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An Adaptive Multiple-Asset Portfolio Strategy with User-Specified Risk Tolerance

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Abstract: We improve the traditional simple moving average strategy by incorporating an investor-specific risk tolerance into the method. We then propose a multiasset generalized moving average crossover (MGMA) strategy. The MGMA strategies allocate wealth between risky assets and risk-free assets in an adaptive manner, with the risk tolerance specified by an investor. We derive the expected log-utility of wealth, which allows us to estimate the optimal allocation parameters. The algorithm using our MGMA strategy is also presented. As the multiple risky assets can have different variability levels and could have various degrees of correlations, this trading strategy is evaluated on both simulated data and global high-frequency exchange-traded fund (ETF) data. It is shown that the MGMA strategies could significantly increase both the investor's expected utility of wealth and the investor's expected wealth.

Keywords: algorithm; high-frequency; exchange-traded fund; moving average; technical analysis; strategy

MSC: 62P05; 91G10

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1. Introduction

This paper provides an optimal and adaptive portfolio allocation strategy based on the technical analysis of a diversified investment portfolio. The investment portfolio often contains more than one risky asset to avert possible significant loss. Portfolio allocation is an important strategy for investors and traders, and they are interested in an optimal allocation when they have enough capital to invest in more than two assets. They might want to allocate the wealth not only between one risk-free asset and another risky asset but also among different risky assets. The common approach is to assign equal weights when allocating the wealth among the risky assets, which, however, is not always optimal. We believe that an allocation amount should be a function of the investor's specified risk tolerance, and this consideration could lead to a more favorable investment outcome. We focus on finding optimal trading strategies based on technical analysis, such as moving averages for building a multiple-asset portfolio. We propose a multiasset generalized moving average crossover (MGMA) strategy. This strategy can allocate wealth not only between one risky asset and another risk-free asset but also among different risky assets, with the risk tolerance specified by the investor. It can also increase both the investor's expected utility of wealth and the investor's expected wealth.

Technical analysis is widely adopted by investors in practice. The empirical evidence, including the predicted performance of a stock return, demonstrates the usefulness of technical analysis (see [1–4]). Among all the technical analysis methods, the moving average strategy is the simplest and most popular trading rule. Ref. [1] appeared to be the first article to provide strong evidence of profitability by using a moving average technique in analyzing daily Dow Jones Industrial Average (DJIA) data. The moving average strategy in technical analysis follows an all-or-nothing investment strategy: when a *buy* signal is triggered by the moving average crossover (MA) strategy, the investor should allocate all of his/her wealth to the stocks of interest; when a *sell* signal is triggered by MA, the investor should allocate none of the wealth into the stocks by selling all the current holdings. Thus, this simple moving average strategy suffers from a well-known drawback, since its allocation is always either 100% or 0%. Ref. [2] provided further evidence based on different time series data obtained from financial markets. These studies have generated further research interest on moving average strategies. However, most of the studies have been focused on validating the strategy using different data sets. The conclusions are mixed and inconclusive (see [5–9]). An increasing number of studies are focused on the predictive power of a moving average technique (see [10–15]). Ref. [16] provided the first theoretical analysis for this simple moving average crossover strategy. Their study focuses on how technical analysis such as this moving average strategy can add value to commonly used allocation rules that invest fixed proportions of wealth in stocks. Ref. [17] provided a general equilibrium model for the MA strategy and argued MA signals in their model are helpful for investors in pricing the asset. In addition, more studies focus on the applications of MA strategy. Ref. [18] combined MA signals to create a factor to explain various term momentum. Ref. [19] combined MA signals to estimate equity returns. Ref. [20] suggested that the effectiveness of technical analysis depends on the level of PIN (probability of informed trading).

Most asset allocation studies focus on finding an optimal portfolio choice under different modeling processes (see [21–25]). Refs. [26,27] incorporated technical indicators into the portfolio construction problem. However, they do not study the optimal allocation in the context of using technical analysis strategies for a multiasset portfolio. In addition, few studies reconciled the technical indicators with a portfolio selection policy that guides investment decisions in a multiasset setting. Ref. [28] bridged the gap by devising a portfolio strategy in which optimal weights are directly parameterized as a function of multiple trend-following signals. However, there is no extension to a multiple-asset setting with an optimal allocation as the objective. Recently, an increasing number of studies are focused on using machine learning for portfolio allocation strategies. Ref. [29] used machine learning to find the optimal portfolio weights between the market index and the risk-free asset and found that a portfolio allocation strategy employing machine learning to reward–risk time in the market gave significant improvements in investor utility and ratios. They use random forest and update the weights on only a monthly basis or over a relatively long period of time, while our methods are instantaneous and much less computationally intensive, since they do not require building a forest or any tree pruning. Ref. [30] developed optimization algorithms with machine learning techniques and assessed the risk characteristics of a large commodity portfolio. Their objective is the prediction of risk measures instead of expected returns. Ref. [31] proposed a novel two-stage method for well-diversified portfolio construction based on stock return prediction using machine learning. They use the mean-variance model for portfolio construction. The challenge of the mean-variance model is its well-known sensitivity to the change in mean return.

We derive the expected log-utility of wealth, which provides the mechanism for the optimal allocation estimates. This provides the theoretical foundations of our strategy. The algorithm using our MGMA strategy is also developed for the application of our methods. Furthermore, motivated by recent years' studies on higher-frequency information on financial market forecasting (see [32,33]), we tested the proposed MGMA strategy using the daily second-level exchange-traded fund in the North American market. In general, the Sharpe index is appropriate for evaluating a trading strategy. We did not feel that this is the best measure for our method, since the expected return of our strategy is clearly not normally distributed, with the median much closer to the minimum than that from the maximum, as suggested in our simulation study. This suggested that there is a long tail on the right-hand side associated with the case of high profitability. The large variation above the median level suggested a desirable significant chance of making a great profit. If we use the standard deviation as the denominator, it will dilute the advantage of our method, since it is unduly penalized by the possibility of extreme profitability.

The rest of the paper is organized as follows. We introduce the general model with the MGMA strategy in Section 2. We present the main theoretical results in Section 3. We provide an investment algorithm for the multiasset portfolio in Section 4. We carry out simulation studies in Section 5. We present the results of real data analysis in Section 6. We conclude the article in Section 7.

2. The Model and the MGMA Strategy

Suppose that there are $n+1$ assets in a financial market. For convenience, we assume that the first one is risk-free, e.g., a cash or money market account with a constant interest rate of r. The other n assets are risky ones, which, for example, can be stocks or indices representing the aggregate equity market. A multiasset portfolio contains n risky assets. The wealth can be allocated not only between the risk-free asset and one risky asset but also among risky assets.

We follow [34] to define a general model for a multiasset portfolio with multiple predictive variables. Suppose that the price of the risk-free asset P_t^f at any time t satisfies that

$$dP_t^f = rP_t^f dt. \tag{1}$$

Moreover, suppose that there are q predictive variables that can be accurately observed at continuous times. Then, the vector of n risky asset prices $\boldsymbol{p}_t$ at any time t satisfies that

$$d\boldsymbol{p}_t = \mathrm{diag}(\boldsymbol{p}_t)\{(\boldsymbol{\alpha} + U\boldsymbol{x}_t)dt + V_p d\boldsymbol{b}_t\}, \tag{2}$$

and the dynamics of the vector of q predictive variables $\boldsymbol{x}_t$ satisfies that

$$d\boldsymbol{x}_t = (\boldsymbol{\beta} + \Theta\boldsymbol{x}_t)dt + V_x d\boldsymbol{z}_t, \tag{3}$$

where

$$\boldsymbol{p}_t = \begin{pmatrix} p_{1t} \\ \vdots \\ p_{nt} \end{pmatrix}, \quad \mathrm{diag}(\boldsymbol{p}_t) = \begin{pmatrix} p_{1t} & 0 & \dots & 0 \\ 0 & p_{2t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_{nt} \end{pmatrix}, \quad \boldsymbol{x}_t = \begin{pmatrix} x_{1t} \\ \vdots \\ x_{qt} \end{pmatrix},$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & \dots & u_{1q} \\ \vdots & \ddots & \vdots \\ u_{n1} & \dots & u_{nq} \end{pmatrix}, \quad V_p = \begin{pmatrix} v_{11}^p & \dots & v_{1n}^p \\ \vdots & \ddots & \vdots \\ v_{n1}^p & \dots & v_{nn}^p \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_q \end{pmatrix},$$

and

$$\Theta = \begin{pmatrix} \theta_{11} & \dots & \theta_{1q} \\ \vdots & \ddots & \vdots \\ \theta_{q1} & \dots & \theta_{qq} \end{pmatrix}, \quad V_x = \begin{pmatrix} v_{11}^x & \dots & v_{1q}^x \\ \vdots & \ddots & \vdots \\ v_{q1}^x & \dots & v_{qq}^x \end{pmatrix}, \quad \boldsymbol{b}_t = \begin{pmatrix} b_{1t} \\ \vdots \\ b_{nt} \end{pmatrix}, \quad \boldsymbol{z}_t = \begin{pmatrix} z_{1t} \\ \vdots \\ z_{qt} \end{pmatrix}.$$

The vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and matrices U, Θ, V_p, and V_x are all unknown. The vectors $\boldsymbol{b}_t$ and $\boldsymbol{z}_t$ are a multidimensional standard Brownian motion, such that

$$\mathrm{Var}(\boldsymbol{b}_t) = tI_n, \quad \mathrm{Var}(\boldsymbol{z}_t) = tI_q, \quad \mathrm{Corr}(\boldsymbol{b}_t, \boldsymbol{z}_t) = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1q} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \cdots & \rho_{nq} \end{pmatrix} \triangleq V_{\boldsymbol{bz}},$$

where I_ℓ denotes an $\ell \times \ell$ identity matrix. Each predictive variable x_{it} is assumed to be a stationary process for $t \geq 0$, $i = 1, \dots, q$. In order to ensure x_{it} is a mean-reverting process,

Θ is assumed to be symmetric negative definite, i.e., $\Theta = \Theta^\top$ and $\boldsymbol{a}^\top \Theta \boldsymbol{a} < 0$ for any $\boldsymbol{a} \in \mathbb{R}^q$.

We first recall the original MA strategy. We define some notations for the kth stock ($k = 1, \ldots, n$) in the market. Let p_{kt} be the real stock price at time t and y_{kt} be its log-transformed stock price, i.e.,

$$y_{kt} = \log p_{kt}. \tag{4}$$

Denote a lag or lookback period by $h > 0$. In view of [16], a continuous time version of the moving average of this log-transformed stock price at any time t is defined as

$$m_{kt}^{(h)} = \frac{1}{h} \int_{t-h}^{t} y_{ku} du, \tag{5}$$

i.e., the average log-transformed stock price over time period $[t-h, t]$. Let $m_{kt}^{(s,l)}$ be the difference between $m_{kt}^{(s)}$ and $m_{kt}^{(l)}$, where $s > 0$ is a short-term lookback period and l is a long-term lookback period ($l > s$), i.e.,

$$m_{kt}^{(s,l)} = m_{kt}^{(s)} - m_{kt}^{(l)}. \tag{6}$$

Define $\tilde{\Omega}_i$ as

$$\tilde{\Omega}_i = \begin{cases} (-\infty, 0), & \text{if } i = 1, \\ [0, \infty), & \text{if } i = 2. \end{cases} \tag{7}$$

Denote the MA strategy for an n-asset portfolio by $\boldsymbol{\tau}_t = (\tau_{1t}, \ldots, \tau_{nt})^\top$. Then, $\boldsymbol{\tau}_t$ for a single-asset portfolio, i.e., $n = 1$, is defined as

$$\tau_{1t} = \begin{cases} 0, & \text{if } m_{1t}^{(s,l)} \in \tilde{\Omega}_1, \\ 1, & \text{if } m_{1t}^{(s,l)} \in \tilde{\Omega}_2, \end{cases} \tag{8}$$

and the MA strategy $\boldsymbol{\tau}_t$ for a two-asset portfolio, i.e., $n = 2$, is defined in Table 1. We follow the common approach to assign equal weights when there is more than one investment signal.

Table 1. MA strategy $\mathcal{T}_t = (\tau_{1t}, \tau_{2t})^\top$ for a two-asset portfolio.

$(\boldsymbol{\tau_{1t}}, \boldsymbol{\tau_{2t}})$	$\boldsymbol{m_{2t}^{(s,l)} \in \tilde{\Omega}_1}$	$\boldsymbol{m_{2t}^{(s,l)} \in \tilde{\Omega}_2}$
$m_{1t}^{(s,l)} \in \tilde{\Omega}_1$	$(0, 0)$	$(0, 1)$
$m_{1t}^{(s,l)} \in \tilde{\Omega}_2$	$(1, 0)$	$(0.5, 0.5)$

We now define the MGMA strategy. The key is to introduce an investor's specific risk tolerance $\epsilon > 0$ into the moving average strategy. Define Ω_i as

$$\Omega_i = \begin{cases} (-\infty, -\epsilon), & \text{if } i = 1, \\ [-\epsilon, 0), & \text{if } i = 2, \\ [0, \epsilon], & \text{if } i = 3, \\ (\epsilon, \infty), & \text{if } i = 4. \end{cases} \tag{9}$$

Let $\boldsymbol{p}_t$ be the vector of n stock prices, $\boldsymbol{y}_t$ be the vector of n log-transformed stock prices, and $\boldsymbol{m}_t^{(s,l)}$ be the vector of n differences between the moving averages, i.e.,

$$\boldsymbol{p}_t = \begin{pmatrix} p_{1t} \\ \vdots \\ p_{nt} \end{pmatrix}, \quad \boldsymbol{y}_t = \begin{pmatrix} y_{1t} \\ \vdots \\ y_{nt} \end{pmatrix}, \quad \boldsymbol{m}_t^{(s,l)} = \begin{pmatrix} m_{1t}^{(s,l)} \\ \vdots \\ m_{nt}^{(s,l)} \end{pmatrix}.$$

Let $\Xi = \{1, 2, 3, 4\}$ and $i_k \in \Xi$, $k = 1, \ldots, n$. Define $\Omega_{(i_1,\ldots,i_n)}$ as

$$\Omega_{(i_1,\ldots,i_n)} = \Omega_{i_1} \times \ldots \times \Omega_{i_n}. \tag{10}$$

Let η_{kt} be the MGMA strategy for the kth risky asset in a multiasset portfolio. Let $\delta_{k,(i_1,\ldots,i_n)}$ be the asset allocation parameter for the kth risky asset in the multiasset portfolio. Suppose that $\boldsymbol{\eta}_t$ is the vector based on the MGMA strategy and $\boldsymbol{\delta}_{(i_1,\ldots,i_n)}$ is the vector of the n asset allocation parameters, i.e.,

$$\boldsymbol{\eta}_t = \begin{pmatrix} \eta_{1t} \\ \vdots \\ \eta_{nt} \end{pmatrix}, \quad \boldsymbol{\delta}_{(i_1,\ldots,i_n)} = \begin{pmatrix} \delta_{1,(i_1,\ldots,i_n)} \\ \vdots \\ \delta_{n,(i_1,\ldots,i_n)} \end{pmatrix}.$$

Then, for $t \geq l$, we define the MGMA strategy $\boldsymbol{\eta}_t$ as

$$\boldsymbol{\eta}_t = \sum_{i_1 \in d,\ldots,i_n \in d} \boldsymbol{\delta}_{(i_1,\ldots,i_n)} \mathbf{1}_{\Omega_{(i_1,\ldots,i_n)}}\left(\boldsymbol{m}_t^{(s,l)}\right), \tag{11}$$

where $\mathbf{1}_{\Omega_{(i_1,\ldots,i_n)}}\left(\boldsymbol{m}_t^{(s,l)}\right)$ is an indicator function such that

$$\mathbf{1}_{\Omega_{(i_1,\ldots,i_n)}}\left(\boldsymbol{m}_t^{(s,l)}\right) = \begin{cases} 1, & \text{if } \boldsymbol{m}_t^{(s,l)} \in \Omega_{(i_1,\ldots,i_n)}, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

To ensure $\boldsymbol{\eta}_t$ is well-defined, for $t < l$, we define $\boldsymbol{\eta}_t$ as a constant vector $\boldsymbol{\lambda}$, i.e., $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^\top$, where λ_k is a constant for $k = 1, \ldots, n$ and $\sum_{k=1}^{n} \lambda_k \leq 1$.

The MGMA strategy $\boldsymbol{\eta}_t$ is a market-timing strategy that allocates wealth not only between one risk-free asset and one risky asset but also among risky assets, with the risk tolerance specified by the investor. Note that the MA strategy is a special case of the MGMA strategy. Theoretically speaking, the asset allocation parameter $\delta_{k,(i_1,\ldots,i_n)}$ can be any number which is interpreted as a long portion of stocks if $\delta_{k,(i_1,\ldots,i_n)} \geq 0$ and a short portion of stocks if $\delta_{k,(i_1,\ldots,i_n)} < 0$. Therefore, there are $n4^n$ parameters for the MGMA strategy on a multiasset portfolio which contains n risky assets.

We give some examples of the MGMA strategy. The MGMA strategy $\boldsymbol{\eta}_t^\top = (\eta_{1t})$ for a signal-asset portfolio $(n = 1)$ is defined as

$$\eta_{1t} = \begin{cases} \delta_{1,(1)}, & \text{if } m_{1t}^{(s,l)} \in \Omega_1, \\ \delta_{1,(2)}, & \text{if } m_{1t}^{(s,l)} \in \Omega_2, \\ \delta_{1,(3)}, & \text{if } m_{1t}^{(s,l)} \in \Omega_3, \\ \delta_{1,(4)}, & \text{if } m_{1t}^{(s,l)} \in \Omega_4, \end{cases} \tag{13}$$

where $\delta_{1,(1)} = 0$ and $\delta_{1,(4)} = 1$. The MGMA strategy $\boldsymbol{\eta}_t^\top = (\eta_{1t}, \eta_{2t})$ for a two-asset portfolio $(n = 2)$ consists of 32 parameters, which is defined in Table 2.

Table 2. MGMA strategy $\boldsymbol{\eta}_t = (\eta_{1t}, \eta_{2t})^\top$ for a two-asset portfolio.

(η_{1t}, η_{2t})	$m_{2t}^{(s,l)} \in \Omega_1$	$m_{2t}^{(s,l)} \in \Omega_2$	$m_{2t}^{(s,l)} \in \Omega_3$	$m_{2t}^{(s,l)} \in \Omega_4$
$m_{1t}^{(s,l)} \in \Omega_1$	$\left(\delta_{1,(1,1)}, \delta_{2,(1,1)}\right)$	$\left(\delta_{1,(1,2)}, \delta_{2,(1,2)}\right)$	$\left(\delta_{1,(1,3)}, \delta_{2,(1,3)}\right)$	$\left(\delta_{1,(1,4)}, \delta_{2,(1,4)}\right)$
$m_{1t}^{(s,l)} \in \Omega_2$	$\left(\delta_{1,(2,1)}, \delta_{2,(2,1)}\right)$	$\left(\delta_{1,(2,2)}, \delta_{2,(2,2)}\right)$	$\left(\delta_{1,(2,3)}, \delta_{2,(2,3)}\right)$	$\left(\delta_{1,(2,4)}, \delta_{2,(2,4)}\right)$
$m_{1t}^{(s,l)} \in \Omega_3$	$\left(\delta_{1,(3,1)}, \delta_{2,(3,1)}\right)$	$\left(\delta_{1,(3,2)}, \delta_{2,(3,2)}\right)$	$\left(\delta_{1,(3,3)}, \delta_{2,(3,3)}\right)$	$\left(\delta_{1,(3,4)}, \delta_{2,(3,4)}\right)$
$m_{1t}^{(s,l)} \in \Omega_4$	$\left(\delta_{1,(4,1)}, \delta_{2,(4,1)}\right)$	$\left(\delta_{1,(4,2)}, \delta_{2,(4,2)}\right)$	$\left(\delta_{1,(4,3)}, \delta_{2,(4,3)}\right)$	$\left(\delta_{1,(4,4)}, \delta_{2,(4,4)}\right)$

It is obvious that the MGMA strategy is very complex, even for a two-asset portfolio. In light of [6], we consider no-borrowing and no-short-sale constrains, i.e., $\delta_{k,(i_1,\ldots,i_n)} \in [0,1]$ and $\sum_{k=1}^{n} \delta_{k,(i_1,\ldots,i_n)} \leq 1$. We use these constrains to reduce parameters to five, i.e., $a_1, a_2, a_3, a_4, a_5 \in [0,1]$, as in Table 3, for implementation.

Table 3. Simplified MGMA strategy $\boldsymbol{\eta}_t = (\eta_{1t}, \eta_{2t})^\top$ for a two-asset portfolio.

(η_{1t}, η_{2t})	$\boldsymbol{m}_{2t}^{(s,l)} \in \boldsymbol{\Omega}_1$	$\boldsymbol{m}_{2t}^{(s,l)} \in \boldsymbol{\Omega}_2$	$\boldsymbol{m}_{2t}^{(s,l)} \in \boldsymbol{\Omega}_3$	$\boldsymbol{m}_{2t}^{(s,l)} \in \boldsymbol{\Omega}_4$
$\boldsymbol{m}_{1t}^{(s,l)} \in \boldsymbol{\Omega}_1$	$(0,0)$	$(0,a_1)$	$(0,a_2)$	$(0,1)$
$\boldsymbol{m}_{1t}^{(s,l)} \in \boldsymbol{\Omega}_2$	$(a_3,0)$	$(a_3[1-a_1(1-a_5)], a_1[1-a_3a_5])$	$(a_3[1-a_2(1-a_5)], a_2[1-a_3a_5])$	$(a_3a_5, 1-a_3a_5)$
$\boldsymbol{m}_{1t}^{(s,l)} \in \boldsymbol{\Omega}_3$	$(a_4,0)$	$(a_4[1-a_1(1-a_5)], a_1[1-a_4a_5])$	$(a_4[1-a_2(1-a_5)], a_2[1-a_4a_5])$	$(a_4a_5, 1-a_4a_5)$
$\boldsymbol{m}_{1t}^{(s,l)} \in \boldsymbol{\Omega}_4$	$(1,0)$	$(1-a_1(1-a_5), a_1(1-a_5))$	$(1-a_2(1-a_5), a_2(1-a_5))$	$(a_5, 1-a_5)$

The MGMA strategy from an asset allocation perspective now becomes finding the optimal $\boldsymbol{\eta}_t$ that maximizes the investor's expected log-utility of wealth

$$\max_{\boldsymbol{\eta}_t} E(\log w_T), \tag{14}$$

subject to a budget constraint

$$\frac{dw_t}{w_t} = rdt + \boldsymbol{\eta}_t^\top(\boldsymbol{\alpha} + U\boldsymbol{x}_t - r\mathbf{1}_n)dt + \boldsymbol{\eta}_t^\top V_p d\boldsymbol{b}_t, \tag{15}$$

given an initial wealth w_0 for a multiasset portfolio, a constant rate of interest r, and an investment horizon T, where $\mathbf{1}_n = (1,\ldots,1)^\top$.

3. The Analytic Results

To focus on the framework and the MGMA strategy, we only present the main analytic results in this section. The lemmas used to derive the formulas are presented in Appendix A.

In order to find an optimal $\boldsymbol{\eta}_t$, we need to derive the investor's expected log-utility of wealth $E(\log w_T)$. To derive it, we need to find the joint distribution of $\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s,l)}\right)$. Let $\boldsymbol{\mu_x}$ be the expectation of $\boldsymbol{x}_t$, $\boldsymbol{\mu_m}$ be the expectation of $\boldsymbol{m}_t^{(s,l)}$, $\Sigma_{\boldsymbol{x}}$ be the variance–covariance matrix of $\boldsymbol{x}_t$, $\Sigma_{\boldsymbol{m}}$ be the variance–covariance of $\boldsymbol{m}_t^{(s,l)}$, and $\Delta_{\boldsymbol{xm}}$ be the covariance matrix between $\boldsymbol{x}_t$ and $\boldsymbol{m}_t^{(s,l)}$. Based on Lemmas A2, A4, A9 and A12 in Appendix A, it is derived that $\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s,l)}\right)$ has a multivariate normal distribution, i.e.,

$$\begin{pmatrix} \boldsymbol{x}_t \\ \boldsymbol{m}_t^{(s,l)} \end{pmatrix} \sim \mathrm{MN}\left[\begin{pmatrix} \boldsymbol{\mu_x} \\ \boldsymbol{\mu_m} \end{pmatrix}, \begin{pmatrix} \Sigma_{\boldsymbol{x}} & \Delta_{\boldsymbol{xm}} \\ \Delta_{\boldsymbol{xm}}^\top & \Sigma_{\boldsymbol{m}} \end{pmatrix}\right], \tag{16}$$

and

$$\begin{aligned}
\boldsymbol{\mu_x} &= -\Theta^{-1}\boldsymbol{\beta}, \\
\boldsymbol{\mu_m} &= \frac{1}{2}(l-s)\left[\boldsymbol{\alpha} - U\Theta^{-1}\boldsymbol{\beta}\right], \\
\Sigma_{\boldsymbol{x}} &= -\frac{1}{2}V_x\Theta^{-1}V_x^\top, \\
\Delta_{\boldsymbol{xm}} &= \left[\frac{1}{s}\left(I_q - e^{s\Theta}\right) - \frac{1}{l}\left(I_q - e^{l\Theta}\right)\right]Q_3^\top, \\
\Sigma_{\boldsymbol{m}} &= Q_4(s,s) - Q_4(s,l) - Q_4^\top(s,l) + Q_4(l,l),
\end{aligned} \tag{17}$$

where

$$Q_1 = -\frac{1}{2}V_xV_x^\top\Theta^{-3}, \quad Q_2 = \Theta^{-2}V_xV_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_3 = UQ_1 + Q_2^\top,$$

and

$$
\begin{aligned}
Q_4(s,l) = & \frac{1}{sl}U\left\{\Theta^{-1}\left[sI_q - \Theta^{-1}\left(e^{l\Theta} - e^{(l-s)\Theta}\right)\right] + \frac{1}{6}\left(3l^2s + s^3\right)\Theta - \frac{1}{2}s^2 I_q\right\}Q_3^\top \\
& + \frac{1}{sl}Q_3\left\{\Theta^{-1}\left[sI_q + \Theta^{-1}\left(I_q - e^{s\Theta}\right)\right] + \frac{1}{6}\left(3l^2s + s^3\right)\Theta - \frac{1}{2}\left(2ls - s^2\right)I_q\right\}U^\top \\
& - \frac{1}{sl}\left[\frac{1}{6}\left(3l^2s + s^3\right)\right]V_p V_p^\top.
\end{aligned}
$$

Note that the distribution of $\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s,l)}\right)$ does not depend on t. From the multivariate normal distribution of $\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s,l)}\right)^\top$, we have

$$
E\left(\boldsymbol{x}_t \mid \boldsymbol{m}_t^{(s,l)}\right) = \boldsymbol{\mu_x} + \Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\left(\boldsymbol{m}_t^{(s,l)} - \boldsymbol{\mu_m}\right). \tag{18}
$$

Denote $\Sigma_{\boldsymbol{m}}$, $\boldsymbol{\mu_m}$ and $\boldsymbol{\sigma_m}$ as

$$
\Sigma_{\boldsymbol{m}} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \dots & \sigma_n^2 \end{pmatrix}, \quad \boldsymbol{\mu_m} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \quad \boldsymbol{\sigma_m} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}.
$$

Thus, we have

$$
\boldsymbol{Z}_{R_{\boldsymbol{m}}} = \begin{pmatrix} \sigma_1^{-1} & 0 & \dots & 0 \\ 0 & \sigma_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n^{-1} \end{pmatrix}\left(\boldsymbol{m}_t^{(s,l)} - \boldsymbol{\mu_m}\right) \sim \mathrm{MN}(\mathbf{0}_n, R_{\boldsymbol{m}}), \tag{19}
$$

where $\mathbf{0}_n^\top = (0, \dots, 0)$ and $R_{\boldsymbol{m}}$ is the correlation matrix for the vector $\boldsymbol{m}_t^{(s,l)}$, i.e.,

$$
R_{\boldsymbol{m}} = \begin{pmatrix} 1 & \frac{\sigma_{12}}{\sigma_1\sigma_2} & \dots & \frac{\sigma_{1n}}{\sigma_1\sigma_n} \\ \frac{\sigma_{12}}{\sigma_1\sigma_2} & 1 & \dots & \frac{\sigma_{2n}}{\sigma_2\sigma_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{1n}}{\sigma_1\sigma_n} & \frac{\sigma_{2n}}{\sigma_2\sigma_n} & \dots & 1 \end{pmatrix}.
$$

Denote the probability density function of $\boldsymbol{Z}_{R_{\boldsymbol{m}}} \sim \mathrm{MN}(\mathbf{0}_n, R_{\boldsymbol{m}})$ by $\phi_{R_{\boldsymbol{m}}}(m_1, \dots, m_n)$. Let $H = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ be any hyper-rectangle. For a simple presentation, let $\mathcal{A}_i = [a_i, b_i]$, $i = 1, \dots, n$. We define

$$
\Phi_{R_{\boldsymbol{m}}}(H) = Pr(\boldsymbol{Z}_{R_{\boldsymbol{m}}} \in H) = \int_{a_1}^{b_1} dm_1 \int_{a_2}^{b_2} dm_2 \dots \int_{a_n}^{b_n} \phi_{R_{\boldsymbol{m}}}(m_1, \dots, m_n) dm_n, \tag{20}
$$

and

$$
\boldsymbol{\Psi}_{R_{\boldsymbol{m}}}(H) = \int_{a_1}^{b_1} dm_1 \int_{a_2}^{b_2} dm_2 \dots \int_{a_n}^{b_n} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \phi_{R_{\boldsymbol{m}}}(m_1, \dots, m_n) dm_n. \tag{21}
$$

Similar to Equation (9), we define $\Omega_i(\mu,\sigma)$ as

$$\Omega_i(\mu,\sigma)=\frac{\Omega_i-\mu}{\sigma}=\begin{cases}\left(-\infty,-\frac{\epsilon+\mu}{\sigma}\right), & \text{if } i=1,\\ \left[-\frac{\epsilon+\mu}{\sigma},-\frac{\mu}{\sigma}\right), & \text{if } i=2,\\ \left[-\frac{\mu}{\sigma},\frac{\epsilon-\mu}{\sigma}\right], & \text{if } i=3,\\ \left(\frac{\epsilon-\mu}{\sigma},\infty\right), & \text{if } i=4.\end{cases} \tag{22}$$

We also define

$$\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})=\Omega_{i_1}(\mu_1,\sigma_1)\times\ldots\times\Omega_{i_n}(\mu_n,\sigma_n), \tag{23}$$

where $i_k\in\{1,2,3,4\}$, $k=1,\ldots,n$.

Given an initial wealth w_0, a constant rate of interest r, and an investment horizon T, let $\epsilon>0$ be the investor-specified risk tolerance, $\delta_{(i_1,\ldots,i_n)}$ be the vector of n asset allocation parameters, and $\boldsymbol{\eta}_t$ be the vector-based multiasset generalized moving average crossover (MGMA) strategy. For the MGMA strategy, we have the following propositions.

Proposition 1. *The expectation of $\boldsymbol{\eta}_t^\top$ is independent of time t and given by*

$$E\left(\boldsymbol{\eta}_t^\top\right)=\sum_{i_1\in d,\ldots,i_n\in d}\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top\Phi_{R_{\boldsymbol{m}}}\left(\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right).$$

Proof. By Equations (11), (19), (22) and (23), we have

$$\begin{aligned}E\left(\boldsymbol{\eta}_t^\top\right)&=\sum_{i_1\in d,\ldots,i_n\in d}\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top P\left(\boldsymbol{m}_t^{(s,l)}\in\Omega_{(i_1,\ldots,i_n)}\right)\\&=\sum_{i_1\in d,\ldots,i_n\in d}\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top P\left(\boldsymbol{Z}_{R_{\boldsymbol{m}}}\in\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right)\\&=\sum_{i_1\in d,\ldots,i_n\in d}\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top\Phi_{R_{\boldsymbol{m}}}\left(\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right),\end{aligned}$$

which concludes the proposition. □

Proposition 2. *The expectation of $\boldsymbol{\eta}_t^\top V_p V_p^\top\boldsymbol{\eta}_t$ is independent of time t and given by*

$$E\left(\boldsymbol{\eta}_t^\top V_pV_p^\top\boldsymbol{\eta}_t\right)=\sum_{i_1\in d,\ldots,i_n\in d}\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top V_pV_p^\top\boldsymbol{\delta}_{(i_1,\ldots,i_n)}\Phi_{R_{\boldsymbol{m}}}\left(\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right).$$

Proof. In light of Equations (11), (19), (22) and (23), we have

$$\begin{aligned}E\left(\boldsymbol{\eta}_t^\top V_pV_p^\top\boldsymbol{\eta}_t\right)&=\sum_{i_1\in d,\ldots,i_n\in d}\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top V_pV_p^\top\boldsymbol{\delta}_{(i_1,\ldots,i_n)}P\left(\boldsymbol{m}_t^{(s,l)}\in\Omega_{(i_1,\ldots,i_n)}\right)\\&=\sum_{i_1\in d,\ldots,i_n\in d}\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top V_pV_p^\top\boldsymbol{\delta}_{(i_1,\ldots,i_n)}P\left(\boldsymbol{Z}_{R_{\boldsymbol{m}}}\in\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right)\\&=\sum_{i_1\in d,\ldots,i_n\in d}\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top V_pV_p^\top\boldsymbol{\delta}_{(i_1,\ldots,i_n)}\Phi_{R_{\boldsymbol{m}}}\left(\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right),\end{aligned}$$

which concludes the proposition. □

Proposition 3. *The expectation of $\boldsymbol{\eta}_t^\top U\boldsymbol{x}_t$ is independent of time t and given by*

$$E\left(\boldsymbol{\eta}_t^\top U\boldsymbol{x}_t\right) = \sum_{i_1\in d,\dots,i_n\in d} \boldsymbol{\delta}^\top_{(i_1,\dots,i_n)} U\boldsymbol{\mu_x}\Phi_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right)$$
$$+ \sum_{i_1\in d,\dots,i_n\in d} \boldsymbol{\delta}^\top_{(i_1,\dots,i_n)} U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\begin{pmatrix}\sigma_1 & 0 & \dots & 0\\ 0 & \sigma_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sigma_n\end{pmatrix}\boldsymbol{\Psi}_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right).$$

Proof. By (18) and Proposition 1, we have

$$E\left(\boldsymbol{\eta}_t^\top U\boldsymbol{x}_t\right) = E\left(\boldsymbol{\eta}_t^\top\right)U\left(\boldsymbol{\mu_x} - \Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\boldsymbol{\mu_m}\right) + E\left(\boldsymbol{\eta}_t^\top U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\boldsymbol{m}_t^{(s,l)}\right),$$

where

$$\begin{aligned}
&E\left(\boldsymbol{\eta}_t^\top\right)U\left(\boldsymbol{\mu_x} - \Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\boldsymbol{\mu_m}\right)\\
&= \sum_{i_1\in d,\dots,i_n\in d} \boldsymbol{\delta}^\top_{(i_1,\dots,i_n)} U\left(\boldsymbol{\mu_x} - \Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\boldsymbol{\mu_m}\right)\Phi_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right)\\
&= \sum_{i_1\in d,\dots,i_n\in d} \boldsymbol{\delta}^\top_{(i_1,\dots,i_n)} U\boldsymbol{\mu_x}\Phi_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right)\\
&- \sum_{i_1\in d,\dots,i_n\in d} \boldsymbol{\delta}^\top_{(i_1,\dots,i_n)} U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\boldsymbol{\mu_m}\Phi_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right),
\end{aligned}$$

and based on Equations (11) and (19),

$$E\left(\boldsymbol{\eta}_t^\top U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\boldsymbol{m}_t^{(s,l)}\right) = \sum_{i_1\in d,\dots,i_n\in d} \boldsymbol{\delta}^\top_{(i_1,\dots,i_n)} U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}E\left(\mathbf{1}_{\Omega_{(i_1,\dots,i_n)}}\left(\boldsymbol{m}_t^{(s,l)}\right)\boldsymbol{m}_t^{(s,l)}\right),$$

where

$$\begin{aligned}
&E\left(\mathbf{1}_{\Omega_{(i_1,\dots,i_n)}}\left(\boldsymbol{m}_t^{(s,l)}\right)\boldsymbol{m}_t^{(s,l)}\right)\\
&= E\left(\mathbf{1}_{\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})}(\boldsymbol{Z}_{R_m})(\begin{pmatrix}\sigma_1 & 0 & \dots & 0\\ 0 & \sigma_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sigma_n\end{pmatrix}\boldsymbol{Z}_{R_m} + \boldsymbol{\mu_m})\right)\\
&= \begin{pmatrix}\sigma_1 & 0 & \dots & 0\\ 0 & \sigma_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sigma_n\end{pmatrix}\boldsymbol{\Psi}_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right) + \boldsymbol{\mu_m}\Phi_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
&E\left(\boldsymbol{\eta}_t^\top U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\boldsymbol{m}_t^{(s,l)}\right)\\
&= \sum_{i_1\in d,\dots,i_n\in d} \boldsymbol{\delta}^\top_{(i_1,\dots,i_n)} U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\begin{pmatrix}\sigma_1 & 0 & \dots & 0\\ 0 & \sigma_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sigma_n\end{pmatrix}\boldsymbol{\Psi}_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right)\\
&+ \sum_{i_1\in d,\dots,i_n\in d} \boldsymbol{\delta}^\top_{(i_1,\dots,i_n)} U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\boldsymbol{\mu_m}\Phi_{R_m}\left(\Omega_{(i_1,\dots,i_n)}(\boldsymbol{\mu_m},\boldsymbol{\sigma_m})\right).
\end{aligned}$$

□

Proposition 4. *Let $\boldsymbol{\lambda}$ be a constant vector for MGMA strategy $\boldsymbol{\eta}_t$ when $t < l$, i.e., $\boldsymbol{\lambda}^\top = (\lambda_1, \ldots, \lambda_n)$, where λ_k is a constant for $k = 1, \ldots, n$ and $\sum_{k=1}^n \lambda_k \leq 1$. Let $\epsilon > 0$ be the investor-specified risk tolerance, then the investor's expected log-utility of wealth at the end of investment period T is*

$$E(\log w_T) = a_6 + (T-l)\left[E\left(\boldsymbol{\eta}_t^\top\right)(\boldsymbol{\alpha} - r\mathbf{1}_n) - \frac{1}{2}E\left(\boldsymbol{\eta}_t^\top V_p V_p^\top \boldsymbol{\eta}_t\right) + E\left(\boldsymbol{\eta}_t^\top U\boldsymbol{x}_t\right)\right], \tag{24}$$

where $\mathbf{1}_n^\top = (1, \ldots, 1)$ and a_6 is a constant depending on l, i.e.,

$$a_6 = \log w_0 + rT + l\left[\boldsymbol{\lambda}^\top(\boldsymbol{\alpha} - r\mathbf{1}_n) - \frac{1}{2}\left(\boldsymbol{\lambda}^\top V_p V_p^\top \boldsymbol{\lambda}\right) - \boldsymbol{\lambda}^\top U\Theta^{-1}\boldsymbol{\beta}\right].$$

By Propositions 1–3, Equation (24) can be rewritten as

$$\begin{aligned}
&E(\log w_T) \\
&= a_6 + \sum_{i_1 \in d, \ldots, i_n \in d} (T-l)\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top \Phi_{R_{\boldsymbol{m}}}\left(\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m}, \boldsymbol{\sigma_m})\right)(\boldsymbol{\alpha} - r\mathbf{1}_n) \\
&- \sum_{i_1 \in d, \ldots, i_n \in d} \frac{1}{2}(T-l)\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top V_p V_p^\top \boldsymbol{\delta}_{(i_1,\ldots,i_n)} \Phi_{R_{\boldsymbol{m}}}\left(\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m}, \boldsymbol{\sigma_m})\right) \\
&+ \sum_{i_1 \in d, \ldots, i_n \in d} (T-l)\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top U\boldsymbol{\mu_x}\Phi_{R_{\boldsymbol{m}}}\left(\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m}, \boldsymbol{\sigma_m})\right) \\
&+ \sum_{i_1 \in d, \ldots, i_n \in d} (T-l)\boldsymbol{\delta}_{(i_1,\ldots,i_n)}^\top U\Delta_{\boldsymbol{xm}}\Sigma_{\boldsymbol{m}}^{-1}\begin{pmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n \end{pmatrix}\boldsymbol{\Psi}_{R_{\boldsymbol{m}}}\left(\Omega_{(i_1,\ldots,i_n)}(\boldsymbol{\mu_m}, \boldsymbol{\sigma_m})\right).
\end{aligned} \tag{25}$$

Proof. Based on Equations (1) and (2), the budget constraint for the multi-asset portfolio follows

$$\frac{dw_t}{w_t} = \boldsymbol{\eta}_t^\top(\mathrm{diag}(\boldsymbol{p}_t))^{-1}d\boldsymbol{p}_t + \left(1 - \boldsymbol{\eta}_t^\top\mathbf{1}_n\right)rdt = rdt + \boldsymbol{\eta}_t^\top(\boldsymbol{\alpha} + U\boldsymbol{x}_t - r\mathbf{1}_n)dt + \boldsymbol{\eta}_t^\top V_p d\boldsymbol{b}_t,$$

Since $(dt)^2 = o(dt), dtd\boldsymbol{b}_t = o(dt)$ and $d\boldsymbol{b}_t d\boldsymbol{b}_t^\top = dtI_n$, where $\mathbf{0}_n^\top = (0, \ldots, 0)$ and I_n is the identity matrix,

$$\left(\frac{dw_t}{w_t}\right)^2 = \left(\boldsymbol{\eta}_t^\top V_p d\boldsymbol{b}_t\right)^2 = \boldsymbol{\eta}_t^\top V_p d\boldsymbol{b}_t d\boldsymbol{b}_t^\top V_p^\top \boldsymbol{\eta}_t = \boldsymbol{\eta}_t^\top V_p I_n dt V_p^\top \boldsymbol{\eta}_t = \boldsymbol{\eta}_t^\top V_p V_p^\top \boldsymbol{\eta}_t dt,$$

which implies that

$$d(\log w_t) = \frac{dw_t}{w_t} - \frac{1}{2}\left(\frac{dw_t}{w_t}\right)^2 = \left(r + \boldsymbol{\eta}_t^\top(\boldsymbol{\alpha} + U\boldsymbol{x}_t - r\mathbf{1}_n) - \frac{1}{2}\boldsymbol{\eta}_t^\top V_p V_p^\top \boldsymbol{\eta}_t\right)dt + \boldsymbol{\eta}_t^\top V_p d\boldsymbol{b}_t,$$

By Equation (11) with $T \geq l$,

$$\begin{aligned}
\log w_T = \log w_0 + rT + \boldsymbol{\lambda}^\top(\boldsymbol{\alpha} - r\mathbf{1}_n)l + \int_l^T \boldsymbol{\eta}_t^\top(\boldsymbol{\alpha} - r\mathbf{1}_n)dt + \int_0^l \boldsymbol{\lambda}^\top U\boldsymbol{x}_t dt \\
+ \int_l^T \boldsymbol{\eta}_t^\top U\boldsymbol{x}_t dt - \frac{1}{2}\left(\boldsymbol{\lambda}^T V_p V_p^\top \boldsymbol{\lambda}\right)l - \frac{1}{2}\int_l^T \boldsymbol{\eta}_t^\top V_p V_p^\top \boldsymbol{\eta}_t dt + \int_0^T \boldsymbol{\eta}_t^\top V_p d\boldsymbol{b}_t,
\end{aligned}$$

which implies that

$$
\begin{aligned}
&E(\log w_T) = \log w_0 + rT \\
&+ \boldsymbol{\lambda}^\top(\boldsymbol{\alpha} - r\mathbf{1}_n)l - \frac{1}{2}\left(\boldsymbol{\lambda}^\top V_p V_p^\top \boldsymbol{\lambda}\right)l + \int_0^l \boldsymbol{\lambda}^\top U E(\boldsymbol{x}_t)dt + \int_l^T E\left(\boldsymbol{\eta}_t^\top\right)dt(\boldsymbol{\alpha} - r\mathbf{1}_n) \\
&- \frac{1}{2}\int_l^T E\left(\boldsymbol{\eta}_t^\top V_p V_p^\top \boldsymbol{\eta}_t\right)dt + \int_l^T E\left(\boldsymbol{\eta}_t^\top U\boldsymbol{x}_t\right)dt + \int_0^T E\left(\boldsymbol{\eta}_t^\top\right)V_p E(d\boldsymbol{b}_t).
\end{aligned}
$$

By Propositions 1–3, we note that $E\left(\boldsymbol{\eta}_t^\top\right)$, $E\left(\boldsymbol{\eta}_t^\top V_p V_p^\top \boldsymbol{\eta}_t\right)$ and $E\left(\boldsymbol{\eta}_t^\top U\boldsymbol{x}_t\right)$ are all independent of time t. Since $E(d\boldsymbol{b}_t) = \mathbf{0}_n$ and $E(\boldsymbol{x}_t) = -\Theta^{-1}\boldsymbol{\beta}$ by Lemma A2 in Appendix A, we derive

$$
E(\log w_T) = a_6 + (T - l)\left[E\left(\boldsymbol{\eta}_t^\top\right)(\boldsymbol{\alpha} - r\mathbf{1}_n) - \frac{1}{2}E\left(\boldsymbol{\eta}_t^\top V_p V_p^\top \boldsymbol{\eta}_t\right) + E\left(\boldsymbol{\eta}_t^\top U\boldsymbol{x}_t\right)\right],
$$

where $\mathbf{1}_n^\top = (1, \ldots, 1)$, and a_6 is a constant depending on l, i.e.,

$$
a_6 = \log w_0 + rT + l\left[\boldsymbol{\lambda}^\top(\boldsymbol{\alpha} - r\mathbf{1}_n) - \frac{1}{2}\left(\boldsymbol{\lambda}^\top V_p V_p^\top \boldsymbol{\lambda}\right) - \boldsymbol{\lambda}^\top U\Theta^{-1}\boldsymbol{\beta}\right],
$$

then Equation (24) is proved. □

Now, we can calculate optimal estimates of the asset allocation parameters for the MGMA strategy by maximizing $E(\log w_T)$ with respect to asset allocation parameters $\boldsymbol{\delta}_{(i_1,\ldots,i_n)}$. Suppose that the investor-specific risk tolerance $\epsilon = \epsilon_0$, then for kth stock, we solve following equation for optimal estimates $\delta^*_{k,(i_1,\ldots,i_n)}$, i.e.,

$$
\left.\frac{\partial E(\log w_T)}{\partial \delta_{k,(i_1,\ldots,i_n)}}\right|_{\epsilon=\epsilon_0, \delta_{k,(i_1,\ldots,i_n)}=\delta^*_{k,(i_1,\ldots,i_n)}} = 0. \tag{26}
$$

We also restrict $\delta_{k,(i_1,\ldots,i_n)} \in [0, 1]$, which means there are no-borrowing and no-short-sale constrains; then, the optimal estimates of $\boldsymbol{\delta}_{(i_1,\ldots,i_n)}$ are

$$
\boldsymbol{\delta}^*_{(i_1,\ldots,i_n)} = \begin{pmatrix} \delta^*_{1,(i_1,\ldots,i_n)} \\ \vdots \\ \delta^*_{n,(i_1,\ldots,i_n)} \end{pmatrix}. \tag{27}
$$

Note that the optimal estimates $\boldsymbol{\delta}^*_{(i_1,\ldots,i_n)}$ are functions of the investor-specified risk tolerance ϵ. The results illustrate that the MGMA is a better investment strategy compared with the MA strategy for the multiasset portfolio, because it has a higher expected utility of wealth for the investor.

4. An Investment Algorithm for Multiasset Portfolio

We propose an investment algorithm using the MGMA strategy for a multiasset portfolio. The algorithm is tested on simulation data and real data to evaluate the performance of the MGMA strategy. The algorithm contains the following steps:

Step 1. Set investment parameters w_0, r and T, ϵ, $\boldsymbol{\lambda}$, s and l.
Step 2. Compute model parameters $\boldsymbol{\mu}_{\boldsymbol{x}}$, $\boldsymbol{\mu}_{\boldsymbol{m}}$, $\Sigma_{\boldsymbol{x}}$, $\Delta_{\boldsymbol{xm}}$, $\Sigma_{\boldsymbol{m}}$, $\boldsymbol{\sigma}_{\boldsymbol{m}}$ and $R_{\boldsymbol{m}}$.
Step 3. Compute $\boldsymbol{\delta}^*_{(i_1,\ldots,i_n)}$ and $E(\log w_T)$.
Step 4. Calculate $\boldsymbol{y}_t$, $\boldsymbol{m}_t^{(s)}$, $\boldsymbol{m}_t^{(l)}$ and $\boldsymbol{m}_t^{(s,l)}$.
Step 5. Allocate the wealth among n risky assets and one risk-free asset according to $\boldsymbol{\delta}^*_{(i_1,\ldots,i_n)}$.
Step 6. The holding risky assets are sold at the end of the investment horizon T.

5. Simulation Studies

We present several numerical examples based on a simulated two-asset portfolio and a simulated three-asset portfolio. Motivated by recent research on higher-frequency information on financial market forecasting, we simulated a daily second-level two-asset portfolio and a daily second-level three-asset portfolio. The investment algorithm is tested and compared with the MA strategy as the benchmark.

5.1. Simulation Results for Two-Asset Portfolio

The simulated two-asset portfolio data are generated using the parameters below.

$$\boldsymbol{\beta} = \begin{pmatrix} 0.0100 \\ 0.6542 \end{pmatrix}, \quad \Theta = \begin{pmatrix} -0.253 & 0 \\ 0 & 0.1438 \end{pmatrix}, \quad V_x = \begin{pmatrix} 0.012 & 0 \\ 0 & 0.3356 \end{pmatrix},$$

and

$$\boldsymbol{\alpha} = \begin{pmatrix} 0.0310 \\ -0.0742 \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad U = \begin{pmatrix} 2.0720 & 0.0150 \\ 0.0235 & 0.0181 \end{pmatrix},$$

and

$$V_p = \begin{pmatrix} 0.195 & 0.100 \\ 0.100 & 0.495 \end{pmatrix}, \quad V_{\boldsymbol{bz}} = \begin{pmatrix} -0.073 & 0.0050 \\ 0.001 & -0.9083 \end{pmatrix},$$

The simulation runs 1000 times. Each time series contains 97,500 observed points.

The simulation studies are performed under two scenarios ($s = 5$ and $l = 30$ vs. $s = 5$ and $l = 10$). We set initial wealth $w_0 = 1,000,000$ and interest rate $r = 0$. Under each scenario, we test the MGMA strategy based on $\epsilon = 0.005, 0.01$ and 0.05 and compare it with the MA strategy. The MGMA strategy performance results are provided in Tables 4 and 5. We first report the theoretical expected log-utility of wealth $E(\log W_T)^*$ based on Equation (25) with the percentage increase in the expected log-utility of wealth compared with the MA strategy. We then report numerical summaries to calculate from the simulation results, including the expected log-utility of wealth $E(\log W_T)$, the expected wealth $E(W_T)$, the expected return on asset ratio $E(ROA\ \%)$, etc.

In the rest of this paper, $\hat{a}_1^*$, $\hat{a}_2^*$, $\hat{a}_3^*$, $\hat{a}_4^*$, and $\hat{a}_5^*$ respectively stand for the estimates of a_1, a_2, a_3, a_4, and a_5, and $E(TRANS\ \#)$ denotes the expected number of transactions.

Table 4. MGMA strategy performance summary for scenario 1 on the simulated two-asset portfolio (1000 run; $s = 5$; $l = 30$).

	MA	$MGMA$ ($\epsilon = 0.005$)	$MGMA$ ($\epsilon = 0.01$)	$MGMA$ ($\epsilon = 0.05$)
$\hat{a}_1$	na	0.03322929	0.03320624	0.033097262
$\hat{a}_2$	na	0.03281636	0.03236742	0.028855955
$\hat{a}_3$	na	1	1	1
$\hat{a}_4$	na	1	1	1
$\hat{a}_5$	na	1	1	1
$E(\log W_T)^*$	13.847667	13.896920	13.905998	13.945960
$\Delta\%\ E(\log W_T)^*$	na	0.36%	0.42%	0.71%
$E(\log W_T)$	13.794745	13.837687	13.845006	13.866623
$\log E(W_T)$	13.829619	13.861375	13.866273	13.886932
$E(W_T)$	1,014,208	1,046,932	1,052,073	1,074,033
$E(ROA\ \%)$	1.42%	4.69%	5.21%	7.40%
$SD(W_T)$	283,524	233,340	223,184	221,949
$MAX(W_T)$	3,058,106	2,273,877	2,186,479	2,176,345
$MIN(W_T)$	541,872	510,691	539,512	629,311
$MEDIAN(W_T)$	959,840	1,010,483	1,017,564	1,050,674
$E(TRANS\ \#)$	25	68	67	52

Table 5. MGMA strategy performance summary for scenario 2 on simulated two-asset portfolio (1000 run; $s = 5$; $l = 10$).

	MA	$MGMA$ $(\epsilon = 0.005)$	$MGMA$ $(\epsilon = 0.01)$	$MGMA$ $(\epsilon = 0.05)$
$\hat{a}_1$	na	0.034401876	0.034369298	0.072961411
$\hat{a}_2$	na	0.033698847	0.032947417	0.113863840
$\hat{a}_3$	na	1	1	1
$\hat{a}_4$	na	1	1	1
$\hat{a}_5$	na	1	1	1
$E(\log W_T)^*$	13.843826	13.916324	13.938047	13.966346
$\Delta\%\ E(\log W_T)^*$	na	0.52%	0.68%	0.89%
$E(\log W_T)$	13.786813	13.847683	13.863594	13.876244
$\log E(W_T)$	13.825506	13.870658	13.885702	13.901359
$E(W_T)$	1,010,045	1,056,696	1,072,714	1,089,641
$E(ROA\ \%)$	1.00%	5.67%	7.27%	8.96%
$SD(W_T)$	298,598	231,436	231,640	250,256
$MAX(W_T)$	3,633,894	2,354,587	2,202,291	2,467,362
$MIN(W_T)$	387,856	535,477	543,935	513,012
$MEDIAN(W_T)$	965,330	1,029,843	1,043,520	1,058,534
$E(TRANS\ \#)$	56	157	144	68

Note that the MGMA strategy for a two-asset portfolio can increase the investor's expected log-utility of wealth and also increase the investor's expected wealth and the expected return on asset ratio from the simulation results. Under scenario 1, the expected log-utility of wealth increases in the range of 0.36% to 0.71%. The expected return ratio increases from benchmark return 1.42% to 4.69%, 5.21% and 7.40%, respectively. Under scenario 2, the expected log-utility of wealth increases in the range 0.52% to 0.89%. The expected return ratio increases from benchmark return 1.00% to 5.67%, 7.27% and 8.96%, respectively.

5.2. Simulation Results for Three-Asset Portfolio

The simulated three-asset portfolio time series data are generated using the parameters below.

$$\boldsymbol{\beta} = \begin{pmatrix} 0.010 \\ 0.065 \\ 0.185 \end{pmatrix}, \quad \Theta = \begin{pmatrix} -0.253 & 0 & 0 \\ 0 & -1.1438 & 0 \\ 0 & 0 & -1.89 \end{pmatrix}, \quad V_x = \begin{pmatrix} 0.012 & 0 & 0 \\ 0 & 0.3356 & 0 \\ 0 & 0 & 0.134 \end{pmatrix},$$

and

$$\boldsymbol{\alpha} = \begin{pmatrix} 0.0310 \\ -0.0742 \\ -0.0945 \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1.2720 & 0.0150 & 1.500 \\ 1.0235 & 1.0181 & 0.512 \\ 0.5000 & 0.0200 & 0.145 \end{pmatrix},$$

and

$$V_p = \begin{pmatrix} 0.195 & 0.100 & 0.200 \\ 0.100 & 0.495 & 0.345 \\ 0.200 & 0.345 & 0.271 \end{pmatrix}, \quad V_{bz} = \begin{pmatrix} -0.073 & 0.001 & -0.10 \\ 0.001 & -0.108 & 0.09 \\ -0.050 & 0.040 & 0.10 \end{pmatrix},$$

The simulation runs 1000 times. Each time series contains 97,500 observed points.

The simulation studies are performed under two scenarios ($s = 5$ and $l = 30$ vs. $s = 5$ and $l = 10$). We set initial wealth $w_0 = 1,000,000$ and interest rate $r = 0$. Under each scenario, we test the MGMA strategy based on $\epsilon = 0.001$ and 0.0005 and compare it with the MA strategy. The MGMA strategy performance results are provided in Tables 6 and 7. We first report the theoretical expected log-utility of wealth $E(\log W_T)^*$ based on Equation (25) and the percentage increase in the expected log-utility of wealth compared with the MA strategy. We then report numerical summaries to calculate from the simulation results, including the expected log-utility of wealth $E(\log W_T)$, the expected wealth $E(W_T)$, the expected return on asset ratio $E(ROA\ \%)$, the standard deviation of wealth $SD(W_T)$, the maximum of wealth $MAX(W_T)$, the minimum of wealth $MIN(W_T)$, the median of wealth

$MEDIAN(W_T)$ and the expected number of transactions $E(TRANS\ \#)$. By using no-borrowing and no-short-sale constraints, we can reduce the parameters of MGMA strategy for a three-asset portfolio from 192 to 37 for implementation. For easy illustration, we do not report the optimal asset allocation parameters in the simulation summary tables.

Table 6. MGMA strategy performance summary for scenario 1 on the simulated three-asset portfolio (1000 run; $s = 5$; $l = 30$).

	MA	$MGMA(\epsilon = 0.001)$	$MGMA(\epsilon = 0.005)$
$E(\log W_T)^*$	13.832331	13.911899	13.918328
$\Delta\%\ E(\log W_T)^*$	na	0.58%	0.62%
$E(\log W_T)$	13.785474	13.863884	13.868901
$\log E(W_T)$	13.834971	13.900779	13.904022
$E(W_T)$	1,019,651	1,089,009	1,092,547
$E(ROA\ \%)$	1.97%	8.90%	9.25%
$SD(W_T)$	339,954	308,718	303,463
$MAX(W_T)$	2,926,612	2,764,480	2,696,956
$MIN(W_T)$	366,703	421,502	458,013
$MEDIAN(W_T)$	946,339	1,038,858	1,043,233
$E(TRANS\ \#)$	36	101	101

Table 7. MGMA strategy performance summary for scenario 2 on simulated three-asset portfolio (1000 run; $s = 5$; $l = 10$).

	MA	$MGMA(\epsilon = 0.001)$	$MGMA(\epsilon = 0.005)$
$E(\log W_T)^*$	13.823737	13.914620	13.934576
$\Delta\%\ E(\log W_T)^*$	na	0.66%	0.80%
$E(\log W_T)$	13.771318	13.857019	13.875335
$\log E(W_T)$	13.827102	13.894495	13.909474
$E(W_T)$	1,011,659	1,082,187	1,098,520
$E(ROA\ \%)$	1.17%	8.22%	9.85%
$SD(W_T)$	356,262	307,575	297,192
$MAX(W_T)$	3,360,435	2,643,002	2,794,930
$MIN(W_T)$	419,986	479,525	504,198
$MEDIAN(W_T)$	937,137	1,025,448	1,053,388
$E(TRANS\ \#)$	82	240	237

Note that the MGMA strategy for a three-asset portfolio can increase the investor's expected log-utility of wealth and also increase the investor's expected wealth and the expected return on asset ratio from the simulation results. Under scenario 1, the expected log-utility of wealth increases in the range 0.58% to 0.62%. The expected return ratio increases from a benchmark return of 1.97% to 8.90% and 9.25%, respectively. Under scenario 2, the expected log-utility of wealth increases in the range of 0.66% to 0.80%. The expected return ratio increases from a benchmark return of 1.17% to 8.22% and 9.85%, respectively.

6. Real Data Applications

We present several real data analyses based on high-frequency exchange-traded fund (ETF) data. The investment algorithm is tested and compared with the benchmark. The simplified MGMA strategy for a two-asset portfolio in Table 3 is used. The MA strategy for a two-asset portfolio in Table 1 is used as the benchmark strategy.

6.1. An Algorithm to Estimate Model Parameters

In order to use the investment algorithm for a multiasset portfolio on real data, we need to estimate model parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, U, Θ, V_p, V_x, and $V_{\boldsymbol{bz}}$. There is no such algorithm in the literature due to complex model settings. We propose an algorithm to fill the gap.

Without loss generality, we describe the algorithm by using a general model for a two-asset portfolio. Based on Equation (2),

$$\frac{dp_{1t}}{p_{1t}} = (\alpha_1 + u_{11}x_{1t} + u_{12}x_{2t})dt + e^p_{1t},$$
$$\frac{dp_{2t}}{p_{2t}} = (\alpha_2 + u_{21}x_{1t} + u_{22}x_{2t})dt + e^p_{2t},$$

and Equation (3),

$$dx_{1t} = (\beta_1 + \theta_{11}x_{1t} + \theta_{12}x_{2t})dt + e^x_{1t},$$
$$dx_{2t} = (\beta_2 + \theta_{21}x_{1t} + \theta_{22}x_{2t})dt + e^x_{2t},$$

where

$$\boldsymbol{e}^p_t = \begin{pmatrix} e^p_{1t} \\ e^p_{2t} \end{pmatrix} = \begin{pmatrix} v^p_{11}db_{1t} + v^p_{12}db_{2t} \\ v^p_{21}db_{1t} + v^p_{22}db_{2t} \end{pmatrix} = V_p d\boldsymbol{b}_t \sim \mathrm{MN}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad dtV_pV_p^\top\right],$$

and

$$\boldsymbol{e}^x_t = \begin{pmatrix} e^x_{1t} \\ e^x_{2t} \end{pmatrix} = \begin{pmatrix} v^x_{11}dz_{1t} + v^x_{12}dz_{2t} \\ v^x_{21}dz_{1t} + v^x_{22}dz_{2t} \end{pmatrix} = V_x d\boldsymbol{z}_t \sim \mathrm{MN}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad dtV_xV_x^\top\right].$$

Let $dtV_pV_p^\top \triangleq \Sigma_{e^p}$; then, it is easy to check the log-likelihood function for $\boldsymbol{e}^p_t$ is

$$l\left(\Sigma_{e^p} \mid \boldsymbol{e}^p_t\right) = -\frac{T}{2}\log|\Sigma_{e^p}| - \frac{1}{2}\sum_{t=1}^{T}\left\{\left(\boldsymbol{e}^p_t\right)^\top \Sigma^{-1}_{e^p}\boldsymbol{e}^p_t\right\} - T\log(2\pi),$$

and let $dtV_xV_x^\top \triangleq \Sigma_{e^x} = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$, then the log-likelihood function for $\boldsymbol{e}^x_t$ is

$$\begin{aligned} l(\Sigma_{e^x} \mid \boldsymbol{e}^x_t) &= -\frac{T}{2}\log|\Sigma_{e^x}| - \frac{1}{2}\sum_{t=1}^{T}\left\{(\boldsymbol{e}^x_t)^\top \Sigma^{-1}_{e^x}\boldsymbol{e}^x_t\right\} - T\log(2\pi) \\ &= -\frac{T}{2}\log(v_1v_2) - \frac{1}{2}\sum_{t=1}^{T}\left\{\frac{(e^x_{1t})^2}{v_1} + \frac{(e^x_{2t})^2}{v_2}\right\} - T\log(2\pi). \end{aligned}$$

Let $\mathrm{Cov}(d\boldsymbol{b}_t, d\boldsymbol{z}_t) \triangleq \Sigma_{\boldsymbol{bz}}$; it is also easy to verify that

$$\Sigma_{\boldsymbol{bz}} = dtV_{\boldsymbol{bz}}.$$

Then, the algorithm contains the following steps:

Step 1. Given a dt, calculate dp_{1t}, dp_{2t}, dx_{1t} and dx_{2t} (for $t > 1$) based on the historical time series.

Step 2. Use least square estimation method to estimate parameters $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{u}_{11}$, $\hat{u}_{12}$, $\hat{u}_{21}$, and $\hat{u}_{22}$ by minimizing

$$\sum_{i=1}^{2}\sum_{t=2}^{T}\left[\frac{dp_{it}}{p_{it}} - (\alpha_i + u_{i1}x_{1t} + u_{i2}x_{2t})dt\right]^2.$$

Step 3. Let $\Theta = \mathrm{diag}(\theta_{11}, \theta_{22})$, and use least square estimation method to estimate parameters $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\theta}_{11}$, and $\hat{\theta}_{22}$ by minimizing

$$\sum_{i=1}^{2}\sum_{t=2}^{T}[dx_{it} - (\beta_i + \theta_{i1}x_{1t} + \theta_{i2}x_{2t})dt]^2.$$

Step 4. Calculate $\hat{\boldsymbol{e}}^p_t$ and $\hat{\boldsymbol{e}}^x_t$ from p_{1t}, p_{2t}, x_{1t} and x_{2t}, $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{u}_{11}$, $\hat{u}_{12}$, $\hat{u}_{21}$, $\hat{u}_{22}$, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\theta}_{11}$, $\hat{\theta}_{12}$, $\hat{\theta}_{21}$, and $\hat{\theta}_{22}$.

Step 5. Use maximum likelihood estimation method and set

$$\frac{\partial l\left(\Sigma_{e^p} \mid \hat{\boldsymbol{e}}_t^p\right)}{\partial \Sigma_{e^p}} = 0,$$

to estimate $\widehat{\Sigma}_{e^p}$ from $\hat{\boldsymbol{e}}_t^p$. Since $\widehat{V}_p = \left(\frac{1}{dt}\widehat{\Sigma}_{e^p}\right)^{\frac{1}{2}}$, we can estimate parameters $\hat{v}_{11}^p$, $\hat{v}_{12}^p$, $\hat{v}_{21}^p$, and $\hat{v}_{22}^p$.

Step 6. Use maximum likelihood estimation method and set

$$\frac{\partial l(\Sigma_{e^x} \mid \hat{\boldsymbol{e}}_t^x)}{\partial \Sigma_{e^x}} = 0,$$

to estimate $\widehat{\Sigma}_{e^x}$, $\hat{v}_{11}^x$, $\hat{v}_{12}^x$, $\hat{v}_{21}^x$ and $\hat{v}_{22}^x$ from $\hat{\boldsymbol{e}}_t^x$.

Step 7. Calculate $d\hat{\boldsymbol{b}}_t$ and $d\hat{\boldsymbol{z}}_t$ from $\widehat{V}_p$, $\widehat{V}_x$, $\hat{\boldsymbol{e}}_t^p$, and $\hat{\boldsymbol{e}}_t^x$.

Step 8. Calculate $\widehat{\Sigma}_{\boldsymbol{bz}}$ from $d\hat{\boldsymbol{b}}_t$ and $d\hat{\boldsymbol{z}}_t$. Then, the estimated parameter is $\widehat{V}_{\boldsymbol{bz}} = \frac{1}{dt}\widehat{\Sigma}_{\boldsymbol{bz}}$.

6.2. Case 1: MGMA Strategy on High-Frequency Exchange-Traded Fund in North American Market

We use PowerShares QQQ Trust Series 1 (QQQ) and SPDR S&P 500 ETF Trust (SPY). These are exchange-traded funds incorporated in the USA. QQQ ETF tracks the performance of the Nasdaq 100 Index. It holds large-cap U.S. stocks and tends to focus on the technology and consumer sector. The holdings are weighted by market capitalization. As of 6 October 2017, there were 107 holding companies. The top three holding companies are Apple Inc., Austin, TX, USA (AAPL, 11.57%), Microsoft Corp., Redmond, WA, USA (MSFT, 8.44%), and Amazon.com, Inc., Seattle, WA, USA (AMZN, 6.86%). SPY ETF tracks the S&P 500 Index. The trust consists of a portfolio representing all 500 stocks in the S&P 500 Index. It holds predominantly large-cap U.S. stocks. It is structured as a unit investment trust and pays dividends on a quarterly basis. The holdings are weighted by market capitalization. As of 6 October 2017, the top three holding companies were Apple Inc., Austin, TX, USA (AAPL, 3.67%), Microsoft Corp., Redmond, WA, USA (MSFT, 2.68%), and Facebook Inc., Menlo Park, CA, USA) Class A (FB, 1.87%).

We collected daily second-level QQQ ETF, SPY ETF, MSFT, and AAPL price time series for this study. The QQQ ETF price time series and SPY ETF price time series are used as the vector-based ETF price $\boldsymbol{p}_t$. The MSFT and AAPL stock price time series are used as the vector-based predictive variable $\boldsymbol{x}_t$. The collection period is the daily trading time from 9:30 a.m. to 4:00 p.m. (Eastern Time) to ensure a high liquid market. We divided QQQ ETF and SPY ETF time series into two data sets: vector-based ETF price $\boldsymbol{p}_t$ training data (9:30 a.m. to 3:00 p.m., which contains 19,800 s) and vector-based ETF price $\boldsymbol{p}_t$ test data (3:00 p.m. to 4:00 p.m., which contains 3601 s). We use the MSFT and AAPL price time series as the vector-based predictive variables' $\boldsymbol{x}_t$ training data (9:30 a.m. to 3:00 p.m., which contains 19,800 s). We set initial wealth $w_0 = 10,000$ and interest rate $r = 0$. Suppose that the investor's risk tolerance is 0.000001. We restrict a_1, a_2, a_3, a_4, and a_5 in $[0, 1]$, s in $5, 10$, and l in $30, 60, 90, 120, 180$, and 240. We use training data to choose model parameters with the highest return. We first report the MGMA strategy performance summary for QQQ ETF and SPY ETF on training data; then, we report the MGMA strategy evaluation summary for QQQ ETF and SPY ETF on test data. Our study spans five days from 10 February 2017 to 10 June 2017. We plot second-level QQQ ETF and SPY ETF price time series on day 1 (10 February 2017) in Figure 1 as an example.

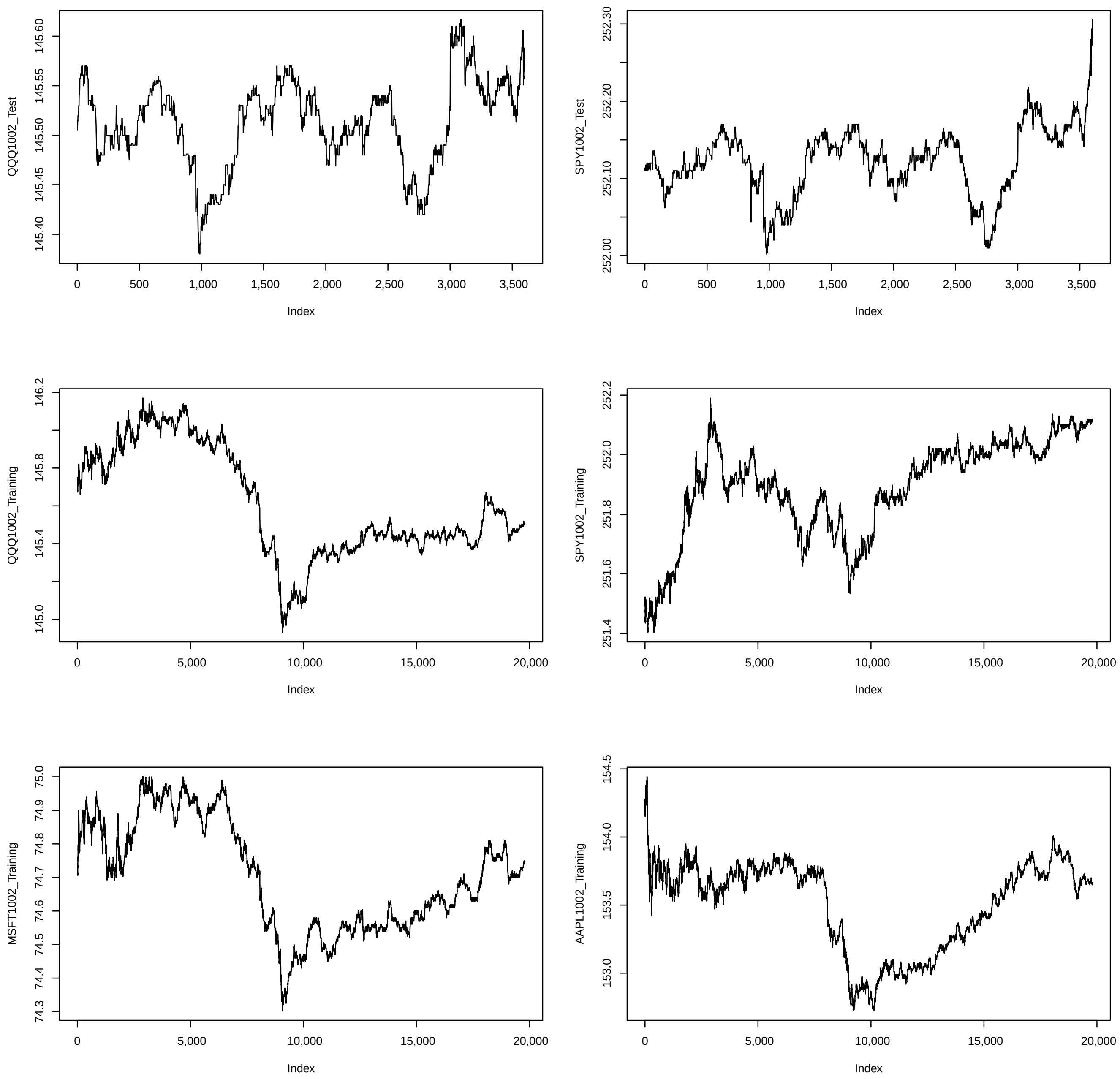

Figure 1. Case 1—Second-level QQQ ETF and SPY ETF prices time series on day 1 (10 February 2017).

The MGMA strategy performance summary for QQQ ETF and SPY ETF on day 1 (10 February 2017) to day 5 (10 June 2017); training data are provided in Table 8. The MGMA strategy evaluation summary for QQQ ETF and SPY ETF on day 1 (10 February 2017) to day 5 (10 June 2017) test data is provided in Table 9.

Note that the MGMA strategy in general can outperform the MA strategy for both backward investments in training data and forward investments in test data. For example, for day 1 (10 February 2017), the MGMA strategy can increase the daily return ratio from 0.09498% to 0.24542% on training data, which equals an increase in annual return ratio of 46.1%; the MGMA strategy can increase the daily return ratio from 0.06668% to 0.08534% on test data, which equals an increase in annual return ratio of 4.8%.

Table 8. Case 1—MGMA strategy performance summary for QQQ ETF and SPY ETF on day 1 (10 February 2017) to day 5 (10 June 2017) training data.

Training Data	Day	10 February 2017	10 March 2017	10 April 2017	10 May 2017	10 June 2017
	time	9:30 a.m.–3:00 p.m.	9:30 a.m.–3:00 p.m.	9:30 a.m.–3:00 p.m.	9:30 a.m.–3:00 p.m.	9:30 a.m.–3:00 p.m.
	T	19,800 s	19,800 s	19,800 s	19,800 s	19,800 s
	dt	1 s	1 s	1 s	1 s	1 s
tuned parameters	s	10	10	10	10	10
	l	180	60	240	180	120
	$\hat{a}_1$	0.3287	0.1250	0.0742	0.0737	0.8537
	$\hat{a}_2$	0.9611	0.2346	0.7111	0	0.2780
	$\hat{a}_3$	0.0352	0	0	1	0
	$\hat{a}_4$	0.7160	0	0	1	0
	$\hat{a}_5$	1	1	1	1	1
backward MA	$E(W_T)$	10,009.49768	9,955.07379	10,001.35849	10,002.15772	9,997.54490
	return ratio (%)	0.09498%	−0.44926%	0.01358%	0.02158%	−0.02455%
	trans num	504	1,117	421	535	589
backward MGMA	$E(W_T)$	10,024.54175	9,974.19921	10,010.97889	10,015.65524	10,016.17125
	return ratio (%)	0.24542%	−0.25801%	0.10979%	0.15655%	0.16171%
	trans num	655	1,557	534	708	794

Table 9. Case 1—MGMA strategy evaluation summary for QQQ ETF and SPY ETF on day 1 (10 February 2017) to day 5 (10 June 2017) test data.

test data	*day*	10 February 2017	10 March 2017	10 April 2017	10 May 2017	10 June 2017
	time	3:00 p.m.–4:00 p.m.	3:00 p.m.–4:00 p.m.	3:00 p.m.–4:00 p.m.	3:00 p.m.–4:00 p.m.	3:00 p.m.–4:00 p.m.
	T	3601 s	3601 s	3601 s	3601 s	3601 s
	dt	1 s	1 s	1 s	1 s	1 s
tuned parameters	s	10	10	10	10	10
	l	180	60	240	180	120
	$\hat{a}_1$	0.3287	0.1250	0.0742	0.0737	0.8537
	$\hat{a}_2$	0.9611	0.2346	0.7111	0	0.2780
	$\hat{a}_3$	0.0352	0	0	1	0
	$\hat{a}_4$	0.7160	0	0	1	0
	$\hat{a}_5$	1	1	1	1	1
forward MA	$E(W_T)$	10,006.66819	9990.21402	9998.84086	10,003.58666	10,001.47801
	return ratio (%)	0.06668%	−0.09786%	−0.01159%	0.03587%	0.01478%
	trans num	67	210	85	92	111
forward MGMA	$E(W_T)$	10,008.53428	9991.06108	10,000.43623	10,008.31762	9999.77159
	return ratio (%)	0.08534%	−0.08939%	0.00436%	0.08318%	−0.00228%
	trans num	82	318	127	120	141

The MGMA strategy performance summary for QQQ ETF and SPY ETF on day 1 (10 February 2017) to day 5 (10 June 2017); training data are provided in Figure 2. The MGMA strategy evaluation summary for QQQ ETF and SPY ETF on day 1 (10 February 2017) to day 5 (10 June 2017) test data is provided in Figure 3.

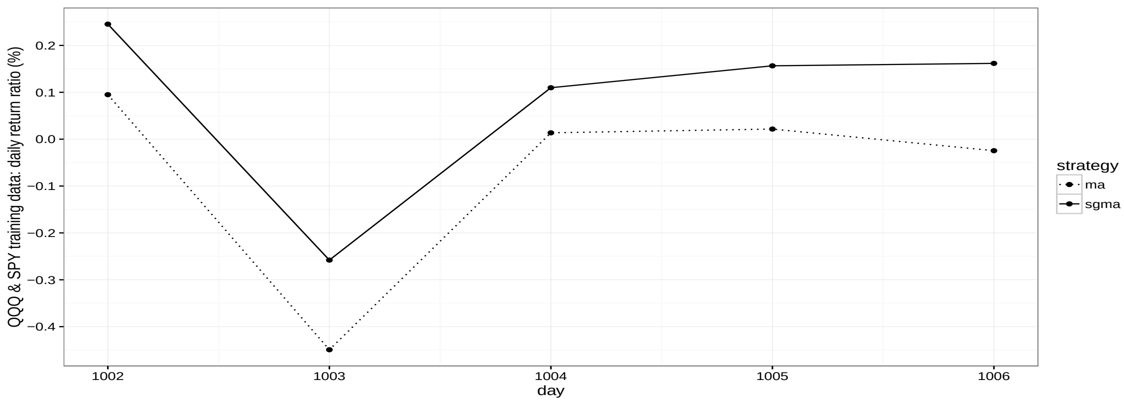

Figure 2. Case 1—MGMA strategy performance summary plot for QQQ ETF and SPY ETF on day 1 (10 February 2017) to day 5 (10 June 2017) training data.

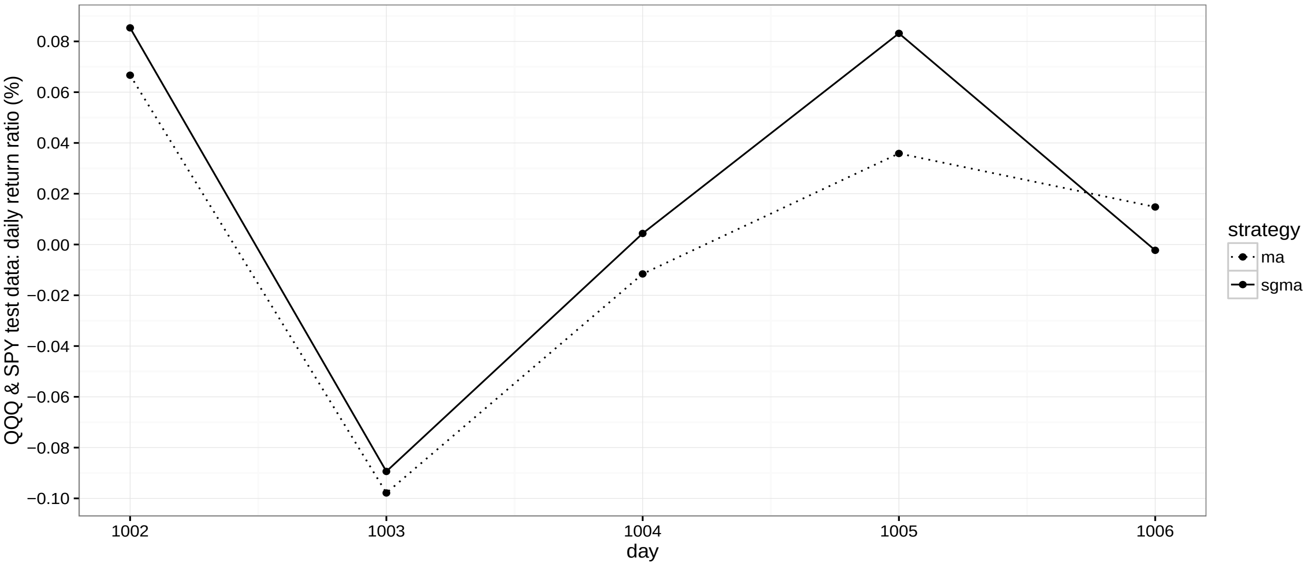

Figure 3. Case 1—MGMA strategy evaluation summary plot for QQQ ETF and SPY ETF on day 1 (10 February 2017) to day 5 (10 June 2017) test data.

6.3. *Case 2: MGMA Strategy on High-Frequency Exchange-Traded Fund in Asian Market*

We use the China 50 ETF (HuaXia) and Huatai-Pinebridge CSI 300 ETF (HuaTai). These are exchange-traded funds incorporated in China. The China 50 ETF (HuaXia) tracks the performance of the Shanghai Stock Exchange 50 Index (the SSE 50 Index). The holdings are weighted by market capitalization. As of 22 March 2018, the top five holding companies are PingAn Insurance Group, Shenzhen, China (PingAn, 12.30%), China Merchants Bank Co., Ltd., Shenzhen, China (CMB, 5.65%), Kweichow Moutai Co Ltd., Zunyi China (600519, 5.41%), Industrial Bank Co Ltd., Fuzhou, China (IndBank, 4.81%), and China Minsheng Banking Corp Ltd., Beijing, China (CMAKY, 4.45%). The Huatai-Pinebridge CSI 300 ETF (HuaTai) is a capitalization-weighted stock market index designed to replicate the performance of the top 300 stocks traded in the Shanghai and Shenzhen stock exchange. As of 22 March 2018, the top three holding companies are PingAn Insurance Group, Shenzhen, China (PingAn, 4.17%), China Merchants Bank Co Ltd., Shenzhen, China (CMB, 2.34%), and Industrial Bank Co Ltd., Fuzhou, China (IndBank, 2.34%).

We collected daily second-level HuaXia ETF, HuaTai ETF, PingAn and IndBank price time series for this study. The HuaXia ETF price time series and HuaTai ETF price time series are used as the vector-based ETF price $\boldsymbol{p}_t$. The PingAn and IndBank stock price time series are used as the vector-based predictive variable $\boldsymbol{x}_t$. The collection period is the daily trading time from 8:30 p.m. to 2:00 a.m. (Eastern Time) to ensure a highly liquid market. Note that there is a break time from 10:30 p.m. to 0:00 a.m. (Eastern Time) for the Asian Market. We divide the HuaXia ETF and HuaTai ETF time series into two data sets: vector-based ETF price $\boldsymbol{p}_t$ training data (8:30 p.m. to 10:30 p.m. and 0:00 a.m. to 1:00 a.m., which contains 10,800 s) and vector-based ETF price $\boldsymbol{p}_t$ test data (1:00 a.m. to 2:00 a.m., which contains 3601 s). We use the PingAn and IndBank price time series as vector-based predictive variable $\boldsymbol{x}_t$ training data (8:30 p.m. to 10:30 p.m. and 0:00 a.m. to 1:00 a.m., which contains 10,800 s). We set initial wealth $w_0 = 10,000$ and interest rate $r = 0$. Suppose that the investor's risk tolerance is 0.0005. We restrict a_1, a_2, a_3, a_4, and a_5 in $[0,1]$, s in $5, 10$, and l in $30, 60, 90, 120, 180$, and 240. We use training data to choose model parameters with the highest return. We first report the MGMA strategy performance summary for HuaXia ETF and HuaTai ETF on training data; then, we report the MGMA strategy evaluation summary for HuaXia ETF and HuaTai ETF on test data. Our study spans five days from 18 March 2018 to 22 March 2018. We plot the second-level HuaTai ETF and HuaXia ETF price time series on day 1 (18 March 2018) in Figure 4 as an example.

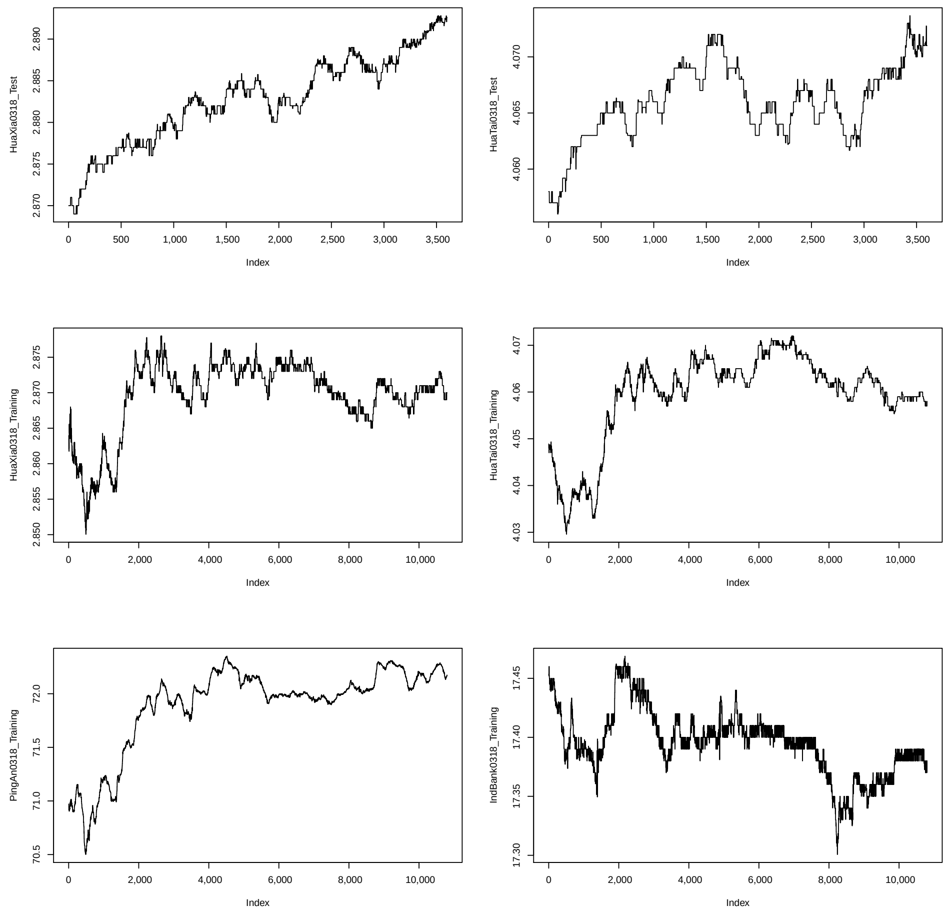

Figure 4. Case 2—Second-level HuaXia ETF and HuaTai ETF prices time series on day 1 (18 March 2018).

The MGMA strategy performance summary for HuaXia ETF and HuaTai ETF on day 1 (18 March 2018) to day 5 (22 March 2018) training data is provided in Table 10. The MGMA strategy evaluation summary for HuaXia ETF and Huatai ETF on day 1 (18 March 2018) to day 5 (22 March 2018) test data is provided in Table 11.

Table 10. Case 2—MGMA strategy performance summary for HuaXia ETF and HuaTai ETF on day 1 (18 March 2018) to day 5 (22 March 2018) training data.

training data	*day*	*18 March 2018*	*19 March 2018*	*20 March 2018*	*21 March 2018*	*22 March 2018*
	time	*8:30 p.m.–1:00 a.m.*	*8:30 p.m.–1:00 a.m.*	*8:30 p.m.–1:00 a.m.*	*8:30 p.m.–1:00 a.m.*	*8:30 p.m.–1:00 a.m.*
	T	$10,800\ s$	$10,800\ s$	$10,800\ s$	$10,800\ s$	$10,800\ s$
	dt	$1\ s$	$1\ s$	$1\ s$	$1\ s$	$1\ s$
tuned parameters	s	10	10	10	10	10
	l	30	30	30	30	30
	$\hat{a}_1$	1	1	1	1	1
	$\hat{a}_2$	1	1	1	1	1
	$\hat{a}_3$	1	1	1	1	1
	$\hat{a}_4$	1	1	1	1	1
	$\hat{a}_5$	1	1	1	1	1
backward MA	$E(W_T)$	9697.42479	9742.02916	9483.70570	9654.20256	9645.70242
	return ratio (%)	−3.02575%	−2.57971%	−5.16294%	−3.45797%	−3.54298%
	trans num	837	833	1043	861	917
backward MGMA	$E(W_T)$	10,015.85838	10,053.51711	9945.74085	9873.99072	9961.93758
	return ratio (%)	0.15858%	0.53517%	−0.54259%	−1.26009%	−0.38062%
	trans num	869	852	1056	914	1013

Table 11. Case 2—MGMA strategy evaluation summary for HuaXia ETF and HuaTai ETF on day 1 (18 March 2018) to day 5 (22 March 2018) test data.

test data	*day*	*18 March 2018*	*19 March 2018*	*20 March 2018*	*21 March 2018*	*22 March 2018*
	time	*1:00 a.m.–2:00 a.m.*	*1:00 a.m.–2:00 a.m.*	*1:00 a.m.–2:00 a.m.*	*1:00 a.m.–2:00 a.m.*	*1:00 a.m.–2:00 a.m.*
	T	3601 s	3601 s	3601 s	3601 s	3601 s
	dt	1 s	1 s	1 s	1 s	1 s
tuned parameters	s	10	10	10	10	10
	l	30	30	30	30	30
	$\hat{a}_1$	1	1	1	1	1
	$\hat{a}_2$	1	1	1	1	1
	$\hat{a}_3$	1	1	1	1	1
	$\hat{a}_4$	1	1	1	1	1
	$\hat{a}_5$	1	1	1	1	1
$forward\ MA$	$E(W_T)$	9958.09171	9873.56029	9825.82477	9832.73198	10,049.94455
	return ratio (%)	−0.41908%	−1.26440%	−1.74175%	−1.67268%	0.49945%
	trans num	270	317	325	311	252
$forward\ MGMA$	$E(W_T)$	10,075.23280	10,001.72750	9913.81940	9988.70100	10,049.94455
	return ratio (%)	0.75233%	0.01728%	−0.86181%	−0.11299%	0.47835%
	trans num	274	321	350	317	289

Note that the MGMA strategy in general can outperform the MA strategy for both backward investments on training data and forward investment on test data. Note that the optimal short-lag s, long-lag l, and allocation parameters a_1^* to a_5^* are all the same for 5 days, which might suggest that it is possible to use fixed parameters for MGMA strategy in reality.

The MGMA strategy performance summary for HuaXia ETF and HuaTai ETF on day 1 (18 March 2018) to day 5 (22 March 2018) training data is provided in Figure 5. The MGMA strategy evaluation summary for HuaXia ETF and HuaTai ETF on day 1 (18 March 2018) to day 5 (22 March 2018) test data is provided in Figure 6.

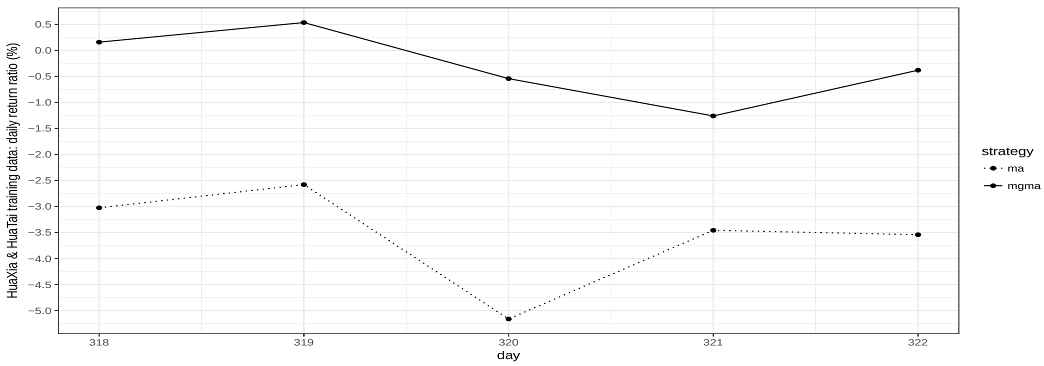

Figure 5. Case 2—MGMA strategy performance summary plot for HuaXia ETF and HuaTai ETF on day 1 (18 March 2018) to day 5 (22 March 2018) training data.

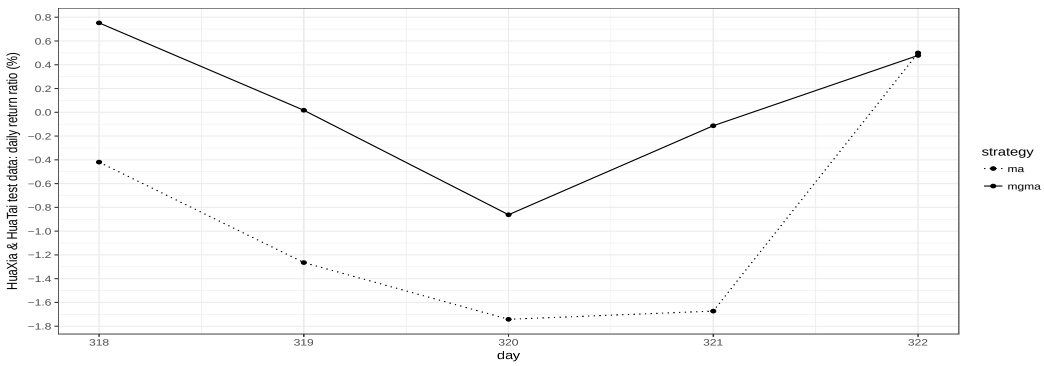

Figure 6. Case 2—MGMA strategy evaluation summary plot for HuaXia ETF and HuaTai ETF on day 1 (18 March 2018) to day 5 (22 March 2018) test data.

7. Conclusions

In this paper, we propose a multiasset generalized moving average crossover (MGMA) strategy. Our study demonstrates that the MGMA strategy can provide more investment options with the investor's risk tolerance. The MGMA strategy can solve the well-known problem for the MA strategy for a multiasset portfolio. Simulation studies demonstrate that the MGMA strategy can increase both the investor's expected utility of wealth and the investor's expected wealth. Two high-frequency ETF real-time examples from the North American market and Asian market demonstrate that the MGMA strategy can outperform the MA strategy for both backward investment on training data and forward investment on test data. The MGMA strategy has built the foundation for reconciling the moving average technique with the portfolio allocation strategy for multiple assets. In the future, we would like to extend the algorithm for adaptive or online prediction in order to go beyond the current reactionary nature. While this approach is proposed for optimizing asset allocation, it can also be used to create a general framework for analyzing the relative importance or impact of a particular repeated measured index in a multiple time series setting. It could have broad applications in climate change or healthcare.

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Appendix A

We provide the preliminary lemmas, which are used to derive the analytical results.

Lemma A1. *Let Θ be symmetric negative definite. If Θ and V_x are exchangeable, i.e., $\Theta V_x = V_x\Theta$, then Θ and $e^{t\Theta}$, $e^{t\Theta}$ and V_x, $V_x^\top$ and $e^{t\Theta}$ are also exchangeable.*

Proof. Since Θ is symmetric negative definite, by the definition of matrix exponential, it follows that

$$\Theta e^{t\Theta} = \Theta \sum_{k=0}^{\infty} \frac{1}{k!}(t\Theta)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(t\Theta)^k\Theta = e^{t\Theta}\Theta.$$

By the assumptions that $\Theta V_x = V_x\Theta$ and Θ is symmetric negative definite, we obtain that $\Theta^k V_x = V_x\Theta^k$, which implies that

$$e^{t\Theta}V_x = \sum_{k=0}^{\infty} \frac{1}{k!}(t\Theta)^k V_x = V_x \sum_{k=0}^{\infty} \frac{1}{k!}(t\Theta)^k = V_x e^{t\Theta}.$$

Therefore, we have

$$V_x^\top e^{t\Theta} = V_x^\top \left(e^{t\Theta}\right)^\top = \left(e^{t\Theta}V_x\right)^\top = \left(V_x e^{t\Theta}\right)^\top = \left(e^{t\Theta}\right)^\top V_x^\top = e^{t\Theta^\top} V_x^\top = e^{t\Theta} V_x^\top.$$

□

Lemma A2. *Let $\boldsymbol{x}_t$ be the vector of the predictive variables in the market satisfying (3), $\boldsymbol{\mu}_{\boldsymbol{x}}$ be the vector of expectation of $\boldsymbol{x}_t$, and $\Sigma_{\boldsymbol{x}}$ be the variance–covariance matrix of $\boldsymbol{x}_t$. Then, $\boldsymbol{x}_t$ is multivariate normal distributed and has the following expression*

$$\boldsymbol{x}_t = e^{t\Theta}\boldsymbol{x}_0 - \left(I_q - e^{t\Theta}\right)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{(t-u)\Theta} V_x d\boldsymbol{z}_u,$$

and

$$\begin{aligned}\boldsymbol{\mu}_{\boldsymbol{x}} &= -\Theta^{-1}\boldsymbol{\beta},\\ \Sigma_{\boldsymbol{x}} &= -\frac{1}{2}V_x\Theta^{-1}V_x^\top,\\ \mathrm{Cov}(\boldsymbol{x}_t,\boldsymbol{x}_s) &= -\frac{1}{2}V_x\Theta^{-1}e^{|t-s|\Theta}V_x^\top.\end{aligned}$$

Proof. Let I_q be an identity matrix and $\mathbf{0}_n^\top = (0,\dots,0)$. We have

$$\begin{aligned}d\Big(e^{-u\Theta}\boldsymbol{x}_u\Big) &= -\Theta e^{-u\Theta}\boldsymbol{x}_u du + e^{-u\Theta}d\boldsymbol{x}_u\\ &= -\Theta e^{-u\Theta}\boldsymbol{x}_u du + e^{-u\Theta}((\boldsymbol{\beta}+\Theta\boldsymbol{x}_u)du + V_x d\boldsymbol{z}_u) = e^{-u\Theta}\boldsymbol{\beta}du + e^{-u\Theta}V_x d\boldsymbol{z}_u.\end{aligned}$$

which, jointly with Lemma A1, yields that

$$\begin{aligned}&e^{-u\Theta}\boldsymbol{x}_u\ |_0^\top = e^{-T\Theta}\boldsymbol{x}_T - \boldsymbol{x}_0 = \int_0^t d\Big(e^{-u\Theta}\boldsymbol{x}_u\Big) = \int_0^t e^{-u\Theta}\boldsymbol{\beta}du + \int_0^t e^{-u\Theta}V_x d\boldsymbol{z}_u\\ &= -\Theta^{-1}e^{-u\Theta}\ |_0^\top\ \boldsymbol{\beta} + \int_0^t e^{-u\Theta}V_x d\boldsymbol{z}_u = \Big(-\Theta^{-1}e^{-t\Theta}+\Theta^{-1}\Big)\boldsymbol{\beta} + \int_0^t e^{-u\Theta}V_x d\boldsymbol{z}_u\\ &= -\Big(e^{-t\Theta}-I_q\Big)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{-u\Theta}V_x d\boldsymbol{z}_u.\end{aligned}$$

Therefore, we have

$$e^{-t\Theta}\boldsymbol{x}_t = \boldsymbol{x}_0 - \Big(e^{-t\Theta}-I_q\Big)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{-u\Theta}V_x d\boldsymbol{z}_u,$$

and

$$\boldsymbol{x}_t = e^{t\Theta}\boldsymbol{x}_0 - \Big(I_q - e^{t\Theta}\Big)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u, \tag{A1}$$

Since $\boldsymbol{z}_t$ is a multidimensional standard Brownian motion, we obtain that $\boldsymbol{x}_t$ is multivariate normal distributed. By the fact that $E(d\boldsymbol{z}_u) = \mathbf{0}_n$, it follows that

$$\begin{aligned}\boldsymbol{\mu}_x = E(\boldsymbol{x}_t) &= e^{t\Theta}E(\boldsymbol{x}_0) - \Big(I_q - e^{t\Theta}\Big)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{(t-u)\Theta}V_x E(d\boldsymbol{z}_u)\\ &= e^{t\Theta}E(\boldsymbol{x}_0) - \Big(I_q - e^{t\Theta}\Big)\Theta^{-1}\boldsymbol{\beta}.\end{aligned}$$

As $\boldsymbol{x}_t$ is a stationary process, we have $E(\boldsymbol{x}_t) = E(\boldsymbol{x}_0)$, and hence obtain that $\boldsymbol{\mu}_x = -\Theta^{-1}\boldsymbol{\beta}$. In view that $E\big(d\boldsymbol{z}_u d\boldsymbol{z}_u^\top\big) = duI_q$, V_x and $e^{t\Theta}$ are exchangeable, and Θ is symmetric, by Lemma A1, the variance–covariance matrix $\Sigma_{\boldsymbol{x}}$ is

$$\begin{aligned}\Sigma_{\boldsymbol{x}} = \mathrm{Var}(\boldsymbol{x}_t) &= \mathrm{Var}\bigg(e^{t\Theta}\boldsymbol{x}_0 - \Big(I_q - e^{t\Theta}\Big)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u\bigg)\\ &= e^{t\Theta}\mathrm{Var}(\boldsymbol{x}_0)e^{t\Theta^\top} + \mathrm{Var}\bigg(\int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u\bigg),\end{aligned}$$

where

$$\begin{aligned}
&\mathrm{Var}\left(\int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u\right)\\
&= E\left[\int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u \int_0^t d\boldsymbol{z}_u^\top V_x^\top e^{(t-u)\Theta^\top}\right] - E\left[\int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u\right]E\left[\int_0^t d\boldsymbol{z}_u^\top V_x^\top e^{(t-u)\Theta^\top}\right]\\
&= \int_0^t e^{(t-u)\Theta}V_x V_x^\top e^{(t-u)\Theta^\top}du = \int_0^t V_x e^{(t-u)\Theta}e^{(t-u)\Theta^\top}du V_x^\top = V_x\int_0^t e^{2(t-u)\Theta}du V_x^\top\\
&= V_x\left[-\frac{1}{2}\Theta^{-1}e^{2(t-u)\Theta}\mid_0^\top\right]V_x^\top = -\frac{1}{2}V_x\Theta^{-1}V_x^\top + \frac{1}{2}V_x\Theta^{-1}e^{t\Theta}e^{t\Theta^\top}V_x^\top\\
&= -\frac{1}{2}V_x\Theta^{-1}V_x^\top + \frac{1}{2}e^{t\Theta}V_x\Theta^{-1}V_x^\top e^{t\Theta^\top},
\end{aligned}$$

which implies that

$$\Sigma_{\boldsymbol{x}} = \mathrm{Var}(\boldsymbol{x}_t) = e^{t\Theta}\mathrm{Var}(\boldsymbol{x}_0)e^{t\Theta^\top} - \frac{1}{2}V_x\Theta^{-1}V_x^\top + \frac{1}{2}e^{t\Theta}V_x\Theta^{-1}V_x^\top e^{t\Theta^\top}. \tag{A2}$$

Since $\boldsymbol{x}_t$ is a stationary process, it follows that $\mathrm{Var}(\boldsymbol{x}_t) = \mathrm{Var}(\boldsymbol{x}_0)$, which, jointly with (A2), yields that $\Sigma_x = -\frac{1}{2}V_x\Theta^{-1}V_x^\top$. Moreover,

$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{x}_t,\boldsymbol{x}_s) &= \mathrm{Cov}\Bigg(e^{t\Theta}\boldsymbol{x}_0 - \left(I_q - e^{t\Theta}\right)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u,\\
&e^{s\Theta}\boldsymbol{x}_0 - \left(I_q - e^{s\Theta}\right)\Theta^{-1}\boldsymbol{\beta} + \int_0^s e^{(s-u)\Theta}V_x d\boldsymbol{z}_u\Bigg)\\
&= \mathrm{Cov}\left(e^{t\Theta}\boldsymbol{x}_0, e^{s\Theta}\boldsymbol{x}_0\right) + \mathrm{Cov}\left(\int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u, \int_0^s e^{(s-u)\Theta}V_x d\boldsymbol{z}_u\right).
\end{aligned} \tag{A3}$$

Note that

$$\mathrm{Cov}\left(e^{t\Theta}\boldsymbol{x}_0, e^{s\Theta}\boldsymbol{x}_0\right) = e^{t\Theta}\mathrm{Var}(\boldsymbol{x}_0)e^{s\Theta} = -\frac{1}{2}e^{t\Theta}V_x\Theta^{-1}V_x^\top e^{s\Theta}. \tag{A4}$$

Since $E\left(d\boldsymbol{z}_a d\boldsymbol{z}_b^\top\right) = 0$ if $a \neq b$, we have

$$\begin{aligned}
&\mathrm{Cov}\left(\int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u, \int_0^s e^{(s-u)\Theta}V_x d\boldsymbol{z}_u\right)\\
&= \int_0^{\min(t,s)} e^{(t-u)\Theta}V_x V_x^\top e^{(s-u)\Theta^\top}du = V_x\int_0^{\min(t,s)} e^{(t+s-2u)\Theta}du V_x^\top\\
&= V_x\left[-\frac{1}{2}\Theta^{-1}\left(e^{(t+s-2\min(t,s))\Theta} - e^{(t+s)\Theta}\right)\right]V_x^\top\\
&= -\frac{1}{2}V_x\Theta^{-1}e^{|t-s|\Theta}V_x^\top + \frac{1}{2}V_x\Theta^{-1}e^{(t+s)\Theta}V_x^\top.
\end{aligned} \tag{A5}$$

By (A3)–(A5), it follows that

$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{x}_t,\boldsymbol{x}_s) &= -\frac{1}{2}e^{t\Theta}V_x\Theta^{-1}V_x^\top e^{s\Theta} - \frac{1}{2}V_x\Theta^{-1}e^{|t-s|\Theta}V_x^\top + \frac{1}{2}V_x\Theta^{-1}e^{(t+s)\Theta}V_x^\top\\
&= -\frac{1}{2}V_x\Theta^{-1}e^{(t+s)\Theta}V_x^\top - \frac{1}{2}V_x\Theta^{-1}e^{|t-s|\Theta}V_x^\top + \frac{1}{2}V_x\Theta^{-1}e^{(t+s)\Theta}V_x^\top\\
&= -\frac{1}{2}V_x\Theta^{-1}e^{|t-s|\Theta}V_x^\top.
\end{aligned}$$

□

Lemma A3. *Let $\boldsymbol{y}_t$ be the vector of the log-transformed stock prices and $\boldsymbol{\mu}_{\boldsymbol{y}}$ be its expectation. Then, $\boldsymbol{y}_t$ follows a multivariate normal distribution and has the following expression*

$$\boldsymbol{y}_t = \boldsymbol{y}_0 + \int_0^t (\boldsymbol{\alpha} + U\boldsymbol{x}_u)du + V_p \boldsymbol{b}_t,$$

whose mean vector is given by

$$\boldsymbol{\mu}_{\boldsymbol{y}} = \boldsymbol{y}_0 + \left(\boldsymbol{\alpha} - U\Theta^{-1}\boldsymbol{\beta}\right)t.$$

Proof. By (2) and (4), we have

$$d(\boldsymbol{y}_t) = d(\log \boldsymbol{p}_t) = (\boldsymbol{\alpha} + U\boldsymbol{x}_t)dt + V_p d\boldsymbol{b}_t.$$

Note that $\boldsymbol{b}_t$ is multidimensional standard Brownian motion. Thus, we obtain that

$$\boldsymbol{y}_t - \boldsymbol{y}_0 = \int_0^t d(\boldsymbol{y}_u) = \int_0^t (\boldsymbol{\alpha} + U\boldsymbol{x}_u)du + \int_0^t V_p d\boldsymbol{b}_u,$$

and hence

$$\boldsymbol{y}_t = \boldsymbol{y}_0 + \int_0^t (\boldsymbol{\alpha} + U\boldsymbol{x}_u)du + V_p \boldsymbol{b}_t,$$

which, jointly with Lemma A2, yields that $\boldsymbol{y}_t$ follows a multivariate normal distribution with the mean vector

$$\begin{aligned}\boldsymbol{\mu}_y = E(\boldsymbol{y}_t) &= E(\boldsymbol{y}_0) + \int_0^t (\boldsymbol{\alpha} + UE(\boldsymbol{x}_u))du + V_p E(\boldsymbol{b}_t) \\ &= \boldsymbol{y}_0 + \left(\boldsymbol{\alpha} - U\Theta^{-1}\boldsymbol{\beta}\right)t.\end{aligned}$$

□

Lemma A4. *Let $\boldsymbol{m}_t^{(h)}$ be the vector of the moving average based on lookback period h and $\boldsymbol{m}_t^{(s,l)}$ be the vector of the difference between the moving averages based on lookback period s and l. Then, $\boldsymbol{m}_t^{(h)}$ follows a multivariate normal distribution with mean $E\left(\boldsymbol{m}_t^{(h)}\right)$, and $\boldsymbol{m}_t^{(s,l)}$ follows a multivariate normal distribution with mean $E\left(\boldsymbol{m}_t^{(s,l)}\right)$, where*

$$E\left(\boldsymbol{m}_t^{(h)}\right) = \boldsymbol{y}_0 + \left(\boldsymbol{\alpha} - U\Theta^{-1}\boldsymbol{\beta}\right)\left(t - \frac{h}{2}\right),$$

and

$$E\left(\boldsymbol{m}_t^{(s,l)}\right) = \frac{1}{2}(l-s)\left(\boldsymbol{\alpha} - U\Theta^{-1}\boldsymbol{\beta}\right),$$

Proof. By the definition of $\boldsymbol{m}_t^{(h)}$ given in (5) and Lemma A3, we have

$$\begin{aligned}E\left(\boldsymbol{m}_t^{(h)}\right) &= E\left(\frac{1}{h}\int_{t-h}^t \boldsymbol{y}_u du\right) = \frac{1}{h}\int_{t-h}^t E(\boldsymbol{y}_u)du \\ &= \frac{1}{h}\int_{t-h}^t \boldsymbol{y}_0 du + \frac{1}{h}\int_{t-h}^t \left(\boldsymbol{\alpha} - U\Theta^{-1}\boldsymbol{\beta}\right)u du \\ &= \boldsymbol{y}_0 + \left(\boldsymbol{\alpha} - U\Theta^{-1}\boldsymbol{\beta}\right)\left(t - \frac{h}{2}\right),\end{aligned}$$

and

$$E\left(\boldsymbol{m}_t^{(s,l)}\right) = E\left(\boldsymbol{m}_t^{(s)} - \boldsymbol{m}_t^{(l)}\right) = E\left(\boldsymbol{m}_t^{(s)}\right) - E\left(\boldsymbol{m}_t^{(l)}\right)$$
$$= \frac{1}{2}(l-s)\left(\boldsymbol{\alpha} - U\Theta^{-1}\boldsymbol{\beta}\right).$$

□

Lemma A5. *Let $\boldsymbol{z}_u$ and $\boldsymbol{b}_v$ be a multidimensional standard Brownian motion. If $Corr(\boldsymbol{b}_v, \boldsymbol{z}_u) = V_{\boldsymbol{bz}}$, then*

$$\mathrm{Cov}(\boldsymbol{z}_u, \boldsymbol{b}_v) = \min(u,v)V_{\boldsymbol{bz}}^{\top},$$

and

$$\mathrm{Cov}(d\boldsymbol{z}_u, \boldsymbol{b}_v) = \begin{cases} V_{\boldsymbol{bz}}^{\top}du, & \text{if } u < v, \\ 0, & \text{if } u \geq v. \end{cases}$$

Proof. Given that z_{iu} and b_{jv} are two-dimensional standard Brownian motions with correlation coefficient ρ_{ji}, we can express z_{iu} by $\rho_{ji}b_{ju} + \sqrt{1-\rho_{ji}^2}b'_{ju}$, where b_{ju} and b'_{ju} are independent. It follows that

$$\begin{aligned}\mathrm{Cov}(z_{iu}, b_{jv}) &= \mathrm{Cov}\left(\rho_{ji}b_{ju} + \sqrt{1-\rho_{ji}^2}b'_{ju}, b_{jv}\right) \\ &= \rho_{ji}\mathrm{Cov}(b_{ju}, b_{jv}) + \sqrt{1-\rho_{ji}^2}\mathrm{Cov}\left(b'_{ju}, b_{jv}\right) \\ &= \rho_{ji}\min(u,v),\end{aligned}$$

which implies that

$$\mathrm{Cov}(\boldsymbol{z}_u, \boldsymbol{b}_v) = \min(u,v)V_{\boldsymbol{bz}}^{\top}.$$

We also have

$$\begin{aligned}\mathrm{Cov}(d\boldsymbol{z}_u, \boldsymbol{b}_v) &= \mathrm{Cov}(\boldsymbol{z}_{u+du} - \boldsymbol{z}_u, \boldsymbol{b}_v) = \mathrm{Cov}(\boldsymbol{z}_{u+du}, \boldsymbol{b}_v) - \mathrm{Cov}(\boldsymbol{z}_u, \boldsymbol{b}_v) \\ &= \min(u+du, v)V_{\boldsymbol{bz}}^{\top} - \min(u,v)V_{\boldsymbol{bz}}^{\top} \\ &= \begin{cases} V_{\boldsymbol{bz}}^{\top}du, & \text{if } u < v, \\ 0, & \text{if } u \geq v. \end{cases}\end{aligned}$$

□

Lemma A6. *Let $\boldsymbol{x}_t$ be the vector of predictive variables in the market and $\boldsymbol{b}_v$ be multidimensional standard Brownian motion. Then, we have*

$$\mathrm{Cov}(\boldsymbol{x}_t, \boldsymbol{b}_v) = \begin{cases} -\Theta^{-1}\left(e^{(t-v)\Theta} - e^{t\Theta}\right)V_xV_{\boldsymbol{bz}}^{\top}, & \text{if } t \geq v, \\ -\Theta^{-1}\left(I_q - e^{t\Theta}\right)V_xV_{\boldsymbol{bz}}^{\top}, & \text{if } t < v. \end{cases}$$

Proof. If $t \geq v$, by Lemmas A2 and A5, we have

$$\begin{aligned}\mathrm{Cov}(\boldsymbol{x}_t, \boldsymbol{b}_v) &= \mathrm{Cov}\left(e^{t\Theta}\boldsymbol{x}_0 - \left(I_q - e^{t\Theta}\right)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u, \boldsymbol{b}_v\right) \\ &= \mathrm{Cov}\left(\int_0^t e^{(t-u)\Theta}V_x d\boldsymbol{z}_u, \boldsymbol{b}_v\right) = \int_0^t e^{(t-u)\Theta}V_x\mathrm{Cov}(d\boldsymbol{z}_u, \boldsymbol{b}_v) \\ &= \int_0^v e^{(t-u)\Theta}V_x\mathrm{Cov}(d\boldsymbol{z}_u, \boldsymbol{b}_v) + \int_v^t e^{(t-u)\Theta}V_x\mathrm{Cov}(d\boldsymbol{z}_u, \boldsymbol{b}_v) \\ &= -\Theta^{-1}\left(e^{(t-v)\Theta} - e^{t\Theta}\right)V_xV_{\boldsymbol{bz}}^{\top},\end{aligned}$$

where $\boldsymbol{z}_u$ is a multidimensional standard Brownian motion satisfying that $Corr(\boldsymbol{b}_v, \boldsymbol{z}_u) = V_{\boldsymbol{bz}}$. If $t < v$, by Lemmas A2 and A5, it follows that

$$
\begin{aligned}
\operatorname{Cov}(\boldsymbol{x}_t, \boldsymbol{b}_v) &= \operatorname{Cov}\left(e^{t\Theta}\boldsymbol{x}_0 - \left(I_q - e^{t\Theta}\right)\Theta^{-1}\boldsymbol{\beta} + \int_0^t e^{(t-u)\Theta} V_x d\boldsymbol{z}_u, \boldsymbol{b}_v\right) \\
&= \int_0^t e^{(t-u)\Theta} V_x \operatorname{Cov}(d\boldsymbol{z}_u, \boldsymbol{b}_v) = -\Theta^{-1}\left(I_q - e^{t\Theta}\right) V_x V_{\boldsymbol{bz}}^\top .
\end{aligned}
$$

□

Lemma A7. *Let $\boldsymbol{x}_t$ be the vector of predictive variables in the market and $\boldsymbol{y}_u$ be the vector of log-transformed stock prices. Then, for $t \geq u$, we have*

$$
\operatorname{Cov}(\boldsymbol{x}_t, \boldsymbol{y}_u) = \Theta^{-1} e^{t\Theta}\left(e^{-u\Theta} - I_q\right) V_x \left(\frac{1}{2}\Theta^{-1} V_x^\top U^\top - V_{\boldsymbol{bz}}^\top V_p^\top\right).
$$

Proof. If $t \geq u$, by Lemmas A2, A3 and A6, we have

$$
\begin{aligned}
&\operatorname{Cov}(\boldsymbol{x}_t, \boldsymbol{y}_u) = \operatorname{Cov}\left(\boldsymbol{x}_t, \boldsymbol{y}_0 + \int_0^u (\boldsymbol{\alpha} + U\boldsymbol{x}_v) dv + V_p \boldsymbol{b}_u\right) \\
&= \int_0^u \operatorname{Cov}(\boldsymbol{x}_t, \boldsymbol{x}_v) U^\top dv + \operatorname{Cov}(\boldsymbol{x}_t, \boldsymbol{b}_u) V_p^\top \\
&= \int_0^u -\frac{1}{2} V_x \Theta^{-1} e^{|t-v|\Theta} V_x^\top U^\top dv - \Theta^{-1}\left(e^{(t-u)\Theta} - e^{t\Theta}\right) V_x V_{\boldsymbol{bz}}^\top V_p^\top \\
&= \int_0^u -\frac{1}{2} V_x \Theta^{-1} e^{(t-v)\Theta} V_x^\top U^\top dv - \Theta^{-1}\left(e^{(t-u)\Theta} - e^{t\Theta}\right) V_x V_{\boldsymbol{bz}}^\top V_p^\top \\
&= -\frac{1}{2} V_x \Theta^{-1}\left(-\Theta^{-1}\left[e^{(t-u)\Theta} - e^{t\Theta}\right]\right) V_x^\top U^\top - \Theta^{-1}\left(e^{(t-u)\Theta} - e^{t\Theta}\right) V_x V_{\boldsymbol{bz}}^\top V_p^\top \\
&= \Theta^{-1} e^{t\Theta}\left(e^{-u\Theta} - I_q\right) V_x \left(\frac{1}{2}\Theta^{-1} V_x^\top U^\top - V_{\boldsymbol{bz}}^\top V_p^\top\right).
\end{aligned}
$$

□

Lemma A8. *Let $\boldsymbol{x}_t$ be the vector of predictive variables in the market and $\boldsymbol{m}_t^{(h)}$ be the vector of moving averages based on lookback period h. Then, we have*

$$
\operatorname{Cov}\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(h)}\right) = \left(\Theta e^{t\Theta} + \frac{1}{h}\left(I_q - e^{h\Theta}\right)\right) Q_3^\top,
$$

where

$$
Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.
$$

Proof. By the definition of $\boldsymbol{m}_t^{(h)}$ and Lemma A7, we have

$$
\begin{aligned}
&\operatorname{Cov}\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(h)}\right) = \operatorname{Cov}\left(\boldsymbol{x}_t, \frac{1}{h}\int_{t-h}^t \boldsymbol{y}_u du\right) = \frac{1}{h}\int_{t-h}^t \operatorname{Cov}(\boldsymbol{x}_t, \boldsymbol{y}_u) du \\
&= \frac{1}{h}\int_{t-h}^t \Theta^{-1} e^{t\Theta}\left(e^{-u\Theta} - I_q\right) V_x \left(\frac{1}{2}\Theta^{-1} V_x^\top U^\top - V_{\boldsymbol{bz}}^\top V_p^\top\right) du \\
&= \frac{1}{h}\Theta^{-1} e^{t\Theta}\left(-\Theta^{-1}\left(e^{-t\Theta} - e^{-(t-h)\Theta}\right) - hI_q\right) V_x \left(\frac{1}{2}\Theta^{-1} V_x^\top U^\top - V_{\boldsymbol{bz}}^\top V_p^\top\right) \\
&= \left(-\frac{1}{h}\Theta^{-2}\left(I_q - e^{h\Theta}\right) - \Theta^{-1} e^{t\Theta}\right) V_x \left(\frac{1}{2}\Theta^{-1} V_x^\top U^\top - V_{\boldsymbol{bz}}^\top V_p^\top\right) \\
&= \left(\Theta e^{t\Theta} + \frac{1}{h}\left(I_q - e^{h\Theta}\right)\right)\left(\Theta^{-2} V_x \left(V_{\boldsymbol{bz}}^\top V_p^\top - \frac{1}{2}\Theta^{-1} V_x^\top U^\top\right)\right) \\
&= \left(\Theta e^{t\Theta} + \frac{1}{h}\left(I_q - e^{h\Theta}\right)\right) Q_3^\top,
\end{aligned}
$$

where

$$Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.$$

□

Lemma A9. *Let $\boldsymbol{x}_t$ be the vector of the predictive variables in the market and $\boldsymbol{m}_t^{(s,l)}$ be the vector of the moving average difference based on lookback periods s and l ($l > s$). Then, we have*

$$\mathrm{Cov}\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s,l)}\right) = \left(\frac{1}{s}\left(I_q - e^{s\Theta}\right) - \frac{1}{l}\left(I_q - e^{l\Theta}\right)\right) Q_3^\top,$$

where

$$Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.$$

Proof. By the definition of $\boldsymbol{m}_t^{(s,l)}$ and Lemma A8, we have

$$\begin{aligned}
\mathrm{Cov}\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s,l)}\right) &= \mathrm{Cov}\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s)} - \boldsymbol{m}_t^{(l)}\right) = \mathrm{Cov}\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s)}\right) - \mathrm{Cov}\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(l)}\right) \\
&= \left(\Theta e^{t\Theta} + \frac{1}{s}\left(I_q - e^{s\Theta}\right)\right) Q_3^\top - \left(\Theta e^{t\Theta} + \frac{1}{l}\left(I_q - e^{l\Theta}\right)\right) Q_3^\top \\
&= \left(\frac{1}{s}\left(I_q - e^{s\Theta}\right) - \frac{1}{l}\left(I_q - e^{l\Theta}\right)\right) Q_3^\top,
\end{aligned}$$

where

$$Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.$$

□

By Lemma A9, $\mathrm{Cov}\left(\boldsymbol{x}_t, \boldsymbol{m}_t^{(s,l)}\right)$ is independent of time t.

Lemma A10. *Let $\boldsymbol{y}_t$ be the vector of the log-transformed stock prices. Then, we have*

$$\mathrm{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) = \begin{cases} UK_1(u,v)Q_3^\top + Q_3 K_2(v) U^\top + v V_p V_p^\top, & \text{if } u \geq v, \\ Q_3 K_1(v,u) U^\top + UK_2(u) Q_3^\top + u V_p V_p^\top, & \text{if } u < v. \end{cases}$$

where

$$K_1(u,v) = -e^{(u-v)\Theta} + e^{u\Theta} - v\Theta, \quad K_2(v) = e^{v\Theta} - v\Theta - I_q,$$

$$Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.$$

Proof. By Lemmas A2, A3 and A6, for $u \geq v$, we have

$$\begin{aligned}
&\mathrm{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) \\
&= \mathrm{Cov}\left(\boldsymbol{y}_0 + \int_0^u (\boldsymbol{\alpha} + U\boldsymbol{x}_a) da + V_p \boldsymbol{b}_u, \boldsymbol{y}_0 + \int_0^v (\boldsymbol{\alpha} + U\boldsymbol{x}_j) dj + V_p \boldsymbol{b}_v\right) \\
&= \mathrm{Cov}\left(U \int_0^u \boldsymbol{x}_a da + V_p \boldsymbol{b}_u, U \int_0^v \boldsymbol{x}_j dj + V_p \boldsymbol{b}_v\right) \\
&= U \int_0^u da \int_0^v \mathrm{Cov}(\boldsymbol{x}_a, \boldsymbol{x}_j) dj U^\top + U \int_0^u \mathrm{Cov}(\boldsymbol{x}_a, \boldsymbol{b}_v) da V_p^\top \\
&+ V_p \int_0^v \mathrm{Cov}(\boldsymbol{b}_u, \boldsymbol{x}_j) dj U^\top + V_p \mathrm{Cov}(\boldsymbol{b}_u, \boldsymbol{b}_v) V_p^\top.
\end{aligned}$$

Since

$$
\begin{aligned}
&\int_0^u da \int_0^v \mathrm{Cov}(\boldsymbol{x}_a, \boldsymbol{x}_j) dj \\
&= \int_0^u da \int_0^v -\frac{1}{2} V_x \Theta^{-1} e^{|a-j|\Theta} V_x^\top dj \\
&= -\frac{1}{2} V_x \Theta^{-1} \left[\int_0^v dj \int_0^u e^{|a-j|\Theta} da \right] V_x^\top \\
&= -\frac{1}{2} V_x \Theta^{-1} \left[\int_0^v dj \left(\int_0^j e^{|a-j|\Theta} da + \int_j^u e^{|a-j|\Theta} da \right) \right] V_x^\top \\
&= -\frac{1}{2} V_x \Theta^{-1} \left\{ \int_0^v \left[\int_0^j e^{(j-a)\Theta} da + \int_j^u e^{(a-j)\Theta} da \right] dj \right\} V_x^\top \\
&= -\frac{1}{2} V_x \Theta^{-1} \left\{ \int_0^v \left[-\Theta^{-1} \left(I_q - e^{j\Theta} \right) + \Theta^{-1} \left(e^{(u-j)\Theta} - I_q \right) \right] dj \right\} V_x^\top \\
&= -\frac{1}{2} V_x \Theta^{-1} \Theta^{-1} \left[\int_0^v e^{j\Theta} dj - 2 \int_0^v I_q dj + \int_0^v e^{(u-j)\Theta} dj \right] V_x^\top \\
&= -\frac{1}{2} V_x \Theta^{-1} \Theta^{-1} \left[\Theta^{-1} \left(e^{v\Theta} - I_q \right) - 2v I_q - \Theta^{-1} \left(e^{(u-v)\Theta} - e^{u\Theta} \right) \right] V_x^\top \\
&= -\frac{1}{2} V_x \Theta^{-3} \left(e^{u\Theta} + e^{v\Theta} - e^{(u-v)\Theta} - 2v\Theta - I_q \right) V_x^\top \\
&= -\frac{1}{2} V_x V_x^\top \Theta^{-3} \left(e^{u\Theta} + e^{v\Theta} - e^{(u-v)\Theta} - 2v\Theta - I_q \right),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^u \mathrm{Cov}(\boldsymbol{x}_a, \boldsymbol{b}_v) da \\
&= \int_0^v \mathrm{Cov}(\boldsymbol{x}_a, \boldsymbol{b}_v) da + \int_v^u \mathrm{Cov}(\boldsymbol{x}_a, \boldsymbol{b}_v) da \\
&= \int_0^v -\Theta^{-1} \left(I_q - e^{a\Theta} \right) V_x V_{\boldsymbol{bz}}^\top da + \int_v^u -\Theta^{-1} \left(e^{(a-v)\Theta} - e^{a\Theta} \right) V_x V_{\boldsymbol{bz}}^\top da \\
&= -\Theta^{-1} \left(\int_0^v \left(I_q - e^{a\Theta} \right) da \right) V_x V_{\boldsymbol{bz}}^\top - \Theta^{-1} \left(\int_v^u \left(e^{(a-v)\Theta} - e^{a\Theta} \right) da \right) V_x V_{\boldsymbol{bz}}^\top \\
&= -\Theta^{-1} \left(v I_q - \Theta^{-1} \left(e^{v\Theta} - I_q \right) \right) V_x V_{\boldsymbol{bz}}^\top - \Theta^{-2} \left(e^{(u-v)\Theta} - I_q - e^{u\Theta} + e^{v\Theta} \right) V_x V_{\boldsymbol{bz}}^\top \\
&= \left(-v\Theta + e^{v\Theta} - I_q - e^{(u-v)\Theta} + I_q + e^{u\Theta} - e^{v\Theta} \right) \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top \\
&= \left(e^{u\Theta} - e^{(u-v)\Theta} - v\Theta \right) \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^v \mathrm{Cov}(\boldsymbol{b}_u, \boldsymbol{x}_j) dj = \int_0^v \mathrm{Cov}(\boldsymbol{x}_j, \boldsymbol{b}_u)^\top dj = \int_0^v -V_{\boldsymbol{bz}} V_x^\top \left(I_q - e^{j\Theta} \right) \Theta^{-1} dj \\
&= -V_{\boldsymbol{bz}} V_x^\top \left(\int_0^v \left(I_q - e^{j\Theta} \right) dj \right) \Theta^{-1} = -V_{\boldsymbol{bz}} V_x^\top \left(v I_q - \Theta^{-1} \left(e^{v\Theta} - I_q \right) \right) \Theta^{-1} \\
&= V_{\boldsymbol{bz}} V_x^\top \Theta^{-2} \left(e^{v\Theta} - I_q - v\Theta \right) = \left(\Theta^{-2} V_x V_{\boldsymbol{bz}}^\top \right)^\top \left(e^{v\Theta} - v\Theta - I_q \right),
\end{aligned}
$$

and

$$
\mathrm{Cov}(\boldsymbol{b}_u, \boldsymbol{b}_v) = \min(u, v) I_q,
$$

for $u \geq v$, we obtain that

$$
\begin{aligned}
\mathrm{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) = {} & U \left[-\frac{1}{2} V_x V_x^\top \Theta^{-3} \left(e^{u\Theta} + e^{v\Theta} - e^{(u-v)\Theta} - 2v\Theta - I_q \right) \right] U^\top \\
&+ U \left(e^{u\Theta} - e^{(u-v)\Theta} - v\Theta \right) \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top \\
&+ V_p \left(\Theta^{-2} V_x V_{\boldsymbol{bz}}^\top \right)^\top \left(e^{v\Theta} - v\Theta - I_q \right) U^\top + V_p \min(u, v) V_p^\top.
\end{aligned}
$$

Let

$$K_1(u,v) = -e^{(u-v)\Theta} + e^{u\Theta} - v\Theta, \quad K_2(v) = e^{v\Theta} - v\Theta - I_q,$$

$$Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.$$

Then, $K_1(u,v), K_2(v)$ and Q_1 are symmetric, $Q_1 K_1(u,v) = K_1(u,v) Q_1$, and Q_1, Q_2, and Q_3 are all independent of time t. Therefore, for $u \geq v$,

$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) &= UQ_1(K_1(u,v) + K_2(v))U^\top + UK_1(u,v)Q_2 + Q_2^\top K_2(v)U^\top + vV_pV_p^\top \\
&= UK_1(u,v)Q_1U^\top + UQ_1K_2(v)U^\top + UK_1(u,v)Q_2 + Q_2^\top K_2(v)U^\top + vV_pV_p^\top \\
&= UK_1(u,v)\left(Q_1U^\top + Q_2\right) + \left(UQ_1 + Q_2^\top\right)K_2(v)U^\top + vV_pV_p^\top \\
&= UK_1(u,v)Q_3^\top + Q_3K_2(v)U^\top + vV_pV_p^\top.
\end{aligned}$$

Similarly, for $u < v$, we can derive that

$$\begin{aligned}
\mathrm{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) = (\mathrm{Cov}(\boldsymbol{y}_v, \boldsymbol{y}_u))^\top &= \left(UK_1(v,u)Q_3^\top + Q_3K_2(u)U^\top + uV_pV_p^\top\right)^\top \\
&= Q_3K_1(v,u)U^\top + UK_2(u)Q_3^\top + uV_pV_p^\top.
\end{aligned}$$

□

Lemma A11. *Let $\boldsymbol{m}_t^{(s)}$ and $\boldsymbol{m}_t^{(l)}$ be the vectors of the moving averages based on lookback periods s and l ($l > s$). Then, we have*

$$\mathrm{Cov}\left(\boldsymbol{m}_t^{(s)}, \boldsymbol{m}_t^{(l)}\right) = J(t;s,l) + Q_4(s,l),$$

where

$$\begin{aligned}
J(t;s,l) = &\frac{1}{s}U\Theta^{-1}\left(e^{t\Theta} - e^{(t-s)\Theta}\right)Q_3^\top - tU\Theta Q_3^\top \\
&+ \frac{1}{l}Q_3\Theta^{-1}\left(e^{t\Theta} - e^{(t-l)\Theta}\right)U^\top - tQ_3\Theta U^\top + tV_pV_p^\top,
\end{aligned}$$

$$\begin{aligned}
Q_4(s,l) = &\frac{1}{sl}U\left\{\Theta^{-1}\left[sI_q - \Theta^{-1}\left(e^{l\Theta} - e^{(l-s)\Theta}\right)\right] + \frac{1}{6}\left(3l^2s + s^3\right)\Theta - \frac{1}{2}s^2I_q\right\}Q_3^\top \\
&+ \frac{1}{sl}Q_3\left\{\Theta^{-1}\left[sI_q + \Theta^{-1}\left(I_q - e^{s\Theta}\right)\right] + \frac{1}{6}\left(3l^2s + s^3\right)\Theta - \frac{1}{2}\left(2ls - s^2\right)I_q\right\}U^\top \\
&- \frac{1}{sl}\left[\frac{1}{6}\left(3l^2s + s^3\right)\right]V_pV_p^\top,
\end{aligned}$$

and

$$Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.$$

Proof. By the definitions of $\boldsymbol{m}_t^{(s)}$ and $\boldsymbol{m}_t^{(l)}$, and Lemma A10, we have

$$
\begin{aligned}
&\operatorname{Cov}\left(\boldsymbol{m}_t^{(s)}, \boldsymbol{m}_t^{(l)}\right) \\
&= \operatorname{Cov}\left(\frac{1}{s}\int_{t-s}^{t} \boldsymbol{y}_u du, \frac{1}{l}\int_{t-l}^{t} \boldsymbol{y}_v dv\right) = \frac{1}{sl}\int_{t-s}^{t} du \int_{t-l}^{t} \operatorname{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) dv \\
&= \frac{1}{sl}\int_{t-s}^{t} du \left(\int_{t-l}^{u} \operatorname{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) dv + \int_{u}^{t} \operatorname{Cov}(\boldsymbol{y}_u, \boldsymbol{y}_v) dv\right) \\
&= \frac{1}{sl}\int_{t-s}^{t} du \left\{ \int_{t-l}^{u} \left[UK_1(u,v)Q_3^\top + Q_3K_2(v)U^\top + vV_pV_p^\top\right] dv \right. \\
&\left. + \int_{u}^{t} \left[Q_3K_1(v,u)U^\top + UK_2(u)Q_3^\top + uV_pV_p^\top\right] dv \right\} \\
&= \frac{1}{sl}\int_{t-s}^{t} du \left\{ \int_{t-l}^{u} UK_1(u,v)Q_3^\top dv + \int_{t-l}^{u} Q_3K_2(v)U^\top dv + \int_{t-l}^{u} vV_pV_p^\top dv \right. \\
&\left. + \int_{u}^{t} Q_3K_1(v,u)U^\top dv + \int_{u}^{t} UK_2(u)Q_3^\top dv + \int_{u}^{\top} uV_pV_p^\top dv \right\}.
\end{aligned}
$$

Since

(1). $\int_{t-l}^{u} UK_1(u,v)Q_3^\top dv = \int_{t-l}^{u} U\left(-e^{(u-v)\Theta} + e^{u\Theta} - v\Theta\right)Q_3^\top dv$
$= U\left\{\Theta^{-1}\left(I_q - e^{(u-t+l)\Theta}\right) + e^{u\Theta}(u-t+l) - \frac{1}{2}(u^2-(t-l)^2)\Theta\right\}Q_3^\top,$

(2). $\int_{t-l}^{u} Q_3K_2(v)U^\top dv = \int_{t-l}^{u} Q_3(e^{v\Theta} - v\Theta - I_q)U^\top dv Q_3\left\{\Theta^{-1}\left(e^{u\Theta} - e^{(t-l)\Theta}\right)\right.$
$\left. -\frac{1}{2}(u^2-(t-l)^2)\Theta - (u-t+l)I_q\right\}U^\top,$

(3). $\int_{t-l}^{u} vV_pV_p^\top dv = \frac{1}{2}(u^2-(t-l)^2)V_pV_p^\top,$

(4). $\int_{u}^{t} Q_3K_1(v,u)U^\top dv = Q_3\int_{u}^{t}\left(-e^{(v-u)\Theta} + e^{v\Theta} - u\Theta\right)dvU^\top$
$= Q_3\left(-\Theta^{-1}\left(e^{(t-u)\Theta} - I_q\right) + \Theta^{-1}(e^{t\Theta} - e^{u\Theta}) - u(t-u)\Theta\right)U^\top,$

(5). $\int_{u}^{t} UK_2(u)Q_3^\top dv = U\int_{u}^{t}(e^{u\Theta} - u\Theta - I_q)dvQ_3^\top = U[(t-u)(e^{u\Theta} - u\Theta - I_q)]Q_3^\top,$

(6). $\int_{u}^{t} uV_pV_p^\top dv = u(t-u)V_pV_p^\top,$

we have

$(1)+(5) = U\left\{\Theta^{-1}\left(I_q - e^{(u-t+l)\Theta}\right) + e^{u\Theta}l + \left(\frac{1}{2}(t-u-l)^2 - ul\right)\Theta - (t-u)I_q\right\}Q_3^\top,$

$(2)+(4) = Q_3\left\{\Theta^{-1}\left(e^{t\Theta} + I_q - e^{(t-u)\Theta} - e^{(t-l)\Theta}\right) + \left(\frac{1}{2}(t-u-l)^2 - ul\right)\Theta\right.$
$\left. -(u-t+l)I_q\right\}U^\top,$

$(3)+(6) = -\left(\frac{1}{2}(t-u-l)^2 - ul\right)V_pV_p^\top,$

which implies that

$$
\begin{aligned}
\int_{t-s}^{t}[(1)+(5)]du &= U\left\{ \int_{t-s}^{t} \Theta^{-1}\left(I_q - e^{(u-t+l)\Theta}\right)du + \int_{t-s}^{t} e^{u\Theta} l du \right. \\
&\left. + \int_{t-s}^{t}\left(\frac{1}{2}(t-u-l)^2 - ul\right)\Theta du - \int_{t-s}^{t}(t-u)I_q du\right\}Q_3^\top \\
&= U\left\{\Theta^{-1}\left[sI_q - \Theta^{-1}\left(e^{l\Theta} - e^{(l-s)\Theta}\right)\right] + \Theta^{-1}\left(e^{t\Theta} - e^{(t-s)\Theta}\right)l \right. \\
&\left. + \left(\frac{1}{6}\left(l^3 - (l-s)^3\right) + \frac{1}{2}s^2l - tsl\right)\Theta - \frac{1}{2}s^2 I_q\right\}Q_3^\top,
\end{aligned}
$$

and

$$\begin{aligned}\int_{t-s}^{t}[(2)+(4)]du &= Q_3\bigg\{\int_{t-s}^{t}\Theta^{-1}\Big(e^{t\Theta}+I_q-e^{(t-u)\Theta}-e^{(t-l)\Theta}\Big)du \\ &\quad+\int_{t-s}^{t}\Big(\frac{1}{2}(t-u-l)^2-ul\Big)\Theta du-\int_{t-s}^{t}(u-t+l)I_q du\bigg\}U^\top \\ &= Q_3\bigg\{\Theta^{-1}\Big(se^{t\Theta}+sI_q+\Theta^{-1}\Big(I_q-e^{s\Theta}\Big)-se^{(t-l)\Theta}\Big) \\ &\quad+\Big(\frac{1}{6}\Big(l^3-(l-s)^3\Big)+\frac{1}{2}s^2l-tsl\Big)\Theta-\frac{1}{2}\Big(l^2-(l-s)^2\Big)I_q\bigg\}U^\top,\end{aligned}$$

and

$$\begin{aligned}\int_{t-s}^{t}[(3)+(6)]du &= \int_{t-s}^{t}-\Big(\frac{1}{2}(t-u-l)^2-ul\Big)V_pV_p^\top du \\ &= -\Big(\frac{1}{6}\Big(l^3-(l-s)^3\Big)+\frac{1}{2}s^2l-tsl\Big)V_pV_p^\top.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{Cov}\Big(\boldsymbol{m}_t^{(s)},\boldsymbol{m}_t^{(l)}\Big) \\ &= \frac{1}{sl}U\bigg\{\Theta^{-1}\Big[sI_q-\Theta^{-1}\Big(e^{l\Theta}-e^{(l-s)\Theta}\Big)\Big]+\Theta^{-1}\Big(e^{t\Theta}-e^{(t-s)\Theta}\Big)l \\ &+\Big(\frac{1}{6}\Big(l^3-(l-s)^3\Big)+\frac{1}{2}s^2l-tsl\Big)\Theta-\frac{1}{2}s^2I_q\bigg\}Q_3^\top \\ &+\frac{1}{sl}Q_3\bigg\{\Theta^{-1}\Big(se^{t\Theta}+sI_q+\Theta^{-1}\Big(I_q-e^{s\Theta}\Big)-se^{(t-l)\Theta}\Big) \\ &+\Big(\frac{1}{6}\Big(l^3-(l-s)^3\Big)+\frac{1}{2}s^2l-tsl\Big)\Theta-\frac{1}{2}\Big(l^2-(l-s)^2\Big)I_q\bigg\}U^\top \\ &+\frac{1}{sl}\Big(-\frac{1}{6}\Big(l^3-(l-s)^3\Big)-\frac{1}{2}s^2l+tsl\Big)V_pV_p^\top \\ &= \frac{1}{s}U\Theta^{-1}\Big(e^{t\Theta}-e^{(t-s)\Theta}\Big)Q_3^\top-tU\Theta Q_3^\top+\frac{1}{l}Q_3\Theta^{-1}\Big(e^{t\Theta}-e^{(t-l)\Theta}\Big)U^\top-tQ_3\Theta U^\top \\ &+tV_pV_p^\top+\frac{1}{sl}U\bigg\{\Theta^{-1}\Big[sI_q-\Theta^{-1}\Big(e^{l\Theta}-e^{(l-s)\Theta}\Big)\Big]+\frac{1}{6}\Big(3l^2s+s^3\Big)\Theta-\frac{1}{2}s^2I_q\bigg\}Q_3^\top \\ &+\frac{1}{sl}Q_3\bigg\{\Theta^{-1}\Big[sI_q+\Theta^{-1}\Big(I_q-e^{s\Theta}\Big)\Big]+\frac{1}{6}\Big(3l^2s+s^3\Big)\Theta-\frac{1}{2}\Big(2ls-s^2\Big)I_q\bigg\}U^\top \\ &-\frac{1}{sl}\Big[\frac{1}{6}\Big(3l^2s+s^3\Big)\Big]V_pV_p^\top \\ &= J(t;s,l)+Q_4(s,l),\end{aligned}$$

where

$$\begin{aligned}J(t;s,l) &= \frac{1}{s}U\Theta^{-1}\Big(e^{t\Theta}-e^{(t-s)\Theta}\Big)Q_3^\top-tU\Theta Q_3^\top \\ &\quad+\frac{1}{l}Q_3\Theta^{-1}\Big(e^{t\Theta}-e^{(t-l)\Theta}\Big)U^\top-tQ_3\Theta U^\top+tV_pV_p^\top,\end{aligned}$$

and

$$\begin{aligned}Q_4(s,l) &= \frac{1}{sl}U\bigg\{\Theta^{-1}\Big[sI_q-\Theta^{-1}\Big(e^{l\Theta}-e^{(l-s)\Theta}\Big)\Big]+\frac{1}{6}\Big(3l^2s+s^3\Big)\Theta-\frac{1}{2}s^2I_q\bigg\}Q_3^\top \\ &\quad+\frac{1}{sl}Q_3\bigg\{\Theta^{-1}\Big[sI_q+\Theta^{-1}\Big(I_q-e^{s\Theta}\Big)\Big]+\frac{1}{6}\Big(3l^2s+s^3\Big)\Theta-\frac{1}{2}\Big(2ls-s^2\Big)I_q\bigg\}U^\top \\ &\quad-\frac{1}{sl}\Big[\frac{1}{6}\Big(3l^2s+s^3\Big)\Big]V_pV_p^\top,\end{aligned}$$

and

$$Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.$$

□

It is noted that Q_1 is symmetric, Q_1, Q_2, and Q_3 are independent of s, l, and t, Q_4 is independent of t but is dependent on s and l, and J is dependent on s, l, and t.

Lemma A12. *Let $\boldsymbol{m}_t^{(s,l)}$ be the vector of the moving average difference based on lookback periods s and l ($l > s$). Then, $\mathrm{Var}\left(\boldsymbol{m}_t^{(s,l)}\right)$ is independent of time t, i.e.,*

$$\mathrm{Var}\left(\boldsymbol{m}_t^{(s,l)}\right) = Q_4(s,s) - Q_4(s,l) - Q_4^\top(s,l) + Q_4(l,l),$$

where

$$\begin{aligned} Q_4(s,l) = &\frac{1}{sl} U \left\{ \Theta^{-1} \left[sI_q - \Theta^{-1} \left(e^{l\Theta} - e^{(l-s)\Theta} \right) \right] + \frac{1}{6} \left(3l^2 s + s^3 \right) \Theta - \frac{1}{2} s^2 I_q \right\} Q_3^\top \\ &+ \frac{1}{sl} Q_3 \left\{ \Theta^{-1} \left[sI_q + \Theta^{-1} \left(I_q - e^{s\Theta} \right) \right] + \frac{1}{6} \left(3l^2 s + s^3 \right) \Theta - \frac{1}{2} \left(2ls - s^2 \right) I_q \right\} U^\top \\ &- \frac{1}{sl} \left[\frac{1}{6} \left(3l^2 s + s^3 \right) \right] V_p V_p^\top, \end{aligned}$$

and

$$Q_3 = UQ_1 + Q_2^\top, \quad Q_2 = \Theta^{-2} V_x V_{\boldsymbol{bz}}^\top V_p^\top, \quad Q_1 = -\frac{1}{2} V_x V_x^\top \Theta^{-3}.$$

Proof. By the definition of $\boldsymbol{m}_t^{(s,l)}$ and Lemma A11, we have

$$\begin{aligned} &\mathrm{Var}\left(\boldsymbol{m}_t^{(s,l)}\right) = \mathrm{Var}\left(\boldsymbol{m}_t^{(s)} - \boldsymbol{m}_t^{(l)}\right) \\ &= \mathrm{Cov}\left(\boldsymbol{m}_t^{(s)}, \boldsymbol{m}_t^{(s)}\right) - \mathrm{Cov}\left(\boldsymbol{m}_t^{(s)}, \boldsymbol{m}_t^{(l)}\right) - \mathrm{Cov}\left(\boldsymbol{m}_t^{(l)}, \boldsymbol{m}_t^{(s)}\right) + \mathrm{Cov}\left(\boldsymbol{m}_t^{(l)}, \boldsymbol{m}_t^{(l)}\right) \\ &= \left[J(t;s,s) - J(t;s,l) - J^\top(t;s,l) + J(t;l,l) \right] \\ &+ \left[Q_4(s,s) - Q_4(s,l) - Q_4^\top(s,l) + Q_4(l,l) \right] \\ &= Q_4(s,s) - Q_4(s,l) - Q_4^\top(s,l) + Q_4(l,l), \end{aligned}$$

in view of the fact that

$$J(t;s,s) - J(t;s,l) - J^\top(t;s,l) + J(t;l,l) = 0.$$

□

Lemma A13. *Let $\boldsymbol{b}_t^{(0)}$ be an n-dimensional standard Brownian motion and $\boldsymbol{z}_t^{(0)}$ be a q-dimensional standard Brownian motion. Assume that $\boldsymbol{b}_t^{(0)}$ and $\boldsymbol{z}_t^{(0)}$ are independent. If there is a symmetric matrix Γ such that $\Gamma\Gamma^\top = \begin{pmatrix} I_n & V_{\boldsymbol{bz}} \\ V_{\boldsymbol{bz}}^\top & I_q \end{pmatrix}$, then $(\boldsymbol{b}_t \ \boldsymbol{z}_t)^\top = \Gamma \left(\boldsymbol{b}_t^{(0)} \ \boldsymbol{z}_t^{(0)} \right)^\top$ are multidimensional standard Brownian motions and $\mathrm{Corr}(\boldsymbol{b}_t, \boldsymbol{z}_t) = V_{\boldsymbol{bz}}$.*

Proof. Since $\boldsymbol{b}_t^{(0)}$ and $\boldsymbol{z}_t^{(0)}$ are independent standard Brownian motions with dimensions n and q, respectively, we have $\mathrm{Var}\begin{pmatrix}\boldsymbol{b}_t^{(0)}\\ \boldsymbol{z}_t^{(0)}\end{pmatrix} = tI_{(n+q)}$. Let $\begin{pmatrix}\boldsymbol{b}_t\\ \boldsymbol{z}_t\end{pmatrix} = \Gamma\begin{pmatrix}\boldsymbol{b}_t^{(0)}\\ \boldsymbol{z}_t^{(0)}\end{pmatrix}$. Then, we have

$$\mathrm{Var}\begin{pmatrix}\boldsymbol{b}_t\\ \boldsymbol{z}_t\end{pmatrix} = \Gamma\mathrm{Var}\begin{pmatrix}\boldsymbol{b}_t^{(0)}\\ \boldsymbol{z}_t^{(0)}\end{pmatrix}\Gamma^\top = \Gamma tI_{(n+q)}\Gamma^\top = t\Gamma\Gamma^\top = t\begin{pmatrix}I_n & V_{\boldsymbol{bz}}\\ V_{\boldsymbol{bz}}^\top & I_q\end{pmatrix},$$

which implies that

$$\mathrm{Var}(\boldsymbol{b}_t) = tI_n, \quad \mathrm{Var}(\boldsymbol{z}_t) = tI_q, \quad \mathrm{Cov}(\boldsymbol{b}_t, \boldsymbol{z}_t) = tV_{\boldsymbol{bz}}\ ,$$

and hence $\mathrm{Corr}(\boldsymbol{b}_t, \boldsymbol{z}_t) = V_{\boldsymbol{bz}}$. In addition, we have

$$\begin{aligned}\mathrm{Var}\begin{pmatrix}\boldsymbol{b}_{t+dt} - \boldsymbol{b}_t\\ \boldsymbol{z}_{t+dt} - \boldsymbol{z}_t\end{pmatrix} &= \mathrm{Var}\left[\Gamma\begin{pmatrix}\boldsymbol{b}_{t+dt}^{(0)} - \boldsymbol{b}_t^{(0)}\\ \boldsymbol{z}_{t+dt}^{(0)} - \boldsymbol{z}_t^{(0)}\end{pmatrix}\right] = \Gamma\mathrm{Var}\left[\begin{pmatrix}\boldsymbol{b}_{t+dt}^{(0)} - \boldsymbol{b}_t^{(0)}\\ \boldsymbol{z}_{t+dt}^{(0)} - \boldsymbol{z}_t^{(0)}\end{pmatrix}\right]\Gamma^\top\\ &= \Gamma dtI_{(n+q)}\Gamma^\top = dt\Gamma\Gamma^\top = dt\begin{pmatrix}I_n & V_{\boldsymbol{bz}}\\ V_{\boldsymbol{bz}}^\top & I_q\end{pmatrix},\end{aligned}$$

which implies that

$$\mathrm{Var}(\boldsymbol{b}_{t+dt} - \boldsymbol{b}_t) = dtI_n, \quad \mathrm{Var}(\boldsymbol{z}_{t+dt} - \boldsymbol{z}_t) = dtI_q,$$

and hence both $\boldsymbol{b}_t$ and $\boldsymbol{z}_t$ are standard Brownian motions. □

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