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Conics from the Cartan Decomposition of $SO(2,1)$

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Abstract: The aim of this paper is to introduce and study the class of conics provided by the symmetric matrices of the Cartan decomposition of the Lie group $SO(2,1)$. This class depends on two real parameters as components of the cylinder $S^1 \times \mathbb{R}$ and we use a deformation inspired by Finsler indicatrices in order to obtain proper ellipses. A complex approach is also included.

Keywords: conic; Cartan decomposition of $SO(2,1)$; complex variable

MSC: 11D09; 51N20; 30C10; 22E47

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1. Introduction

After more than two thousand years,conics continue to be a versatile object of mathematics and the very recent book, [1], is veritable proof of this fact. A lot of techniques, from analytical to projective, have been developed to handle these remarkable curves.

The starting point of note is the article [2], where symmetric Pythagorean triple preserving (PTP) matrices are used to generate conics. Hence, we continue this line of research with the following other classes of symmetric matrices of order 3: (i) those produced by the adjoint representation of the 3-dimensional matrix Lie group $SU(2) = S^3$ in [3]; and (ii) the magic matrices in [4].

A fourth class of remarkable matrices are provided by the Cartan decomposition of the Lie group $SO(2,1)$, and the present paper studies the associated conics. More precisely, we obtain only degenerated conics and, hence, we perform a translation inspired by the notion of indicatrix from Finsler geometry. In Section 2, ellipses indexed by the product $S^1 \times \mathbb{R}$, but having a rotational symmetry, more precisely, having a canonical form depending only on the parameter $\beta \in \mathbb{R}$ are presented. Their eccentricity depends bijectively on β, and, hence, we can express the canonical form only in terms of eccentricity.

Two classes of examples are discussed in Section 3, namely self-complementary ellipses, i.e., with eccentricity $\frac{1}{\sqrt{2}}$, and symmetric ellipses, i.e., with a common coefficient for x^2 and y^2. A canonical conic function is also computed for our eccentricity, depending on β. We finish the Section 2 with the expression of the fixed points of the linear fractional (or Möbius) function associated to the two by two symmetric $SO(2,1)$-matrices. These fixed points do not depend on β.

In the Section 4 we discuss our conic in terms of its three Hermitian coefficients and another complex number, called *affix*. Another remarkable class of examples appears when its two complex Hermitian coefficients are pure real. We note that some hard computations were performed with WolframAlpha.

2. Conics Provided by the Cartan Decomposition of $SO(2,1)$

In the setting of two-dimensional Euclidean space $(\mathbb{R}^2, g_{can} = diag(1,1))$ let us consider the conic Γ implicitly defined by $f \in C^\infty(\mathbb{R}^2)$ as: $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid f_\Gamma(x,y) = 0\}$ where f_Γ is a quadratic function of the form $f_\Gamma(x,y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$.

The study of Γ is based on the symmetric matrices (e means extended):

$$\Gamma := \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix} \in Sym(2), \quad \Gamma^e := \begin{pmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{pmatrix} \in Sym(3). \tag{1}$$

In fact, the algebraic invariants associated to Γ are:

$$I := r_{11} + r_{22} = Tr\Gamma, \delta := \det \Gamma, \Delta := \det \Gamma^e, D := \delta + r_{11}r_{00} - r_{10}^2 + r_{22}r_{00} - r_{20}^2. \tag{2}$$

The necessity to search for remarkable symmetric matrices of order three follows.

A very useful three-dimensional matrix Lie group in both mathematics and physics is $SO(2,1)$ with its Lie algebra $so(2,1) \simeq su(1,1)$. Recall that $O(2,1)$ is the matrix group preserving the 3D Minkowski–Lorentz norm $\|\bar{v} = (x,y,z)\|^2 = x^2 + y^2 - z^2$ and $SO(2,1)$ is its subgroup with determinant 1. An important property of $SO(2,1)$ is that it is semisimple. On the Lie algebra level we have the isomorphisms: $so(2,1) \simeq sp(2,\mathbb{R}) \simeq sl(2,\mathbb{R}) \simeq su(1,1)$. More precisely, with $\gamma = diag(1,1,-1)$ we have:

$$SO(2,1) = \{S \in SL(3,\mathbb{R}); S \cdot \gamma \cdot S^t = \gamma\}, \quad so(2,1) = \{\Gamma \in gl(3,\mathbb{R}); \Gamma^t = -\gamma \cdot \Gamma \cdot \gamma\}.$$

For $SO(2,1)$ the Cartan decomposition $SO(2,1) = K \cdot A \cdot K$ is well known, where the groups $K \simeq SO(2) \times \{1\}$ and $A \subset Sym(3)$ are given, respectively, by:

$$K(\alpha) := \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K, \quad A(\beta) := \begin{pmatrix} \cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & \cosh\beta \end{pmatrix} \in A. \tag{3}$$

Let H^+ be the positive sheet of the hyperboloid $\mathcal{H} : \|\bar{v}\|^2 = -1$ i.e., $z(v) > 0$. Then H^+ is exactly the homogeneous space $SO(2,1)/K$ since K is precisely the stabilizer of the point $(0,0,1) \in H^+$. The group K is isomorphic to the unit circle group $(S^1, \cdot)$, which is the 1-dimensional torus.

The KAK decomposition of $SO(2,1)$ means that every matrix $X \in SO(2,1)$ is a product:

$$X := X(\alpha_1, \beta, \alpha_2) = K(\alpha_1) \cdot A(\beta) \cdot K(\alpha_2), \quad \alpha_1, \alpha_2, \beta \in \mathbb{R} \tag{4}$$

and a straightforward computation gives the $SO(2,1)$-matrix:

$$X(\alpha_1,\beta,\alpha_2) = \begin{pmatrix} \cos\alpha_1 \cos\alpha_2 \cosh\beta - \sin\alpha_1 \sin\alpha_2 & -\cos\alpha_1 \sin\alpha_2 \cosh\beta - \sin\alpha_1 \cos\alpha_2 & \cos\alpha_1 \sinh\beta \\ \sin\alpha_1 \cos\alpha_2 \cosh\beta + \cos\alpha_1 \sin\alpha_2 & -\sin\alpha_1 \sin\alpha_2 \cosh\beta + \cos\alpha_1 \cos\alpha_2 & \sin\alpha_1 \sinh\beta \\ \cos\alpha_2 \sinh\beta & -\sin\alpha_2 \sinh\beta & \cosh\beta \end{pmatrix}. \tag{5}$$

The generic matrix X is symmetric if, and only if, $\alpha_2 = -\alpha_1$ and then we arrive at the symmetric $SO(2,1)$-matrix:

$$X(\alpha,\beta) := X(\alpha,\beta,-\alpha) = \begin{pmatrix} \cos^2\alpha \cosh\beta + \sin^2\alpha & \cos\alpha \sin\alpha(\cosh\beta - 1) & \cos\alpha \sinh\beta \\ \sin\alpha \cos\alpha(\cosh\beta - 1) & \sin^2\alpha \cosh\beta + \cos^2\alpha & \sin\alpha \sinh\beta \\ \cos\alpha \sinh\beta & \sin\alpha \sinh\beta & \cosh\beta \end{pmatrix} \tag{6}$$

which has the trace $TrX = 1 + 2\cosh\beta \geq 3$ and the eigenvalues $\lambda_1 = 1, \lambda_2 = \cosh\beta - \sinh\beta = e^{-\beta}, \lambda_3 = \cosh\beta + \sinh\beta = e^{\beta}$. For the sake of comparison, we mention that the set $\{Y \in SO(3) \cap Sym(3); TrY = -1\}$ is described by:

$$Y(\alpha,\beta) := \begin{pmatrix} \cos^2\alpha \cos 2\beta - \sin^2\alpha & \cos^2\alpha \sin 2\beta & \sin 2\alpha \cos\beta \\ \cos^2\alpha \sin 2\beta & -\cos^2\alpha \cos 2\beta - \sin^2\alpha & \sin 2\alpha \sin\beta \\ \sin 2\alpha \cos\beta & \sin 2\alpha \sin\beta & -\cos 2\alpha \end{pmatrix}, \lambda_1 = \lambda_2 = -1, \lambda_3 = 1.$$

Let us consider the universal (i.e., independent of α and β) not special and not symmetrical orthogonal matrix:

$$S = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \in SO(3), \quad S^{-1} = S^t = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and then $S^{-1} \cdot X(\alpha, \beta) \cdot S$ is:

$$\begin{pmatrix} \sin^2 \alpha \cosh \beta & \frac{\sin \alpha}{\sqrt{2}}[\cos \alpha(\cosh \beta - 1) - \sinh \beta] & \frac{\sin \alpha}{\sqrt{2}}[\cos \alpha(\cosh \beta - 1) + \sinh \beta] \\ \frac{\sin \alpha}{\sqrt{2}}[\cos \alpha(\cosh \beta - 1) - \sinh \beta] & \frac{1}{2}[(\cos^2 \alpha + 1)\cosh \beta + \sin^2 \alpha] & -\frac{\sin^2 \alpha}{2}(\cosh \beta - 1) \\ \frac{\sin \alpha}{\sqrt{2}}[\cos \alpha(\cosh \beta - 1) + \sinh \beta] & -\frac{\sin^2 \alpha}{2}(\cosh \beta - 1) & \frac{1}{2}[(\cos^2 \alpha + 1)\cosh \beta + \sin^2 \alpha] \end{pmatrix}$$
$$+ \cos \alpha \cdot diag(\cos \alpha, -\sinh \beta, \sinh \beta).$$

Since $X(0, \beta) = A(\beta)$, S gives the diagonal form of $A(\beta)$: $S^t \cdot A(\beta) \cdot S = diag(1, e^{-\beta}, e^{\beta}) = Exp(diag(0, -\beta, \beta))$. The associated spectral curve is:

$$R(\beta, \mu) = (\mu - e^{-\beta})(\mu - 1)(\mu - e^{\beta})$$

is studied as Example 9.3 in ([5], p. 93). $X(\alpha, \beta)$ yields the following class of conics:

Definition 1. *A $SO(2,1)$-conic is a conic depending on Cayley–Klein parameters $(\alpha, \beta) \in S^1 \times \mathbb{R}$ in the form:*

$$\begin{aligned} \Gamma(\alpha, \beta) : [\cos^2 \alpha(\cosh \beta - 1) + 1]x^2 + \sin 2\alpha(\cosh \beta - 1)xy + [\sin^2 \alpha(\cosh \beta - 1) + 1]y^2 \\ +2 \sinh \beta[\cos \alpha x + \sin \alpha y] + \cosh \beta = 0. \end{aligned} \tag{7}$$

As functions we have $f_{\Gamma(\alpha,\beta)}(x, y) = f_{\Gamma(\frac{\pi}{2}-\alpha,\beta)}(y, x)$.

We immediately have:

Proposition 1. *All invariants of $\Gamma(\alpha, \beta)$ depend only on β:*

$$\delta = \delta(\beta) = \cosh \beta \geq 1, I = I(\beta) = \cosh \beta + 1 \geq 2, D = D(\beta) = 2\cosh \beta + 1 \geq 3, \Delta = 1. \tag{8}$$

Hence, any $SO(2,1)$-conic Γ is an imaginary ellipse with eccentricity e_Γ depending, again, only on β:

$$e^2_\Gamma(\beta) = 1 - \cosh \beta \leq 0. \tag{9}$$

The only $SO(2,1)$-circle is the void circle $x^2 + y^2 + 1 = 0$ characterized by $\beta = 0$ and $X(\alpha, 0) = I_3$.

This negative result inspired us to translate the given conic in the following way: if $\Gamma : f(x, y) = 0$ is the initial conic, then it is natural to study its associated indicatrix $f(x, y) = 1$. The conic figuratrix of Finslerian functions were studied in [6]

3. The Translated $SO(2,1)$-Conics

Following the discussion above, we introduce:

Definition 2. *A translated $SO(2,1)$-conic is a conic depending on $(\alpha, \beta) \in S^1 \times \mathbb{R}$ in the form:*

$$\begin{aligned} \tilde{\Gamma}(\alpha, \beta) : [\cos^2 \alpha(\cosh \beta - 1) + 1]x^2 + \sin 2\alpha(\cosh \beta - 1)xy + [\sin^2 \alpha(\cosh \beta - 1) + 1]y^2 \\ +2 \sinh \beta[\cos \alpha x + \sin \alpha y] + \cosh \beta - 1 = 0. \end{aligned} \tag{10}$$

We immediately have:

Proposition 2. *All invariants of $\tilde{\Gamma}(\alpha, \beta)$ depend only on β:*

$$\tilde{\delta}(\beta) = \cosh\beta \geq 1, \quad \tilde{I}(\beta) = \cosh\beta + 1 \geq 2, \quad \tilde{D}(\beta) = -\tilde{\Delta}(\beta) = \cosh\beta - 1 \geq 0. \tag{11}$$

Hence, any translated $SO(2,1)$-conic $\tilde{\Gamma}$ is an ellipse with eccentricity $\tilde{e} = e_{\tilde{\Gamma}}$ depending, again, only on β:

$$\tilde{e}(\beta) = \sqrt{1 - \frac{1}{\cosh\beta}} = \frac{\sqrt{2}\sinh\frac{\beta}{2}}{\sqrt{\cosh\beta}}. \tag{12}$$

The only translated $SO(2,1)$-circle is the only degenerated one of this type, namely the double point $O(x = 0, y = 0)$, characterized by $\beta = 0$.

In fact, excepting this double point, the trigonometrical rotation of angle $\frac{\pi}{2} - \alpha$ with the inverse is:

$$x = (\sin\alpha)\tilde{x} + (\cos\alpha)\tilde{y}, \quad y = (-\cos\alpha)\tilde{x} + (\sin\alpha)\tilde{y} \tag{13}$$

which gives the canonical form, independent of α, and symmetric with respect to the $O\tilde{y}$-axis i.e., invariant with respect to the map $\tilde{x} \to -\tilde{x}$:

$$\tilde{\Gamma}(\beta) : \frac{\cosh\beta}{\cosh\beta - 1}\tilde{x}^2 + \frac{\cosh^2\beta}{\cosh\beta - 1}(\tilde{y} + \tanh\beta)^2 - 1 = 0. \tag{14}$$

Remark 1. *(i) The function $\beta \to \tilde{e}(\beta)$, given by (12), is a bijective one, so, then, we can express the canonical form (14) entirely in terms of $\tilde{e}$:*

$$\tilde{\Gamma}(\tilde{e}) : \frac{\tilde{x}^2}{\tilde{e}^2} + \frac{(\tilde{y} + \tilde{e}\sqrt{2 - \tilde{e}^2})^2}{\tilde{e}^2(1 - \tilde{e}^2)} - 1 = 0. \tag{15}$$

Furthermore: $\tilde{\delta}(\tilde{e}) = \frac{1}{1-\tilde{e}^2}$, $\tilde{I}(\tilde{e}) = \frac{2-\tilde{e}^2}{1-\tilde{e}^2}$, $\tilde{D}(\tilde{e}) = -\tilde{\Delta}(\tilde{e}) = \frac{\tilde{e}^2}{1-\tilde{e}^2}$. Two proper ellipses of canonical equation:

$$\tilde{\Gamma}(\tilde{e}) : \frac{X^2}{\tilde{e}^2} + \frac{Y^2}{\tilde{e}^2(1 - \tilde{e}^2)} - 1 = 0$$

cannot be confocal since the two conditions:

$$\tilde{e}_2^2 - \tilde{e}_1^2 = \lambda, \quad \tilde{e}_2^2(1 - \tilde{e}_2^2) - \tilde{e}_1^2(1 - \tilde{e}_1^2) = \lambda$$

yield an impossible relation $\tilde{e}_1^4 + \tilde{e}_2^4 = 0$. The area enclosed by the ellipse $\tilde{\Gamma}(\tilde{e})$ is:

$$\mathcal{A}(\tilde{e}) = \pi\tilde{a}\tilde{b} = \pi\tilde{e}^2\sqrt{1 - \tilde{e}^2} = \frac{\pi(\cosh\beta - 1)}{(\cosh\beta)^{\frac{3}{2}}}.$$

(ii) In ([7], p. 360) a function is defined, called the canonical conic function, on a set of ellipses with the same eccentricity e as:

$$C_{ellipse}(e) := \frac{1}{8(1 - e^2)^2}\left[\pi\sqrt{1 - e^2} - 2e + 2e^3 - 2\sqrt{1 - e^2}\arcsin e\right]. \tag{16}$$

For our translated $SO(2,1)$-ellipse $\tilde{\Gamma}$ we obtain:

$$C_{ellipse}(\tilde{e}(\beta)) = \frac{(\cosh\beta)^{\frac{3}{2}}}{8}\left[\pi - \frac{2\tilde{e}(\beta)}{\sqrt{\cosh\beta}} - 2\arcsin\sqrt{1 - \frac{1}{\cosh\beta}}\right]. \tag{17}$$

(iii) In [8] a quaternion-inspired (but non-internal) product is considered on the set $(0,1]$:

$$u_1 \odot_c u_2 = \sqrt{\frac{(1 - (u_1 u_2)^2}{u_1^2 + u_2^2}}. \tag{18}$$

The $\odot_c$-square of our eccentricity is:

$$\tilde{e}^2_{\odot_c} = \sqrt{\frac{2\cosh\beta - 1}{2\cosh\beta(\cosh\beta - 1)}}. \tag{19}$$

(iv) A notion specific to hyperbolic geometry is Lobachevsky's angle of parallelism function Π *defined by ([4], p. 141):* $\sin\Pi(x) = \frac{1}{\cosh x}$. *Then, the eccentricity is expressed in terms of* Π*, as:*

$$\tilde{e}(\beta) = |\cos\frac{\Pi(\beta)}{2} - \sin\frac{\Pi(\beta)}{2}|. \tag{20}$$

(v) Recall that, in ([9], p. 16), the motion in Newton's gravitational potential is governed by the effective potential:

$$V_{eff}(r) := V(r) + \frac{L^2}{2r^2} = -\frac{M}{r} + \frac{L^2}{2r^2}$$

and that the bounded motions are ellipses with the eccentricity $e = \frac{kL^2}{M}$ *for a constant* $k \geq 0$. *For* $L > 0$ *and* $k > 0$ *let us call Kepler ellipses these bounded trajectories. Hence, for a translated* $SO(2,1)$*-Kepler ellipse we have* $M = \frac{kL^2\sqrt{\cosh\beta}}{\sqrt{\cosh\beta - 1}}$ *and the effective potential is:*

$$V_{eff}(r) = \frac{L^2}{2r^2}\left(1 - \frac{2kr\sqrt{\cosh\beta}}{\sqrt{\cosh\beta - 1}}\right).$$

(vi) It is well known that the locus of points with orthogonal tangents to the conic Γ *is the Monge (or director) circle with general equation:*

$$\mathcal{C}(\Gamma) : \delta(x^2 + y^2) - 2\begin{vmatrix} r_{12} & r_{10} \\ r_{22} & r_{20} \end{vmatrix} x + 2\begin{vmatrix} r_{11} & r_{10} \\ r_{12} & r_{20} \end{vmatrix} y + \begin{vmatrix} r_{12} & r_{10} \\ r_{10} & r_{00} \end{vmatrix} + \begin{vmatrix} r_{22} & r_{20} \\ r_{20} & r_{00} \end{vmatrix} = 0.$$

For our translated $SO(2,1)$*-conic it results in the circle:*

$$\mathcal{C}(\Gamma) : x^2 + y^2 + 2\tanh\beta(x\cos\alpha + y\sin\alpha) + \frac{(\cosh\beta - 1)^2}{\cosh\beta}(\sin\alpha\cos\alpha + \sin^2\alpha) = \cosh\beta - 1.$$

(vii) The Joukowski map $J : \mathbb{C}^* \to \mathbb{C}$, $J(z) := z + \frac{1}{r}$ *transforms the circle of radius* $r > 1$ *into the ellipse* $E(r)$ *with* $a = r + \frac{1}{r}$ *and* $b = r - \frac{1}{r}$*; hence, the eccentricity of* $E(r)$ *is* $\frac{2r}{r^2+1}$. *The last ratio is equal to* $\tilde{e}(\beta)$ *if, and only if,:*

$$r \in \left\{\frac{\sqrt{\cosh\beta} - 1}{\sqrt{\cosh\beta - 1}}, \frac{\sqrt{\cosh\beta} + 1}{\sqrt{\cosh\beta - 1}}\right\}$$

but only the second value is greater than 1. *Hence:*

$$r = r(\beta) = \frac{\sqrt{\cosh\beta} + 1}{\sqrt{\cosh\beta - 1}} > 1, a = a(\beta) := \frac{2\sqrt{\cosh\beta}}{\sqrt{\cosh\beta - 1}} = \frac{2}{\tilde{e}(\beta)}, b = b(\beta) := \frac{2}{\sqrt{\cosh\beta - 1}}.$$

Example 1. *A special class of ellipses is called self-complementary and given by* $\tilde{e} = \frac{1}{\sqrt{2}} = 0.7071\ldots$*; see details in [4]. It follows* $\cosh\beta = 2$, $\sinh\beta = \sqrt{3}$ *which means* $\beta = \ln(2 + \sqrt{3})$, $\Pi(\beta) = \frac{\pi}{6}$, $r = \sqrt{2} + 1 > 1$ *and the self-complementary ellipse depends on* α*:*

$$\begin{cases} \tilde{\Gamma}^s(\alpha) : (\cos^2\alpha + 1)x^2 + \sin 2\alpha xy + (\sin^2\alpha + 1)y^2 + 2\sqrt{3}[(\cos\alpha)x + (\sin\alpha)y] + 1 = 0, \\ \tilde{\delta} = 2, \tilde{I} = 3, \tilde{D} = -\tilde{\Delta} = 1, \mathcal{C} : x^2 + y^2 + \sqrt{3}(x\cos\alpha + y\sin\alpha) + \frac{\sin\alpha\cos\alpha + \sin^2\alpha}{2} - 1 = 0. \end{cases} \tag{21}$$

In particular:

$$\tilde{\Gamma}^s(0) : 2x^2 + y^2 + 2\sqrt{3}x + 1 = 0, \quad \tilde{\Gamma}^s(\pi) : 2x^2 + y^2 - 2\sqrt{3}x + 1 = 0. \tag{22}$$

Another form of the general ellipse $\tilde{\Gamma}^s$ *is:*

$$\tilde{\Gamma}^s(\alpha) : x^2 + y^2 + [(\cos\alpha)x + (\sin\alpha)y + \sqrt{3}]^2 - 2 = 0. \tag{23}$$

and the canonical form (14)*, equivalently* (15)*, is:*

$$\tilde{\Gamma}^s(\alpha) : 2\tilde{x}^2 + 4\left(\tilde{y} + \frac{\sqrt{3}}{2}\right)^2 - 1 = 0 \tag{24}$$

with center $\tilde{O}(0, -\frac{\sqrt{3}}{2})$ *in* $(\tilde{x}, \tilde{y})$*-coordinates. In the Equation (23) we observe the circle* $\mathcal{C}(O, r = \sqrt{2})$ *and the double line* $d_\alpha : (\cos\alpha)x + (\sin\alpha)y + \sqrt{3} = 0$*, so, in terms of [10], the conic* $\tilde{\Gamma}^s(\alpha)$ *belongs to a bitangent pencil. We point out that a tube of radius* $\beta = \ln(2 + \sqrt{3})$ *appears in the classification result of [11].*

Remark 2. *For* $\beta \neq 0$ *we can divide the Equation* (10) *to* $\cosh\beta - 1$ *and get the limit* $\beta \to +\infty$*. The double line follows:*

$$\tilde{\Gamma}(\alpha, +\infty) : (\cos\alpha)x + (\sin\alpha)y + 1 = 0. \tag{25}$$

The rotation (13) *yields the horizontal line* $\tilde{\Gamma}(\alpha, +\infty) : \tilde{y} = -1$.

Example 2. *The general conic* Γ *is called symmetric if* $r_{11} = r_{22}$*. For* (10) *this means* $\alpha \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$ *and then we have two symmetrically-translated* $SO(2,1)$*-ellipses:*

$$\begin{cases} \tilde{\Gamma}(\frac{\pi}{4}, \beta) : (\cosh\beta + 1)(x^2 + y^2) + 2(\cosh\beta - 1)(xy + 1) + 2\sqrt{2}\sinh\beta(x + y) = 0, \\ \tilde{\Gamma}(\frac{3\pi}{4}, \beta) : (\cosh\beta + 1)(x^2 + y^2) + 2(\cosh\beta - 1)(xy + 1) + 2\sqrt{2}\sinh\beta(x - y) = 0. \end{cases} \tag{26}$$

or in the form of (15)*:*

$$\begin{cases} \tilde{\Gamma}(\frac{\pi}{4}, \tilde{e}) : (2 - \tilde{e}^2)(x^2 + y^2) + 2\tilde{e}^2(xy + 1) + 2\tilde{e}\sqrt{2(2 - \tilde{e}^2)}(x + y) = 0, \\ \tilde{\Gamma}(\frac{3\pi}{4}, \tilde{e}) : (2 - \tilde{e}^2)(x^2 + y^2) + 2\tilde{e}^2(xy + 1) + 2\tilde{e}\sqrt{2(2 - \tilde{e}^2)}(x - y) = 0. \end{cases} \tag{27}$$

In particular, we have the self-complementary symmetrically-translated $SO(2,1)$*-ellipses:*

$$\begin{cases} \tilde{\Gamma}(\frac{\pi}{4}, \tilde{e} = \frac{1}{\sqrt{2}}) : 3(x^2 + y^2) + 2(xy + 1) + 2\sqrt{6}(x + y) = 0, \\ \tilde{\Gamma}(\frac{3\pi}{4}, \tilde{e} = \frac{1}{\sqrt{2}}) : 3(x^2 + y^2) + 2(xy + 1) + 2\sqrt{6}(x - y) = 0. \end{cases} \tag{28}$$

The first ellipse (26) *is symmetric with respect to the first bisectrix, due to invariance with respect to the linear transform* $(x, y) \to (y, x)$.

To the 2×2 matrix $\Gamma \in Sym(2)$ we associate the linear fractional (or Möbius) function $f_\Gamma : \mathbb{R}P^1 = \mathbb{R} \cup \{\infty\} \to \mathbb{R}P^1$:

$$f_\Gamma(t) := \frac{r_{11}t + r_{12}}{r_{12}t + r_{22}} \tag{29}$$

which is an involution if, and only if, $\Gamma \in sl(2, \mathbb{R})$. For our $SO(2,1)$-matrix we obtain:

$$f_\Gamma(t) = \frac{[\cos^2\alpha(\cosh\beta - 1) + 1]t + \sin\alpha\cos\alpha(\cosh\beta - 1)}{\sin\alpha\cos\alpha(\cosh\beta - 1)t + [\sin^2\alpha(\cosh\beta - 1) + 1]} \tag{30}$$

and, since $I = Tr\Gamma = \cosh\beta + 1 \geq 2$, it follows that f_Γ is not an involution for $\beta \neq 0$. Indeed, a direct computation gives the square:

$$\Gamma^2 = \begin{pmatrix} 1+\cos^2\alpha(\cosh^2\beta - 1) & \sin\alpha\cos\alpha\sinh^2\beta \\ \sin\alpha\cos\alpha\sinh^2\beta & 1+\sin^2\alpha(\cosh^2\beta - 1) \end{pmatrix}$$

with $Tr\Gamma^2 = \cosh^2\beta + 1$ and the eigenvalues $\lambda_1 = 1$, $\lambda_2 = \cosh^2\beta$.

Another important issue involves determining the fixed points of f_Γ, which are the solutions of the equation:

$$\sin\alpha\cos\alpha(\cosh\beta - 1)t^2 - \cos 2\alpha(\cosh\beta - 1)t - \sin\alpha\cos\alpha(\cosh\beta - 1) = 0. \quad (31)$$

Supposing that $\beta \neq 0$ we must consider the equation:

$$\sin\alpha\cos\alpha t^2 - \cos 2\alpha t - \sin\alpha\cos\alpha = 0 \quad (32)$$

having the universal (i.e., not dependent on α) discriminant:

$$\Delta_{f_\Gamma} = 1. \quad (33)$$

In conclusion, for $\alpha \in (0, \frac{\pi}{2})$ the function f_Γ has exactly two fixed points:

$$t_1 = -\tan\alpha \neq t_2 = \cot\alpha.$$

4. A Complex Approach to Translated $SO(2,1)$-Conics

The aim of this section is to study the translated $SO(2,1)$-conic $\tilde{\Gamma}$ by using the complex structure of the plane. More precisely, with the usual notation $z = x + iy \in \mathbb{C}$ we derive the complex expression of a general conic Γ:

$$\tilde{\Gamma} : F(z, \bar{z}) := Az^2 + Bz\bar{z} + \bar{A}\bar{z}^2 + Cz + \bar{C}\bar{z} + r_{00} = 0 \quad (34)$$

with:

$$A = \frac{r_{11} - r_{22}}{4} - \frac{r_{12}}{2}i \in \mathbb{C}, \quad 2B = r_{11} + r_{22} = I \in \mathbb{R}, \quad C = r_{10} - r_{20}i \in \mathbb{C}. \quad (35)$$

It follows that the usual rotation performed to eliminate the mixed term xy means to reduce/rotate A in the real line, while the translation which eliminates the term y has a similar meaning with respect to C. The inverse relationship between f and F is:

$$r_{11} = B + 2\Re A, \quad r_{22} = B - 2\Re A, \quad r_{12} = -2\Im A, \quad r_{10} = \Re C, \quad r_{20} = -\Im C \quad (36)$$

with $\Re$ and $\Im$ being, respectively, the real and imaginary parts. The conic is symmetric if, and only if, A is pure imaginary.

The linear invariant I and the quadratic invariant δ are the traces which, respectively, are the determinants of the Hermitian matrix:

$$\Gamma^c = \begin{pmatrix} B & 2\bar{A} \\ 2A & B \end{pmatrix} \quad (37)$$

which is a special one, the entries of the main diagonal being equal; hence, their set is the three-dimensional subspace $Sym(2)$ of the four-dimensional real linear space of 2×2 Hermitian matrices. The square of the eccentricity is:

$$e^2 = \frac{4|A|}{2|A| - Bsign(\Delta)} \quad (38)$$

where $sign$ means the signum function and $|z|$ is the modulus of the complex number z.

For our translated $SO(2,1)$-conic (10) we have the new coefficients, which we call *Hermitian*:

$$\begin{cases} A(\alpha,\beta) = \frac{\cosh\beta-1}{4}e^{-2\alpha i}, \quad B = B(\beta) = \frac{\cosh\beta+1}{2} \geq 1, \quad C(\alpha,\beta) = \sinh\beta e^{-i\alpha}, \\ A(\alpha,\tilde{e}) = \frac{\tilde{e}^2}{4(1-\tilde{e}^2)}e^{-2\alpha i}, \quad B = B(\tilde{e}) = \frac{2-\tilde{e}^2}{2(1-\tilde{e}^2)}, \quad C(\alpha,\tilde{e}) = \frac{\tilde{e}\sqrt{2-\tilde{e}^2}}{1-\tilde{e}^2}e^{-i\alpha}. \end{cases} \tag{39}$$

which satisfy the quadratic relation:

$$8AB = C^2. \tag{40}$$

Hence, a multiplication with $8B$ of the Equation (10) gives a new relation for the translated $SO(2,1)$-conic, expressed only in B and C:

$$\tilde{\Gamma} : (Cz)^2 + 2(2B|z|)^2 + (\bar{C}\bar{z})^2 + 8B(Cz + \bar{C}\bar{z} + 2B - 2) = 0. \tag{41}$$

Example 3. *Both A and C from (39) are real if, and only if, $\alpha \in \mathbb{Z}\pi$ and, then, we obtain the real–translated $SO(2,1)$-conic:*

$$\begin{cases} \tilde{\Gamma}^r = \tilde{\Gamma}(k\pi,\beta) : (\cosh\beta)x^2 + y^2 + 2[(-1)^k\sinh\beta]x + (\cosh\beta - 1) = 0, \quad k \in \{0,1\} \\ A^r = A^r(\beta) = \frac{\cosh\beta-1}{4}, \quad C^r = C^r(k,\beta) = (-1)^k\sinh\beta. \end{cases} \tag{42}$$

The ellipse $\tilde{\Gamma}^r$ is symmetric with respect to the Ox-axis due to the invariance with respect to the symmetry $(x,y) \to (x,-y)$. We have two families:

$$\begin{cases} \tilde{\Gamma}^r_0(\beta) = \tilde{\Gamma}^r(k=0,\beta) : (\cosh\beta)x^2 + y^2 + 2\sinh\beta x + (\cosh\beta - 1) = 0, \\ \tilde{\Gamma}^r_1(\beta) = \tilde{\Gamma}^r(k=1,\beta) : (\cosh\beta)x^2 + y^2 - 2\sinh\beta x + (\cosh\beta - 1) = 0. \end{cases} \tag{43}$$

With $\cosh\beta = 2$ and $\sinh\beta = \sqrt{3}$ we re-obtain the translated $SO(2,1)$-self-complementary ellipses of the previous section: $\tilde{\Gamma}^r_0(self) = \tilde{\Gamma}^s(0)$, $\tilde{\Gamma}^r_1(self) = \tilde{\Gamma}^s(\pi)$.
The triple $(x,y,z) = (A^r, B, C^r)(\beta)$ is a curve on the infinite elliptic cone:

$$\mathcal{C} : 8xy = z^2. \tag{44}$$

Considering (x,y,z) as homogeneous coordinates corresponding to $X = \frac{x}{z}$, $Y = \frac{y}{z}$, it follows that the equilateral hyperbola is: $XY = \frac{1}{8}$.

Example 4. *Inspired by the expression of A and C let us call the unit complex number:*

$$z_{\tilde{\Gamma}} := e^{-\alpha i} \in S^1 \tag{45}$$

as being the affix of $\tilde{\Gamma}$. Then $z_{\tilde{\Gamma}}$ belongs to $\tilde{\Gamma}$ if, and only if,:

$$(\cosh\beta - 1)(\cos 2\alpha)^2 + 2\sinh\beta\cos 2\alpha + \cosh\beta = 0 \tag{46}$$

with the unique solution:

$$\cos 2\alpha = \frac{\sqrt{\cosh\beta - 1} - \sinh\beta}{\cosh\beta - 1} = \frac{\sqrt{1-\tilde{e}^2} - \sqrt{2-\tilde{e}^2}}{\tilde{e}} < 0. \tag{47}$$

For the example of self-complementary ellipses we obtain:

$$\alpha = \frac{1}{2}\arccos(1-\sqrt{3}) = 68.55^\circ. \tag{48}$$

To any $\hat{z} \in \mathbb{C}^*$ we can associate two binary quadratic forms with null determinant:

$$F^{\pm}_{\hat{z}}(x,y) := (|\hat{z}| + \Re\hat{z})x^2 \pm 2(\Im\hat{z})xy + (|\hat{z}| - \Re\hat{z})y^2. \tag{49}$$

For our affix we have:

$$F^{\pm}_{\tilde{\Gamma}}(x,y) = F^{\pm}(\alpha)(x,y) = (1+\cos\alpha)x^2 \pm 2\sin\alpha xy + (1-\cos\alpha)y^2 = 2\Big(\cos\frac{\alpha}{2}x \pm \sin\frac{\alpha}{2}y\Big)^2. \quad (50)$$

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