**, *4*, 1–51. [CrossRef]
32. Øksendal, B. *Stochastic Differential Equations: An Introduction with Applications*, 6th ed.; Springer: New York, NY, USA, 2003; ISBN 978-3-642-14394-6.
33. Burkholder, D.L.; Davis, B.J.; Gundy, R.F. Integral inequalities for convex functions of operators on martingales. *Proc. Sixth Berkeley Symp. Math. Stat. Prob.* **1972**, *2*, 223–240. [CrossRef]
34. Granas, A.; Dugundji, J. *Fixed Point Theory*, 2003rd ed.; Springer: New York, NY, USA, 2003; ISBN 978-0387001739.
35. Zhou, Z.; Bambos, N.; Glynn, P. Deterministic and Stochastic Wireless Network Games: Equilibrium, Dynamics, and Price of Anarchy. *Oper. Res.* **2018**, *66*, 1498–1516. [CrossRef]