Continuous Differentiability in the Context of Generalized Approach to Differentiability

Nikola Koceić-Bilan * and Snježana Braić

Faculty of Science, University of Split, 21000 Split, Croatia
* Correspondence: koceic@pmfst.hr; Tel.: +385-2161-9202

Abstract: Recently, in their paper, the authors generalized the notion of differentiability by defining it for all points of the functional domain (not only interior points) in which the notion of differentiability can be considered meaningful. In this paper, the notion of continuous differentiability is introduced for the differentiable function \( f : X \rightarrow \mathbb{R}^m \) with a not necessarily open domain \( X \subseteq \mathbb{R}^n \); i.e., the continuity of the mapping \( df : X \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) is considered. In addition to introducing continuous differentiability in the context of this generalized approach to differentiability, its characterization is also given. It is proved that the continuity of derivatives at some not necessarily interior points of the functional domain in the direction of \( n \) linearly independent vectors implies (continuous) differentiability.

Keywords: (continuous) differentiability; derivatives in the direction; set of linear contribution; linearization space; raylike neighbourhood; raylike set

MSC: 26B05; 26B12

1. Introduction

Differentiability has almost always been considered in mathematical analysis only for functions \( f : \Omega \rightarrow \mathbb{R}^m \) with an open domain \( \Omega \subseteq \mathbb{R}^n \) (see [1–3]). In the paper [4], the authors made a natural generalization of differentiability by defining it at some non-inner points of the functional domain, which include not only the boundary points of the domain but also all points where the notion of differentiability is meaningful (points admitting neighborhood ray, i.e., points that allow a segment line beginning in them and belonging to the domain). They were motivated by some problems and shortcomings encountered in some applications of the traditional approach to differentiability. In the literature, there are some other generalizations and approaches to differentiability (derivability) such as the fractional derivative [5] or the derivative at the endpoints of a segment [6]. However, the generalization given in [4] allows applications in all areas where standard differentiability can be applied. Since some unexpected phenomena may appear in this generalized approach to differentiability in special cases (such as the non-uniqueness of the differential in some special cases, a function discontinuity at a point where a function is differentiable), the results published in [4] are also interesting from a purely mathematical point of view. Therefore, it is natural and challenging to extend the well-known concept of differentiability with this extended and generalized approach. In this paper, we present some new results from this research, the first results of which have been published in [4].

The focus of this work is the mapping \( df : X \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^m), P \rightarrow \text{df}(P) \), which associates with each point \( P \in X \) a unique differential (a linear operator) \( df(P) : \mathbb{R}^n \rightarrow \mathbb{R}^m \), where \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) denotes the vector space of all linear operators \( A : \mathbb{R}^n \rightarrow \mathbb{R}^m \). To ensure the correctness of this mapping, some conditions are imposed on the set \( X \). To ensure the existence of the differential of the function \( f \) at each point \( P \in X \), we assume that each point admits an nbd ray into \( X \) and that the function \( f \) is differentiable on \( X \). To
ensure the uniqueness of the differential of the function \( f \) at each point, we assume that its linearization space at each point \( P \in X \) is equal to \( \mathbb{R}^n \). A reminder of all these notions, already introduced in [4], is given in the preliminaries section. The main goal of this paper is to establish the notion of continuous differentiability, i.e., continuity of the mapping \( df \). However, the continuity of \( df \) can be regarded as reasonable only if we endow the vector space \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) of linear operators with a topological structure. In order to do that, we first introduce the operator norm to obtain \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) as a normed vector space. In addition to continuously differentiable functions, we consider functions of the class \( C^1 \). This notion is somewhat stronger than continuous differentiability, and it avoids possible pathologies such as the discontinuity of continuously differentiable functions. In the last section, we present a characterization of continuous differentiability. We also establish a sufficient condition for the differentiability of function at a point. It is widely known that the continuity of partial derivatives at interior points of the function domain implies the differentiability of the functions at these points. However, here, we provide a more general and stronger result with a broader possible application. We have proved that the continuity of the derivatives at some not necessarily interior points of the functional domain in the direction of \( n \) linear independent vectors implies (continuous) differentiability.

2. Preliminaries

In this section, we give an overview of the main concepts, notations, and statements needed for the generalized approach to differentiability that we will use in this paper. Since the reader can find all of these with much more details, examples, and explanations in [4], we will list them here only briefly to complete this work. We say that a point \( P_0 \in X \subset \mathbb{R}^n \) admits a neighborhood ray (or simply admits an nbd ray) in \( X \) if there exists \( H \in \mathbb{R}^n \setminus \{0\} \) such that the line segment \( P_0 + H \in X \) is contained in \( X \). For a given \( V \in \mathbb{R}^n \), if there exists \( \lambda_0 \in \mathbb{R}^+ \) such that \( P_0 + \lambda_0 V \subseteq X \), then we say that the point \( P_0 \) admits a neighborhood ray in \( X \) in the direction of \( V \). The set

\[
\Delta_{X,P_0} := \{ H \in \mathbb{R}^n \setminus \{0\} \mid P_0 + H \subseteq X \}
\]

is said to be the set of linear contributions at \( P_0 \) in \( X \), and its linear hull

\[
\Sigma_{X,P_0} := [\Delta_{X,P_0}]
\]

is called the linearization space at \( P_0 \) with respect to \( X \). In addition, \( \Sigma_{X,P_0} \) is also called the linearization space of the function \( f \) at the point \( P_0 \) whenever the function \( f : X \to \mathbb{R}^m \) is given.

The function \( f : X \to \mathbb{R}^m \) is said to be differentiable at the point \( P_0 \) admitting an nbd ray in \( X \) if

\[
\lim_{H \to 0 \atop H \in \Delta_{X,P_0}} \frac{f(P_0 + H) - f(P_0) - A(H)}{||H||} = 0
\]

for some linear operator \( A : \mathbb{R}^n \to \mathbb{R}^m \). If such a linear operator exists, we call it the differential of the function \( f \) at the point \( P_0 \). Every linear operator \( B : \mathbb{R}^n \to \mathbb{R}^m \) that coincides with \( A \) on the subspace \( \Sigma_{X,P_0} \) is the differential of the function \( f \) at \( P_0 \). Therefore, the differential of a function at a point need not be unique, but they all agree on \( \Sigma_{X,P_0} \). However, if \( \Sigma_{X,P_0} = \mathbb{R}^n \), then it is unique and is denoted by \( df(P_0) \).

Let \( X \subseteq \mathbb{R}^n \) and \( P_0 \in X \) be a point admitting an nbd ray in \( X \). A neighborhood \( U \) of the point \( P_0 \) in \( X \) is said to be a raylike neighborhood of the point \( P_0 \) in \( X \) provided \( \overline{P_0P} \subseteq U \) holds for every \( P \in U \). If there exists at least one raylike nbd in \( X \) of the point \( P_0 \), we say that the point \( P_0 \) admits a raylike nbd in \( X \). If \( f : X \to \mathbb{R}^m \) is differentiable at \( P_0 \in X \subseteq \mathbb{R}^n \) admitting a raylike nbd in \( X \), then it is also continuous at \( P_0 \). Differentiability does not imply continuity in general.

Since every point of an open set in \( \mathbb{R}^n \) admits a raylike nbd in it, all phenomena and pathologies listed before are not possible for functions defined on an open domain.
Let $P_0 \in X \subseteq \mathbb{R}^n, n \geq 2$ be a point admitting an nbd ray in $X$ in the direction of the vector $V \in \mathbb{R}^n \setminus \{0\}$. The set
\[ \Delta_{V(X,P_0)} := \{ tV \mid t \in \mathbb{R} \} \cap \Delta_{X,P_0} \]
is said to be the set of linear contributions at $P_0$ in the direction of $V$ into $X$. If the limit
\[ \lim_{hV \in h \to 0} \frac{f(P_0 + hV) - f(P_0)}{h} \]
exists, we call it the derivative at $P_0$ of the function $f : X \to \mathbb{R}$ in the direction of $V$ and denote it by $\partial_V f(P_0)$. The derivative at $P_0$ in the direction of the $i$-th basis vector of the standard ordered basis for $\mathbb{R}^n$ is called the $i$-th partial derivative of $f$ at $P_0$ and is denoted by $\partial_i f(P_0)$.

Let $P_0 \in X \subseteq \mathbb{R}^n$ be a point admitting an nbd ray in $X$ in the direction of $V \in \mathbb{R}^n \setminus \{0\}$. If $f : X \to \mathbb{R}$ is differentiable at $P_0$, then $f$ has a derivative at $P_0$ in the direction of $V$ and the value of each differential of the function $f$ at $P_0$, at $V$ is equal to $\partial_V f(P_0)$. Especially, if $\Sigma_{X,P_0} = \mathbb{R}^n$ then $\partial_V f(P_0) = d f(P_0)(V)$.

A function $f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$ is differentiable at a point $P_0 \in X$ if and only if all its coordinate functions $f_i = p_i \circ f$, $i = 1, \ldots, m$ ($p_i : \mathbb{R}^m \to \mathbb{R}$ denotes the projection map) are differentiable at $P_0$. If $P_0$ admits an nbd ray in $X$ in the direction of $e_1, \ldots, e_n$, then the linear operator $d f(P_0) : \mathbb{R}^n \to \mathbb{R}^m$ in the pair of standard bases is represented by the well-known Jacobi matrix determined by the numbers $\partial_j f_i(P_0)$, $i = 1, \ldots, n, j = 1, \ldots, m$. If $P_0$ admits an nbd ray in $X$ in the direction of $n$ linearly independent vectors $V_1, \ldots, V_n$, then $d f(P_0)$ is represented by the following matrix
\[ \begin{bmatrix}
\partial_{V_1} f_1(P_0) & \ldots & \partial_{V_n} f_1(P_0) \\
\vdots & & \vdots \\
\partial_{V_1} f_m(P_0) & \ldots & \partial_{V_n} f_m(P_0)
\end{bmatrix} \]
in the pair of ordered bases $(V_1, \ldots, V_n)$ and $(e_1, \ldots, e_m)$.

We end this section with two theorems (Theorem 6 and Theorem 9 in [4]), which we will hereinafter use.

**Theorem 1.** Let $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m, f : X \to \mathbb{R}^m, f(X) \subseteq Y$, and $g : Y \to \mathbb{R}^p$. Let $P_0 \in X$ be a point admitting a raylike nbd in $X$, and let $Q_0 = f(P_0)$ be the point admitting a raylike nbd in $Y$. If $f$ is differentiable at $P_0$ and $g$ is differentiable at $Q_0$, then the composition $g \circ f : X \to \mathbb{R}^p$ is differentiable at $P_0$ and $B \circ A$ is its differential at the point $P_0$, where $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^m \to \mathbb{R}^p$ are differentials at the points $P_0$ and $Q_0$ of the functions $f$ and $g$, respectively.

**Theorem 2.** Let $X \subseteq \mathbb{R}^n, f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$, and $P_0 \in X$ be a point admitting an nbd ray in $X$. The function $f$ is differentiable at $P_0$ if and only if $f_i$ is differentiable at $P_0$ for every $i = 1, \ldots, m$. A linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$ is the differential of the function $f$ at $P_0$ if and only if $p_i \circ A : \mathbb{R}^n \to \mathbb{R}$ is the differential of the function $f_i$ at $P_0$, for $i = 1, \ldots, m$.

3. **Vector Space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$**

In this section, we study the space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ of all linear operators $A : \mathbb{R}^n \to \mathbb{R}^m$. As we will show, this space can be organized in the structure of a normed vector space. Since a differential of a function at a point is a linear operator, obtaining a topological structure (induced by the norm) on $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is the necessary step so that the continuity of the mapping $d f : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, $P \cdot d f(P)$ can be regarded as reasonable for a function $f : X \to \mathbb{R}^m$, which is differentiable at every point of its domain and whose linearization space is $\mathbb{R}^n$ at every $P \in X$. 

The well-known fact in linear algebra (see, e.g., [7–9]) is that the sets \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) and \( M(m, n) \) (set of all real \( m \times n \) matrices) are vector spaces. We recall that every linear operator on every basis \( \{b_1, \ldots, b_n\} \) of \( \mathbb{R}^n \) is uniquely determined by its action. If for a linear operator \( A : \mathbb{R}^n \to \mathbb{R}^m \) holds \( A(b_j) = \sum_{i=1}^{m} a_{ij} b'_i, j = 1, \ldots, n \), where \( \{b'_1, \ldots, b'_m\} \) is a basis of \( \mathbb{R}^m \), then \( A \) can be represented as a linear combination

\[
A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} A_{ij}
\]

of the linear operators \( A_{ij} : \mathbb{R}^n \to \mathbb{R}^m, i = 1, \ldots, m, j = 1, \ldots, n \), defined as follows

\[
A_{ij}(b_k) = \begin{cases} b'_{ij}, k = j \\ 0, k \in \{1, \ldots, n\} \setminus \{j\} \end{cases}
\]

Therefore, the set

\[
\{ A_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n \}
\]

constitutes a basis of the vector space \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \). The set \( \{ M_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n \} \) consisting of all matrices having all 0 values except the value 1 at the position \((i, j)\) constitutes a basis of vector space \( M(m, n) \). Indeed, for any matrix

\[
M = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}
\]

it holds

\[
M = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} M_{ij}.
\]

For every linear operator \( A : \mathbb{R}^n \to \mathbb{R}^m \), let us denote by

\[
\phi_{(b_1, \ldots, b_n), (b'_1, \ldots, b'_m)}(A) := \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}
\]

its matrix representation in the ordered pair of bases \( (b_1, \ldots, b_n) \) and \( (b'_1, \ldots, b'_m) \). This defines a mapping

\[
\phi_{(b_1, \ldots, b_n), (b'_1, \ldots, b'_m)} : \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to M(m, n)
\]

of vector spaces which bijectively maps the basis \( \{ A_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n \} \) to the basis \( \{ M_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n \} \). Consequently, \( \phi_{(b_1, \ldots, b_n), (b'_1, \ldots, b'_m)} \) is an isomorphism of vector spaces ([7]).

Furthermore, a function which associates with the vector \( e_{(i-1)m+j} \) of a canonical basis \( \{e_1, \ldots, e_{mn}\} \) of \( \mathbb{R}^{mn} \) the matrix \( M_{ij} \), for all \( i = 1, \ldots, m, j = 1, \ldots, n \), induces a unique isomorphism \( \psi_{m,n} : M(m, n) \to \mathbb{R}^{mn} \) that maps the matrix \( M = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \) to the vector

\[
\psi_{m,n}(A) = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn}) \in \mathbb{R}^{mn}.
\]
We will now introduce a norm on the space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. One way to do this is quite simple and natural. It is suffices to inherit the Euclidean norm $\|\cdot\|_2$ given on $\mathbb{R}^n$ by this formula

$$||A|| = \left\| \Psi_{m,n} \circ \varphi_{(b_1, \ldots, b_k)}(\varphi_{(b_{k+1}, \ldots, b_n)}(A)) \right\|_2$$

(1)

$A \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. Since $\varphi_{m,n}$ and $\varphi_{(b_1, \ldots, b_k)}(\varphi_{(b_{k+1}, \ldots, b_n)})$ are isomorphisms, it is trivial to check that this indeed defines a norm. However, a standard norm on the space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, usually studied in functional analysis (see [10–12]), is called the operator norm and is defined as follows:

$$||A||_{op} = \sup\{|A(x)|_{2} | x \in \mathbb{R}^n, \|x\|_2 = 1\}.$$  

Properties of the operator norm can be found in [10–13], but for the sake of completeness of this paper, we bring the following proposition omitting its proof (see Proposition 3, Chapter 6. in [13]).

**Proposition 1.** For every pair of linear operators $A : \mathbb{R}^n \to \mathbb{R}^k$ and $B : \mathbb{R}^k \to \mathbb{R}^m$, it holds $||B \circ A||_{op} \leq ||B||_{op} \cdot ||A||_{op}$.

**Theorem 3.** A composition of linear operators $\circ : \text{Hom}(\mathbb{R}^n, \mathbb{R}^k) \times \text{Hom}(\mathbb{R}^k, \mathbb{R}^m) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function.

**Proof.** Let $g$ denote linear operators composition $\circ : \text{Hom}(\mathbb{R}^n, \mathbb{R}^k) \times \text{Hom}(\mathbb{R}^k, \mathbb{R}^m) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, $g(A, B) = B \circ A$. Let $(A_0, B_0) \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^k) \times \text{Hom}(\mathbb{R}^k, \mathbb{R}^m)$ be an arbitrary point of its domain and let $\varepsilon > 0$. Notice that for every $(A, B) \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^k) \times \text{Hom}(\mathbb{R}^k, \mathbb{R}^m)$, the following equality holds

$$||g(A, B) - g(A_0, B_0)||_{op} = ||B \circ A - B_0 \circ A_0||_{op} =$$

$$= ||(B - B_0) \circ (A - A_0) + B_0 \circ (A - A_0) + (B - B_0) \circ A_0||_{op}.$$  

By Proposition 1 and triangle inequality, it follows

$$||g(A, B) - g(A_0, B_0)||_{op} \leq$$

$$\leq ||B - B_0||_{op} \cdot ||A - A_0||_{op} + ||B_0||_{op} \cdot ||A - A_0||_{op} + ||B - B_0||_{op} \cdot ||A_0||_{op}.$$  

Let $\delta = \min\{1, \frac{\varepsilon}{1 + ||B_0||_{op} + ||A_0||_{op}}\} > 0$ and let $(A, B) \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^k) \times \text{Hom}(\mathbb{R}^k, \mathbb{R}^m)$ be any point such that

$$d((A, B), (A_0, B_0)) = \sqrt{||A - A_0||_{op}^2 + ||B - B_0||_{op}^2} < \delta,$$

where $d$ denotes a standard product metric. It follows $||A - A_0||_{op} < \delta$ and $||B - B_0||_{op} < \delta$ and consequently

$$||g(A, B) - g(A_0, B_0)||_{op} < \delta + ||A_0||_{op} \cdot \delta + ||B_0||_{op} \cdot \delta < \varepsilon,$$

which proves that the function $g$ is continuous at the point $(A_0, B_0)$. □

It is widely known that all norms given on a finite-dimensional real vector space are equivalent, i.e., all induced topologies are the same (see Theorem 1.21. in [11,14]).

**Corollary 1.** All norms given on the space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ induce the same topology structure.
Corollary 2. Let \( \{b_1, \ldots, b_n\} \) and \( \{b'_1, \ldots, b'_m\} \) be bases of spaces \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively. The mapping \( \psi_{mn} \circ \varphi_{(b_1, \ldots, b_n), (b'_1, \ldots, b'_m)} : \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{mn} \) between the topological space \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \), whose topology is induced by the operator norm and the Euclidean space \( \mathbb{R}^{mn} \) whose topology is induced by the Euclidean norm is a homeomorphism.

**Proof.** If we take on the space \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) the norm \( \| \cdot \| \) defined by the Formula (1), then the mapping \( \psi_{mn} \circ \varphi_{(b_1, \ldots, b_n), (b'_1, \ldots, b'_m)} \) is the isometry between metric spaces \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) and \( \mathbb{R}^{mn} \) whose metrics are induced by the norms \( \| \cdot \| \) and \( \| \cdot \|_2 \), respectively. Consequently, \( \psi_{mn} \circ \varphi_{(b_1, \ldots, b_n), (b'_1, \ldots, b'_m)} \) is a homeomorphism between induced topological spaces \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) and \( \mathbb{R}^{mn} \) (induced metrics define topologies on them). By Corollary 1, topological space \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) endowed with the topology induced by the metric 
\[
d_{op}(A, B) = \| A - B \|_{op} \quad \text{for all} \quad A, B \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m).
\]
Therefore, by taking the latest topological space, the mappings \( \psi_{mn} \circ \varphi_{(b_1, \ldots, b_n), (b'_1, \ldots, b'_m)} \) and \( (\psi_{mn} \circ \varphi_{(b_1, \ldots, b_n), (b'_1, \ldots, b'_m)})^{-1} \) remain continuous, which proves the statement. \( \square \)

Corollary 3. For every \( j = 1, \ldots, m \), the function \( \mathcal{L}_j : \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}) \) which maps every linear operator \( A = (A_1, \ldots, A_m) : \mathbb{R}^n \to \mathbb{R}^m \) in its coordinate function \( \mathcal{L}_j(A) = A_j = p_j \circ A : \mathbb{R}^n \to \mathbb{R} \) (\( p_j : \mathbb{R}^m \to \mathbb{R} \) is the projection mapping) is continuous.

**Proof.** The function \( \mathcal{L}_j \) can be viewed as:
\[
\text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \bigg( \text{const}_{p_j} \bigg) \text{Hom}(\mathbb{R}^n, \mathbb{R}) 
\]

The statement now follows from Theorem 3. \( \square \)

Corollary 4. Let \( X \) be a topological space. A function \( F : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) is continuous if and only if \( \mathcal{L}_j \circ F : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}) \) is continuous, for every \( j = 1, \ldots, m \).

**Proof.** Notice that the continuity of the function \( \mathcal{L}_j \circ F : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}) \) is equivalent to the continuity of the function \( \psi_{mn} \circ \varphi_{(e_1, \ldots, e_n), (1)} \circ \mathcal{L}_j \circ F : X \to \mathbb{R}^m \), which is equivalent to the continuity of the function \( \psi_{mn} \circ \varphi_{(e_1, \ldots, e_n), (1)} \circ \mathcal{L}_j \circ F : X \to \mathbb{R}^{mn} \). Since \( \psi_{mn} \) and \( \varphi_{(e_1, \ldots, e_n), (1)} \) are homeomorphisms (Corollary 2), it follows that \( F : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) is continuous. The necessity is the consequence of the previous corollary. \( \square \)

Corollary 5. Let \( X \) be a topological space. If the functions \( f, g : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) are continuous, then the function \( \lambda f + \mu g : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) is continuous, for every \( \lambda, \mu \in \mathbb{R} \).

4. Continuously Differentiable Functions

The main goal of this section is to explore the nature of the mapping \( df : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) for a differentiable function \( f : X \to \mathbb{R}^m \) where \( X \subseteq \mathbb{R}^n \) is the set in which each point admits an nbd ray in it and \( \Sigma_X = \mathbb{R}^n \), for every \( P \in X \). The mapping \( df \) associates with each point \( P \in X \) the unique differential \( df(P) \) belonging to \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \). Since \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \) is endowed with a topological structure induced by the operator norm and \( X \) already has a topological structure inherited from the Euclidean space \( \mathbb{R}^n \), it makes sense to consider the continuity of \( df \). If \( df \) is a continuous mapping, then it means that \( f \) is a continuously differentiable function. However, for such a function, some undesirable phenomena might occur (see Example 2). To avoid pathologies (such as the discontinuity of continuously differentiable functions), we will strengthen the condition of continuous
differentiability to obtain the notion of class $C^1$ functions. These functions are considered the most beautiful in this context.

**Definition 1.** Let $f : X \to \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$ be a function and let every point $P \in X$ admit an nbd $\Sigma_{X,P} = \mathbb{R}^n$. The function $f$ is said to be **continuously differentiable** provided it is differentiable on $X$ and the mapping $d f : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous. If in addition every point $P \in X$ admits a raylike nbd in $X$, then we say that $f$ is of the class $C^1$. The set of all functions $X \to \mathbb{R}^m$ of the class $C^1$ is denoted by $C^1(X, \mathbb{R}^m)$ or simply $C^1(X)$ in the case $m = 1$.

**Example 1.** Since the differential of the linear operator $f$ at each point is equal to the same linear operator, the mapping $d f$ from the previous definition is constant and thus continuous. Therefore, every linear operator is a function of class $C^1$. For the same reason, every constant function on the set $X$ is continuously differentiable as in the previous definition.

Since every point $P$ of an open set $\Omega \subseteq \mathbb{R}^n$ admits a raylike nbd in it and the linearization space $\Sigma_{X,P}$ at $P$ with respect to $X$ is equal to $\mathbb{R}^n$, it holds:

**Corollary 6.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A function $f : \Omega \to \mathbb{R}^m$ is continuously differentiable if and only if it is a function of the class $C^1$.

The notion of continuous differentiability coincides with the notion of “being of class $C^1$” for functions with an open domain. In this case, the two terms are synonymous, which can be found in almost all university textbooks dealing with the traditional approach to differentiability. However, for functions with a non-open domain, these terms differ, as we can see in the following example.

**Example 2.** Let $P_n = \left(0, \frac{1}{n}\right)$, $P'_n = \left(0, \frac{2n+1}{2n(n+1)}\right)$, $Q_n = \left(1 + \frac{1}{n}\right)$, $Q'_n = \left(1 + \frac{2n+1}{2n(n+1)}\right)$, $n \in \mathbb{N}$, be points of $\mathbb{R}^2$. Let us denote by $X_n$ a compact set from $\mathbb{R}^2$ bounded by lengths $P_n Q_n$, $P'_n Q'_n$, $P_n P'_n$ and $Q_n Q'_n$, $n \in \mathbb{N}$. Let $X_0$ be a compact set from $\mathbb{R}^2$ bounded by lengths $(0,0)(1,0)$, $(0,-1)(1,-1)$, $(0,0)(0,-1)$ and $(1,0)(1,-1)$. Let $X = \bigcup_{n \in \mathbb{N}_0} X_n$. Let us define the function $f : X \to \mathbb{R}$ where

$$f(T) = n$$

and $T \in X_n$, $n \in \mathbb{N}_0$. The function $f$ is differentiable and the differential is unique at every point in its domain since the linearization space of the function $f$ at every point is equal to $\mathbb{R}^2$. Since $d f(T) = 0$, for every $T \in X$, the function $d f : X \rightarrow \text{Hom}(\mathbb{R}^2, \mathbb{R})$ is continuous, so $f$ is continuously differentiable. However, all the points $T \in (0,0)(1,0)$ are the points of discontinuity of the function $d f$, thus it is not of class $C^1$. If we transform the function $f$ to be continuous by setting $f(T) = \frac{1}{n}$, $T \in X_n$, for each $n \in \mathbb{N}$, it would still not be of class $C^1$ because points of length $(0,0)(1,0)$ do not admit a raylike nbd in $X$.

**Proposition 2.** Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq X$ be the sets for which each point admits a raylike nbd in $X$, i.e., in $Y$, and let $\Sigma_{X,P} = \mathbb{R}^n$, for every $P \in X$, and $\Sigma_{Y,Q} = \mathbb{R}^n$, for every $Q \in Y$. If the function $f : X \to \mathbb{R}^m$ is of class $C^1$, then the restriction $f|_Y : Y \to \mathbb{R}^m$ is a function of class $C^1$.

**Proof.** The statement follows from the continuity of the function $d f|_Y$ as a restriction of the continuous function $d f$ and the equality $d f(P) = d(f|_Y)(P)$, for every $P \in Y$ (Proposition 3 in [4]).

In the context of continuous differentiability on the dual space $\text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, we can use either the operator norm or the induced norm after identifying the linear operator $A$ with a vector $\psi_m \circ \varphi_{(b_1, \ldots, b_n)}(b'_1, \ldots, b'_m)$ where $(b_1, \ldots, b_n)$ and $(b'_1, \ldots, b'_m)$ is any pair of ordered space bases $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively (Corollary 1).

In the case of $m = n = 1$, for the differentiable function $f : X \subseteq \mathbb{R} \rightarrow \mathbb{R}$, a mapping $d f : X \rightarrow \text{Hom}(\mathbb{R}, \mathbb{R})$ is defined by the association $x \rightarrow f'(x)$, i.e., by the function $f' : X \rightarrow \mathbb{R}$.
Note that a function \( \psi_{1,1} \circ \varphi_{(1),(1)} : \text{Hom}(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R} \) is a homeomorphism (Corollary 2); thus, the mapping \( df \) can be identified with the association

\[
\psi_{1,1} \circ \varphi_{(1),(1)} \circ df : X \rightarrow \mathbb{R}, \quad \psi_{1,1} \circ \varphi_{(1),(1)}(df(x)) = f'(x).
\]

Therefore, it holds \( \psi_{1,1} \circ \varphi_{(1),(1)} \circ df = f' \). Consequently, in addition to the notion of continuous differentiability of such functions, we use the notion of continuous derivability of the function \( f : X \rightarrow \mathbb{R} \), which denotes a differentiable function whose derivative function \( f' : X \rightarrow \mathbb{R} \) is continuous. Since by Theorem 7 in [4] the derivability of the function \( f \) is equivalent to differentiability, it follows from Corollary 2 in [4] that:

**Corollary 7.** Let \( X \subseteq \mathbb{R} \) be the set on which every point admits an nbd ray. The function \( f : X \rightarrow \mathbb{R} \) is continuously differentiable if and only if it is continuously derivable.

A counterexample in which a real function of a real variable can be continuously differentiable but not belong to class \( C^1 \) is given in Example 9 in [4].

**Corollary 8.** Let \( X \subseteq \mathbb{R} \) be a set for which every point admits an nbd ray. The function \( f : X \rightarrow \mathbb{R} \) is of class \( C^1 \) if and only if it is continuously differentiable.

The following example shows that a differentiable function need not necessarily be continuously differentiable.

**Example 3.** The function \( f : (0, \infty) \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} \frac{\epsilon}{x}, & x < e \\ \frac{\epsilon}{x^2}, & x \geq e \end{cases} \) is differentiable but not continuously differentiable. Indeed, \( f \) is differentiable at every point of the domain, and its derivative is

\[
f'(x) = \begin{cases} -\frac{\epsilon}{x^2}, & x < e \\ \frac{\epsilon}{x^2}, & x \geq e \end{cases},
\]

however, \( f' \) has discontinuity at the point \( x = e \) because at this point, the left and right limits \((-\frac{1}{e} \text{ and } \frac{1}{e})\) of the function differ.

**Theorem 4.** Let \( X \subseteq \mathbb{R}^n \) and \( Y \subseteq \mathbb{R}^m \) be sets whose linearization spaces are \( \Sigma_{X,P} = \mathbb{R}^n \) and \( \Sigma_{Y,Q} = \mathbb{R}^m \), for all \( P \in X \) and \( Q \in Y \), and every point in \( X \) admits an nbd ray neighborhood in \( X \). Let \( f : X \rightarrow \mathbb{R}^m \) be a function such that \( f(X) \subseteq Y \). If \( f : X \rightarrow \mathbb{R}^m \) is the function of class \( C^1 \) and \( g : Y \rightarrow \mathbb{R}^p \) is a continuously differentiable function, then \( g \circ f : X \rightarrow \mathbb{R}^p \) is a function of class \( C^1 \).

**Proof.** According to the definition of continuous differentiability and functions of class \( C^1 \), the functions \( f \) and \( g \) satisfy the conditions of Theorem 1 for each \( P \in X \); thus, \( g \circ f \) is a differentiable function for every \( P \in X \) and \( d(g \circ f)(P) = dg(f(P)) \circ df(P) \) holds. Let us prove that the function \( d(g \circ f) : X \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^p) \) defined by a rule

\[
d(g \circ f)(P) = dg(f(P)) \circ df(P), \quad P \in X,
\]

is continuous. Notice that the function \( d(g \circ f) \) is a composition of functions

\[
X \xrightarrow{(df, dg \circ f)} \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \times \text{Hom}(\mathbb{R}^m, \mathbb{R}^p) \xrightarrow{\circ} \text{Hom}(\mathbb{R}^n, \mathbb{R}^p).
\]

Since the function \((df, dg \circ f)\) is continuous, and according to Theorem 3, the mapping

\[
\text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \times \text{Hom}(\mathbb{R}^m, \mathbb{R}^p) \xrightarrow{\circ} \text{Hom}(\mathbb{R}^n, \mathbb{R}^p),
\]

is continuous, it follows that \( d(g \circ f) \) is a continuous function.  \( \Box \)
5. Characterization Theorems of Continuous Differentiability

We have seen that the differentiability of the function \( f : X \to \mathbb{R}^m \), at the point \( P_0 \in X \subseteq \mathbb{R}^n \), implies the existence of derivatives in the directions of all vectors in whose direction there exists an nbd ray of \( P_0 \) in \( X \) (Corollary 5 in [4]), while the converse is not true in general (Example 8 in [4]). However, if along with the existence of the derivatives in the directions of \( n \) linearly independent vectors the continuity of the corresponding derivatives functions on a special set containing the point \( P_0 \) is fulfilled, then this implies also the differentiability of the function. It is known that the existence and continuity of the partial derivatives of a function given on an open domain implies the differentiability of this function:

**Theorem 5.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set and let \( f : \Omega \to \mathbb{R}^m \) be a mapping such that there exists an \( i \)-th partial derivative \( \partial_i f(P) \) of \( f \) at \( P \), for every \( i = 1, \ldots, n \) and every \( P \in \Omega \). If the partial derivatives \( \partial_i f : \Omega \to \mathbb{R}^m, i = 1, \ldots, n \), are continuous functions at \( P_0 \), then \( f \) is differentiable at \( P_0 \).

This theorem can be found in any university handbook of mathematical analysis (see, e.g., Theorem 9.21 in [1]). We generalize this result, which is a part of common mathematical knowledge, in two ways. First, we consider functions given on much more general domains than open sets. Second, we establish a relation between the existence and continuity of derivatives in the direction of “admissible” vectors (partial derivatives are only a special case of this).

Let \( X \subseteq \mathbb{R}^n \) and \( P_0 \in X \) be the point admitting an nbd ray in \( X \) in the direction of the vector \( V_i, i = 1, \ldots, n \), where \( V_1, \ldots, V_n \in \mathbb{R}^n \setminus \{0\} \) are linearly independent vectors and let \( \Delta_{X,P_0} \) be the set of linear contributions at \( P_0 \) in \( X \). If there exist \( h_0, h_i \in \mathbb{R}, h_i \leq 0, h_i > 0, i = 1, \ldots, n \), such that set \( \Delta_{V_1,\ldots,V_n}^{h_0,h_1,\ldots,h_n} := \{ \sum_{i=1}^{n} h_i V_i \in \mathbb{R}^n \mid h_i \in [h_i,0) \} \) is in the neighborhood of the point \( 0 \) in \( \Delta_{X,P_0} \cup \{0\} \), then the set \( P_0 + \Delta_{V_1,\ldots,V_n}^{h_0,h_1,\ldots,h_n} \) is said to be a raylike set in \( X \) at \( P_0 \) in the direction of the linear combinations of vectors \( V_1, \ldots, V_n \). It is easy to see that any point of \( P_0 + \Delta_{V_1,\ldots,V_n}^{h_0,h_1,\ldots,h_n} \) admits an nbd ray in \( X \) in the direction of the vector \( V_i, i = 1, \ldots, n \), and that the linearization space at any point with respect to \( X \) is equal to \( \mathbb{R}^n \). In addition, notice that the set \( P_0 + \Delta_{V_1,\ldots,V_n}^{h_0,h_1,\ldots,h_n} \) is a raylike nbd of \( P_0 \) in \( X \). Indeed, for every \( P = \sum_{i=1}^{n} h_i V_i \in P_0 + \Delta_{V_1,\ldots,V_n} \), it is \( \overline{0P} \subseteq X \) because it holds \( t \left( \sum_{i=1}^{n} h_i V_i \right) \in \Delta_{V_1,\ldots,V_n} \) for every \( t \in [0,1] \), and consequently \( 0(P - P_0) \subseteq \Delta_{V_1,\ldots,V_n}^{h_0,h_1,\ldots,h_n} \).

**Example 4.** Let \( A \in \mathbb{R}^2 \). If \( V_1, V_2 \in \mathbb{R}^2 \) are non-collinear vectors and if \( X \subseteq \mathbb{R}^2 \) is a closed triangle with vertices \( A + V_1, A + V_1 + V_2 \) and \( A + V_2 \), then the (closed) parallellogram with vertices \( A, A + \frac{1}{2} V_1, A + \frac{1}{2} V_2 \) and \( A + \frac{1}{2} V_1 + \frac{1}{2} V_2 \) is a raylike set in \( X \) at \( A \) in the direction of the linear combinations of the vectors \( V_1, V_2 \). In fact, the set of linear combinations \( \Delta_{X,A} \) of \( A \) in \( X \) coincides with a triangle with vertices \( 0, V_1 \) and \( V_2 \) but without a vertex in \( 0 \), and the parallellogram \( \Pi = \left\{ \sum_{i=1}^{2} h_i V_i \in \mathbb{R}^2 \mid h_i \in [0,\frac{1}{2}] \right\} \) with vertices \( 0, \frac{1}{2} V_1, \frac{1}{2} V_2 \) and \( \frac{1}{2} (V_1 + V_2) \) is the neighborhood of the origin \( 0 \) in \( \Delta_{X,A} \cup \{0\} \). Therefore, the parallellogram \( A + \Pi \) is the raylike set in \( X \) at \( A \) in the direction of the linear combinations of the vectors \( V_1, V_2 \).

**Theorem 6.** Let \( X \subseteq \mathbb{R}^n \), \( P_0 \in X \) and \( V_1, \ldots, V_n \in \mathbb{R}^n \) be linearly independent vectors such that \( P_0 \) admits an nbd ray in \( X \) in the direction of the vector \( V_i, i = 1, \ldots, n \). Let there exist a raylike set \( W \) in \( X \) at \( P_0 \) in the direction of the linear combinations of vectors \( V_1, \ldots, V_n \). Let \( f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m \) be a mapping for which there exist all derivatives in the direction of the vector \( V_1, \ldots, V_n \) at every point in \( W \). If all directional derivatives \( \partial_{V_i f_j} : W \to \mathbb{R}, i = 1, \ldots, n, j = 1, \ldots, m \), are continuous functions at \( P_0 \), then \( f \) is differentiable at \( P_0 \).
Proof. By Theorem 2, it suffices to prove the statement for the scalar case \( m = 1 \). Let \( \Delta \) be the set of linear contributions at \( P_0 \) in \( X \). Then, \( W - P_0 \) is the neighborhood of the point \( 0 \in \mathbb{R}^n \) in \( \Delta \cup \{0\} \), and there are numbers \( h_i^0, h_i^0 \in \mathbb{R}, h_i^0 \leq 0, h_i^0 > 0, i = 1, \ldots, n \), such that

\[
W - P_0 = \left\{ \sum_{i=1}^{n} h_i V_i \in \mathbb{R}^n \mid h_i \in [h_i^0, h_i^0] \right\}.
\]

Then, for each \( H \in W - P_0 \), there are numbers \( h_i \in [h_i^0, h_i^0], i = 1, \ldots, n \) such that \( H = h_1 V_1 + \cdots + h_n V_n \) and holds

\[
f\left( P_0 + H \right) - f(P_0) = f\left( P_0 + h_1 V_1 + \cdots + h_n V_n \right) - f(P_0) = f\left( P_0 + h_1 V_1 + \cdots + h_n V_n \right) + \\
+ f\left( P_0 + h_1 V_1 + \cdots + h_n V_n \right) - f(P_0) = f(P_0 + h_1 V_1 + \cdots + h_n V_n) - f(P_0).
\]

Let us introduce auxiliary functions

\[
\xi_i : [h_i^0, h_i^0] \to \mathbb{R}, \xi_i(t) = f(P_0 + t V_i + h_i V_{i+1} + \cdots + h_n V_n),
\]

for every \( i = 1, \ldots, n \). Now, it follows from the previous equality

\[
f\left( P_0 + H \right) - f(P_0) = \xi_1(h_1) - \xi_1(0) + \xi_2(h_2) - \xi_2(0) + \cdots + \xi_n(h_n) - \xi_n(0).
\]

Note that \( \xi_i \) is differentiable by the assumption of the theorem, and it holds

\[
\xi_i'(t) = \partial V_i f(P_0 + t V_i + h_i V_{i+1} + \cdots + h_n V_n)
\]

for every \( t \in [h_i^0, h_i^0] \). According to Lagrange’s theorem [15], there are \( \theta_i \in (0, 1) \) such that

\[
\xi_i(h_i) - \xi_i(0) = \xi_i'(\theta_i^i h_i) h_i, i = 1, \ldots, n.
\]

Now, it is

\[
f\left( P_0 + H \right) - f(P_0) = f\left( P_0 + \theta_i^i h_i V_i + h_i V_{i+1} + \cdots + h_n V_n \right) = \\
= \sum_{i=1}^{n} \partial V_i f\left( P_0 + \theta_i^i h_i V_i + h_i V_{i+1} + \cdots + h_n V_n \right) h_i = \\
= \sum_{i=1}^{n} \partial V_i f(P_0) h_i + \sum_{i=1}^{n} \left( \partial V_i f\left( P_0 + \theta_i^i h_i V_i + h_i V_{i+1} + \cdots + h_n V_n \right) - \partial V_i f(P_0) \right) h_i.
\]

Since \( (W - P_0) \setminus \{0\} \) is an open set in \( \Delta \), by Theorem 1 in [4], it is enough to prove that it is

\[
\lim_{\substack{H \in (W - P_0) \setminus \{0\} \\ |H| \to 0}} \sum_{i=1}^{n} \left( \partial V_i f\left( P_0 + \theta_i^i h_i V_i + h_i V_{i+1} + \cdots + h_n V_n \right) - \partial V_i f(P_0) \right) h_i = 0.
\]

Since \( |h_i| \leq 1 \), \( i = 1, \ldots, n \), it follows exactly the required if

\[
\lim_{\substack{H \in (W - P_0) \setminus \{0\} \\ |H| \to 0}} \sum_{i=1}^{n} \left( \partial V_i f\left( P_0 + \theta_i^i h_i V_i + h_i V_{i+1} + \cdots + h_n V_n \right) - \partial V_i f(P_0) \right) = 0.
\]
However, this holds true because of the continuity of the partial derivatives \( \partial V_j f \) at point \( P_0 \). Since \( W - P_0 \) is a neighborhood of 0 in \( \Delta \cup \{0\} \) by Theorem 1 in [4], it follows

\[
\lim_{H \to 0} \frac{\sum_{i=1}^{n} \left( \partial V_i f \left( P_0 + \theta_i^H h_1 V_i + h_{i+1} V_{i+1} + \cdots + h_n V_n \right) - \partial V_i f (P_0) \right) h_i}{||H||} = 0.
\]

Thus, we have shown that the function \( f \) is differentiable at \( P_0 \) and that its differential \( df(P_0) : \mathbb{R}^n \to \mathbb{R} \) is given by

\[
df(P_0)(H) = \sum_{i=1}^{n} \partial V_i f (P_0) h_i,
\]

for every \( H = h_1 V_1 + \cdots + h_n V_n \). \( \square \)

**Proposition 3.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set, \( P_0 \in \Omega \) and \( V_1, \ldots, V_n \in \mathbb{R}^n \) linearly independent vectors. Then, there exists a raylike set in \( \Omega \) there at \( P_0 \) in the direction of the linear combinations of the vectors \( V_1, \ldots, V_n \).

**Proof.** There exists a ball \( B(P_0, r) \subseteq \Omega \) and it holds \( B(0, r) \subseteq \Delta \cup \{0\} \) where \( \Delta \) is the set of linear contributions at \( P_0 \) in \( \Omega \). Suffice it to note that for \( h^0 = -\frac{r}{2n||V_i||} \) and \( h^1 = -\frac{r}{2n||V_i||} \), a set \( B_{j=m}^0 h^0 \in \mathbb{R}^n \):

\[
\left\{ \left. h_i V_i \in \mathbb{R}^n \right| h_i \in \left[ h^0, h^1 \right] \right\}
\]

is a neighborhood of point 0 in \( B(0, r) \), so \( P_0 + \Delta \) is a raylike set in \( \Omega \) at \( P_0 \) in the direction of the linear combinations of vectors \( V_1, \ldots, V_n \). \( \square \)

**Corollary 9.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set \( P_0 \in \Omega \) and \( V_1, \ldots, V_n \in \mathbb{R}^n \) linearly independent vectors. Let \( f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m \) be a mapping for which there exist derivatives \( \partial V_j f_i (P) \) in direction of the vector \( V_i \), for each \( P \in \Omega \) and all \( i = 1, \ldots, n \), \( j = 1, \ldots, m \). If all directional derivatives \( \partial V_j f_i : \Omega \to \mathbb{R} \), \( i = 1, \ldots, n \), \( j = 1, \ldots, m \), are continuous functions at \( P_0 \), then \( f \) is differentiable at \( P_0 \).

**Proof.** The proof follows from the previous theorem and proposition. \( \square \)

Let \( X \subseteq \mathbb{R}^n \) be a set of \( V_1, \ldots, V_n \in \mathbb{R}^n \) linearly independent vectors, such that every point \( P \in X \) admits an nbd ray in \( X \) in the direction of the vectors \( V_i, i = 1, \ldots, n \). Let \( f : X \to \mathbb{R}^m \) be a continuously differentiable function. Then, by Corollary 5 in [4], there exist derivatives \( \partial V_j f_i (P) \) in direction of the vectors \( V_i, i = 1, \ldots, n \), \( j = 1, \ldots, m \). Since \( \psi_{m,n} \circ \psi_{(V_1, \ldots, V_n), (\varepsilon_1, \ldots, \varepsilon_m)} : \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^m \) is a homeomorphism, the function \( \psi_{m,n} \circ \psi_{(V_1, \ldots, V_n), (\varepsilon_1, \ldots, \varepsilon_m)} \circ df \) is continuous if and only if \( df \) is continuous. However, \( \psi_{m,n} \circ \psi_{(V_1, \ldots, V_n), (\varepsilon_1, \ldots, \varepsilon_m)} \circ df : X \to \mathbb{R}^m \) is a vector function whose coordinate functions are the directional derivatives:

\[
\partial V_1 f_1, \ldots, \partial V_n f_1, \ldots, \partial V_1 f_m, \ldots, \partial V_n f_m.
\]

Thus, if \( df \) is continuous, then all these derivatives exist, and they are continuous mappings at every point in the domain of the function \( f \). On the other hand, if all directional derivatives \( \partial V_j f_i \), \( i = 1, \ldots, n \), \( j = 1, \ldots, m \), exist on \( X \) and are continuous at every point of the domain, and there exists a raylike set in \( X \) at every point \( P \in X \) (and consequently raylike nbd of it) in the direction of the linear combinations of the vectors \( V_1, \ldots, V_n \), then, by the previous theorem, the function \( f \) is differentiable, and the function \( \psi_{m,n} \circ \psi_{(V_1, \ldots, V_n), (\varepsilon_1, \ldots, \varepsilon_m)} \circ df \) (and then also \( df \)) is continuous. So, we have shown that it holds:

**Corollary 10.** Let \( X \subseteq \mathbb{R}^n \) be a set and \( V_1, \ldots, V_n \in \mathbb{R}^n \) be linearly independent vectors such that there exists a raylike set in \( X \) at every point \( P \in X \) in the direction of the linear combinations of
the vectors $V_1, \ldots, V_n$. A function $f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ is of the class $C^1$ if and only if there exist directional derivatives $\partial V_i f_j(P)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, for each $P \in X$ and all the functions $\partial V_i f_j : X \to \mathbb{R}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, are continuous.

**Corollary 11.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $V_1, \ldots, V_n \in \mathbb{R}^n$ be linearly independent vectors. Mapping $f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m$ is of the class $C^1$ if and only if there exist directional derivatives $\partial V_i f_j(P)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, for every $P \in \Omega$, and all the functions $\partial V_i f_j : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, are continuous.

**Proof.** If $f$ is of class $C^1$, then by Theorem 1, $f_i = p_i \circ f$ is of class $C^1$, because according to Example 1, $p_i : \mathbb{R}^n \to \mathbb{R}$ is of the class $C^1$, for every $i = 1, \ldots, n$. If $f_i : X \to \mathbb{R}$ is of the class $C^1$, for every $i = 1, \ldots, m$, then $f$ is a differentiable function by Theorem 2. Notice that by the assumption all the functions $\partial f_i : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R})$, $i = 1, \ldots, m$, are continuous. Since it holds $L_i \circ df = df_i$, $i = 1, \ldots, m$, the statement follows from Corollary 4. □

**Corollary 12.** Let $X \subseteq \mathbb{R}^n$ be a set, $f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m$ a function, and let every point of the set $X$ admit an nbd ray in $X$, and let the space of linearization of the function $f$ at every point of its domain be equal to $\mathbb{R}^n$. The function $f : X \to \mathbb{R}^m$ is of the class $C^1$ if and only if $f_i : X \to \mathbb{R}$ is of the class $C^1$, for every $i = 1, \ldots, m$.

**Proof.** The differentiability of a linear combination of differentiable functions follows from Theorem 4 in [4], and the continuous differentiability of a linear combination of functions of class $C^1$ follows from Corollary 5 for $d(\lambda f + \mu g) = \lambda df + \mu dg : X \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. □

**6. Conclusions**

In the traditional approach to differentiability, this notion is studied only for functions with an open domain. The generalization of differentiability made in [4] allows us to consider differentiability at some non-interior points of the functional domain which include not only the boundary points of the domain but also all points where the notion of differentiability is meaningful (points admitting neighborhood ray). Since this approach allows applications in all domains where standard differentiability can be applied, it is interesting to consider continuously differentiable functions in this generalized context. By introducing the continuity of the mapping $df : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, which assigns to each point $P \in X$ the linear operator $df(P) : \mathbb{R}^n \to \mathbb{R}^m$, the notion of continuously differentiable functions and functions of class $C^1$ is introduced for not necessary open domain $X$. In addition to a characterization of continuously differentiable functions given on the not necessary open domain, it was proved that the continuity of derivatives in the direction of $n$ linear independent vectors implies (continuous) differentiability. This result provides sufficient conditions for differentiability of the function at the non-interior points of its domain where only derivatives in the direction of $n$ linearly independent vectors exist. Since the sufficient conditions for differentiability given in Theorem 5 as a standard result in mathematical analysis are given only in terms of partial derivatives and only for the functions defined on the open domain, Theorem 6 generalizes known results considerably, and we can consider it as the main contribution of this paper. This can be applied in many different areas of mathematics and engineering, or in short, in all areas where standard differentiability can be applied. Although many functions modeling some problems in engineering (or elsewhere) are given on non-open sets, most engineering applications simply ignore the topology of domains by considering their interior, where standard calculus with partial derivatives can be applied. The results of this paper provide calculus techniques where derivatives in the direction of “admissible” vectors take the role of partial derivatives, allowing mathematicians and
engineers to consider functions given on non-open domains. Since examples of direct application outside of pure mathematics are not the goal of this work, we will only point out that these results can be applied whenever one needs to analyze a function given on an “odd” domain and has no way to use partial derivatives. From a purely mathematical point of view, this work is a resumption of the research recently published in [4]. The fact that it is an essential mathematical notion makes this research and the generalized approach to differentiability sufficiently interesting for mathematicians. In recent years, one can also find some other approaches and results (see [16–18]) on this elementary notion.

Author Contributions: Conceptualization, N.K.-B.; Investigation, N.K.-B.; Data curation, S.B.; Writing—original draft, N.K.-B.; Writing—review & editing, N.K.-B. and S.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References
18. Todorović, V. Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics; Springer Nature Switzerland: Cham, Switzerland, 2019.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.