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Exact and Numerical Analysis of the Pantograph Delay Differential Equation via the Homotopy Perturbation Method

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Abstract: The delay differential equations are of great importance in real-life phenomena. A special type of these equations is the Pantograph delay differential equation. Generally, solving a delay differential equation is a challenge, especially when the complexity of the delay terms increases. In this paper, the homotopy perturbation method is proposed to solve the Pantograph delay differential equation via two different canonical forms; thus, two types of closed-form solutions are determined. The first gives the standard power series solution while the second introduces the exponential function solution. It is declared that the current solution agrees with the corresponding ones in the literature in special cases. In addition, the properties of the solution are provided. Furthermore, the results are numerically validated through performing several comparisons with the available exact solutions. Moreover, the calculated residuals tend to zero, even in a huge domain, which reflects the high accuracy of the current analysis. The obtained results reveal the effectiveness and efficiency of the current analysis which can be further extended to other types of delay equations.

Keywords: pantograph equation; delay equation; homotopy perturbation method; series solution; exact solution

MSC: 34k06

1. Introduction

The pantograph is a particular device which collects the current in electric trains. The pantograph problem has been extensively studied in the literature. Andrews [1] calculated the behavior of an overhead catenary system for railway electrification. Abbott [2] introduced a numerical method for calculating the dynamic behavior of a trolley wire overhead contact system for electric railways. The Pantograph motion on a nearly uniform railway overhead line and system has been discussed in [3–5], respectively. The Pantograph’s model has been investigated by Kato and McLeod [6], Iserles [7], Derfel and Iserles [8], Patade and Bhaulekar [9], Fox et al. [10], and recently by Alenazy et al. [11]. However, this problem still has undiscovered properties as will be declared in this paper. Although a considerable amount of real-world problems were modeled by means of ordinary differential equations (ODEs), it is observed that such ODEs cannot model the actual behavior of natural systems. This is because the ordinary derivative is a local operator which cannot be used to model the memory/hereditary properties in real-life phenomena. Hence, physical phenomena can be accurately modeled by incorporating nonlocal components such as delays. Obtaining an exact/approximate solution of an ODE is regularly easier in contrast with a delay differential equation (DDE). As simple examples, it is easy to find the exact solution of the ODE \( y'(t) - y(t) = 0 \) by any of the standard methods while finding solutions for the very simple DDEs \( y'(t) - y(-t) = 0, \ y'(t) - y(2t) = 0, \ y'(t) - y\left(\frac{t}{2}\right) = 0 \) is not an easy task. Actually, there are no standard methods for solving...
DDEs and, accordingly, the DDEs are difficult to solve when compared with ODEs. Despite the excellency of the obtained results in the literature for the pantograph delay differential equation (PDDE) [6]:

\[
y'(t) = a\, y(t) + b\, y(ct), \quad y(0) = \lambda, \quad b \neq 0, \quad c \in \mathbb{R} - \{1\},
\]

the authors believe that the PDDE (1) still needs an effort to determine its exact solution for all possible real values of parameters \(a, b, c, \) and \(\lambda\). As a special case, Equation (1) reduces to the standard Ambartsumian delay differential equation (ADDE) when \(a = -1\) and \(b = c = \frac{1}{q}\) \((q > 1)\) which has been used for describing the theory of surface brightness in the Milky Way [12]. The ADDE has been investigated in the literature using different methodologies such as the Taylor series [13] and the exponential function solution [14–16].

It was also generalized using different approaches by Khaled et al. [17] and Kumar et al. [18]. Moreover, Ebaid and Al-Jeaid [19] obtained the exact periodic solution for Equation (1) for the special case \(c = -1\) such that \(b > a\). In addition, they determined the exact solution when \(a = b\) and \(c = -1\). The purpose of this paper is to obtain the closed-form solution of the PDDE for arbitrary values of the proportional delay parameter \(c\).

In order to contribute to an improved series solution of the PDDE (1), it may be reasonable to implement the Adomian decomposition method (ADM) [20] which was effectively applied on BVPs [21], Fisher’s equation [22], delay-type equation [23], nonlinear equations [24,25] with the bibliography [26], and Kepler’s equation [27]. However, the ADM needs to calculate what is called the Adomian’s polynomials.

On the other hand, the homotopy perturbation method (HPM) is a relatively recent method, see [28–34]. It will be shown that the HPM is an effective tool to deal with the current PDDE through utilizing two different canonical forms. The first provides the standard power series solution (PSS) while the second gives the exponential function solution (EFS). In addition, it will be declared that the current PSS agrees with the corresponding PSS in [10] when \(\lambda = 1\) and also agrees with the EFS in [15,16] when \(a = -1, b = c = \frac{1}{q}\) \((q > 1)\) for the ADDE. Furthermore, the properties of the EFS, in terms of exponential functions, will be provided. Moreover, several comparisons between the EFS and the available exact ones in the literature are to be performed and, accordingly, the advantages of the present analysis over those in previously published papers will be demonstrated.

The paper has the following structure. In Section 2, the PSS is derived, where the existing results in the literature are obtained for particular cases. Section 3 focuses on evaluating the EFS while its characteristics are discussed in Section 4. Section 5 applies the quantum calculus notations on a general-component formula. Section 6 is devoted to analyze and discuss the results which are concluded in Section 7.

2. The first Canonical Form: PSS

In order to apply the HPM on the PDDE (1), we first rewrite Equation (1) in the following canonical form:

\[
y'(t) = p[ay(t) + by(ct)],
\]

where \(0 < p \leq 1\) is an embedding parameter which is used to construct the homotopy series solution:

\[
y(t) = \sum_{n=0}^{\infty} p^n y_n(t).
\]

On substituting Equation (3) into Equation (2), we have

\[
\sum_{n=0}^{\infty} p^n y'_n(t) = \sum_{n=0}^{\infty} p^{n+1}[ay_n(t) + by_n(ct)],
\]

or

\[
y_0(t) + \sum_{n=1}^{\infty} p^n y'_n(t) = \sum_{n=0}^{\infty} p^{n+1}[ay_n(t) + by_n(ct)],
\]
i.e.,
\[ y'_0(t) + \sum_{n=1}^{\infty} p^n y'_n(t) = \sum_{n=1}^{\infty} p^n [a y_{n-1}(t) + b y_{n-1}(c t)], \]  
(6)

Equation (6) leads to the following systems of initial value problems (IVPs):
\[ y'_0(t) = 0, \quad y_0(0) = \lambda, \]
(7)

and
\[ y'_n(t) = a y_{n-1}(t) + b y_{n-1}(c t), \quad y_n(0) = 0, \quad n \geq 1. \]
(8)
The solution of Equation (7) is given by
\[ y_0(t) = \lambda. \]
(9)

From Equation (8), the \( n \)th-order component can be obtained as
\[ y_n(t) = y_n(0) + \int_0^t [a y_{n-1}(\tau) + b y_{n-1}(c \tau)] d\tau, \quad n \geq 1. \]
(10)

Since \( y_n(0) = 0, \forall n \geq 1 \), then
\[ y_n(t) = \int_0^t [a y_{n-1}(\tau) + b y_{n-1}(c \tau)] d\tau, \quad n \geq 1. \]
(11)

Utilizing Equation (11), we obtain
\[ y_1(t) = \int_0^t [a y_0(\tau) + b y_0(c \tau)] d\tau = \int_0^t (a \lambda + b \lambda) d\tau = \lambda(a + b) t, \]
(12)

\[ y_2(t) = \int_0^t [a y_1(\tau) + b y_1(c \tau)] d\tau = \int_0^t [a \lambda(a + b) \tau + b \lambda(a + b) c \tau] d\tau \]
\[ = \lambda(a + b)(a + b c) t^2 / 2! = \lambda \prod_{k=1}^{2} (a + b c^{k-1}) \frac{t^2}{2!}, \]
(13)

\[ y_3(t) = \lambda(a + b)(a + b c)(a + b c^2) t^3 / 3! = \lambda \prod_{k=1}^{3} (a + b c^{k-1}) \frac{t^3}{3!}, \]
(14)

\[ y_4(t) = \lambda(a + b)(a + b c)(a + b c^2)(a + b c^3) t^4 / 4! = \lambda \prod_{k=1}^{4} (a + b c^{k-1}) \frac{t^4}{4!}, \]
(15)

\[ \vdots \]

\[ y_n(t) = \lambda(a + b)(a + b c)(a + b c^2) \ldots (a + b c^{n-1}) t^n / n! = \lambda \prod_{k=1}^{n} (a + b c^{k-1}) \frac{t^n}{n!}. \]
(16)
The HPM gives the solution as \( p \to 1 \) by
\[ y(t) = \lim_{p \to 1} \sum_{n=1}^{\infty} p^n y_n = \sum_{n=0}^{\infty} y_n = y_0 + \sum_{n=1}^{\infty} y_n, \]
\[ = \lambda + \lambda \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} (a + b c^{k-1}) \frac{t^n}{n!} \right), \]
\[ = \lambda \left[ 1 + \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} (a + b c^{k-1}) \frac{t^n}{n!} \right) \right]. \]
(17)
At \( \lambda = 1 \), the solution (17) becomes
\[
y(t) = 1 + \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} (a + bc^{k-1}) \right) \frac{t^n}{n!},
\]
which is the same closed-form series solution obtained by Fox et al. [10]. In addition, if \( a = -1 \) and \( b = c = \frac{1}{q} \) \( q > 1 \), then the PDDE (1) becomes the ADDE:
\[
y'(t) = -y(t) + \frac{1}{q} y \left( \frac{1}{q} \right), \quad y(0) = \lambda.
\]
Substituting \( a = -1 \) and \( b = c = \frac{1}{q} \) into Equation (17) gives
\[
y(t) = \lambda \left[ 1 + \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} (1 + q^{-k}) \right) \frac{t^n}{n!} \right],
\]
i.e.,
\[
y(t) = \lambda \left[ 1 + \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} (q^{-k} - 1) \right) \frac{t^n}{n!} \right],
\]
which is the same solution obtained in [15,16].

3. The Second Canonical Form: EPS

In this section, we start with transforming Equation (1) into the following equivalent integral equation:
\[
y(t) = \lambda e^{at} + b e^{at} \int_{0}^{t} e^{-a\tau} y(c\tau) d\tau.
\]
The canonical form of the HPM, in this case, is of the form:
\[
y(t) = \lambda e^{at} + p \left( b e^{at} \int_{0}^{t} e^{-a\tau} y(c\tau) d\tau \right).
\]
Inserting Equation (3) into Equation (23) yields
\[
\sum_{n=0}^{\infty} p^n y_n(t) = \lambda e^{at} + b e^{at} \int_{0}^{t} e^{-a\tau} \sum_{n=0}^{\infty} p^{n+1} y_n(c\tau) d\tau,
\]
or
\[
y_0(t) + \sum_{n=1}^{\infty} p^n y_n(t) = \lambda e^{at} + \sum_{n=1}^{\infty} p^n \left( b e^{at} \int_{0}^{t} e^{-a\tau} y_{n-1}(c\tau) d\tau \right),
\]
which gives the recurrence scheme:
\[
y_0(t) = \lambda e^{at},
\]
\[
y_n(t) = b e^{at} \int_{0}^{t} e^{-a\tau} y_{n-1}(c\tau) d\tau, \quad n \geq 1.
\]
For \( n = 1 \), we obtain
\[
y_1(t) = b e^{at} \int_{0}^{t} e^{-a\tau} y_0(c\tau) d\tau = \lambda b e^{at} \int_{0}^{t} e^{a(c-1)\tau} d\tau,
\]
where \( y_0(c\tau) = \lambda e^{at} \). Evaluating the last integral gives
\[
y_1(t) = \frac{\lambda b}{a(c - 1)} (e^{at} - e^{at}), \quad a \neq 0, \ c \neq 1.
\]
Similarly, at \( n = 2 \) and \( n = 3 \), we can get

\[
y_2(t) = \frac{\lambda b^2}{a^2(c - 1)(c^2 - 1)} \left( e^{c^2 t} - (c + 1)e^{ct} + ce^{at} \right), \quad a \neq 0, c \neq \pm 1,
\]
and

\[
y_3(t) = \frac{\lambda b^3}{a^2(c - 1)(c^2 - 1)(c^3 - 1)} \left( e^{c^3 t} - (c^2 + c + 1)e^{c^2 t} + (c^3 + c^2 + c)e^{ct} - c^2 e^{at} \right).
\]

From above, the calculation of the first few components of the homotopy series is indicated, however, the components of higher-order can be easily derived via any software. It will be shown later in a subsequent section that it is possible to derive a unified formula for the \( n \)th-component \( y_n \), \( \forall n \geq 1 \). To do so, at first, we have to list the main characteristics of the preceding components \( y_1 \), \( y_2 \), and \( y_3 \). These characteristics are investigated in the next section and, hence, a unified formula for \( y_n \) will be possible.

### 4. Characteristics of the Components and a Unified Formula for \( y_n(t) \)

#### 4.1. Characteristics of the Components

From the expressions for the components \( y_1(t) \), \( y_2(t) \), and \( y_3(t) \), it can be noticed that

- \( y_1(t) \) involves two terms and can be written as

\[
y_1(t) = \mu_{1,0}e^{at} + \mu_{1,1}e^{ct} = \sum_{m=0}^{1} \mu_{1,m}e^{mc^m},
\]

where

\[
\mu_{1,0} = -\frac{\lambda b}{a(c - 1)},
\]
\[
\mu_{1,1} = \frac{\lambda b}{a(c - 1)}.
\]

- \( y_2(t) \) consists of three terms in the form:

\[
y_2(t) = \mu_{2,0}e^{at} + \mu_{2,1}e^{ct} + \mu_{2,2}e^{c^2t} = \sum_{m=0}^{2} \mu_{2,m}e^{mc^m},
\]

where

\[
\mu_{2,0} = \frac{\lambda b^2 c}{a^2(c - 1)(c^2 - 1)},
\]
\[
\mu_{2,1} = -\frac{\lambda b^2 (c + 1)}{a^2(c - 1)(c^2 - 1)} = -\frac{\lambda b^2}{a^2(c - 1)^2},
\]
\[
\mu_{2,2} = \frac{\lambda b^2}{a^2(c - 1)(c^2 - 1)}.
\]

- \( y_3(t) \) contains four terms and we can write

\[
y_3(t) = \mu_{3,0}e^{at} + \mu_{3,1}e^{ct} + \mu_{3,2}e^{c^2t} + \sigma_{3,3}e^{c^3t} = \sum_{m=0}^{3} \mu_{3,m}e^{mc^m},
\]
The general component $y$

**Theorem 1.** The general component $y_n(t)$ of the homotopy series is given by

$$y_n(t) = \sum_{m=0}^{n} \mu_{n,m} e^{\lambda c^m t},$$

where

$$\mu_{n,m} = \frac{\lambda c^m}{\prod_{j=1}^{n-m}(c^j - 1) \prod_{k=1}^{m}(c^k - 1)}, \quad n, m \geq 0.$$  

**Proof.** From the previous section, the coefficients $\mu_{1,0}$, $\mu_{2,0}$, $\mu_{3,0}$, and $\mu_{4,0}$ can be written as

$$\mu_{1,0} = -\frac{\lambda b^3 c^3}{a^3(c-1)(c^2 - 1)(c^3 - 1)},$$

$$\mu_{2,0} = \frac{\lambda b^3 c^4}{a^3(c-1)^2(c^2 - 1)},$$

$$\mu_{3,0} = -\frac{\lambda b^3 c^4}{a^3(c-1)(c^2 - 1)},$$

$$\mu_{4,0} = \frac{\lambda b^4 e^6}{a^4(c-1)^2(c^2 - 1)(c^3 - 1)}.$$  

Similarly, $y_4(t)$ can be evaluated from (26) and is given by

$$y_4(t) = u_{4,0} e^{\lambda t} + \mu_{4,1} e^{\lambda c t} + \mu_{4,2} e^{\lambda c^2 t} + \mu_{4,3} e^{\lambda c^3 t} + \mu_{4,4} e^{\lambda c^4 t} = \sum_{m=0}^{4} \mu_{4,m} e^{\lambda c^m t},$$

where

$$\mu_{4,0} = \frac{\lambda b^4 e^6}{a^4(c-1)(c^2 - 1)(c^3 - 1)},$$

$$\mu_{4,1} = -\frac{\lambda b^4 c^3}{a^4(c-1)^2(c^2 - 1)(c^3 - 1)},$$

$$\mu_{4,2} = \frac{\lambda b^4 c^4}{a^4(c-1)^2(c^2 - 1)},$$

$$\mu_{4,3} = -\frac{\lambda b^4 c^4}{a^4(c-1)^2(c^2 - 1)},$$

$$\mu_{4,4} = \frac{\lambda b^4 e^6}{a^4(c-1)(c^2 - 1)(c^3 - 1)(c^4 - 1)}.$$  

In view of (31), (34), (38), and (44), $y_n(t)$ takes the form $y_n(t) = \sum_{m=0}^{n} \mu_{n,m} e^{\lambda c^m t}$ \forall $n \geq 1$. Moreover, the initial component $y_0(t) = \sum_{m=0}^{0} \mu_{0,m} e^{\lambda c^m t} = \mu_{0,0} e^{\lambda t}$ where $\mu_{0,0} = \lambda$. Based on the above characteristics, unified formulas for the coefficients $\mu_{n,m}$ and, hence, the $n$th-order component $y_n(t)$ will be determined. This is the issue of the next section.

### 4.2. Unified Formula for $y_n(t)$

**Theorem 1.** The general component $y_n(t)$ of the homotopy series is given by

$$y_n(t) = \sum_{m=0}^{n} \mu_{n,m} e^{\lambda c^m t},$$

where

$$\mu_{n,m} = \frac{\lambda (-1)^{n-m} (b/a)^n c^{1(n-m)(n-m-1)}}{\prod_{j=1}^{n-m}(c^j - 1) \prod_{k=1}^{m}(c^k - 1)}, \quad n, m \geq 0,$$

**Proof.** From the previous section, the coefficients $\mu_{1,0}$, $\mu_{2,0}$, $\mu_{3,0}$, and $\mu_{4,0}$ can be written as

$$\mu_{1,0} = -\frac{\lambda (b/a)^2 c}{\prod_{k=1}^{2}(c^k - 1)},$$

$$\mu_{2,0} = \frac{\lambda (b/a)^2 c}{\prod_{k=1}^{2}(c^k - 1)},$$

$$\mu_{3,0} = -\frac{\lambda (b/a)^3 c^3}{\prod_{k=1}^{3}(c^k - 1)},$$

$$\mu_{4,0} = \frac{\lambda (b/a)^4 c^6}{\prod_{k=1}^{4}(c^k - 1)}.$$
It is noticed from (51) and (52) that the numerators and denominators of these coefficients follow the rules \(\lambda(-1)^n(b/a)^n c^{n(n-1)/2}\) and \(\prod_{k=1}^{n} (c^k-1)\), respectively, \(n = 1, 2, 3, 4\).

It can also be checked that the other higher-order coefficients \((n \geq 5)\) follow the same patterns. Hence, a unified formula for \(\mu_{n,0} \forall n \geq 1\) may be given as

\[
\mu_{n,0} = \frac{\lambda(-1)^n(b/a)^n c^{n(n-1)/2}}{\prod_{k=1}^{n} (c^k-1)}, \quad n \geq 1.
\]  

Similarly, the coefficients \(\mu_{1,1}, \mu_{2,2}, \mu_{3,3}, \text{ and } \mu_{4,4}\) can be written as

\[
\begin{align*}
\mu_{1,1} &= \frac{\lambda(b/a)}{\prod_{k=1}^{1} (c^k-1)}, \quad \mu_{2,2} = \frac{\lambda(b/a)^2}{\prod_{k=1}^{2} (c^k-1)}, \\
\mu_{3,3} &= \frac{\lambda(b/a)^3}{\prod_{k=1}^{3} (c^k-1)}, \quad \mu_{4,4} = \frac{\lambda(b/a)^4}{\prod_{k=1}^{4} (c^k-1)},
\end{align*}
\]  

and, thus,

\[
\mu_{n,n} = \frac{\lambda(b/a)^n}{\prod_{k=1}^{n} (c^k-1)}, \quad n \geq 1.
\]  

For \(n \neq m\), we observe that \(\mu_{n,m}\) satisfies

\[
\begin{align*}
\mu_{2,1} &= -\frac{\lambda(b/a)^2}{\prod_{j=1}^{n-m=1} (c^j-1) \prod_{k=1}^{m} (c^k-1)}, \quad \mu_{3,1} = \frac{\lambda(b/a)^3 c^4}{\prod_{j=1}^{n-m=2} (c^j-1) \prod_{k=1}^{m} (c^k-1)}, \\
\mu_{3,2} &= -\frac{\lambda(b/a)^3 c^3}{\prod_{j=1}^{n-m=1} (c^j-1) \prod_{k=1}^{m} (c^k-1)}, \quad \mu_{4,1} = -\frac{\lambda(b/a)^4 c^3}{\prod_{j=1}^{n-m=3} (c^j-1) \prod_{k=1}^{m} (c^k-1)}, \\
\mu_{4,2} &= \frac{\lambda(b/a)^4 c}{\prod_{j=1}^{n-m=2} (c^j-1) \prod_{k=1}^{m} (c^k-1)}, \quad \mu_{4,3} = -\frac{\lambda(b/a)^4}{\prod_{j=1}^{n-m=4} (c^j-1) \prod_{k=1}^{m} (c^k-1)},
\end{align*}
\]  

and, accordingly, we have

\[
\mu_{n,m} = \frac{\lambda(-1)^{n-m}(b/a)^n c^{n(n-m)(n-m-1)/2}}{\prod_{j=m}^{n-m} (c^j-1) \prod_{k=1}^{m} (c^k-1)}.
\]  

At \(m = 0\), Formula (60) reduces to Formula (53). In addition, when \(n = m\), Formula (60) gives Formula (56). In addition, for \(n = m = 0\), we have from (60) that \(\mu_{0,0} = \lambda\) which leads to the initial component \(y_0(t) = \lambda e^{\lambda t}\) (from Equation (49)). This means that (60) is valid as a unified formula for the coefficients \(\mu_{m,n} \forall n, m \geq 0\), which completes the proof. \(\square\)

5. Closed-Form Solution via Quantum Calculus

The objective of this section is to obtain a closed-form solution by means of some properties in the quantum calculus [35]. We start with the product:

\[
(\alpha : \beta)_i = \prod_{k=0}^{i-1} \left(1 - \alpha \beta^k\right),
\]  

where \((\alpha : \beta)_i\) denotes the Pochhammer symbol. For \(\alpha = \beta = c\), we have

\[
(c : c)_i = \prod_{k=0}^{i-1} \left(1 - c^{k+1}\right) = \prod_{k=1}^{i} \left(1 - c^k\right).
\]  

Using the above property, we are able to construct the closed-form solution of Equation (1) by means of quantum calculus notations as addressed in the theorem below.
Theorem 2. The solution \( y(t) \) of the PDDE (1) is given by
\[
y(t) = \lambda \sum_{n=0}^{\infty} (b/a)^n \sum_{m=0}^{n} (-1)^n c_1^{(n-m)(n-m-1)} e^{ae^{mt}} (c : c)_{n-m}(c : c)_m.
\] (63)

Proof. Let us start the proof by rewriting the general component \( y_n(t) \), from Theorem 1, in the form:
\[
y_n(t) = \sum_{m=0}^{n} \frac{(-1)^{n-m} (b/a)^n c_1^{(n-m)(n-m-1)} e^{ae^{mt}}}{\prod_{j=1}^{m} (c_j - 1) \prod_{k=1}^{n-m} (c_k - 1)}.
\] (64)

The products in the denominator of (64) can be written in view of quantum calculus notations as
\[
\prod_{j=1}^{n-m} (c_j - 1) \prod_{k=1}^{n} (c_k - 1) = (-1)^{n-m} \prod_{j=1}^{m} (1-c_j) \prod_{k=1}^{n} (1-c_k)
\] = \((-1)^n (c : c)_{n-m}(c : c)_m.
\] (65)

Substituting (65) into (64) and simplifying, then
\[
y_n(t) = \sum_{m=0}^{n} \frac{(-1)^m (b/a)^n c_1^{(n-m)(n-m-1)} e^{ae^{mt}}}{(c : c)_{n-m}(c : c)_m}.
\] (66)

The homotopy series solution is given in terms of the auxiliary parameter \( p \) as
\[
y(t) = \sum_{n=0}^{\infty} p^n y_n(t),
\] (67)
and, hence, the solution \( y(t) \) can be obtained by letting \( p \to 1 \). Thus,
\[
y(t) = \sum_{n=0}^{\infty} y_n(t).
\] (68)

From (66) and (68), we obtain
\[
y(t) = \lambda \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^m (b/a)^n c_1^{(n-m)(n-m-1)} e^{ae^{mt}}}{(c : c)_{n-m}(c : c)_m},
\] (69)
i.e.,
\[
y(t) = \lambda \sum_{n=0}^{\infty} (b/a)^n \sum_{m=0}^{n} \frac{(-1)^m c_1^{(n-m)(n-m-1)} e^{ae^{mt}}}{(c : c)_{n-m}(c : c)_m},
\] (70)
which completes the proofs. \( \square \)

6. Discussion

In previous sections, two types of closed-form solutions were established. In this discussion, the behaviors of the two types of solutions, mainly the PSS and the EFS, are explored. Before launching, it is better to mention that the PSS in Equation (17) reduces to several forms of exact solutions at \( c = -1 \), as pointed out in [19]. In this case, the PDDE (1) becomes
\[
y'(t) = ay(t) + by(-t), \quad y(0) = \lambda.
\] (71)

For \( a = b = \alpha \), the exact solution of Equation (71) reads [20]
\[
y(t) = \lambda (1 + 2\alpha t).
\] (72)
For $b > a$, the periodic solution of Equation (71) was obtained in the form [19]:

$$y(t) = \lambda \left( \cos \left( \sqrt{b^2 - a^2} t \right) + \sqrt{\frac{b + a}{b - a}} \sin \left( \sqrt{b^2 - a^2} t \right) \right).$$  \hspace{1cm} (73)

The $r$-term approximation $\Theta_r(t)$ of the EFS solution is given from Equation (70) as

$$\Theta_r(t) = \lambda \sum_{n=0}^{r-1} \left( \frac{b}{a} \right)^n \sum_{m=0}^{n} \frac{(-1)^m c^{1/2} (n-m)(n-m-1) \sqrt{a} \lambda^m t}{(c : c)_{n-m}(c : c)_m}, \quad r \geq 2. \hspace{1cm} (74)$$

At the special case $c = -1$, the approximations $\Theta_r(t)$ can be obtained by taking the limit of Equation (74) as $c \to -1$:

$$\Theta_r(t) = \lambda \lim_{c \to -1} \left( \sum_{n=0}^{\infty} \left( \frac{b}{a} \right)^n \sum_{m=0}^{n} \frac{(-1)^m c^{1/2} (n-m)(n-m-1) \sqrt{a} \lambda^m t}{(c : c)_{n-m}(c : c)_m} \right). \hspace{1cm} (75)$$

So, we focus in the first part of this discussion on performing comparisons between the approximations $\Theta_r(t)$ in Equation (75) and the exact solutions given by Equation (72) ($a = b = a$) and Equation (73) ($b > a$). Figures 1 and 2 show the comparisons between the approximations $\Theta_5(t), \Theta_6(t), \Theta_7(t)$ (Figure 1) and the approximations $\Theta_{15}(t), \Theta_{20}(t), \Theta_{25}(t)$ (Figure 2) with the exact solution (72) when $\lambda = 1, a = b = a = 1$. It is seen from these figures that the domain of coincidence between the approximations $\Theta_r(t)$ and the exact solution increases as the number of terms $r$ increases. Additional comparisons are depicted in Figures 3 and 4 between the approximations $\Theta_{15}(t), \Theta_{20}(t), \Theta_{25}(t)$ and the periodic solution (73) when $\lambda = 1, a = 1, b = \sqrt{2}$ (Figure 3) and $\lambda = 1, a = 2, b = 3$ (Figure 4). It is also observed from these figures that $\Theta_{25}(t)$ is in full agreement with the exact periodic solution (73) for a complete period indicated by the points in pink and yellow. For $c \neq -1$, the accuracy of the approximations $\Theta_r(t)$ is validated through calculating the residuals $RE_r(t)$ defined by

$$RE_r(t) = \left| \Theta_r'(t) - a\Theta_r(t) - b\Theta_r(ct) \right|, \quad r \geq 2. \hspace{1cm} (76)$$

The numerical results in Figures 5–10 for the obtained residuals reveal the high accuracy of the present analysis, even in a huge domain.
Figure 2. Comparisons between the approximate solutions $\Theta_r(t)$, $r = 15, 20, 25$ in Equation (75) and the exact solution (72) at $\lambda =, a = b = \alpha = 1$ ($c = -1$).

Figure 3. Comparisons between the approximate solutions $\Theta_r(t)$, $r = 15, 20, 25$ in Equation (75) and the exact periodic solution (73) at $\lambda = 1, a = 1, b = \sqrt{2}$ ($c = -1$).

Figure 4. Comparisons between the approximate solutions $\Theta_r(t)$, $r = 15, 20, 25$ in Equation (75) and the exact periodic solution (73) at $\lambda = 1, a = 2, b = 3$ ($c = -1$).
Figure 5. The residuals $R_{E_r}(t)$, $r = 10, 11, 12, 13$ in Equation (76) at $\lambda = 1$, $a = -2$, $b = 1$, and $c = \frac{1}{2}$.

Figure 6. The residuals $R_{E_r}(t)$, $r = 10, 11, 12, 13$ in Equation (76) at $\lambda = 1$, $a = -2$, $b = 1$, and $c = \frac{3}{4}$.

Figure 7. The residuals $R_{E_r}(t)$, $r = 10, 11, 12, 13$ in Equation (76) at $\lambda = 1$, $a = -2$, $b = 1$, and $c = \frac{3}{2}$. 
Figure 8. The residuals $R_{E_r}(t)$, $r = 10, 11, 12, 13$ in Equation (76) at $\lambda = 1$, $a = -2$, $b = 1$, and $c = 2$.

Figure 9. The residuals $R_{E_r}(t)$, $r = 10, 11, 12, 13$ in Equation (76) at various values of $c$ ($0 < c < 1$) when $\lambda = 1$, $a = -1$, and $b = \frac{1}{4}$.

Figure 10. The residuals $R_{E_r}(t)$, $r = 10, 11, 12, 13$ in Equation (76) at various values of $c$ ($c > 1$) when $\lambda = 1$, $a = -1$, and $b = \frac{1}{4}$.
7. Conclusions

In this paper, the HPM is applied to solve the PDDE using two different canonical forms. Accordingly, two types of closed-form solutions, the PSS and EFS, were obtained. It was shown that the present PSS reduces to the corresponding ones in the literature [12,15] for the ADDE in special cases. A unified formula for the homotopy components was successfully obtained for the EFS. The results were numerically validated through several comparisons with the available exact solutions. The calculated residuals reflect the high accuracy of the current analysis, especially, residuals were close to zero even in a huge domain of the studied PDDE. The obtained results reveal that the current approach is effective and straightforward and can be further applied to include other DDEs of higher orders [36–39].


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