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# Qualitative Properties of Solutions of Equations and Inequalities with KPZ-Type Nonlinearities

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**Abstract:** For quasilinear partial differential and integrodifferential equations and inequalities containing nonlinearities of the Kardar–Parisi–Zhang type, various (old and recent) results on qualitative properties of solutions (such as the stabilization of solutions, blow-up phenomena, long-time decay of solutions, and others) are presented. Descriptive examples demonstrating the Bitsadze approach (the technique of monotone maps) applied in this research area are provided.

**Keywords:** quasilinear equations and inequalities; KPZ-nonlinearities; qualitative properties of solutions; methods of monotonous maps; blow-up; stabilization

**MSC:** 35R45; 35K55; 35B40

## 1. Introduction

### 1.1. History and Motivation

The unflinching worldwide interest to quasilinear differential operators with the so-called KPZ-type nonlinearities (i.e., operators containing the second power of the first derivative) is mainly caused by the following two circumstances. The first one is purely theoretical: it is known (see, e.g., [1–3]) that the second power is the greatest one such that Bernstein-type conditions for the corresponding boundary-value problem guarantee the validity of a priori  $L_\infty$ -estimates of first-order derivatives of the solution via the  $L_\infty$ -estimate of the solution itself. On the other hand, such operators arise in applications to various areas not covered by classical linear differential equations: multidimensional interface dynamics (see [4–6]), directed polymer growth (see [5,7–9]), fractional diffusion models (see [10,11]), game-addiction models with unsustainable control (see [12]), models of finite-temperature free fermions (see [13]), stochastic heat propagation under subcritical regimes (see [14]), etc. For the first time, this phenomenon is seemingly noted in the famous paper [4] (and the used abbreviation goes from the names of its authors). Nowadays, the number of publications of those applications cannot be estimated; it suffices to provide several most recent remarkable examples: [6,9–29].

It is worth to note that an efficient tool to investigate such nonlinearities is proposed in [30]: one has to construct monotonous maps taking solutions of quasilinear equations (this Bitsadze approach is easily extended to inequalities as well) to solutions of linear ones. A clear example of the constructing of such a map is provided in the next section. The remaining part of the present paper is devoted to various results on qualitative properties of nonlinear problems, obtained by means of this tool.

Results of Section 5 deserve a special attention: up to the knowledge of the author, KPZ-nonlinearities in the functional-differential case were not considered earlier. Taking into account that the variety of functional-differential operators is much broader than differential-convolutional operators considered in the specified section (e.g., differential-difference operators, operators with contractions and extensions of independent variables, and general-kind integrodifferential operators are quite important for the theory and



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applications), one can reasonably treat functional-differential inequalities (equations) with KPZ-nonlinearities as a very significant and promising area of further investigations.

1.2. Illustrative Example: Gidas–Spruck Theorem and Its Generalizations

The main idea of the Bitsadze approach can be clearly explained on the following simple example.

It is known from [31] that the semilinear equation  $-\Delta u = u^q$  has no global positive solutions provided that  $1 < q < \frac{n+2}{n-2}$ .

Applying advanced nonlinear capacity methods, one can extend this pioneering result to the case of the following quasilinear inequality:

$$-\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} a_{i,j}(x, u) \geq b(x)|u|^q, \tag{1}$$

where  $a_{i,j}$  are Caratheodory functions of  $n + 1$  variables such that

$$|a_{i,j}(x, s)| \leq a(x)|s|^p, \quad x \in \mathbb{R}^n, \quad s \in (-\infty, +\infty), \quad i, j = 1, 2, \dots, n, \tag{2}$$

with a positive  $p$  and nonnegative  $a(x)$  (see [32]).

Note that this is a very substantial generalization: apart from the passage from the semilinear case to the quasilinear one, we pass from equations to nonstrict inequalities. Such results are always stronger: if an inequality has no solutions, then the corresponding equation has no solutions a fortiori.

However, the question whether Condition (2) is essential or it is just a technical restriction remains open. Let us show how to resolve it, using the Bitsadze approach.

In  $\mathbb{R}^n$ , consider the inequality

$$\Delta u + \alpha|\nabla u|^2 + e^{\gamma u} \leq 0, \tag{3}$$

where  $\frac{\gamma}{\alpha} > 0$ . To prove that no global solutions of that inequality exist, assume, to the contrary, that a function  $u(x)$  satisfies inequality (3) in  $\mathbb{R}^n$ . Then introduce the following function in  $\mathbb{R}^n$ :

$$v(x) \stackrel{\text{def}}{=} \frac{1}{\alpha} e^{\alpha u(x)-1}. \tag{4}$$

Then, for each  $j$ ,

$$\frac{\partial v}{\partial x_j} = e^{\alpha u-1} \frac{\partial u}{\partial x_j}, \quad \frac{\partial^2 v}{\partial x_j^2} = \frac{\partial^2 u}{\partial x_j^2} e^{\alpha u-1} + \frac{\partial u}{\partial x_j} \alpha \frac{\partial u}{\partial x_j} e^{\alpha u-1} = e^{\alpha u-1} \left[ \frac{\partial^2 u}{\partial x_j^2} + \alpha \left( \frac{\partial u}{\partial x_j} \right)^2 \right],$$

and, therefore,

$$\Delta v = e^{\alpha u-1} [\Delta u + \alpha|\nabla u|^2].$$

Hence,

$$-e^{1-\alpha u} \Delta v \geq e^{\gamma u} \Rightarrow -\Delta v \geq e^{(\alpha+\gamma)u-1}.$$

From (4), it follows that  $\alpha v(x)$  is always positive, i.e.,  $v(x)$  is a constant-sign function,  $\text{sgn}[v(x)] = \text{sgn}(\alpha)$ , and  $\alpha e v = e^{\alpha u} > 0$ . Then

$$e^{\gamma u} = (e^{\alpha u})^{\frac{\gamma}{\alpha}} = (\alpha e v)^{\frac{\gamma}{\alpha}}$$

and, therefore,

$$e^{(\alpha+\gamma)u-1} = (\alpha e v)^{\frac{\gamma}{\alpha}+1} e^{-1} = (\alpha v)^{\frac{\gamma}{\alpha}+1} e^{\frac{\gamma}{\alpha}} = (|\alpha||v|)^{\frac{\gamma}{\alpha}+1} e^{\frac{\gamma}{\alpha}}$$

because  $\alpha$  and  $v$  always have a same sign. The last expression is equal to  $|\alpha|^{\frac{\gamma}{\alpha}+1} e^{\frac{\gamma}{\alpha} v} |v|^{\frac{\gamma}{\alpha}+1}$ , which means that the introduced function  $v(x)$  satisfies the inequality

$$-\Delta v \geq |\alpha|^{\frac{\gamma}{\alpha}+1} e^{\frac{\gamma}{\alpha} v} |v|^{\frac{\gamma}{\alpha}+1}, \tag{5}$$

which is inequality (1) with  $q = \frac{\gamma}{\alpha} + 1$  and  $a_{i,j}(x, s) = \delta_i^j s$ . Thus, Condition (2) is satisfied. Then, due to the above Mitidieri–Pohozaev generalization, inequality (5) has no global solutions, which yields a contradiction.

Now, represent inequality (3) in the form

$$-\sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \left( \delta_i^j \frac{1}{\alpha} e^{\alpha u} \right) \geq e^{(\alpha+\gamma)u}.$$

The left-hand part of the last inequality is a special case of the left-hand part of inequality (1), but Condition (2) is satisfied for no  $p$ . This confirms that the above Mitidieri–Pohozaev generalization (as well as the corresponding general theory) are actually restricted neither by the growth speed of coefficients nor by the power-like shape of nonlinearities.

### 2. Parabolic Stabilization

Full proofs of the results of this section are provided in [33].

#### 2.1. Regular Case

Let a bounded function  $u(x, t)$  satisfy the equation

$$\frac{\partial u}{\partial t} = \Delta u + g(u) |\nabla u|^2, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{6}$$

where  $g$  is continuous.

Define the function  $f(s)$  as follows:

$$f(s) = \int_0^s e^{\int_0^x g(\tau) d\tau} dx. \tag{7}$$

Then  $f'(s) = e^{\int_0^s g(\tau) d\tau} > 0$  and  $f''(s) = g(s) e^{\int_0^s g(\tau) d\tau}$  so  $g(s) = \frac{f''(s)}{f'(s)}$ , where  $f$  is strictly monotone.

Denoting  $f(u)$  by  $v(x, t)$ , we see that

$$\frac{\partial v}{\partial t} = f'(u) \frac{\partial u}{\partial t}, \quad \frac{\partial v}{\partial x_j} = f'(u) \frac{\partial u}{\partial x_j} \quad \text{and} \quad \frac{\partial^2 v}{\partial x_j^2} = f''(u) \left( \frac{\partial u}{\partial x_j} \right)^2 + f'(u) \frac{\partial^2 u}{\partial x_j^2}.$$

Then  $\frac{\partial v}{\partial t} = \Delta v$ . On the other hand, from the continuity of  $f$  and boundedness of  $u$ , it follows that the function  $v(x, t)$  is bounded as well (as a continuous function  $f$  on the segment  $[\inf u, \sup u]$ ). Thus,  $v(x, t)$  is a bounded solution of the heat equation. Hence, the following stabilization criterion is valid (see [34]):

for any  $x \in \mathbb{R}^n$ ,  $\lim_{t \rightarrow \infty} v(x, t)$  exists if and only if  $\lim_{t \rightarrow \infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} t^{\frac{n}{2}}} \int_{|x|<t} v(x, 0) dx$  exists; if those limits exist, then they are equal to each other.

Taking into account that  $f$  is invertible due its strong monotonicity and  $f^{-1}$  is continuous due the smoothness of  $f$ , we obtain the following assertion:

**Theorem 1.** Let  $g$  be continuous and  $u(x, t)$  be a bounded solution of the Cauchy problem for Equation (6) with a continuous and bounded initial-value function  $u_0(x)$ . Then for any  $x \in \mathbb{R}^n$ ,

$\lim_{t \rightarrow \infty} u(x, t)$  exists if and only if  $\lim_{t \rightarrow \infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}} t^n} \int_{|x| < t} f[u_0(x)] dx$  exists, where  $f$  is defined by (7); if

those limits exist, then the latter limit is equal to  $f\left[\lim_{t \rightarrow \infty} u(x, t)\right]$ .

**Remark 1.** It is not necessary to assume that  $g$  is continuous on the whole real axis; it suffices to assume that it is continuous (and even defined) only in the closure of the range of  $u_0(x)$ . However, any function continuous in  $[\inf u_0, \sup u_0]$  can be extended, preserving the continuity, to the whole axis. Thus, the initial approach does not restrict the generality.

### 2.2. Singular Case

In Equation (6), assume that  $g(s) = \alpha s^\beta$ , where  $\beta \in (-1, 0)$ . Then ansatz (7) is still applicable, but, to guarantee the monotonicity, we have to add the restriction of the positivity of the solution. This yields the following result for the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u + \alpha u^\beta |\nabla u|^2, \quad x \in \mathbb{R}^n, t > 0, \tag{8}$$

$$u \Big|_{t=0} = u_0(x), \quad x \in \mathbb{R}^n. \tag{9}$$

**Theorem 2.** Let  $\alpha \in \mathbb{R}^1, \beta \in (-1, 0)$ , and  $u_0$  be continuous, bounded, nonnegative, and nontrivial in  $\mathbb{R}^n$ . Then there exists a unique positive bounded solution of problem (8) and (9) and the assertion of Theorem 1 holds for it.

If  $\beta = -1$ , i.e., the equation takes the form

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\alpha}{u} |\nabla u|^2, \tag{10}$$

then we cannot use ansatz (7), but we can define  $f(s)$  as  $s^{\alpha+1}$ .

This yields the following result.

**Theorem 3.** Let  $\alpha > -1$  and  $u_0(x)$  be bounded and nonnegative. Then there exists a unique positive bounded solution of problem (10), (9) and the following criterion is valid: for any  $x \in \mathbb{R}^n$ ,

$\lim_{t \rightarrow \infty} u(x, t)$  exists if and only if  $\lim_{t \rightarrow \infty} \frac{1}{t^n} \int_{|x| < t} u_0^{\alpha+1}(x) dx$  exists. If those limits exist, then

$$\lim_{t \rightarrow \infty} u(x, t) = \left[ \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \lim_{t \rightarrow \infty} \frac{1}{t^n} \int_{|x| < t} u_0^{\alpha+1}(x) dx \right]^{\frac{1}{\alpha+1}}.$$

**Remark 2.** If we change the condition of the nonnegativity of the initial-value function by the stronger condition of its positive definiteness, i.e., assume that there exists a positive  $a$  such that  $u_0(x) \geq a$  for each  $x$ , then the assertion of Theorem 3 is valid for  $\alpha < -1$  as well. However, it is not valid for  $\alpha = -1$ .

### 3. Elliptic Stabilization

The nonclassical nature of the half-space Dirichlet problem for elliptic equations is known for a long time (see, e.g., [35,36]): the independent variable varying within a half-line possesses the so-called *timelike* properties, which means, e.g., that the resolving operator possesses the semigroup property with respect to that special variable (though all

independent variables remain to be spatial). Thus, the question about the stabilization of the solution with respect to the timelike variable becomes reasonable. The said phenomenon is explained in this section; full proofs are provided in [37,38].

Denoting  $x = (x_1, \dots, x_n, x_{n+1})$  by  $(x', x_{n+1})$ , consider the problem

$$\Delta u + g(u)|\nabla u|^2 = 0, \quad x' \in \mathbb{R}^n, \quad x_{n+1} > 0, \tag{11}$$

$$u \Big|_{x_{n+1}=0} = \varphi(x), \quad x' \in \mathbb{R}^n, \tag{12}$$

where  $\varphi$  is continuous and bounded in  $\mathbb{R}^n$ .

The following assertions are valid.

**Theorem 4.** *Let  $g$  be continuous in  $(-\infty, +\infty)$ . Then there exists a unique bounded solution  $u(x, t)$  of problem (11) and (12) and the following equivalence takes place for each  $x' \in \mathbb{R}^n$  and each  $l \in (-\infty, +\infty)$ :*

$$\lim_{x_{n+1} \rightarrow +\infty} u(x) = l \text{ if and only if } \lim_{R \rightarrow +\infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}R^n} \int_{|y| < R} f[\varphi(y)] dy = f(l).$$

**Theorem 5.** *Let  $g = \alpha u^\beta$ , where  $-1 < \beta < 0$ , and  $0 \leq \varphi \not\equiv 0$ . Then there exists a unique bounded positive solution  $u(x, t)$  of problem (11) and (12) and the following equivalence takes place for each  $x' \in \mathbb{R}^n$  and each nonnegative  $l$ :*

$$\lim_{x_{n+1} \rightarrow +\infty} u(x) = l \text{ if and only if } \lim_{R \rightarrow +\infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}R^n} \int_{|y| < R} \tilde{f}[\varphi(y)] dy = \tilde{f}(l),$$

where

$$\tilde{f}(s) = \int_0^s e^{\frac{\alpha}{1-\beta}\tau^{1-\beta}} d\tau.$$

**Theorem 6.** *Let  $g = \frac{\alpha}{u}$ , where  $\alpha > -1$ , and  $0 \leq \varphi \not\equiv 0$ . Then there exists a unique bounded positive solution  $u(x, t)$  of problem (11) and (12) and the following equivalence takes place for each  $x' \in \mathbb{R}^n$  and any nonnegative  $l$ :*

$$\lim_{x_{n+1} \rightarrow +\infty} u(x) = l \text{ if and only if } \lim_{R \rightarrow +\infty} \frac{n\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}R^n} \int_{|y| < R} \varphi^{\alpha+1}(y) dy = l^{\alpha+1}.$$

**Remark 3.** *The stabilization phenomenon takes place for parabolic and elliptic equations with the singular Bessel operator*

$$B_{k,y} := \frac{\partial^2}{\partial y^2} + \frac{k}{y} \frac{\partial}{\partial y} = \frac{1}{y^k} \frac{d}{dy} \left( y^k \frac{d}{dy} \right), k > 0,$$

acting with respect to spatial variables, as well. Pertinent results can be found in [39–42].

#### 4. Blow-Up for Partial Differential Inequalities

This section is devoted to the blow-up phenomenon both for stationary and nonstationary problems with KPZ-type nonlinearities. In this section (as well as in the next one), we follow the Pokhozhaev paradigm (see, e.g., [32]): *blow-up phenomena are equivalent to the absence of global solutions*. Note that all presented results refer to differential inequalities instead of differential equations, i.e., the maximal generality is guaranteed. Full proofs

of the results of this section as well as local (instantaneous) results for inequalities with KPZ-type nonlinearities are provided in [43–46].

4.1. Elliptic Case

**Theorem 7.** Let  $g$  be continuous on  $(-\infty, +\infty)$  and  $\beta$  be measurable and a. e. positive in  $\mathbb{R}^n$ . Let there exist  $q > 1$  such that  $\beta^{\frac{1}{1-q}}(x)$  is locally summable in  $\mathbb{R}^n$ , excluding, perhaps, a bounded set, and  $\omega(s) \geq \left| \int_0^s e^{\int_0^\tau g(t)dt} d\tau \right|^q - \int_0^s g(\tau)d\tau$  on  $(-\infty, +\infty)$ . Then the inequality

$$\Delta u + \sum_{j=1}^n g_j(u) \left( \frac{\partial u}{\partial x_j} \right)^2 + \beta(x)\omega(u) \leq 0 \tag{13}$$

has no classical global solutions provided that  $g_j(x) \geq g(x)$ ,  $j = 1, 2, \dots, n$ .

**Example 1.** Suppose that  $\beta > 0$  and  $0 < \frac{\gamma}{\alpha} \leq \frac{2}{n-2}$ . Then the inequality  $\Delta u + \alpha|\nabla u|^2 + \beta e^{\gamma u} \leq 0$  has no global solutions. The critical value  $\frac{2}{n-2}$  is exact.

Note that inequality (13) is a particular case of inequality (3) considered in Section 1.2. Consider the inequality

$$\Delta u - \sum_{j=1}^n \alpha_j(x, u) \left( \frac{\partial u}{\partial x_j} \right)^2 \geq \omega(x, u), \quad x \in \mathbb{R}^n, \tag{14}$$

under the assumption that  $\alpha_j(x, s) \geq \frac{1}{s}$ ,  $j = 1, 2, \dots, n$ .

The following assertions are valid.

**Theorem 8.** Let there exist a nonnegative function  $k(x)$ , a positive constant  $R_0$ , and a positive function  $\theta$  defined on  $[R_0, +\infty)$  such that  $\lim_{t \rightarrow \infty} \theta(t) = \infty$ ,  $\frac{\theta(t)}{t^2}$  is a nonincreasing function, and  $k(x) \geq \frac{\theta(|x|)}{|x|^2}$  provided that  $|x| \geq R_0$ . Suppose that there exists a constant  $p$  from the interval  $(1, +\infty)$  such that  $\omega(x, s) \geq k(x)s^p$ . Then inequality (14) has no positive solutions.

**Theorem 9.** Let there exist a nonnegative function  $k(x)$  and constants  $R_0$  from  $[1, +\infty)$ ,  $C_1$  from  $(0, +\infty)$ , and  $C_0, b$ , and  $d$  from  $\mathbb{R}$  such that  $k(x) \geq 0$  provided that  $|x| \leq R_0$  and  $\frac{C_1}{|x|^b} \leq k(x) \leq C_0|x|^d$  provided that  $|x| > R_0$ . Suppose that there exists a constant  $p$  from the interval  $(-\infty, 1)$  such that  $\omega(x, s) \geq k(x)s^p$ . Then, for any real  $C_2$  and any  $a$  from the interval  $(-\infty, \frac{2-b}{1-p})$ , inequality (14) has no positive solutions satisfying the inequality  $u(x) \leq C_2|x|^a$  outside the ball  $|x| < R_0$ .

4.2. Parabolic Case

Consider the inequality

$$\frac{\partial u}{\partial t} \geq \Delta u + \sum_{j=1}^n g_j(u) \left( \frac{\partial u}{\partial x_j} \right)^2 + \beta(x, t)\omega(u), \quad x \in \mathbb{R}^n, t > 0, \tag{15}$$

where  $g_1, \dots, g_n$  are continuous on  $\mathbb{R}^1$ ,  $\beta(x, t)$  is a measurable and a. e. positive function, there exists  $q > 1$  such that  $\beta^{\frac{1}{1-q}}(x, t)$  is locally summable in  $\mathbb{R}^n \times (0, \infty)$ , excluding, perhaps,

a bounded set, and  $\omega(s) \geq \left| \int_0^s e^{\int_0^\tau g_0(\theta) d\theta} d\tau \right|^q e^{-\int_0^s g_0(\tau) d\tau}$  on  $\mathbb{R}^1$ , where  $g_0(s) = \min_{j=1,2,\dots,n} g_j(s)$ .

Assume that  $u_0$  is continuous and nonnegative on  $\mathbb{R}^1$  and define on  $(0, \infty)$  the following function depending on parameters  $\varkappa$  and  $\mu$ :

$$C_{\varkappa,\mu}(R) \stackrel{\text{def}}{=} R^{n+\frac{\mu}{\varkappa}} \left[ R^{\frac{2q}{1-q}} + R^{\frac{\mu q}{\varkappa(1-q)}} \right] \int_{1 < \tau^\varkappa + |\xi|^\mu < 2} \beta^{\frac{1}{1-q}} \left( R\xi, R^{\frac{\mu}{\varkappa}} \tau \right) d\xi d\tau. \tag{16}$$

The following assertion is valid:

**Theorem 10.** *If there exist  $\varkappa \geq 1$  and  $\mu \geq 2$  such that  $\liminf_{R \rightarrow \infty} C_{\varkappa,\mu}(R)$  is finite, then problem (15), (9) has no classical nontrivial solutions.*

**Remark 4.** *The last assertion remains to be valid if the nonnegativity assumption for  $u_0(x)$  is replaced by the following weaker assumption:*

$$\liminf_{R \rightarrow \infty} \int_{|x| < R} \int_0^{u_0(x)} e^{\int_0^\tau g_0(\theta) d\theta} d\tau dx \geq 0.$$

Consider the inequality

$$\frac{\partial u}{\partial t} \geq \Delta u + \frac{\alpha}{u^\gamma} |\nabla u|^2 + \beta(x, t)\omega(u), \quad x \in \mathbb{R}^n, t > 0, \tag{17}$$

where  $\beta$  is as above,  $\alpha \neq 0, 0 < \gamma \neq 1$ ,  $\omega$  satisfies the inequality  $\omega(s) \geq \left( \int_0^s e^{\frac{\alpha}{1-\gamma} \tau^{1-\gamma}} d\tau \right)^q e^{\frac{\alpha}{\gamma-1} s^{1-\gamma}}$  in  $(0, +\infty)$ , and  $0 \neq u_0 \in C(\mathbb{R}^n)$ . Then problem (17), (9) has no classical positive solutions under the assumptions of Theorem 10.

For the inequality

$$\frac{\partial u}{\partial t} \geq \Delta u + \frac{\alpha}{u} |\nabla u|^2 + \beta(x, t)\omega(u), \quad x \in \mathbb{R}^n, t > 0, \tag{18}$$

the following assumptions are imposed:  $\alpha > -1$  and there exists  $\rho > 1$  such that  $\omega(s) \geq s^\rho$  on  $(0, +\infty)$  and  $\beta^{\frac{1+\alpha}{1-\rho}}(x, t)$  is locally summable in  $\mathbb{R}^n \times (0, \infty)$ , excluding, perhaps, a bounded set. Then, for continuous and different from the identical zero initial-value functions, the nonexistence of classical positive solutions for the Cauchy problem is guaranteed by the following condition: *there exist  $\varkappa \geq 1$  and  $\mu \geq 2$  such that*

$$\liminf_{R \rightarrow \infty} R^{n+\frac{\mu}{\varkappa}} \left[ R^{\frac{2(\alpha+\rho)}{1-\rho}} + R^{\frac{(\alpha+\rho)\mu}{(1-\rho)\varkappa}} \right] \int_{1 < \tau^\varkappa + |\xi|^\mu < 2} \beta^{\frac{1+\alpha}{1-\rho}} \left( R\xi, R^{\frac{\mu}{\varkappa}} \tau \right) d\xi d\tau < \infty.$$

The inequality

$$\frac{\partial u}{\partial t} \geq \Delta u - \frac{|\nabla u|^2}{u} + \omega(x, t, u), \quad x \in \mathbb{R}^n, t > 0, \tag{19}$$

can be considered under the following assumptions: there exist  $q > 1, \sigma > 0$ , and a measurable and a.e. positive function  $\beta(x, t)$  such that  $\beta^{\frac{1}{1-q}}(x, t)$  is locally summable in  $\mathbb{R}^n \times (0, \infty)$ , excluding, perhaps, a bounded set,  $\omega(x, t, s) \geq \beta(x, t) s \left| \ln \frac{s}{\sigma} \right|^q$  on  $\mathbb{R}^n \times (0, +\infty) \times (0, +\infty)$ , and there exist  $\varkappa \geq 1$  and  $\mu \geq 2$  such that  $\liminf_{R \rightarrow \infty} C_{\varkappa,\mu}(R)$  is finite, where  $C_{\varkappa,\mu}(R)$  is defined by (16).

Then, for continuous initial-value functions, problem (19), (9) has no solutions such that  $\sigma \leq u(x, t) \neq \sigma$ .

**Remark 5.** If we additionally restrict the behavior of the initial-value function by the inequality  $\liminf_{R \rightarrow \infty} \int_{|x| < R} \ln u_0(x) dx \geq 0$  and assume that  $\omega(x, t, s) \geq \beta(x, t)s |\ln s|^q$  in  $\mathbb{R}^n \times (0, +\infty) \times (0, +\infty)$ , then problem (19), (9) has no positive solutions at all.

**Example 2.** For  $\alpha > -1, \rho > 1$ , and nontrivial initial-value functions, the Cauchy problem for the inequality

$$\frac{\partial u}{\partial t} \geq \Delta u + \frac{\alpha}{u} |\nabla u|^2 + u^\rho$$

has no positive solutions if  $\frac{\alpha + 1}{\rho - 1} \geq \frac{n}{2}$ , and this critical value is exact.

### 5. Integrodifferential Blow-Up

In this section, the investigation gets out the framework of differential inequalities: we deal with functional-differential ones, i.e., with inequalities containing other operators (apart from differential ones) acting on the desired functions. In our case, the operators different from differential ones, are convolution ones.

Full proofs of the results of this section are provided in [47].

#### 5.1. Stationary Case

The following assertions are valid.

**Proposition 1.** Let there exist from  $(-1, \infty)$  such that  $a_j(x, s) \geq \frac{1}{s}, j = 1, 2, \dots, n$ , and  $b(x, s) \geq \frac{1}{(+1)s}$ . Assume that  $K(x)$  is bounded from below by the function  $|x|^{\beta-n}, \beta \in (0, n)$ , and  $\gamma > +1$ . For  $\beta < n - 2$ , assume (additionally) that  $\gamma \leq \frac{(+1)n}{n - 2 - \beta}$ . Then the inequality

$$\Delta u + \sum_{j=1}^n a_j(x, u) \left( \frac{\partial u}{\partial x_j} \right)^2 + b(x, u) K * u^\gamma \leq 0$$

has no global positive solutions.

**Proposition 2.** Assume that  $K(x)$  is bounded from above by the function  $|x|^{\beta-n}, \beta \in (0, n)$ , and  $\gamma < +1$ . For  $\beta < n - 2$ , assume (additionally) that  $\gamma \geq \frac{(+1)n}{n - 2 - \beta}$ . Let there exist from  $(-\infty, -1)$  such that  $a_j(x, s) \leq \frac{1}{s}, j = 1, 2, \dots, n$ , and  $b(x, s) \leq \frac{1}{(+1)s}$ . Then the inequality

$$\Delta u + \sum_{j=1}^n a_j(x, u) \left( \frac{\partial u}{\partial x_j} \right)^2 + b(x, u) K * u^\gamma \geq 0$$

has no global positive solutions.

**Proposition 3.** Let real numbers  $q$  and  $\beta$  and a positive integer  $n$  satisfy the inequalities

$$0 < \beta < n, q > 1, \text{ and } q(n - 2 - \beta) \leq n$$



and there exist a function  $g$  continuous on the real line and such that the inequalities

$$a_j(x, s) \geq g(s), b(x, s) \geq e^{-\int_0^s g(\tau) d\tau}, \text{ and } K(s) \geq \left| \int_0^s e^{\int_0^t g(\tau) d\tau} dt \right|^q$$

are satisfied for  $j = 1, 2, \dots, n, s \in \mathbb{R}^1$ , and  $x \in \mathbb{R}^n$ . Then the inequality

$$\Delta u + \sum_{j=1}^n a_j(x, u) \left( \frac{\partial u}{\partial x_j} \right)^2 + b(x, u) K[u(x)] * |x|^{\beta-n} \leq 0$$

has no classical solutions in  $\mathbb{R}^n$ .

### 5.2. Nonstationary Case

The following assertions are valid.

**Proposition 4.** Let  $\varkappa > -1, 0 < \beta < n, u_0^{\varkappa+1} \in L_{1,loc}(\mathbb{R}^n)$ , and there exist positive constants  $C$  and  $R_0$  and a nonnegative constant  $\gamma$  such that inequality

$$\int_{|x|<R} u_0^{\varkappa+1}(x) dx \geq CR^\gamma$$

holds for each  $R$  from  $(R_0, +\infty)$ . Let  $a_j(x, t, s) \geq \frac{\varkappa}{s}, j = 1, 2, \dots, n, b(x, t, s) \geq \frac{1}{(\varkappa + 1)s^\varkappa}$ , and the function  $K(x)$  be bounded from below by the Riesz kernel  $|x|^{\beta-n}$ . Then, for  $\gamma \geq n$ , the Cauchy problem for the inequality

$$\frac{\partial u}{\partial t} \geq \Delta u + \sum_{j=1}^n a_j(x, t, u) \left( \frac{\partial u}{\partial x_j} \right)^2 + b(x, t, u) K * u^\omega$$

has no classical positive solutions provided that  $\omega$  satisfies the inequality  $\omega > \varkappa + 1$ , while, for  $\gamma < n$ , it has no classical positive solutions provided that  $\omega$  satisfies the inequality  $1 < \frac{\omega}{\varkappa + 1} < 1 + \frac{\beta + 2}{n - \max\{\gamma, \beta\}}$ .

**Proposition 5.** Let real numbers  $q$  and  $\beta$  and a positive integer  $n$  satisfy the inequalities

$$0 < \beta < n, q > 1, \text{ and } 1 < q \leq \frac{n + 2}{n - \beta}.$$

Let there exist a function  $g$  continuous on the real line and such that the inequalities

$$a_j(x, t, s) \geq g(s), b(x, t, s) \geq e^{-\int_0^s g(\tau) d\tau}, \text{ and } K(s) \geq \left| \int_0^s e^{\int_0^t g(\tau) d\tau} dt \right|^q$$

are satisfied for  $j = 1, 2, \dots, n, s \in \mathbb{R}^1, x \in \mathbb{R}^n$ , and  $t > 0$ , while the function  $u_0(x)$  satisfies the conditions

$$\int_0^{u_0(x)} e^{\int_0^s g(\tau) d\tau} ds \in L_{1,loc}(\mathbb{R}^n) \text{ and } \lim_{R \rightarrow \infty} \frac{1}{R^\beta} \int_{|x|<R} \int_0^{u_0(x)} e^{\int_0^s g(\tau) d\tau} ds dx \geq 0.$$

Then the Cauchy problem for the inequality

$$\frac{\partial u}{\partial t} \geq \Delta u + \sum_{j=1}^n a_j(x, t, u) \left( \frac{\partial u}{\partial x_j} \right)^2 + b(x, t, u) K(u) * |x|^{\beta-n}$$

has no nontrivial classical solutions.

### 6. Qualitative Properties of Solutions

#### 6.1. Parabolic Equations Admitting Degenerations at Infinity

Full proofs of the results of this section are provided in [48].

Consider the equation

$$\rho(x) \frac{\partial u}{\partial t} = \Delta u + g(u) |\nabla u|^2, \quad x \in \mathbb{R}^n, \quad t \in (0, +\infty), \tag{20}$$

under the assumption that the equation  $\Delta w + \rho(x) = 0$  has a solution bounded in  $\mathbb{R}^n$ .

The following assertions are valid.

**Theorem 11.** Let  $u(x, t)$  be a bounded solution of the Cauchy problem for Equation (20), where  $g$  is continuous and there exists a constant  $\theta, 0 < \theta < 1$ , such that  $\rho \in C_{loc}^{+\theta}(\mathbb{R}^n)$  and  $f(u_0) \in C_{loc}(\mathbb{R}^n)$ , where  $f$  is defined by relation (7). Then there exists a Lipschitz on  $[0, +\infty)$  function  $A$  such that the relation

$$\lim_{R \rightarrow \infty} \frac{1}{R^{n-1}} \int_{|x|=R} \int_0^t f[u(x, \tau)] d\tau d\sigma_x = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} A(t)$$

is satisfied for any positive  $t$  and the relation

$$\lim_{R \rightarrow \infty} \frac{1}{R^{n-1}} \int_{|x|=R} \left( \int_0^t f[u(x, \tau)] d\tau - A(t) \right) d\sigma_x = 0$$

is satisfied uniformly with respect to  $t$  from  $[0, T]$  for any positive  $T$ .

**Theorem 12.** Let  $u(x, t)$  be a bounded positive solution of the Cauchy problem for Equation (20), where  $g(s) = \alpha s^\beta, \beta \in (-1, 0), \alpha \in \mathbb{R}^1$ , and the coefficient  $\rho(x)$  and the initial-value function  $u_0(x)$  satisfy the assumptions of Theorem 11. Then the assertion of Theorem 11 holds.

**Theorem 13.** Let  $u(x, t)$  be a bounded positive solution of the Cauchy problem for the equation

$$\rho(x) \frac{\partial u}{\partial t} = \Delta u + \frac{\alpha}{u} |\nabla u|^2, \quad x \in \mathbb{R}^n, \quad t \in (0, +\infty), \tag{21}$$

where  $\alpha > -1$ , the coefficient  $\rho(x)$  satisfies the assumptions of Theorem 11, and the initial-value function  $u_0(x)$  is such that  $u_0^{\alpha+1} \in C_{loc}(\mathbb{R}^n)$ . Then there exists a Lipschitz on  $[0, +\infty)$  function  $A$  such that the relation

$$\lim_{R \rightarrow \infty} \frac{1}{R^{n-1}} \int_{|x|=R} U_{\alpha+1}(x, t) dx = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} A(t)$$

is satisfied for any positive  $t$  and the relation

$$\lim_{R \rightarrow \infty} \frac{1}{R^{n-1}} \int_{|x|=R} [U_{\alpha+1}(x, t) - A(t)] dx = 0$$

is satisfied uniformly with respect to  $t$  from  $[0, T]$  for any positive  $T$ , where

$$U_s(x, t) = \int_0^t u^s(x, \tau) d\tau, \quad s > 0.$$

**Proposition 6.** Let  $u(x, t)$  be a bounded solution of the Cauchy problem for Equation (21), where  $\alpha \neq -1$ ,  $\inf u \geq B > 0$ , the coefficient  $\rho(x)$  satisfies the assumptions of Theorem 11, and the initial-value function  $u_0(x)$  is such that  $u_0^{\alpha+1} \in C_{loc}(\mathbb{R}^n)$ . Then the assertion of Theorem 13 holds.

**Proposition 7.** Let  $u(x, t)$  be a bounded solution of the Cauchy problem for the equation

$$\rho(x) \frac{\partial u}{\partial t} = \Delta u - \frac{1}{u} |\nabla u|^2,$$

where  $\inf u \geq B > 0$ , the coefficient  $\rho(x)$  satisfies the assumptions of Theorem 11, and the initial-value function  $u_0(x)$  is such that  $\ln u_0 \in C_{loc}(\mathbb{R}^n)$ . Then there exists a Lipschitz on  $[0, +\infty)$  function  $A$  such that the relation

$$\lim_{R \rightarrow \infty} \frac{1}{R^{n-1}} \int_{|x|=R} \int_0^t \ln u(x, \tau) d\tau d\sigma_x = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} A(t)$$

is satisfied for any positive  $t$  and the relation

$$\lim_{R \rightarrow \infty} \frac{1}{R^{n-1}} \int_{|x|=R} \left( \int_0^t \ln u(x, \tau) d\tau - A(t) \right) d\sigma_x = 0$$

is satisfied uniformly with respect to  $t$  from  $[0, T]$  for any positive  $T$ .

## 6.2. Extinction Phenomena

### 6.2.1. Parabolic Inequalities

Full proofs of the results of this section are provided in [49].

The Cauchy problem for the inequality

$$\Delta u + \sum_{j=1}^n a_j(x, u) \left( \frac{\partial u}{\partial x_j} \right)^2 - \frac{\partial u}{\partial t} \geq f(u), \tag{22}$$

where  $u_0(x)$  is continuous, bounded, and nonnegative in  $\mathbb{R}^n$  and  $\lim_{|x| \rightarrow \infty} u_0(x) = 0$ , is considered. The following functions are introduced:

$$F(s) \stackrel{\text{def}}{=} \int_0^s \int_0^x g(\tau) d\tau dx \quad \text{and} \quad \beta(s) \stackrel{\text{def}}{=} e^{\int_0^{F^{-1}(s)} g(\tau) d\tau} f[F^{-1}(s)].$$

Under the above assumptions, the following assertions are valid.

**Theorem 14.** Let there exist a continuous function  $g(s)$  such that  $a_j(x, s) \leq g(s)$ ,  $j = 1, 2, \dots, n$ , in  $[0, +\infty)$ . Let  $f(0) = 0$ ,  $f(s) > 0$  in  $(0, +\infty)$ ,  $\beta$  is a nonincreasing function on  $(0, +\infty)$ , and

$$\int_0^1 \frac{ds}{\sqrt{s\beta(s)}} < \infty. \text{ Then any nonnegative solution of the above Cauchy problem for Equation (22)}$$

possesses the following properties:

- (i)  $\text{supp } u(x, t)$  is bounded for any positive  $t$ ;

- (ii)  $\lim_{|x| \rightarrow \infty} u(x, t) = 0$  uniformly with respect to  $t$  from  $[0, +\infty)$ ;
- (iii) there exists a positive  $T$  such that  $u(x, t) \equiv 0$  in the half-space  $\mathbb{R}^n \times [T, +\infty)$ .

**Theorem 15.** Let there exist  $a$  from  $(-1, +\infty)$  such that  $a_j(x, s) \leq \frac{a}{s}$  in  $\mathbb{R}^n \times (0, +\infty)$ ,  $j = 1, 2, \dots, n$ ,  $b$  from  $(0, +\infty)$ , and  $p$  from  $(-a, 1)$  such that  $f(s) \geq bs^p$  in  $(0, +\infty)$ . Then the above Cauchy problem for Equation (22) has no positive solutions.

### 6.2.2. Parabolic Equations with Potentials

Full proofs of the results of this section are provided in [50].

The Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\beta}{u} |\nabla u|^2 + C(x, t)u, \tag{23}$$

where  $n \geq 3$  and  $u_0(x)$  is continuous and bounded in  $\mathbb{R}^n$ , is considered.

Under the above assumptions, the following assertions are valid.

**Theorem 16.** If  $\beta < -1$  and

$$C(x, t) \geq \alpha \min\left(1, \frac{1}{|x|^2}\right),$$

then the above Cauchy problem for Equation (23) has no positively definite solutions.

**Theorem 17.** If  $\beta > -1$ , then

- (i) there exists at most one classical bounded nonnegative solution of the Cauchy problem for Equation (23);
- (ii) if

$$C(x, t) \leq -\alpha \min\left(1, \frac{1}{|x|^2}\right), \tag{24}$$

then  $u \xrightarrow{t \rightarrow \infty} 0$  uniformly with respect to  $x$  in each compactum of the space  $\mathbb{R}^n$  provided that  $u(x, t)$  exists;

- (iii) if  $C(x, t) \leq -\alpha$  (this inequality is stronger), then there exists a positive constant  $a$  such that the inequality

$$u(x, t) \leq \sup_{x \in \mathbb{R}^n} u_0(x) e^{-at}$$

holds in  $\mathbb{R}^n \times (0, \infty)$  provided that  $u(x, t)$  exists.

**Theorem 18.** If  $\beta > -1$ ,  $u(x, t)$  is a classical bounded nonnegative solution of the Cauchy problem for Equation (23), and there exists  $\alpha$  such that  $\frac{\alpha}{\beta + 1} > n - 1$  and  $C(x, t)$  satisfies Condition (24).

Then for each compactum  $K$  of the space  $\mathbb{R}^n$  there exist constants  $M$  and  $T$  such that

$$u(x, t) \leq \frac{M}{t}, = \frac{2 - n + \sqrt{(2 - n)^2 + 4\alpha}}{6(\beta + 1)},$$

for each  $(x, t)$  from  $K \times (T, +\infty)$ .

### 6.3. Singular Equations with Nonclassical Neumann Conditions

Full proofs of the results of this section are provided in [51].

### 6.3.1. Stationary Case

The problem

$$\begin{aligned} \Delta u + \frac{\alpha}{u} |\nabla u|^2 - a(x')u^p &= 0, \quad x' := (x_1, \dots, x_n) \in \Omega, \quad x_{n+1} \geq 0, \\ u^\alpha \frac{\partial u}{\partial n} &= 0, \quad x' \in \partial\Omega, \quad x_{n+1} \geq 0, \end{aligned} \tag{25}$$

where  $\Omega$  is a bounded domain with a Lipschitz boundary in  $\mathbb{R}^n$  and  $a$  is a bounded measurable function, is considered.

Under the above assumptions, the following assertion is valid.

**Theorem 19.** *Let a positive function  $u(x)$  satisfy problem (25) and the condition*

$$\lim_{x_{n+1} \rightarrow \infty} \frac{1}{x_{n+1}} \int_{\Omega} u^{\alpha+1}(x') dx' = 0.$$

Let one of the systems of inequalities

$$\left\{ \begin{array}{l} \alpha > -1, \\ p > 1, \\ \int_{\Omega} a(y) dy > 0, \\ a(y) \geq 0 \text{ in } \Omega \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \alpha < -1, \\ p < 1, \\ \int_{\Omega} a(y) dy < 0, \\ a(y) \leq 0 \text{ in } \Omega \end{array} \right.$$

be satisfied. Then  $\lim_{x_{n+1} \rightarrow \infty} x_{n+1}^{\frac{2(\alpha+1)}{p-1}} u^{\alpha+1}(x)$  exists for each  $x'$  from  $\Omega$ , this limit is uniform with respect to  $x'$  from  $\overline{\Omega}$ , and it is equal either to zero or to

$$\left( \frac{2(2\alpha + p + 1) \text{mes}\Omega}{(\alpha + 1)^2 \int_{\Omega} a(y) dy [(\alpha + 1)a(x') - 1]^2} \right)^{\frac{\alpha+1}{p-1}}.$$

### 6.3.2. Nonstationary Case: Blow-Up

The problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + \frac{\alpha}{u} |\nabla u|^2 - a(x)u^p, \quad x \in \Omega, \quad 0 < t < T, \\ u^\alpha \frac{\partial u}{\partial n} &= 0, \quad x \in \partial\Omega, \quad 0 < t < T, \end{aligned} \tag{26}$$

where  $T$  is a positive constant,  $\Omega$  is a bounded domain with a Lipschitz boundary in  $\mathbb{R}^n$ , the function  $a$  is bounded and measurable, and the constant  $\alpha$  is different from  $-1$ , is considered.

Under the above assumptions, the following assertions are valid.

**Theorem 20.** *For each (arbitrarily small) positive  $T$ , problem (26) has no positive solutions if one of the following two collections of conditions is satisfied:*

- (i)  $\int_{\Omega} a(x) dx < 0$ ,  $\alpha > -1$ , and  $p > 1$ ;
- (ii)  $\int_{\Omega} a(x) dx > 0$ ,  $\alpha < -1$ , and  $p < 1$ .

**Theorem 21.** Problem (26) in the cylinder  $\Omega \times (0, \infty)$  has no positive bounded solutions if  $\int_{\Omega} a(x)dx = 0$ , the function  $a(x)$  is different from the identical zero,  $\alpha > -1$ , and  $p > 1$ .

**Theorem 22.** Problem (26) in the cylinder  $\Omega \times (0, \infty)$  has no positively definite bounded solutions provided that  $\int_{\Omega} a(x)dx = 0$ , the function  $a(x)$  is different from the identical zero,  $\alpha < -1$ , and  $p < 1$ .

6.3.3. Nonstationary Case: Large-Time Behavior

Under the above assumptions of Section 6.3.2, the following assertion is valid.

**Theorem 23.** If a positive function  $u(x, t)$  satisfies problem (17) in the cylinder  $\Omega \times (0, \infty)$  and one of the systems of inequalities

$$\begin{cases} a(x) \geq 0 \text{ in } \Omega, \\ \int_{\Omega} a(x)dx > 0, \\ p > 1 \end{cases} \quad \text{or} \quad \begin{cases} a(x) \leq 0 \text{ in } \Omega, \\ \int_{\Omega} a(x)dx < 0, \\ p < 1 \end{cases}$$

is satisfied, then  $\lim_{t \rightarrow \infty} t^{\frac{\alpha+1}{p-1}} u^{\alpha+1}(x, t)$  exists, this limit is uniform in  $\bar{\Omega}$ , and it is equal either to zero

or to the constant  $\left[ \frac{p-1}{\text{mes } \Omega} \int_{\Omega} a(y)dy \right]^{\frac{1+\alpha}{1-p}}$ .

7. Conclusions

In this paper, we provide various results on qualitative properties of solutions of inequalities with KPZ-nonlinearities. Since they are nonstrict, the corresponding equations can be treated as particular cases of inequalities and, therefore, the provided results are valid for those equations as well. The provided results (obtained by methods based on properties of monotonic maps) refer to the global nonexistence, stabilization and extinction phenomena, compactification of solution supports, and other significant properties. The considered inequalities and equations contain not only partial differential operators; a number of results is obtained for integrodifferential (more exactly, convolutional-differential) inequalities and equations as well.

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