Article

Fuzzy Triple Controlled Metric Like Spaces with Applications

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Abstract: In this article, we introduce the concept of a fuzzy triple controlled metric like space in the sense that the self distance may not be equal to one. We have used three functions in our space that generalize fuzzy controlled rectangular, extended fuzzy rectangular, fuzzy $b-$rectangular and fuzzy rectangular metric like spaces. Various examples are given to justify our definitions and results. As for the topological aspect, we prove a fuzzy triple controlled metric like space is not Hausdorff. We also apply our main result to solve the uniqueness of the solution of a fractional differential equation.

Keywords: fuzzy metric like space; controlled metric space; fractional differential equation

MSC: 47H10; 47H04; 47H07

1. Introduction

Fixed point theory provides powerful tools for proving the existence and uniqueness of solutions to various types of problems and Banach fixed point theorem is a fundamental result in fixed point theory [1]. There are several variations of the Banach fixed point theorem that use different types of contractions. For example, the Kannan fixed point theorem [2] is a generalization of the Banach fixed point theorem that uses a more relaxed contraction condition known as Kannan contraction. Similarly, the Nadler fixed point [3] theorem uses a weaker contraction condition called Nadler contraction. The $F-$contraction [4] is a more general class of contractions than the Banach contraction and it allows for a wider range of functions to be used as contractions. Many variations of the Banach contraction theorem have been developed over the years, and most of them require the self-map to be continuous. The Suzuki-type contraction [5] has been used to prove the existence and uniqueness of fixed points for self-maps of metric spaces that are not necessarily complete or continuous. However, it should be noted that the Suzuki-type contraction has more restrictive conditions than the $F-$contraction. In particular, the Suzuki-type contraction requires the function $\phi$ to satisfy certain growth conditions, while the $F-$contraction only requires $\phi$ to be a nondecreasing function with $\phi(0) = 0$. In 2008, Berinde et al. [6] introduced the concept of an almost contraction, which is a self-map on a metric space that is continuous at its fixed points. This is useful in applications where the self-map may have discontinuities or singularities. The authors in [7] introduced a new type of contraction called the generalized Suzuki-type $F-$contraction for fuzzy mappings, which is a generalization of the Suzuki-type contraction and the $F-$contraction introduced by Wardowski. In 2019, Saleem et al. [8] introduced the Suzuki-type generalized multi-valued almost contraction mapping that combines the concepts of Suzuki-type contraction and almost contraction. The authors in [9] introduced Suzuki-type $(\alpha, \beta, \gamma)$-generalized and modified proximal contractive mapping. This new type of contraction generalizes the concept of proximal contractive mapping that combines the concepts of Suzuki-type contraction, almost contraction and modified proximal contraction. The authors in [10] used $F-$contraction, $F-$Suzuki
contraction, and $F$-expanding mappings to prove the existence and uniqueness of fixed points for self-mappings of complete metric spaces. The results generalize some of the well-known fixed point theorems in the literature, such as the Banach fixed point theorem and the Suzuki fixed point theorem. In 2020, Fatemah et al. [11] utilized multi-valued mapping and showed that their results can be applied to linear systems, which have important applications in various areas of engineering and control theory.

In 1965, Zadeh [12] generalized the definition of a crisp set by defining the concept of a fuzzy set that takes the membership value of the elements in the interval $[0, 1]$. Since a fuzzy set addresses the uncertainty and gives more accurate results as compared to crisp set, so researchers have used fuzzy sets in almost every branch of mathematics ([13–15]). In 1975, Kramosil and Michálek [16] utilized fuzzy sets and gave the notion of a fuzzy metric space, which is considered a generalization of Menger’s statistical metric spaces [17]. As the topological properties of metric spaces play a vital role so, George and Veeramani [18] generalized the definition of a fuzzy metric space. They discussed the topology of a fuzzy metric space and proved that a fuzzy metric space is Hausdorff. In 1983, Grabiec [19] established the fuzzy version of Banach fixed point theorem.

Hitzler et al. [20] introduced dislocated topologies that can be characterized in terms of certain axioms that generalize the usual axioms of a metric space. They also introduced the concept of dislocated convergence, which is a generalization of the usual notion of convergence in metric spaces. In 2013, the author in [21] introduced the idea of a metric like space. Alghamdi et al. [22] introduced $b$-metric like space that is a generalization of metric like space. Mlaiki et al. [23] used the controlled function and introduced the concept of controlled metric type spaces. The work of Mlaiki et al. [24] and Asim et al. [25] deals with the mathematical concept of rectangular metric like and extended rectangular $b$-metric spaces respectively. Abdeljawad et al. [26] used two controlled functions, $a$ and $y$ and defined the idea of double controlled metric type spaces.

The concept of fuzzy metric like space is introduced by Shukla et al. [27] which is a generalization of [18]. They proved fixed point results by using fuzzy contractive mappings. Javed et al. [28] introduced the notion of a fuzzy $b$-metric like space in the sense of Kramosil and Michálek. Saleem et al. [29] introduced the notion of fuzzy double controlled metric space and proved Banach fixed point results while Furqan et al. [30] gave the notion of a fuzzy triple controlled metric space. In addition, they show that these spaces are not necessarily Hausdorff, which is an interesting result.

The homotopy method is a powerful numerical technique that can be used to solve a wide range of nonlinear problems ([31–36]). The authors in ([37,38]) have utilized the homotopy method in estimation of thermal parameters within annular fins. Homotopy method has also been used in inverse analysis of flow problem [39]. In [40], the author has investigated the weighted homotopy analysis method in the solution of one dimensional wave equation and showed the accuracy of the method by means of examples. Liu [41] proposed the multigrid homotopy technique for nonlinear inverse problem. In [42], the author combined the wavelet multiscale method and homotopy method and introduced multiscale-homotopy method for the parameter identification problem in partial differential equation. An other homotopy method, homotopy continuation technique, is applied to crack identification of beam structures [43]. In 2021, Courbot and Colicchio [44] applied the homotopy technique to find the solution of constrained BLASSO and applied their results to 3D tomographic diffractive microscopy images. Slota et al. [45] applied the homotopy technique to the one face fractional inverse Stefan design problem, while in [46], the authors utilized the homotopy method in porosity reconstruction on the basis of Biot elastic model [47].

The concept of like spaces in fuzzy triple controlled metric spaces seems to be a generalization of the notion of like spaces in metric spaces, where the self-distance between points is fixed to be one. By allowing the self-distance to vary in fuzzy triple controlled metric spaces, we have introduced a new level of flexibility in the concept of like spaces, which could have potential applications in various fields. Our proposed fuzzy triple
controlled metric like space generalizes many existing results. For example, it generalizes rectangular metric like, $b-$rectangular metric like, extended and controlled metric like spaces in fuzzy environment, that can be regarded as the main advantage of our proposed methods. We discuss the topology properties and prove that a fuzzy triple controlled metric like space is not Hausdorff. We prove the Banach contraction principle in the settings of newly defined space and apply our results to fractional differential equation.

2. Preliminaries

In this section, we will cover the essential concepts related to metric like spaces.

Definition 1 ([21]). A mapping $d : F \times F \rightarrow \mathbb{R}^+ \cup \{0\}$ on a non-empty set $F$ is called a metric-like if $d$ satisfies:

$ML1 \ d(k_1, k_2) = 0 \Rightarrow k_1 = k_2$;
$ML2 \ d(k_1, k_2) = d(k_2, k_1)$;
$ML3 \ d(k_1, k_3) \leq d(k_1, k_2) + d(k_2, k_3)$.

The pair $(F, d)$ is called a metric-like space.

Example 1 ([21]). Let $F = \{0, 1\}$ and $d$ is given by $d(k_1, k_2) = 0$ if $k_1 = k_2 = 0$, and $d(k_1, k_2) = 1$, otherwise. Then $(F, d)$ is a metric-like space.

Definition 2 ([22]). Let $F$ be a non-empty set and $b \geq 1$, a function $d : F \times F \rightarrow \mathbb{R}^+ \cup \{0\}$ is called $b-$metric like if $d$ satisfies:

$bML1 \ d(k_1, k_2) = 0 \Rightarrow k_1 = k_2$;
$bML2 \ d(k_1, k_2) = d(k_2, k_1)$;
$bML3 \ d(k_1, k_3) \leq b[d(k_1, k_2) + d(k_2, k_3)]$.

The pair $(F, d)$ is called $b-$metric like space.

Example 2 ([22]). Let $F = \mathbb{R}$ and $d : (\mathbb{R}, \infty) \times (\mathbb{R}, \infty) \rightarrow [0, \infty)$ be defined as $d(k_1, k_2) = (k_1 + k_2)^2$. Then $(F, d)$ is $b-$metric like space with $b = 2$.

Mlaiki et al. [23] generalized the Definition (1) by introducing the rectangular metric like space as follows:

Definition 3 ([23]). Let $F$ be non-empty set and $d : F \times F \rightarrow [0, \infty)$ be a function, then $d$ is said to be a rectangular metric-like if it satisfies:

$RML1 \ d(k_1, k_2) = 0 \Rightarrow k_1 = k_2$;
$RML2 \ d(k_1, k_2) = d(k_2, k_1)$;
$RML3 \ d(k_1, k_4) \leq d(k_1, k_2) + d(k_2, k_3) + d(k_3, k_4)$, for all distinct $k_2, k_3, k_4 \in F \setminus \{k_1, k_4\}$.

The pair $(F, d)$ is called rectangular metric-like space.

Definition 4 ([48]). A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfies the following conditions for all $\alpha, \beta, \gamma, \delta \in [0, 1]$, then it is a continuous triangular norm or $t-$norm:

(+1) $*(k_1, k_2) = *(k_2, k_1)$;
(+2) $*(k_1, *(k_2, k_3)) = *(k_1, k_2), k_3)$;
(+3) $*$ is continuous;
(+4) $*(k_1, 1) = k$ for every $k \in [0, 1]$;
(+5) $*(k_1, k_2) \leq *(k_3, k_4)$ whenever $k_1 \leq k_3, k_2 \leq k_4$.

The most commonly used $t-$norms are product $t-$norm $(k_1 * k_2 = k_1 k_2)$, minimum $t-$norm $(k_1 * k_2 = \min\{k_1, k_2\})$ and Lukasewicz $t-$norm $(k_1 * k_2 = \max\{k_1 + k_2 - 1, 0\})$. 
In 2012, Samet et al. [49] introduced a new type of contraction, called $\alpha - \psi -$ contraction. They used a function $\psi : F \rightarrow F$ with the properties:

1. $\psi$ is non-decreasing;
2. $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t$, where $\psi^n$ is the $n$–th iterative of $\psi$.

**Definition 5 ([49]).** Let $(F , d)$ be a metric space and $T : F \rightarrow F$ be a mapping. Then $T$ is an $\alpha - \psi -$ contraction, if there exists two functions, $\alpha : F \times F \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$a(k_1, k_2) d(Tk_1, Tk_2) \leq \psi(d(k_1, k_2))$$

(1)

for all $k_1, k_2 \in F$.

**Example 3.** Consider the function defined by

$$\psi(t) = \begin{cases} t - \frac{t^2}{2}, & \text{if } t \in [0, 1], \\ \frac{1}{2}, & \text{for } t > 1, \end{cases}$$

clearly $\psi \in \Psi$.

**Definition 6 ([50]).** Let $(F , M, *)$ be a fuzzy metric space with $\alpha : F \times F \times (0, \infty) \rightarrow (0, \infty)$ be a function. The mapping $T : F \rightarrow F$ is called an $\alpha -$admissible if,

$$\alpha(k_1, k_2, t) \geq 1 \Rightarrow \alpha(Tk_1, Tk_2, t) \geq 1 \text{ for all } t > 0, k_1, k_2 \in F.$$  

3. Main Results

This section consists the definition of fuzzy triple controlled metric like space and some of its topological aspects. In this section, we will prove fixed point results with the help of $\alpha - \psi -$ contraction and later with the help of standard Banach contraction. Some remarks and examples are also given that illustrates our results. We start with the definition of a fuzzy triple controlled metric like space as follows:

**Definition 7.** Let $f_1, f_2, f_3 : F \times F \rightarrow [1, \infty)$ be three non-comparable functions. A fuzzy set

$$M : F \times F \times (0, \infty) \rightarrow [0, 1],$$

together with a continuous $t$–norm $*$, is called a fuzzy triple controlled metric like, if for any $k_1, k_2 \in F$ and all distinct $k_3, k_4 \in F \setminus \{k_1, k_2\}$, $M$ satisfies:

(M1) $M(k_1, k_2, t) > 0$;
(M2) if $M(k_1, k_2, t) = 1$ for all $t > 0$, then $k_1 = k_2$;
(M3) $M(k_1, k_2, t) = M(k_2, k_1, t)$;
(M4) $M(k_1, k_3, t + s + w) \geq M(k_1, k_2, \frac{t}{f_1(k_1, k_2)}) * M(k_2, k_3, \frac{w}{f_2(k_2, k_3)}) * M(k_3, k_4, \frac{s}{f_3(k_3, k_4)})$, for all $t, s, w > 0$;
(M5) $M(k_1, k_2, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Then $(F , M, *)$ is called a fuzzy triple controlled metric like space.

**Remark 1.**

(i) If we take $f_1(k_1, k_2) = f_2(k_2, k_3) = f_3(k_3, k_4) = f_1(k_1, k_4)$ in condition (M4) of a fuzzy triple controlled metric like space, then it reduces to an extended fuzzy $b -$ rectangular metric like space [51].

(ii) If we take $f_1(k_1, k_2) = f_2(k_2, k_3) = f_3(k_3, k_4) = b \geq 1$ in condition (M4) of a fuzzy triple controlled metric like space, then it reduces to fuzzy $b -$ rectangular metric like space [52].

(iii) If we take $f_1(k_1, k_2) = f_2(k_2, k_3) = f_3(k_3, k_4) = 1$ in condition (M4) of a fuzzy triple controlled metric like space, then it reduces to fuzzy rectangular metric like space [52].
According to Saleem et al. in [51], not every extended fuzzy \( b \)–rectangular metric like space is Hausdorff. If we restrict the fuzzy triple controlled metric like space for case (ii) and (iii) (as in Remark 1), then the topology of fuzzy \( b \)–rectangular and fuzzy rectangular metric like space is also not Hausdorff.

**Remark 2.** In (M4), if \( M(\kappa_3, \kappa_4, \frac{w}{f_1(\kappa_3, \kappa_4)}) = 1 \), then we have the following observations:

(i) If we substitute \( s + w = t' \) and \( f_1(\kappa_1, \kappa_2) = f_2(\kappa_2, \kappa_3) = b \geq 1 \) in (M4) of a fuzzy triple-controlled metric-like space, then it reduces to a fuzzy \( b \)–metric like space [28].

(ii) If we substitute \( s + w = t' \) and \( f_1(\kappa_1, \kappa_2) = f_2(\kappa_2, \kappa_3) = 1 \), in (M4) of fuzzy triple controlled metric like space, then it reduces to a fuzzy metric like space [27].

**Remark 3.** In Definition 7, the non-comparable functions means that they are not equal. For example, if we put them equal, say \( f_1 = f_2 = f_3 = f \), then Definition 7 reduces to the definition of extended fuzzy \( b \)–rectangular metric like space [51]. For more detail, see Remarks 1 and 2.

The authors in [27,28] have not discussed the topological properties of fuzzy \( b \)–metric like and fuzzy metric like space respectively. If we restrict ourselves and take \( s + w = t' \) and \( f_1(\kappa_1, \kappa_2) = f_2(\kappa_2, \kappa_3) = b \geq 1 \), then the results in [28] becomes the special cases of fuzzy triple controlled metric like space. Similarly if we take \( s + w = t' \) and \( f_1(\kappa_1, \kappa_2) = f_2(\kappa_2, \kappa_3) = 1 \), then our results generalizes the results in [27].

Following example justifies the Definition (7).

**Example 4.** Consider \( F = \{0, 1, 2, 3\} \) and let \( f_1, f_2, f_3 : F \times F \rightarrow [1, \infty) \) be three non-comparable functions defined as \( f_1(\kappa_1, \kappa_2) = 1 + \kappa_1 + \kappa_2, \ f_2(\kappa_1, \kappa_2) = \kappa_1^2 + \kappa_2 + 2 \) and \( f_3(\kappa_1, \kappa_2) = \kappa_1^3 + \kappa_2^2 + 3 \). Define a rectangular metric-like space by \( d(\kappa_1, \kappa_2) = \max\{\kappa_1, \kappa_2\} \).

Now define \( M : F \times F \times (0, \infty) \rightarrow [0, 1] \) as:

\[
M(\kappa_1, \kappa_2, t) = \frac{t}{t + d(\kappa_1, \kappa_2)}.
\]

Then \( (F, M, \ast) \) is a fuzzy triple controlled metric like space with product \( t \)–norm.

Now, \( f_1(0, 1) = f_1(1, 0) = 2, f_1(0, 2) = f_1(2, 0) = 3, f_1(0, 3) = f_1(3, 0) = 4, f_1(1, 2) = f_1(2, 1) = 4, f_1(1, 3) = f_1(3, 1) = 5, f_1(2, 3) = f_1(3, 2) = 6 \), \( f_1(0, 0) = 1, f_1(1, 1) = 3, f_1(2, 2) = 5, f_1(3, 3) = 7 \),

\[
f_2(0, 1) = 3, f_2(1, 0) = 4, f_2(0, 2) = 4, f_2(2, 0) = 6, f_2(0, 3) = 5, f_2(3, 0) = 11, f_2(1, 2) = 5, f_2(2, 1) = 7, f_2(1, 3) = 6, f_2(3, 1) = 12, f_2(2, 3) = 9, f_2(3, 2) = 13, f_2(0, 0) = 2, f_2(1, 1) = 4, f_2(2, 2) = 8, f_2(3, 3) = 14,
\]

\[
f_3(0, 1) = f_3(1, 0) = 4, f_3(0, 2) = f_3(2, 0) = 7, f_3(0, 3) = f_3(3, 0) = 12, f_3(1, 2) = f_3(2, 1) = 8, f_3(1, 3) = f_3(3, 1) = 13, f_3(2, 3) = f_3(3, 2) = 17, f_3(0, 0) = 3, f_3(1, 1) = 5, f_3(2, 2) = 11, f_3(3, 3) = 21.
\]

Axioms (M1) to (M3) and (M5) can easily be verify, we check only (M4).

**Case 1.** Let \( \kappa_1 = 0, \kappa_4 = 3 \), then

\[
M(0, 3, t + s + w) = \frac{t + s + w}{t + s + w + \max\{0, 3\}} = \frac{t + s + w}{t + s + w + 3}.
\]

Now,

\[
M(0, 1, \frac{t}{f_1(0, 1)}) = \frac{t}{f_1(0, 1)} \frac{\frac{t}{f_1(0, 1)}}{\max\{0, 1\}} = \frac{\frac{t}{f_1(0, 1)}}{\frac{t}{f_1(0, 1)} + \max\{0, 1\}} = \frac{\frac{t}{f_1(0, 1)}}{\frac{t}{f_1(0, 1)} + 1} = \frac{\frac{t}{f_1(0, 1)}}{\frac{t}{f_1(0, 1)} + 1} = \frac{t}{t + 2}.
\]
\[
M\left(1, 2, \frac{s}{f_2(1, 2)}\right) = \frac{s}{f_2(1, 2)} + \max\{1, 2\} = \frac{s}{\frac{5}{2}} + 2 = \frac{s}{s + 10},
\]
and
\[
M\left(2, 3, \frac{w}{f_3(2, 3)}\right) = \frac{w}{f_3(2, 3)} + \max\{2, 3\} = \frac{w}{\frac{17}{2}} + 3 = \frac{w}{w + 51}.
\]

Clearly,
\[
M(0, 3, t + s + w) \geq M(0, 1, \frac{t}{f_1(0, 1)}) \ast M(1, 2, \frac{s}{f_2(1, 2)}) \ast M(2, 3, \frac{w}{f_3(2, 3)}).
\]

**Case 2.** Let \(\kappa_1 = 0, \kappa_4 = 2\), then
\[
M(0, 2, t + s + w) = \frac{t + s + w}{t + s + w + \max\{0, 2\}} = \frac{t + s + w}{t + s + w + 2}.
\]
Now,
\[
M\left(0, 1, \frac{t}{f_1(0, 1)}\right) = \frac{t}{f_1(0, 1)} + \max\{0, 1\} = \frac{t}{\frac{1}{2}} + 1 = \frac{t}{t + 2},
\]
\[
M\left(1, 3, \frac{s}{f_2(1, 3)}\right) = \frac{s}{f_2(1, 3)} + \max\{1, 3\} = \frac{s}{\frac{6}{3}} + 3 = \frac{s}{s + 18},
\]
and
\[
M\left(3, 2, \frac{w}{f_3(3, 2)}\right) = \frac{w}{f_3(3, 2)} + \max\{2, 3\} = \frac{w}{\frac{16}{3}} + 3 = \frac{w}{w + 48}.
\]

Clearly,
\[
M(0, 2, t + s + w) \geq M(0, 1, \frac{t}{f_1(0, 1)}) \ast M(1, 3, \frac{s}{f_2(1, 3)}) \ast M(3, 2, \frac{w}{f_3(3, 2)}).
\]

**Case 3.** Let \(\kappa_1 = 0, \kappa_4 = 1\), then
\[
M(0, 1, t + s + w) = \frac{t + s + w}{t + s + w + \max\{0, 1\}} = \frac{t + s + w}{t + s + w + 1}.
\]
Now,

\[ M\left(0,2, \frac{t}{f_1(0,2)} \right) = \frac{t}{f_1(0,2)} \frac{t}{f_1(0,2)} + \max\{0,2\} = \frac{t}{\frac{t}{3} + 2} = \frac{t}{t + 6}. \]

\[ M\left(2,3, \frac{s}{f_2(2,3)} \right) = \frac{s}{f_2(2,3)} \frac{s}{f_2(2,3)} + \max\{2,3\} = \frac{s}{\frac{s}{9} + 3} = \frac{s}{s + 27}. \]

and

\[ M\left(3,1, \frac{w}{f_3(3,1)} \right) = \frac{w}{f_3(3,1)} \frac{w}{f_3(3,1)} + \max\{3,1\} = \frac{w}{\frac{w}{13} + 3} = \frac{w}{w + 39}. \]

Clearly,

\[ M(0,2,t+s+w) \geq M(0,1, \frac{t}{f_1(0,1)}) \ast M(1,3, \frac{s}{f_2(1,3)}) \ast M(3,2, \frac{w}{f_3(3,2)}). \]

Working in the similar manner, one can prove remaining results. Hence \((F, M, \ast)\) is a fuzzy triple controlled metric like space.

The following example demonstrates that a fuzzy triple controlled metric space may not satisfy the conditions of a fuzzy triple controlled metric-like space.

**Example 5.** Consider \( F = \{1,2,3,4\} \) and let \( f,g,h : F \times F \to [1,\infty) \) be three non-comparable functions defined as \( f(k_1,k_2) = 1 + k_1 + k_2, \ g(k_1,k_2) = k_1^2 + k_2 + 1 \) and \( h(k_1,k_2) = k_1^2 + k_2^2 - 1 \). Now define \( M : F \times F \times (0,\infty) \to [0,1] \) as:

\[ M(k_1,k_2,t) = \frac{\min\{k_1,k_2\} + t}{\max\{k_1,k_2\} + t}. \]

Then \((F, M, \ast)\) is a fuzzy triple controlled metric space with product \( t \)-norm \([31]\). However it is not a fuzzy triple controlled metric like space because if we take, \( k_1 = k_2 = 3 \), then \( M(3,3,t) = \frac{\min\{3,3\} + t}{\max\{3,3\} + t} = 1 \), which contradicts (M2), because in like spaces we only have one sided condition, i.e if \( M(k_1,k_2,t) = 1 \), then \( k_1 = k_2 \). On the other hand, if \( k_1 = k_2 \), then \( M(k_1,k_2,t) \) can not always be equal to 1 but in this example, for all \( k_1,k_2 \in F \), we have \( M(k_1,k_2,t) = 1 \). Hence a fuzzy triple controlled metric space is not a fuzzy triple controlled metric like space.

Next example shows a fuzzy triple controlled metric like space need not to be a fuzzy triple controlled metric space.

**Example 6.** Let \( F = (0,\infty) \) and \( t_1 \ast t_2 = t_1 t_2 \). Define \( M : F \times F \times (0,\infty) \to [0,1] \) as

\[ M(k_1,k_2,t) = \exp \frac{(k_1 + k_2)}{t}. \]
for all \( k_1, k_2 \in F \) and \( t > 0 \). Further assume \( f_1, f_2, f_3 : F \times F \rightarrow [1, \infty) \) be three functions. Then \((F, M, \ast)\) is a fuzzy triple controlled metric like space. We will prove only \((M4)\). Consider
\[
(k_1 + k_4) \leq (k_1 + k_2) + (k_2 + k_3) + (k_3 + k_4)
\]

so we have
\[
\frac{(k_1 + k_4)}{t + s + w} \leq \frac{(k_1 + k_2)}{f_1(k_1, k_2)} + \frac{(k_2 + k_3)}{f_2(k_2, k_3)} + \frac{(k_3 + k_4)}{f_3(k_3, k_4)}
\]

i.e.
\[
\exp\left(-\frac{(k_1 + k_4)}{t + s + w}\right) \geq \exp\left(-\frac{(k_1 + k_2)}{f_1(k_1, k_2)}\right) \cdot \exp\left(-\frac{(k_2 + k_3)}{f_2(k_2, k_3)}\right) \cdot \exp\left(-\frac{(k_3 + k_4)}{f_3(k_3, k_4)}\right)
\]

Thus
\[
M(k_1, k_4, t + s + w) \geq M(k_1, k_2, t) \cdot M(k_2, k_3, s) \cdot M(k_3, k_4, w)
\]

Hence \((F, M, \ast)\) is a fuzzy triple controlled metric like space. However it is not a fuzzy triple controlled metric space. For this we will show that the self distance is not equal to one for all \( k \in F \).

Consider
\[
M(k, k, t) = \exp\left(-\frac{(k + k)}{t}\right) = \frac{1}{(k + k)} \neq 1, \text{ for all } k \in F, \ t > 0.
\]

Hence \((F, M, \ast)\) is not a fuzzy triple controlled metric space.

The following are the definitions of a convergent sequence and a Cauchy sequence in the context of a fuzzy triple controlled metric like space:

**Definition 8.** Let \( \{k_n\} \) be a sequence in a fuzzy triple controlled metric like space \((F, M, \ast)\). Then the sequence is:

1. a convergent sequence, if for all \( t > 0 \), there exists \( \kappa \) in \( F \) such that
   \[
   \lim_{n \to \infty} M(k_n, \kappa, t) = M(\kappa, \kappa, t),
   \]

2. a Cauchy sequence if for all \( t > 0, p \geq 1, \)
   \[
   \lim_{n \to \infty} M(k_{n+p}, k_n, t) \text{ exists and finite.}
   \]

A fuzzy triple controlled metric space \((F, M, \ast)\) is said to be complete if every Cauchy sequence in \( F \) has a limit point in \( F \).

Now we define the open ball in a fuzzy triple controlled metric like space as follows:
Definition 9. Let \((F, M, *)\) be a fuzzy triple controlled metric space. Then, an open ball \(B(\kappa, r, t)\) with \(\kappa\) as its center and \(r\) as its radius is defined as follows:

\[
B(\kappa, r, t) = \{ \kappa' \in F : M(\kappa, \kappa', t) > 1 - r \},
\]

and the topology associated with the fuzzy triple controlled metric like space \((F, M, *)\) is defined as follows:

\[
\tau_M = \{ C \subset F : B(\kappa, r, t) \subset C \}.
\]

The following example demonstrates that a fuzzy triple controlled metric like space may not satisfy the Hausdorff property.

Example 7. Let us consider the fuzzy triple controlled metric like space described in Example (4). We can define an open ball \(B_{\kappa_1}(\kappa_1, r_1, t_1)\) with \(\kappa_1 = 2\), radius \(r_1 = 0.4\) and \(t_1 = 4\) as

\[
B_{\kappa_1}(2, 0.4, 4) = \{ \kappa' \in F : M(2, \kappa', 4) > 0.6 \},
\]

Let 0 \(\in\) \(F\), then \(M(2, 0, 4) = \frac{4}{4 + \text{max}(2, 0)} = \frac{4}{4 + 2} = 0.6666\), so 0 \(\notin\) \(B_{\kappa_1}(\kappa_1, r_1, t_1)\).

Let 1 \(\in\) \(F\), then \(M(2, 1, 4) = \frac{4}{4 + \text{max}(2, 1)} = \frac{4}{4 + 2} = 0.6666\), so 1 \(\in\) \(B_{\kappa_1}(\kappa_1, r_1, t_1)\).

Let 2 \(\in\) \(F\), then \(M(2, 2, 4) = \frac{4}{4 + \text{max}(2, 2)} = \frac{4}{4 + 2} = 0.6666\), so 2 \(\in\) \(B_{\kappa_1}(\kappa_1, r_1, t_1)\).

Let 3 \(\in\) \(F\), then \(M(2, 3, 4) = \frac{4}{4 + \text{max}(2, 3)} = \frac{4}{4 + 3} = 0.5714\), so 3 \(\notin\) \(B_{\kappa_1}(\kappa_1, r_1, t_1)\).

Hence,

\[
B_{\kappa_1}(\kappa_1, r_1, t_1) = \{ 0, 1, 2 \}.
\]

Now define an open ball \(B_{\kappa_2}(\kappa_2, r_2, t_2)\) with center \(\kappa_2 = 1\), radius \(r_2 = 0.2\) and \(t_2 = 6\) as

\[
B_{\kappa_2}(2, 0.2, 6) = \{ \kappa' \in F : M(1, \kappa', 6) > 0.8 \},
\]

Let 0 \(\in\) \(F\), then \(M(1, 0, 6) = \frac{6}{6 + \text{max}(1, 0)} = \frac{6}{6 + 1} = 0.8571\), so 0 \(\notin\) \(B_{\kappa_2}(\kappa_2, r_1, t_2)\).

Let 1 \(\in\) \(F\), then \(M(1, 1, 6) = \frac{6}{6 + \text{max}(1, 1)} = \frac{6}{6 + 1} = 0.8570\), so 1 \(\in\) \(B_{\kappa_2}(\kappa_2, r_1, t_2)\).

Let 2 \(\in\) \(F\), then \(M(1, 2, 6) = \frac{6}{6 + \text{max}(1, 2)} = \frac{6}{6 + 2} = 0.75\), so 2 \(\notin\) \(B_{\kappa_2}(\kappa_2, r_1, t_2)\).

Let 3 \(\in\) \(F\), then \(M(1, 3, 6) = \frac{6}{6 + \text{max}(1, 3)} = \frac{6}{6 + 3} = 0.6666\), so 3 \(\notin\) \(B_{\kappa_2}(\kappa_2, r_1, t_2)\).

Hence,

\[
B_{\kappa_2}(\kappa_2, r_2, t_2) = \{ 0, 1 \}.
\]

Now \(B_{\kappa_1}(\kappa_1, r_1, t_1) \cap B_{\kappa_2}(\kappa_2, r_2, t_2) = \{ 0, 1, 2 \} \cap \{ 0, 1 \} = \{ 0, 1 \} \neq \emptyset\). Thus a fuzzy triple controlled metric like space is not Hausdorff.

Remark 4. According to Remarks (1) and (2), it can be concluded that extended fuzzy rectangular \(b\)-metric like spaces, fuzzy rectangular \(b\)-metric like spaces, fuzzy rectangular metric like spaces, \(\psi\)-metric like spaces, and fuzzy metric like spaces are also not Hausdorff.

Definition 10. Let \((F, M, *)\) be a fuzzy triple controlled metric like space, a mapping \(T : F \longrightarrow F\) is called an \(\alpha - \psi\)-contractive mapping, if there exists two functions, \(\alpha : F \times F \times (0, \infty) \longrightarrow (0, \infty)\) and \(\psi : [0, \infty) \longrightarrow [0, \infty)\) such that

\[
a(\kappa_1, \kappa_2, t)\left(\frac{1}{M(T\kappa_1, T\kappa_2, t)} - 1 \right) \leq \psi\left(\frac{1}{M(\kappa_1, \kappa_2, t)} - 1 \right) \text{ for all } t > 0, \kappa_1, \kappa_2 \in F.\quad (2)
\]
Now we prove Banach contraction principle in the frame work of fuzzy triple controlled metric like space using $a-\psi-\text{contraction.}$

**Theorem 1.** Let $f_1, f_2, f_3 : F \times F \to [1, \infty)$ be three non-comparable functions and $(F, M, \ast)$ be a complete fuzzy triple controlled metric like space. Further, let $T : F \to F$ be an $a-\psi-\text{contrative mapping}$ which satisfies the following conditions:

1. $T$ is $a$-admissible;
2. there exists $\kappa_0 \in F$ such that $a(\kappa_0, T\kappa_0, t) \geq 1$ for all $t$;
3. if $\{\kappa_n\}$ is a sequence in $F$ such that $a(\kappa_n, \kappa_{n+1}, t) \geq 1$ for all $n \geq 1, t \geq 0$, and $\kappa_n \to \kappa$ as $n \to \infty$, then $a(\kappa_n, \kappa, t) \geq 1$ for all $n \geq 1, t \geq 0$.

Then $T$ has a fixed point.

**Proof.** By assumption there exists $\kappa_0 \in F$ such that $a(\kappa_0, T\kappa_0, t) \geq 1$ for all $t > 0$. Let $\kappa_0$ be an arbitrary element. If $\kappa_0 = T\kappa_0$, then $\kappa_0$ is the required fixed point and we are done. If $T\kappa_0 \neq \kappa_0$, then $T\kappa_0 = \kappa_1 \in F$ (say). Continuing in this way, we have

$$T\kappa_1 = T(T\kappa_0) = T^2\kappa_0 = \kappa_2,$$

and

$$T\kappa_2 = T(T\kappa_1) = T^2\kappa_1 = T^2(T\kappa_0) = T^3\kappa_0 = \kappa_3,$$

and so on,

$$T\kappa_n = T(T\kappa_{n-1}) = T^2\kappa_{n-1} = \cdots = T^{n+1}\kappa_0 = \kappa_{n+1}.$$

So, we have iterative sequence $\kappa_n = T\kappa_{n-1} = T^n\kappa_0$ with $\kappa_n \neq \kappa_{n+1}$. Since $T$ is $a$-admissible, for all $t > 0$, we have

$$a(\kappa_0, T\kappa_0, t) = a(\kappa_0, \kappa_1, t) \geq 1 \Rightarrow a(T\kappa_0, T\kappa_1, t) = a(\kappa_1, \kappa_2, t) \geq 1$$

which implies

$$a(\kappa_1, T\kappa_1, t) = a(\kappa_1, \kappa_2, t) \geq 1 \Rightarrow a(T\kappa_1, T\kappa_2, t) = a(\kappa_2, \kappa_3, t) \geq 1$$

continuing in this way, we have

$$a(\kappa_{n-1}, T\kappa_{n-1}, t) = a(\kappa_{n-1}, \kappa_n, t) \geq 1 \Rightarrow a(T\kappa_{n-1}, T\kappa_n, t) = a(\kappa_n, \kappa_{n+1}, t) \geq 1.$$

Now

$$\left(\frac{1}{M(\kappa_1, \kappa_2, t)} - 1\right) = \left(\frac{1}{M(T\kappa_0, T\kappa_1, t)} - 1\right) \leq a(\kappa_0, \kappa_1, t)\left(\frac{1}{M(T\kappa_0, T\kappa_1, t)} - 1\right) \text{ since } a(\kappa_0, \kappa_1, t) \geq 1$$

$$\leq \psi\left(\frac{1}{M(\kappa_0, \kappa_1, t)} - 1\right),$$

so we have

$$\left(\frac{1}{M(\kappa_1, \kappa_2, t)} - 1\right) \leq \psi\left(\frac{1}{M(\kappa_0, \kappa_1, t)} - 1\right). \quad (3)$$

Now

$$\left(\frac{1}{M(\kappa_2, \kappa_3, t)} - 1\right) = \left(\frac{1}{M(T\kappa_1, T\kappa_2, t)} - 1\right) \leq a(\kappa_1, \kappa_2, t)\left(\frac{1}{M(T\kappa_1, T\kappa_2, t)} - 1\right) \text{ since } a(\kappa_1, \kappa_2, t) \geq 1$$

$$\leq \psi\left(\frac{1}{M(\kappa_1, \kappa_2, t)} - 1\right).$$
from (3), we have
\[
\left( \frac{1}{M(\kappa_2, \kappa_3, t)} - 1 \right) \leq \psi \left( \frac{1}{M(\kappa_0, \kappa_1, t)} - 1 \right)
\]

\[
= \psi^2 \left( \frac{1}{M(\kappa_0, \kappa_1, t)} - 1 \right)
\]
similarly
\[
\left( \frac{1}{M(\kappa_3, \kappa_4, t)} - 1 \right) \leq \psi^3 \left( \frac{1}{M(\kappa_0, \kappa_1, t)} - 1 \right)
\]
continuing in this way, we have
\[
\left( \frac{1}{M(\kappa_n, \kappa_{n+1}, t)} - 1 \right) \leq \psi^n \left( \frac{1}{M(\kappa_0, \kappa_1, t)} - 1 \right).
\]
Taking limit \( n \to \infty \), we have
\[
\lim_{n \to \infty} \left( \frac{1}{M(\kappa_n, \kappa_{n+1}, t)} - 1 \right) \leq \lim_{n \to \infty} \psi^n \left( \frac{1}{M(\kappa_0, \kappa_1, t)} - 1 \right) \to 0.
\]
\[
\Rightarrow \lim_{n \to \infty} \left( \frac{1}{M(\kappa_n, \kappa_{n+1}, t)} - 1 \right) = 0
\]
\[
\Rightarrow \lim_{n \to \infty} (M(\kappa_n, \kappa_{n+1}, t) = 1, \text{ for all } t > 0. \tag{4}
\]
On the same line we can prove
\[
\left( \frac{1}{M(\kappa_1, \kappa_3, t)} - 1 \right) \leq \psi \left( \frac{1}{M(\kappa_0, \kappa_2, t)} - 1 \right). \tag{5}
\]
Now
\[
\left( \frac{1}{M(\kappa_2, \kappa_4, t)} - 1 \right) = \left( \frac{1}{M(T\kappa_1, T\kappa_3, t)} - 1 \right)
\]
\[
\leq \alpha(\kappa_1, \kappa_3, t) \left( \frac{1}{M(T\kappa_1, T\kappa_3, t)} - 1 \right) \text{ since } \alpha(\kappa_1, \kappa_3, t) \geq 1
\]
\[
\leq \psi \left( \frac{1}{M(\kappa_1, \kappa_3, t)} - 1 \right).
\]
from (5), we have
\[
\left( \frac{1}{M(\kappa_2, \kappa_4, t)} - 1 \right) \leq \psi \left( \frac{1}{M(\kappa_0, \kappa_2, t)} - 1 \right)
\]

\[
= \psi^2 \left( \frac{1}{M(\kappa_0, \kappa_2, t)} - 1 \right)
\]
similarly
\[
\left( \frac{1}{M(\kappa_3, \kappa_5, t)} - 1 \right) \leq \psi^3 \left( \frac{1}{M(\kappa_0, \kappa_2, t)} - 1 \right)
\]
continuing in this way, we have
\[
\left( \frac{1}{M(\kappa_{n-2}, \kappa_n, t)} - 1 \right) \leq \psi^{n-2} \left( \frac{1}{M(\kappa_0, \kappa_2, t)} - 1 \right).
Taking limit $n \to \infty$, we have
\[
\lim_{n \to \infty} \left( \frac{1}{M(k_{n-2}, k_n, t)} - 1 \right) \leq \lim_{n \to \infty} \psi^n \left( \frac{1}{M(k_0, k_2, t)} - 1 \right) \to 0.
\]
\[
\Rightarrow \lim_{n \to \infty} \left( \frac{1}{M(k_{n-2}, k_n, t)} - 1 \right) = 0
\]
\[
\Rightarrow \lim_{n \to \infty} (M(k_{n-2}, k_n, t)) = 1, \text{ for all } t > 0. \tag{6}
\]

Let $\{k_n\}$ be a sequence in $F$, then we have following cases:

**Case 1.** If $p = 2m + 1$ (say), then

\[
M(k_n, k_{n+2m+1}, t)
\]
\[
\geq M\left(k_n, k_{n+1}, \frac{1}{f_1(k_n, k_{n+1})}\right) \times M\left(k_{n+1}, k_{n+2}, \frac{1}{f_2(k_{n+1}, k_{n+2})}\right)
\]
\[
\times M\left(k_{n+2}, k_{n+2m+1}, \frac{1}{f_3(k_{n+2}, k_{n+2m+1})}\right)
\]
\[
\geq M\left(k_n, k_{n+1}, \frac{1}{f_1(k_n, k_{n+1})}\right) \times M\left(k_{n+1}, k_{n+2}, \frac{1}{f_2(k_{n+1}, k_{n+2})}\right)
\]
\[
\times M\left(k_{n+2}, k_{n+2m+1}, \frac{1}{f_3(k_{n+2}, k_{n+2m+1})}\right)
\]
\[
\times M\left(k_{n+2m+1}, k_{n+2m+3}, \frac{1}{f_3(k_{n+2m+1}, k_{n+2m+3})}\right)
\]
\[
\times M\left(k_{n+2m+3}, k_{n+2m+5}, \frac{1}{f_3(k_{n+2m+3}, k_{n+2m+5})}\right)
\]
\[
\times M\left(k_{n+2m+5}, k_{n+2m+7}, \frac{1}{f_3(k_{n+2m+5}, k_{n+2m+7})}\right)
\]
\[
\times \cdots
\]
\[
\times M\left(k_{n+2m+2}, k_{n+2m+2}, \frac{1}{f_3(k_{n+2m+2}, k_{n+2m+2})}\right)
\]
\[
\times M\left(k_{n+2m+2}, k_{n+2m+1}, \frac{1}{f_3(k_{n+2m+2}, k_{n+2m+1})}\right)
\]
\[
\times M\left(k_{n+2m+1}, k_{n+2}, \frac{1}{f_3(k_{n+2m+1}, k_{n+2})}\right)
\]
\[
\times M\left(k_{n+2}, k_{n+1}, \frac{1}{f_3(k_{n+2}, k_{n+1})}\right)
\]
\[
\times M\left(k_{n+1}, k_{n}, \frac{1}{f_3(k_{n+1}, k_n)}\right).
\]
Taking limit $n \to \infty$ and using (4), we have

$$
\lim_{n \to \infty} M(\kappa_n, \kappa_{n+2n+1}, 1) \\
\geq \lim_{n \to \infty} M\left(\kappa_n, \kappa_{n+1}, \frac{1}{3} f_1(\kappa_{n+1})\right) \cdot \lim_{n \to \infty} M\left(\kappa_{n+1}, \kappa_{n+2}, \frac{1}{3} f_2(\kappa_{n+1}, \kappa_{n+2})\right) \\
\times \lim_{n \to \infty} M\left(\kappa_{n+2}, \kappa_{n+2n+1}, \frac{1}{3} f_3(\kappa_{n+1}, \kappa_{n+2})\right) \\
\geq \lim_{n \to \infty} M\left(\kappa_n, \kappa_{n+1}, \frac{1}{3} f_1(\kappa_{n+1})\right) \cdot \lim_{n \to \infty} M\left(\kappa_{n+1}, \kappa_{n+2}, \frac{1}{3} f_2(\kappa_{n+1}, \kappa_{n+2})\right) \\
\times \lim_{n \to \infty} M\left(\kappa_{n+2}, \kappa_{n+2n+1}, \frac{1}{3} f_3(\kappa_{n+1}, \kappa_{n+2})\right) \\
\geq \lim_{n \to \infty} M\left(\kappa_n, \kappa_{n+1}, \frac{1}{3} f_1(\kappa_{n+1})\right) \cdot \lim_{n \to \infty} M\left(\kappa_{n+1}, \kappa_{n+2}, \frac{1}{3} f_2(\kappa_{n+1}, \kappa_{n+2})\right) \\
\times \lim_{n \to \infty} M\left(\kappa_{n+2}, \kappa_{n+2n+1}, \frac{1}{3} f_3(\kappa_{n+1}, \kappa_{n+2})\right) \\
\times \cdots \\
\geq \lim_{n \to \infty} M\left(\kappa_n, \kappa_{n+1}, \frac{1}{3} f_1(\kappa_{n+1})\right) \cdot \lim_{n \to \infty} M\left(\kappa_{n+1}, \kappa_{n+2}, \frac{1}{3} f_2(\kappa_{n+1}, \kappa_{n+2})\right) \\
\times \lim_{n \to \infty} M\left(\kappa_{n+2}, \kappa_{n+2n+1}, \frac{1}{3} f_3(\kappa_{n+1}, \kappa_{n+2})\right) \\
\times \cdots \\
\geq 1 \cdot 1 \cdot 1 \cdots \cdot 1 = 1
$$
Hence \( \lim_{n \to \infty} M(K_{n+2m}, l) = 1. \)

**Case 2.** If \( p = 2m \) (say) is even, then

\[
M(K_n, K_{n+2m}, l) \\
\geq M(K_n, K_{n+1}, \frac{t}{3} \frac{f_1(K_n, K_{n+1})}{f_3(K_n, K_{n+2})}) * M(K_{n+1}, K_{n+2}, \frac{t}{3} \frac{f_2(K_{n+1}, K_{n+2})}{f_3(K_{n+1}, K_{n+2})}) \\
* M(K_{n+2}, K_{n+2m}, \frac{t}{3} \frac{f_3(K_{n+2}, K_{n+2m})}{f_3(K_{n+2}, K_{n+2m})}) \\
\geq M(K_n, K_{n+1}, \frac{t}{3} \frac{f_1(K_n, K_{n+1})}{f_3(K_n, K_{n+2})}) * M(K_{n+1}, K_{n+2}, \frac{t}{3} \frac{f_2(K_{n+1}, K_{n+2})}{f_3(K_{n+1}, K_{n+2})}) \\
* M(K_{n+2}, K_{n+3}, \frac{t}{3} \frac{f_3(K_{n+2}, K_{n+3})}{f_3(K_{n+2}, K_{n+3})}) \\
* M(K_{n+3}, K_{n+4}, \frac{t}{3} \frac{f_3(K_{n+3}, K_{n+4})}{f_3(K_{n+3}, K_{n+4})}) \\
* M(K_{n+4}, K_{n+5}, \frac{t}{3} \frac{f_3(K_{n+4}, K_{n+5})}{f_3(K_{n+4}, K_{n+5})}) \\
* M(K_{n+5}, K_{n+6}, \frac{t}{3} \frac{f_3(K_{n+5}, K_{n+6})}{f_3(K_{n+5}, K_{n+6})}) \\
* M(K_{n+6}, K_{n+7}, \frac{t}{3} \frac{f_3(K_{n+6}, K_{n+7})}{f_3(K_{n+6}, K_{n+7})}) \\
* M(K_{n+7}, K_{n+8}, \frac{t}{3} \frac{f_3(K_{n+7}, K_{n+8})}{f_3(K_{n+7}, K_{n+8})}) \\
* M(K_{n+8}, K_{n+9}, \frac{t}{3} \frac{f_3(K_{n+8}, K_{n+9})}{f_3(K_{n+8}, K_{n+9})}) \\
* M(K_{n+9}, K_{n+10}, \frac{t}{3} \frac{f_3(K_{n+9}, K_{n+10})}{f_3(K_{n+9}, K_{n+10})}) \\
* \vdots \\
* M(K_{n+2m-4}, K_{n+2m-3}, \frac{t}{3} \frac{f_3(K_{n+2m-4}, K_{n+2m-3})}{f_3(K_{n+2m-4}, K_{n+2m-3})}) \\
* M(K_{n+2m-3}, K_{n+2m-2}, \frac{t}{3} \frac{f_3(K_{n+2m-3}, K_{n+2m-2})}{f_3(K_{n+2m-3}, K_{n+2m-2})}) \\
* M(K_{n+2m-2}, K_{n+2m}, \frac{t}{3} \frac{f_3(K_{n+2m-2}, K_{n+2m})}{f_3(K_{n+2m-2}, K_{n+2m})}) \\
* M(K_{n+2m}, l). 
\]

Taking limit \( n \to \infty \) and using (4) and (6), we have
Now, using (7) and (8), we have

\[
\lim_{n \to \infty} M(\kappa_n, \kappa_{n+1}, t) \\
\geq \lim_{n \to \infty} M \left( \frac{k_n}{f_1(k_n, \kappa_{n+1})} \right) \ast \lim_{n \to \infty} M \left( \frac{k_n}{f_2(k_n, \kappa_{n+2})} \right)
\]

\[
\leq \lim_{n \to \infty} M \left( \frac{k_n}{f_3(k_n, \kappa_{n+3})} \right) \ast \lim_{n \to \infty} M \left( \frac{k_n}{f_4(k_n, \kappa_{n+4})} \right)
\]

\[
\leq \lim_{n \to \infty} M \left( \frac{k_n}{f_5(k_n, \kappa_{n+5})} \right) \ast \lim_{n \to \infty} M \left( \frac{k_n}{f_6(k_n, \kappa_{n+6})} \right)
\]

\[
\leq \lim_{n \to \infty} M \left( \frac{k_n}{f_7(k_n, \kappa_{n+7})} \right) \ast \lim_{n \to \infty} M \left( \frac{k_n}{f_8(k_n, \kappa_{n+8})} \right)
\]

\[
\vdots
\]

\[
\leq \lim_{n \to \infty} M \left( \frac{k_n}{f_n(k_n, \kappa_{n+n})} \right) \ast \lim_{n \to \infty} M \left( \frac{k_n}{f_{n+1}(k_n, \kappa_{n+n+1})} \right)
\]

\[
\geq 1 \times 1 \times \cdots \times 1 = 1.
\]

Thus in both cases, we have

\[
\lim_{n \to \infty} M(\kappa_n, \kappa_n, t) = 1,
\]

(7)

which shows \(\{\kappa_n\}\) is Cauchy in \(F\) and converges to some \(\kappa\) in \(F\) (as \(F\) is complete), so

\[
\lim_{n \to \infty} M(\kappa_n, \kappa, t) = M(\kappa, \kappa, t).
\]

(8)

Now, using (7) and (8), we have

\[
\lim_{n \to \infty} M(\kappa_n, \kappa, t) = M(\kappa, \kappa, t) = \lim_{n \to \infty} M(\kappa_n + p, \kappa_n, t) = 1, \text{ for all } t > 0, \ p \geq 1,
\]
thus we have, \( \lim_{n \to \infty} M(\kappa_n, \kappa, t) = 1 \). Now we have to show that \( \kappa \) is the fixed point of \( T \).

\[
\frac{1}{M(\kappa_{n+1}, T\kappa, t)} - 1 = \frac{1}{M(T\kappa_n, T\kappa, t)} - 1 \\
\leq \alpha(\kappa_n, \kappa, t) \left( \frac{1}{M(T\kappa_n, T\kappa, t)} - 1 \right) \\
\leq \psi \left( \frac{1}{M(\kappa_n, \kappa, t)} - 1 \right).
\]

Taking limit \( n \to \infty \)

\[
\lim_{n \to \infty} \frac{1}{M(\kappa_{n+1}, T\kappa, t)} - 1 \leq \psi \left( \lim_{n \to \infty} \frac{1}{M(\kappa_n, \kappa, t)} - 1 \right) \\
\leq \psi \left( \frac{1}{1} - 1 \right) \leq 0
\]

hence

\[
\lim_{n \to \infty} \frac{1}{M(\kappa_{n+1}, T\kappa, t)} - 1 = 0
\]

\[
\lim_{n \to \infty} M(\kappa_{n+1}, T\kappa, t) = 1.
\]

By using the continuity of \( M \) and \( \kappa_n \to \kappa \), we get

\[
M(\kappa, T\kappa, t) = 1.
\]

Which shows \( \kappa \) is the fixed point of \( T \). \( \square \)

**Example 8.** Let \( F = [0, \infty] \) and \( M : F \times F \times (0, \infty) \to [0, 1] \) be defined as \( M(\kappa_1, \kappa_2, t) = e^{-\frac{(\kappa_1 + \kappa_2)^2}{t}} \) for all \( t > 0 \), further let \( f_1, f_2, f_3 : F \times F \to [0, \infty] \) be three functions defined by \( f_1(\kappa_1, \kappa_2) = \frac{1}{4}(\kappa_1^2 + \kappa_2 + 3) \), \( f_2(\kappa_1, \kappa_2) = \frac{1}{4}(\kappa_1^2 + \kappa_2^2 + 4) \) and \( f_3(\kappa_1, \kappa_2) = \frac{1}{4}(\kappa_1 + \kappa_2 + 5) \). Then \((F, M, *)\) is a complete fuzzy triple controlled metric like space. Let \( T : F \to F \) be given by

\[
T(\kappa) = \begin{cases} 
\frac{x}{3} & \text{if } \kappa, \kappa' \in [0, 1] \\
1 & \text{if } \kappa, \kappa' \in (1, \infty),
\end{cases}
\]

and \( \alpha : F \times F \times (0, \infty) \to (0, \infty) \) be defined as

\[
\alpha(\kappa, \kappa', t) = \begin{cases} 
1 & \text{if } \kappa, \kappa' \in [0, 1] \\
0 & \text{otherwise},
\end{cases}
\]

Let \( \kappa, \kappa' \in F \) be such that \( \alpha(\kappa, \kappa', t) \geq 1 \) for all \( t > 0 \), then \( \kappa, \kappa' \in [0, 1] \). Now \( T\kappa = \frac{x}{3}, T\kappa' = \frac{x'}{3} \) and \( \alpha(T\kappa, T\kappa', t) = 1 \) for all \( t > 0 \), which shows \( F \) is \( \alpha \)-admissible. Now let \( \kappa_0 \in [0, 1] \), then \( \alpha(\kappa_0, T\kappa_0, t) \geq 1 \) and condition (ii) of Theorem (1) is satisfied. Let \( \{\kappa_n\} \) be any sequence in \( F \) such that \( \alpha(\kappa_n, \kappa_{n+1}, t) \geq 1 \) for all \( n \geq 1, t > 0 \) and \( \kappa_n \to \kappa \) as \( n \to \infty \), then \( \{\kappa_n\} \subset [0, 1] \) and hence \( \kappa \in [0, 1] \). Which implies \( \alpha(\kappa_n, \kappa, t) \geq 1 \) for all \( n \geq 1, t > 0 \) and condition (iii) of Theorem 1 is satisfied. Hence \( T \) has at least one fixed point, here 0, 3 are two fixed points of \( T \).

We will now demonstrate the Banach contraction principle in a fuzzy triple controlled metric like space. This principle will be utilized in the application section to establish the existence and uniqueness of a fractional differential equation.
**Theorem 2.** Let \( f_1, f_2, f_3 : \mathcal{F} \times \mathcal{F} \to [1, \frac{1}{K}) \) be three non-comparable functions \((K \in (0, 1))\) and \((\mathcal{F}, M, \ast)\) be a complete fuzzy triple controlled metric like space such that

\[
\lim_{t \to \infty} M(\kappa_1, \kappa_2, t) = 1. \tag{9}
\]

Further let \( T : \mathcal{F} \to \mathcal{F} \) be a self-mapping such that for all \( \kappa_1, \kappa_2 \in \mathcal{F} \),

\[
M(T\kappa_1, T\kappa_2, Kt) \geq M(\kappa_1, \kappa_2, t). \tag{10}
\]

Then \( T \) has a unique fixed point.

**Proof.** Choose \( \kappa_0 \) an arbitrary point in \( \mathcal{F} \) and consider the iterative sequence \( \kappa_0 = T\kappa_0 = T^n\kappa_0 \). Applying (10) successively, we get

\[
M(\kappa_n, \kappa_{n+1}, t) = M(T\kappa_{n-1}, T\kappa_n, t),
\]

\[
\geq M(\kappa_{n-1}, \kappa_n, \frac{t}{K}),
\]

\[
= M(T\kappa_{n-2}, T\kappa_{n-1}, \frac{t}{K^2}),
\]

\[
\geq M(\kappa_{n-2}, \kappa_{n-1}, \frac{t}{K^2}),
\]

\[
\geq M(\kappa_{n-3}, \kappa_{n-2}, \frac{t}{K^3}),
\]

\[
\vdots
\]

\[
\geq M(\kappa_0, \kappa_1, \frac{t}{K^n}),
\]

hence

\[
M(\kappa_n, \kappa_{n+1}, t) \geq M(\kappa_0, \kappa_1, \frac{t}{K^n}). \tag{11}
\]

on the same lines we can prove

\[
M(\kappa_{n-2}, \kappa_n, t) \geq M(\kappa_0, \kappa_2, \frac{t}{K^{n-2}}). \tag{12}
\]

Now writing \( t = \frac{t}{4} + \frac{t}{2} + \frac{t}{4} \) and for any sequence \( \{\kappa_n\} \) in \( \mathcal{F} \), we have following cases:

**Case 1.** If \( p = 2m + 1 \) (say) is odd, then
\[ M(k_0, k_{n+2m+1}, 1) \]
\[ \geq M\left( k_0, k_{n+1}, \frac{i}{f_1(k_0, k_{n+1})} \right) \cdot M\left( k_{n+1}, k_{n+2}, \frac{i}{f_2(k_{n+1}, k_{n+2})} \right) \]
\[ \geq M\left( k_0, k_{n+1}, \frac{i}{f_1(k_0, k_{n+1})} \right) \cdot M\left( k_{n+1}, k_{n+2}, \frac{i}{f_2(k_{n+1}, k_{n+2})} \right) \]
\[ \geq M\left( k_{n+1}, k_{n+2}, \frac{i}{f_1(k_{n+1}, k_{n+2})} \right) \cdot M\left( k_{n+1}, k_{n+2}, \frac{i}{f_2(k_{n+1}, k_{n+2})} \right) \]
\[ \geq M\left( k_{n+2}, k_{n+3}, \frac{i}{f_1(k_{n+2}, k_{n+3})} \right) \cdot M\left( k_{n+2}, k_{n+3}, \frac{i}{f_2(k_{n+2}, k_{n+3})} \right) \]
\[ \geq M\left( k_{n+3}, k_{n+4}, \frac{i}{f_1(k_{n+3}, k_{n+4})} \right) \cdot M\left( k_{n+3}, k_{n+4}, \frac{i}{f_2(k_{n+3}, k_{n+4})} \right) \]
\[ \geq M\left( k_{n+4}, k_{n+5}, \frac{i}{f_1(k_{n+4}, k_{n+5})} \right) \cdot M\left( k_{n+4}, k_{n+5}, \frac{i}{f_2(k_{n+4}, k_{n+5})} \right) \]
\[ \vdots \]
\[ \geq M\left( k_{n+2m-2}, k_{n+2m-1}, \frac{i}{f_1(k_{n+2m-2}, k_{n+2m-1})} \right) \cdot M\left( k_{n+2m-2}, k_{n+2m-1}, \frac{i}{f_2(k_{n+2m-2}, k_{n+2m-1})} \right) \]
\[ \geq M\left( k_{n+2m-1}, k_{n+2m}, \frac{i}{f_1(k_{n+2m-1}, k_{n+2m})} \right) \cdot M\left( k_{n+2m-1}, k_{n+2m}, \frac{i}{f_2(k_{n+2m-1}, k_{n+2m})} \right) \]
\[ \geq M\left( k_{n}, k_{n+2m+1}, 1 \right) \]
Now applying (11) on right hand side, we deduce

\[
M\left(\kappa_n^1, \kappa_n^{2m+1}, t\right) \geq M\left(\kappa_0^1, K_n, f_1(K_n + 1)\right) * M\left(\kappa_0^1, f_2(K_n + 1, \kappa_n^3)\right) * M\left(\kappa_0^1, f_3(K_n + 2, \kappa_n + 3)\right) * \ldots
\]

\[
* M\left(\kappa_0^1, f_m(K_n + 2m, \kappa_n + 2m + 1)\right)
\]

\[
\geq M\left(\kappa_0^1, K_n, \kappa_n + 2m + 1, f_1(K_n, \kappa_n + 1)\right) * M\left(\kappa_0^1, f_2(K_n + 1, \kappa_n + 2)\right) * M\left(\kappa_0^1, f_3(K_n + 2, \kappa_n + 2m + 1)\right) * \ldots
\]

\[
* M\left(\kappa_0^1, f_m(K_n + 2m, \kappa_n + 2m + 1)\right).
\]
Case 2. If \( p = 2m \) (say) is even, then
\[
M(k_n, k_{n+2m+1}) \geq M\left(k_n, k_{n+1}, \frac{\frac{1}{3}}{f_3(k_n, k_{n+1})}\right) \cdot M\left(k_{n+1}, k_{n+2}, \frac{\frac{1}{3}}{f_2(k_{n+1}, k_{n+2})}\right) \\
\cdot M\left(k_{n+2}, k_{n+2m}, \frac{\frac{1}{3}}{f_3(k_{n+2}, k_{n+2m})}\right) \\
\geq M\left(k_n, k_{n+1}, \frac{\frac{1}{3}}{f_1(k_n, k_{n+1})}\right) \cdot M\left(k_{n+1}, k_{n+2}, \frac{\frac{1}{3}}{f_2(k_{n+1}, k_{n+2})}\right) \\
\cdot M\left(k_{n+2}, k_{n+3}, \frac{\frac{1}{3}}{f_2(k_{n+2}, k_{n+3})}\right) \\
\cdot M\left(k_{n+3}, k_{n+4}, \frac{\frac{1}{3}}{f_2(k_{n+3}, k_{n+4})}\right) \\
\cdot M\left(k_{n+4}, k_{n+2m}, \frac{\frac{1}{3}}{f_3(k_{n+4}, k_{n+2m})}\right) \\
\geq M\left(k_n, k_{n+1}, \frac{\frac{1}{3}}{f_1(k_n, k_{n+1})}\right) \cdot M\left(k_{n+1}, k_{n+2}, \frac{\frac{1}{3}}{f_2(k_{n+1}, k_{n+2})}\right) \\
\cdot M\left(k_{n+2}, k_{n+3}, \frac{\frac{1}{3}}{f_1(k_{n+2}, k_{n+3})}\right) \\
\cdot M\left(k_{n+3}, k_{n+4}, \frac{\frac{1}{3}}{f_2(k_{n+3}, k_{n+4})}\right) \\
\cdot M\left(k_{n+4}, k_{n+5}, \frac{\frac{1}{3}}{f_1(k_{n+4}, k_{n+5})}\right) \\
\cdot M\left(k_{n+5}, k_{n+6}, \frac{\frac{1}{3}}{f_2(k_{n+5}, k_{n+6})}\right) \\
\cdot M\left(k_{n+6}, k_{n+7}, \frac{\frac{1}{3}}{f_1(k_{n+6}, k_{n+7})}\right) \\
\cdot M\left(k_{n+7}, k_{n+8}, \frac{\frac{1}{3}}{f_2(k_{n+7}, k_{n+8})}\right) \\
\vdots \\
\cdot M\left(k_{n+2m-4}, k_{n+2m-3}, \frac{\frac{1}{3}}{f_1(k_{n+2m-4}, k_{n+2m-3})}\right) \\
\cdot M\left(k_{n+2m-3}, k_{n+2m-2}, \frac{\frac{1}{3}}{f_2(k_{n+2m-3}, k_{n+2m-2})}\right) \\
\cdot M\left(k_{n+2m-2}, k_{n+2m-1}, \frac{\frac{1}{3}}{f_3(k_{n+2m-2}, k_{n+2m-1})}\right).
Now applying (11) and (12) on right hand side, we deduce

\[ M(\kappa_n, \kappa_{n+2m+1}, t) \]

\[ \geq M\left(\frac{1}{3} f_1(\kappa_n, \kappa_{n+1})\right) * M\left(\frac{1}{3} f_2(\kappa_{n+1}, \kappa_{n+2})\right) \]

\[ M\left(\frac{1}{3} f_1(\kappa_{n+2}, \kappa_{n+3}) f_3(\kappa_{n+2}, \kappa_{n+2m})\right) \]

\[ M\left(\frac{1}{3} f_2(\kappa_{n+3}, \kappa_{n+4}) f_3(\kappa_{n+2}, \kappa_{n+2m})\right) \]

\[ M\left(\frac{1}{3} f_1(\kappa_{n+4}, \kappa_{n+5}) f_3(\kappa_{n+2}, \kappa_{n+2m}) f_3(\kappa_{n+4}, \kappa_{n+2m+1})\right) \]

\[ M\left(\frac{1}{3} f_2(\kappa_{n+5}, \kappa_{n+6}) f_3(\kappa_{n+2}, \kappa_{n+2m}) f_3(\kappa_{n+4}, \kappa_{n+2m+1})\right) \]

\[ M\left(\kappa_{n+6}, \kappa_{n+2m+1}\right) \]

Using (9) for each case, we obtain

\[ \lim_{n \to \infty} M(\kappa_n, \kappa_{n+p}, t) = 1, \quad (13) \]

which shows \( \{\kappa_n\} \) is Cauchy in \( \Gamma \) and converges to some \( \kappa \) in \( \Gamma \) (as \( \Gamma \) is complete), so

\[ \lim_{n \to \infty} M(\kappa_n, \kappa, t) = M(\kappa, \kappa, t). \quad (14) \]

Now, using (13) and (14), we have

\[ \lim_{n \to \infty} M(\kappa_n, \kappa, t) = M(\kappa, \kappa, t) = \lim_{n \to \infty} M(\kappa_{n+p}, \kappa, t) = 1, \text{ for all } t > 0, \ p \geq 1, \]
thus we have, \( \lim_{n \to \infty} M(\kappa_n, \kappa, t) = 1 \). Now we show that \( \kappa \) is the fixed point of \( T \).

\[
M(\kappa, T\kappa, t) \geq M\left(\kappa, \kappa_n, \frac{\kappa}{f_1(\kappa, \kappa_n)}\right) \ast M\left(\kappa_n, \kappa_{n+1}, \frac{\kappa}{f_2(\kappa_n, \kappa_{n+1})}\right)
\]

\[
\ast M\left(\kappa_{n+1}, T\kappa, \frac{\kappa}{f_3(\kappa_{n+1}, T\kappa)}\right),
\]

\[
\geq M\left(\kappa, \kappa_n, \frac{\kappa}{f_1(\kappa, \kappa_n)}\right) \ast M\left(\kappa_n, \kappa, \frac{\kappa}{f_2(\kappa_n, \kappa)}\right)
\]

\[
\ast M\left(T\kappa_n, T\kappa, \frac{\kappa}{f_3(\kappa_{n+1}, T\kappa)}\right),
\]

\[
\geq M\left(\kappa, \kappa_n, \frac{\kappa}{f_1(\kappa, \kappa_n)}\right) \ast M\left(\kappa_n, \kappa_n, \frac{\kappa}{f_2(\kappa_n, \kappa_n)}\right) \ast M\left(\kappa_n, \kappa, \frac{\kappa}{f_3(\kappa_{n+1}, T\kappa)}\right),
\]

\[
\to 1 \ast 1 \ast 1 = 1,
\]

as \( n \to \infty \) which shows \( \kappa \) is a fixed point of \( T \). For uniqueness, assume \( \kappa' \) is also a fixed point of \( T \), i.e., \( T\kappa' = \kappa' \).

\[
M\left(\kappa, \kappa', t\right) = M\left(\kappa, T\kappa', T\kappa', t\right) \geq M\left(\kappa, T\kappa', \frac{\kappa'}{K}\right),
\]

\[
= M\left(\kappa, T\kappa', \frac{\kappa'}{K}\right) \geq M\left(\kappa, \kappa', \frac{\kappa'}{K^2}\right),
\]

\[
\geq \cdots \geq M\left(\kappa, \kappa', \frac{\kappa'}{K^n}\right),
\]

\[
\to 1,
\]

as \( n \to \infty \). Hence \( \kappa = \kappa' \), so the fixed point of the mapping \( T \) is unique. \( \square \)

4. Application

Fixed point theory has been utilised in many fields of mathematics, especially in the uniqueness of the solution of certain differential and integral equations. Recently, Khater et al. [53] provided unique solitary wave solutions to the \( RR \) problem. They used innovative soliton wave solutions and their interactions to understand how dispersion affects the electric field and pulse propagation in optical fibers. In [54], the authors investigated the analytical and semi-analytical solutions of the nonlinear phi-four (PF) equation by applying the sech-tanh expansion method, the modified \( \left( \frac{G'}{G} \right) \)—expansion method and Adomian decomposition method. Fractional calculus has applications in diverse and widespread fields of engineering, medical and other scientific disciplines such as signals processing, viscoelasticity, fluid mechanics, biological population models etc. Due to variable order derivative, a fractional differential equation gives more accuracy in modeling of certain phenomena. Mustafa et al. [55] investigated the stable analytical solutions accuracy of the nonlinear fractional time space telegraph (FNLTST) equation along with applying the trigonometric-quantic-B-spline (TQBS) method. In [56], the authors introduced a modified auxiliary equation method to obtain explicit wave solutions for three nonlinear fractional biological models.

In this section, we will utilize our main results regarding the uniqueness of the solution of a fractional order differential equation that comprises the Caputo fractional derivative.

Consider the space \( \mathcal{F} \) consisting of all continuous functions defined on the interval \( C([0,1], \mathbb{R}) \). Define \( M : F \times (0, \infty) \to [0,1] \) as

\[
M(k_1, k_2, t) = \frac{t}{t + \max |k_1 + k_2|^2}
\]
with \( f_1 = k_1 + k_2 + 1, f_2 = k_1^2 + k_2^2 + 2, f_3 = 3k_1 + k_2 + 1 \) are controlled functions and \( t_1 \ast t_2 = \min \{ t_1, t_2 \} \). We first will show \((F, M, \ast)\) is a triple controlled metric like space. Let \( k_1, k_2 \in F \), then

**(M1)**

\[
M(k_1, k_2, t) = \frac{t}{t + \max |k_1 + k_2|^2},
\]

since \( t > 0 \), so \( M(k_1, k_2, t) > 0 \).

**(M2)** Let \( M(k_1, k_2, t) = 1 \), then

\[
\frac{t}{t + \max |k_1 + k_2|^2} = 1,
\]

\[
t = t + \max |k_1 + k_2|^2,
\]

\[
0 = \max |k_1 + k_2|^2,
\]

\[
0 = \max |k_1 + k_2|,
\]

which implies that \( k_1 = k_2 \).

**(M3)** Now

\[
M(k_1, k_2, t) = \frac{t}{t + \max |k_1 + k_2|^2},
\]

\[
= \frac{t}{t + \max |k_2 + k_1|^2},
\]

\[
= M(k_2, k_1, t).
\]

**(M4)** As

\[
M(k_1, k_4, t + s + w) = \frac{t + s + w}{t + s + w + d(k_1, k_4)}.
\]

Without loss of generality, assume

\[
M(k_1, k_2, \frac{t}{f_1}) \leq M(k_2, k_3, \frac{s}{f_2}),
\]

and

\[
M(k_1, k_2, \frac{t}{f_1}) \leq M(k_3, k_4, \frac{w}{f_3}),
\]

so

\[
\frac{\frac{t}{f_1}}{d(k_1, k_2)} \leq \frac{\frac{s}{f_2}}{d(k_2, k_3)},
\]

and

\[
\frac{\frac{t}{f_1}}{d(k_1, k_2)} \leq \frac{\frac{w}{f_3}}{d(k_3, k_4)},
\]

thus we have

\[
\frac{t}{f_1}d(k_2, k_3) \leq \frac{s}{f_2}d(k_1, k_2) \quad \text{and} \quad \frac{t}{f_1}d(k_3, k_4) \leq \frac{w}{f_3}d(k_1, k_2),
\]
that is
\[
\frac{t}{f_1}(d(k_2, k_3) + d(k_3, k_4)) \leq \left( \frac{s}{f_2} + \frac{w}{f_3} \right) d(k_1, k_2).
\] (15)

Note also that
\[
M(k_1, k_4, \frac{t}{f_1} + \frac{s}{f_2} + \frac{w}{f_3}) \geq M(k_1, k_2, \frac{t}{f_1}),
\]
so
\[
\frac{\frac{t}{f_1} + \frac{s}{f_2} + \frac{w}{f_3}}{\frac{t}{f_1} + d(k_1, k_2)} \geq \frac{\frac{t}{f_1}}{\frac{t}{f_1} + d(k_1, k_2)},
\]
hence
\[
\frac{\frac{t}{f_1} + \frac{s}{f_2} + \frac{w}{f_3}}{\frac{t}{f_1} + d(k_1, k_2)} \geq \frac{\frac{t}{f_1} + d(k_1, k_2)}{\frac{t}{f_1} + d(k_1, k_2)},
\]
after simplification, we have
\[
\frac{t}{f_1}(d(k_2, k_3) + d(k_3, k_4)) \leq \left( \frac{s}{f_2} + \frac{w}{f_3} \right) d(k_1, k_2).
\] (16)

Equations (15) and (16) are identical, so
\[
M(k_1, k_4, t + s + w) \geq M(k_1, k_2, \frac{t}{f_1}) \ast M(k_2, k_3, \frac{s}{f_2}) \ast M(k_3, k_4, \frac{w}{f_3}),
\]
for all \(k_1, k_2, k_3, k_4 \in F\).

\textbf{(M5)} Clearly the function \(M(k_1, k_2) : (0, \infty) \rightarrow [0, 1]\) is continuous. Since all the axioms are satisfied, so \((F, M, \ast)\) is a fuzzy triple controlled metric like space.

\textbf{Definition 11} \([57]\). \textit{Let} \(u\) \textit{be a function define on the interval} \([0, 1]\), \textit{then the Caputo fractional derivative of the function} \(u\) \textit{of order} \(\beta > 0\) \textit{is defined as}
\[
(^cD^\beta_{a+}) u(r) = \frac{1}{\Gamma(n - \beta)} \int_a^r (r - s)^{n-\beta-1} u^{(n)}(s)ds, n - 1 \leq \beta < n, n = [\beta] = 1,
\]
where \([\beta]\) \textit{denotes the integer part of the positive real number} \(\beta\).

Now consider the fractional differential equation
\[
\begin{align*}
(^cD^\beta_{0+}) x(r) &= h(r, x(r)), r \in [0, 1], \\
x(0) &= c_0, x'(0) = c_1, x''(1) = c_2
\end{align*}
\] (17)

\textit{where} \(^cD^\beta_{0+}\) \textit{is the Caputo fractional derivative of order} \(\beta (\beta \leq 3)\), \(h : [0, 1] \rightarrow \mathbb{R}\) \textit{is a continuous function and} \(c_0, c_1, c_2\) \textit{are real constants}.

\textbf{Definition 12} \([58]\). A \textit{function} \(x \in C^3([0, 1], \mathbb{R})\) \textit{with its} \(\beta\)-\textit{derivative existing on} \([0, 1]\) \textit{is said to be a solution of} (17), \textit{if} \(x\) \textit{satisfies the equation} \(^cD^\beta_{0+} (x(r)) = h(r, x(r))\) \textit{and the boundary conditions} \(x(0) = c_0, x'(0) = c_1, x''(1) = c_2\).
Lemma 1 ([57]). Let $2 < \beta \leq 3$ and $h : [0, 1] \rightarrow \mathbb{R}$ be continuous. A function $x$ is the solution of a fractional integral equation

$$x(r) = \frac{1}{\Gamma(\beta)} \int_0^r (r-s)^{\beta-1} u(s)\,ds - \frac{r^2}{2\Gamma(\beta-2)} \int_0^1 (1-s)^{\beta-3} u(s)\,ds$$

$$+ c_0 + c_1 r + c_2 \frac{r^2}{2}$$

if and only if $x$ is the solution of the fractional boundary value problems

$$cD_0^\beta x(r) = u(r),$$

$$x(0) = c_0, x'(0) = c_1, x''(1) = c_2$$

where $x''(1) = 2c_3 + \frac{1}{\Gamma(\beta-2)} \int_0^1 (1-s)^{\beta-3} u(s)\,ds = c_2, c_0, c_1, c_2, c_3 \in \mathbb{R}$.

Theorem 3. Let $T : F \rightarrow F$ be defined as

$$Tx(r) = \frac{1}{\Gamma(\beta)} \int_0^r (r-s)^{\beta-1} h(s, x(s))\,ds - \frac{r^2}{2\Gamma(\beta-2)} \int_0^1 (1-s)^{\beta-3} h(s, x(s))\,ds$$

$$+ c_0 + c_1 r + c_2 \frac{r^2}{2}$$

and suppose for $x, y \in F, r \in [0, 1]$, following conditions satisfies:

(i)

$$|h(r, x(r)) + h(r, y(r))| \leq |x(r) + y(r)|,$$

(ii)

$$\sup_{r \in (0, 1)} \left| \frac{r^\beta}{\Gamma(\beta + 1)} - \frac{r^2}{2\Gamma(\beta-1)} + 2c_0 + 2c_1 r + c_2 r^2 \right|^2 = K, K \in (0, 1).$$

Then the boundary value problem (17) has a unique solution.
Proof. Let \( x, y \in F, r \in [0, 1] \) and consider
\[
|Tx(r) + Ty(r)|^2
\]
\[
= \left| \frac{1}{\Gamma(\beta)} \int_0^r (r-s)^{\beta-1} h(s, x(s)) ds - \frac{r^2}{2\Gamma(\beta-2)} \int_0^1 (1-s)^{\beta-3} h(s, x(s)) ds + c_0 + c_1 r + c_2 r^2 \right|^2
\]
\[
+ \frac{1}{\Gamma(\beta)} \int_0^r (r-s)^{\beta-1} h(s, y(s)) ds - \frac{r^2}{2\Gamma(\beta-2)} \int_0^1 (1-s)^{\beta-3} h(s, y(s)) ds + c_0 + c_1 r + c_2 r^2 \right|^2
\]
\[
\leq \left( \frac{1}{\Gamma(\beta)} \int_0^r (r-s)^{\beta-1} h(s, x(s) + h(s, y(s))) ds \right)^2
\]
\[
= |x(r) + y(r)|^2 \left( \frac{1}{\Gamma(\beta)} \int_0^r (r-s)^{\beta-1} ds - \frac{r^2}{2\Gamma(\beta-2)} \int_0^1 (1-s)^{\beta-3} ds + 2c_0 + 2c_1 r + c_2 r^2 \right)^2
\]
\[
= |x(r) + y(r)|^2 \left( \frac{1}{\Gamma(\beta+1)} - \frac{r^2}{2\Gamma(\beta-1)} \right)^2 + 2c_0 + 2c_1 r + c_2 r^2 \right)^2
\]
\[
\leq |x(r) + y(r)|^2 K
\]
thus, we have
\[
\sup_{r \in [0,1]} |Tx(r) + Ty(r)|^2 \leq \sup_{r \in [0,1]} |x(r) + y(r)|^2
\]
\[
t + \sup_{r \in [0,1]} |Tx(r) + Ty(r)|^2 \leq t + \sup_{r \in [0,1]} |x(r) + y(r)|^2
\]
\[
\frac{1}{t + \sup_{r \in [0,1]} |Tx(r) + Ty(r)|^2} \geq \frac{1}{t + \sup_{r \in [0,1]} |x(r) + y(r)|^2}
\]
\[
\frac{t}{t + \sup_{r \in [0,1]} |Tx(r) + Ty(r)|^2} \geq \frac{t}{t + \sup_{r \in [0,1]} |x(r) + y(r)|^2}
\]
\[
\frac{Kt}{t + \sup_{r \in [0,1]} |Tx(r) + Ty(r)|^2} \geq \frac{t}{t + \sup_{r \in [0,1]} |x(r) + y(r)|^2}
\]
Hence
\[
M(Tx(r), Ty(r), Kt) \geq M(x(r), y(r), t).
\]

Using Theorem (2), we can conclude that the fractional differential Equation (17) possesses a unique solution. \( \square \)

Now we give a numerical example that illustrate Theorem (3).
Example 9. Consider the fractional order differential equation
\[ cD_0^{\frac{5}{2}}(x(r)) = h(r,x(r)), r \in [0,1], \]
with \( x(0) = c_0 = 0, x'(0) = c_1 = \frac{1}{2}, x''(1) = c_2 = 0 \) and \( h(r,x(r)) = x(r) - e^r \). Let \( T \) be the integral operator defined in Theorem (3). Now
(i)
\[ |h(r,x(r)) + h(r,y(r))| = |x(r) - e^r + y(r) - e^r| = |x(r) + y(r) - 2e^r| \leq |(x(r) + y(r))| \]
and (ii)
\[ K = \sup_{r \in (0,1)} \left| \frac{r^5}{2\Gamma(\frac{5}{2} + 1)} - \frac{r^2}{2\Gamma(\frac{5}{2} - 1)} + 2c_0 + 2c_1r + c_2r^2 \right|^2 \]
\[ = \sup_{r \in (0,1)} \left| \frac{r^5}{2\Gamma(\frac{5}{2} + 1)} - \frac{r^2}{2\Gamma(\frac{5}{2} - 1)} + 2(0) + 2\frac{1}{2}r + (0)r^2 \right|^2 \]
\[ = \sup_{r \in (0,1)} \left| \frac{r^5}{2\Gamma(\frac{5}{2} + 1)} - \frac{r^2}{2\Gamma(\frac{5}{2} - 1)} + r \right|^2 < 1 \]
As the conditions of Theorem (3) are met, there exists a unique solution to Equation (18).

5. Conclusions

We have proposed the idea of a fuzzy triple controlled metric like space involving three functions in the rectangular inequality which is a generalization of many metric like spaces in fuzzy set theory. We have demonstrated that a fuzzy triple controlled metric like space does not necessarily satisfy the Hausdorff axiom. Various example are given to validate our definition and results. By utilizing fuzzy \( \alpha - \psi \)-contraction, we have demonstrated the validity of Banach’s fixed point theorem. We have utilized our results to establish the uniqueness of the solution for a fractional differential equation. The newly obtained results can be applied to explore and analyze various existing results in the literature. For example, one can find fixed point by using fuzzy proximal quasi contraction [59] and \((\alpha, \beta)\) implicit contractions [60] in our defined space. In future, scholars may also discover the optimal proximity points using the contraction defined in [61] and can use the fuzzy technique in dislocated quasi-metric spaces [62] to find the fixed points. From an application point of view, Our proposed techniques can be employed to establish the singularity of the solution for a nonlinear potential Kadomtsev-Petviashvili and Calogero-Degasperis equations [53], as well as a nonlinear fractional time-space telegraph (FNLTST) equation [55]. The obtained results can also be apply to the domain of words [63] in which the authors have apply their approach to establish the presence of a solution for certain recurrence equations that are linked to the examination of Quicksort algorithms and Divide and Conquer algorithms, respectively. There are other applications to computer sciences, see also ([64,65]). As for applications in economics, our results can also be apply to dynamic market equilibrium [51], which is a sub-class of our defined fuzzy triple controlled metric like space.

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