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General Fractional Calculus in Multi-Dimensional Space: Riesz Form

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Abstract: An extension of the general fractional calculus (GFC) is proposed as a generalization of the Riesz fractional calculus, which was suggested by Marsel Riesz in 1949. The proposed Riesz form of GFC can be considered as an extension GFC from the positive real line and the Laplace convolution to the m-dimensional Euclidean space and the Fourier convolution. To formulate the general fractional calculus in the Riesz form, the Luchko approach to construction of the GFC, which was suggested by Yuri Luchko in 2021, is used. The general fractional integrals and derivatives are defined as convolution-type operators. In these definitions the Fourier convolution on m-dimensional Euclidean space is used instead of the Laplace convolution on positive semi-axis. Some properties of these general fractional operators are described. The general fractional analogs of first and second fundamental theorems of fractional calculus are proved. The fractional calculus of the Riesz potential and the fractional Laplacian of the Riesz form are special cases of proposed general fractional calculus of the Riesz form.

Keywords: general fractional calculus; fractional derivatives; fractional integrals; Riesz potential; nonlocality

MSC: 26A33

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1. Introduction

Fractional calculus of integrals and derivatives of arbitrary order is an analogous to standard mathematical calculus of integer-order integrals and derivatives [1–8]. Fractional calculus is used to describe systems and processes with non-locality in space and time (for example see books [9–18]), in which one can see the application of fractional calculus in various fields of sciences from mechanics to economics. Handbooks [19,20] contain descriptions of the application of fractional calculus in 25 different areas of physics.

The operators of fractional calculus have different interpretations [21–30], which include the following: (GI) Interpretations in terms of geometry [31–38], (PhI) Interpretations in terms of physics [25–30,35–38], (EcI) Interpretations in terms of economics [39,40], (PrI) Interpretations in terms of probability theory [41–45], and (InI) Interpretations in terms of information theory [46,47].

The fractional calculus is usually used to describe systems and processes with non-locality of the power-law type. To describe different forms of nonlocalities, a general fractional calculus should be formulated. Note that in order for some general operators to form a general fractional calculus, they must satisfy some fractional analogues of the first and second fundamental theorems of standard calculus [48]. These theorems allow us to interpret general fractional operators as some analogues of integrals and derivatives of integer order [49]. It should be noted that these theorems lead to the fact that at least one of the two operator kernels, which describe the respectively a general integral and a general derivative, should be singular [50,51].

The general fractional calculus actually arose in Sonin's work published in 1884 [52] (see also [53,54]). However, the name "general fractional calculus" began to be used starting from the work of Kochubei in 2011 [55]. In the last decade, the general fractional calculus has been actively developed and applied in various fields of science [56–75]. The important form of the general fractional calculus was proposed by Luchko in works of 2021–2022 [76–85]. Different applications of the Luchko general fractional calculus has been considered in works [86–94].

Problems and trends in the development and application of the general fractional calculus were described in work [95]. One of the problems is the extension of the general fractional calculus to the entire real axis and the entire m-dimensional Euclidean space. An extension of Luchko type of GFC on multi-dimensional case is proposed in [87], where a general fractional vector calculus is considered. However, this GFC was proposed for the regions from the set $\mathbb{R}^m_+ = \{(x_1, \ldots, x_m) : x_j \geq 0 \text{ for all } j = 1 \ldots m\}$. The entire m-dimensional Euclidean space is not considered in [87].

One of the important types of fractional calculus in multi-dimensional Euclidean spaces is the theory proposed by Marcel Riesz. The Riesz fractional derivatives and potentials were first suggested in works [96,97] (see also [1,98]). Note that important interpretation of the Riesz fractional Laplacianis noted in paper [98]. These operators also are considered in different works (see [1,4,99–118]). Note that the Riesz fractional derivative are connected with the Liouville fractional derivatives (for example, see Section 12 in [1] for $\mathbb{R}$). The Riesz derivative is also related to Grünwald–Letnikov fractional derivatives (Section 20 in [1]), and the Marchaud fractional derivative (Section 5.4 in [1]). The Riesz fractional derivative can be connected with the Caputo fractional derivatives (see Equations 2.4.6 and 2.4.7 of [4]). The Grünwald–Letnikov–Riesz derivative of this type is considered and applied in [119–122].

This article proposes the construction of an extension of the Riesz fractional calculus to a wide class of operator kernels. In this extension GFC, the m-dimensional Euclidean space and the Fourier convolution are used instead of the positive real line and the Laplace convolution. In the formulation of the GFC in the Riesz form, the Luchko approach to construction of the GFC, which was proposed in [76–85], is used. Therefore this extension can be considered as an extension of the Luchko's general fractional calculus for m-dimensional Euclidean spaces. The sets of pairs of operator kernels for general fractional operators in the Riesz form are defined. The spaces of functions, for which the proposed operators exist, are also defined. Some basic properties of the proposed Riesz general fractional integrals and the Riesz general fractional operators are described. The first and second fundamental theorems of general fractional calculus in the Riesz form are proved. The well-known fractional calculus of the Riesz potential and the fractional Laplacian of the Riesz form are special cases of proposed general fractional calculus of the Riesz form.

It should be note that there are many different definitions of fractional Laplacian [123–125]. In this case, the fractional derivatives of the Riesz form are usually interpreted as a fractional Laplacian. The fractional integrals of the Riesz form are usually interpreted as a fractional Riesz potential. In the framework of these interpretations, the general fractional derivatives and integrals of the Riesz form can be interpreted as a general fractional Laplacian and general fractional Riesz potential.

The content of the article is following. In Section 2, the Fourier convolution and its properties are described as preliminaries. In Section 3, sets of functions and kernel pairs are defined. In Section 4, general fractional integrals, general fractional derivatives of the Caputo type and Riemann–Liouville types are defined in the framework of the GFC in the Riesz form. In Section 5, semi-group properties of GF integration of the Riesz form are proved. In Section 6, action of the Laplacian on GF integrals of the Riesz form is described. In Section 7, first amd second fundamental theorems of GFC in Riesz form are proved. A brief conclusion is given in Section 8.

2. Preliminaries: Fourier Convolution and Its Properties

For m-dimensional Euclidean space $\mathbb{R}^m$, the distance between points $P(x_1, \ldots, x_m)$ and $Q(y_1, \ldots, y_m)$ is described by the equation

$$r_{PQ} = r_{QP} = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{j=1}^{m} (x_j - y_j)^2} \quad (1)$$

that can be considered as the length of the vector

$$\mathbf{r}_{QP} = -\mathbf{r}_{PQ} = \mathbf{x} - \mathbf{y}, \quad (2)$$

where $\mathbf{x}(x_1, \ldots, x_m)$ and $\mathbf{y}(y_1, \ldots, y_m)$.

The Riesz fractional integrals (the Riesz potential) and its generalization can be defined by using the Fourier convolution (see Section 25.3 in [1], pp. 494–495). Therefore, definition of this convolution and some well-known properties will be given below.

Let $f(\mathbf{x})$ and $g(\mathbf{x})$ belong to the space $L_1(\mathbb{R}^m)$. The Fourier convolution is defined by the equation

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^m} g(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d^m\mathbf{y} = \int_{\mathbb{R}^m} f(\mathbf{x} - \mathbf{y})\, g(\mathbf{y})\, d^m\mathbf{y}. \quad (3)$$

The Fourier convolution exists only if functions $f(\mathbf{x})$ and $g(\mathbf{x})$ decay sufficiently rapidly at infinity. Conditions for the existence of the convolution can include different conditions on the functions, since a blow-up in $g(\mathbf{x})$ at infinity can be compensated by sufficiently rapid decay in $f(\mathbf{x})$ at infinity.

For example, convolution (3) of $f(\mathbf{x})$ and $g(\mathbf{x})$ exists, if $f(\mathbf{x})$ and $g(\mathbf{x})$ are Lebesgue integrable in $L_1(\mathbb{R}^m)$. TIn this case, the Fourier convolution $(f * g)(\mathbf{x})$ is also integrable (for example, see Theorem 1.3 in [126], p. 3).

As another example, one can consider $f(\mathbf{x}) \in L_1(\mathbb{R}^m)$ and $g(\mathbf{x}) \in L_p(\mathbb{R}^m)$, where $1 \leq p \leq \infty$. In this case, $(f * g)(\mathbf{x}) \in L_p(\mathbb{R}^m)$ and the following inequality is satisfied

$$\|(f * g)(\mathbf{x})\|_p \leq \|f(\mathbf{x})\|_1 \, \|g(\mathbf{x})\|_p. \quad (4)$$

For the case $p = 1$, inequality (4) gives that the space $L_1(\mathbb{R}^m)$ is a Banach algebra under the convolution.

In the general case, the Young's inequality for convolution [127] states the following property of the Fourier convolution. If $f(\mathbf{x}) \in L_p(\mathbb{R}^m)$ and $g(\mathbf{x}) \in L_q(\mathbb{R}^m)$, where $1 \leq p, q, r \leq \infty$, then $(f * g)(\mathbf{x}) \in L_r(\mathbb{R}^m)$ and the following inequality is satisfied

$$\|(f * g)(\mathbf{x})\|_r \leq \|f(\mathbf{x})\|_p \, \|g(\mathbf{x})\|_q, \quad (5)$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1. \quad (6)$$

The Fourier convolution can be defined such that the associativity property is satisfied

$$(f * (g * h))(\mathbf{x}) = ((f * g) * h)(\mathbf{x}). \quad (7)$$

Note that one can to define the Fourier convolution of a function with a generalized function (distribution). The Fourier convolution of two generalized functions (distributions) can also be defined. Let $f(\mathbf{x})$ be a function with compact support and $g(\mathbf{x})$ a generalized function (distribution). Then $(f * g)(\mathbf{x})$ is a smooth function defined by equation analogous to Equation (3), [1]. For a wide class of functions, for which the Fourier convolution is performed, one can consider convolution of these functions with the Dirac delta function $\delta(\mathbf{x})$ in the form

$$(f * \delta)(\mathbf{x}) = f(\mathbf{x}). \tag{8}$$

For distributions $f(\mathbf{x})$ and $g(\mathbf{x})$ the convolution is defined by the equation

$$(f * g) = (f(\mathbf{x}) \times g(\mathbf{y}), \varphi(\mathbf{x} + \mathbf{y}), \tag{9}$$

where $\times$ is direct product, φ is a function that belongs to the space of infinitely differentiable finite functions. Equation (9) is valid if at least one of $f(\mathbf{x})$ and $g(\mathbf{x})$ has compact support.

3. Sets of Functions and Kernel Pairs

Let us first define sets of functions on the Euclidean space $\mathbb{R}^m$.

Definition 1. *Let a function $f(\mathbf{x})$ on the space $\mathbb{R}^m$ can be represented in the form*

$$f(\mathbf{x}) = |\mathbf{x}|^a \, A(\mathbf{x}), \tag{10}$$

where $a > -m$ and $A(\mathbf{x}) \in C(\mathbb{R}^m)$.
Then the set of such functions will be denoted by the symbol $C_{-m}(\mathbb{R}^m)$.

The set $C_{-m}(\mathbb{R}^m)$ is an analog (m-dimensional analog) of the set $C_{-1}(0, \infty)$ that is used in the general fractional calculus in the Luchko form [76,77,80,81], which was formulated for the positive semi-axis.

Definition 2. *Let $p \in \mathbb{N}_0$ and let a function $f(\mathbf{x})$ on the space $\mathbb{R}^m$ satisfy the condition*

$$(-\Delta)^p \, f(\mathbf{x}) \in C_{-m}(\mathbb{R}^m), \tag{11}$$

where Δ is the Laplace operator

$$\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}. \tag{12}$$

Then, the set of such functions is denoted as $C^{2p}_{-m}(\mathbb{R}^m)$.

Definition 3. *Let $p \in \mathbb{N}_0$ and let a function $f(\mathbf{x})$ satisfy the conditions:*
(1) $(-\Delta)^p \, f(\mathbf{x}) \in L_1(\mathbb{R}^m)$,
(2) $(-\Delta)^p \, f(\mathbf{x}) \in C_{-m}(\mathbb{R}^m)$.
Then, the set of such functions is denoted as $\mathcal{C}^{2p}_{-m}(\mathbb{R}^m)$.

Let us now define sets of pairs of operator kernels for operators on the Euclidean space $\mathbb{R}^m$.

Definition 4. *Let pair of two functions $M(\mathbf{x}) = M(|\mathbf{x}|) \in L_1(\mathbb{R}^m)$, and $K(\mathbf{x}) = K(|\mathbf{x}|) \in L_1(\mathbb{R}^m)$ satisfy the following conditions.*
(1) The functions $M(|\mathbf{x}|)$ and $K(|\mathbf{x}|)$ belong to the set $C_{-m}(\mathbb{R}^m)$.
(2) The Fourier convolution of these functions has the form

$$(M * K)(|\mathbf{x}|) = M_{2-m}(|\mathbf{x}|), \tag{13}$$

where for $m \neq 2$,

$$M_{2-m}(|\mathbf{x}|) = \frac{|\mathbf{x}|^{2-m}}{H_m(2)} = \frac{1}{4\,\pi^{m/2}} \Gamma\left(\frac{m-2}{2}\right) |\mathbf{x}|^{2-m}. \tag{14}$$

Then, the set of such pairs is denoted as $\mathcal{R}^m$.

As an example of kernel pair (M, K) that belongs to the set $\mathcal{R}^m$, one can consider the functions

$$M(|\mathbf{x}|) = M_{\alpha-m}(|\mathbf{x}|) = \frac{|\mathbf{x}|^{\alpha-m}}{H_m(\alpha)}, \tag{15}$$

$$K(|\mathbf{x}|) = M_{2-\alpha-m}(|\mathbf{x}|) = \frac{|\mathbf{x}|^{2-\alpha-m}}{H_m(2-\alpha)}, \tag{16}$$

where

$$H_m(\alpha) = \frac{\pi^{m/2}\, 2^{\alpha}\, \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{m-\alpha}{2}\right)} \tag{17}$$

with $0 < \alpha < m$ and $0 < 2 - \alpha < m$.

Definition 5. *Let a pair of functions $M(\mathbf{x}) = M(|\mathbf{x}|) \in L_1(\mathbb{R}^m)$, and $K(\mathbf{x}) = K(|\mathbf{x}|) \in L_1(\mathbb{R}^m)$ satisfy the following conditions.*

(1) The functions $M(|\mathbf{x}|)$ and $K(|\mathbf{x}|)$ belong to the set $C_{-m}(\mathbb{R}^m)$.

(2) The Fourier convolution of these functions has the form

$$(M * K)(|\mathbf{x}|) = M_{2p-m}(|\mathbf{x}|). \tag{18}$$

Then, the set of such functions is denoted as $\mathcal{R}^m_{2p}$.

The set $\mathcal{R}^m_{2p}$ can be interpreted as an analog of the Luchko set $\mathcal{L}_n$ that is proposed in works [76,77,80,81] for the GFC on $(0, \infty)$.

As an example of kernel pair (M, K) that belongs to the set $\mathcal{R}^m_{2p}$, one can consider the functions

$$M(|\mathbf{x}|) = M_{\alpha-m}(|\mathbf{x}|) = \frac{|\mathbf{x}|^{\alpha-m}}{H_m(\alpha)}, \tag{19}$$

$$K(|\mathbf{x}|) = M_{2p-\alpha-m}(|\mathbf{x}|) = \frac{|\mathbf{x}|^{2p-\alpha-m}}{H_m(2p-\alpha)}, \tag{20}$$

where $0 < \alpha < m$ and $0 < 2\,p - \alpha < m$, the function $H_m(\alpha)$ is defined by Equation (17). The convolution of kernels (19) and (20) is proved in [97] by the following transformations

$$(M * K)(|\mathbf{x}|) = (M_{\alpha-m} * M_{2p-\alpha-m})(|\mathbf{x}|) = M_{\alpha+(2p-\alpha)-m}(|\mathbf{x}|) = M_{2p-m}(|\mathbf{x}|). \tag{21}$$

As another example of a kernel pair (M, K) one can consider the following

$$M(|\mathbf{x}|) = (M_{2p-m} * M_G)(|\mathbf{x}|), \tag{22}$$

$$K(|\mathbf{x}|) = K_G(|\mathbf{x}|), \tag{23}$$

where

$$(M_G * K_G)(|\mathbf{x}|) = \delta^m(|\mathbf{x}|) \tag{24}$$

with m-dimensional Dirac delta function δ^m. Condition (24) means that $M_G = S$ and $K_G = S^{-1}$, where S is a distributions (generalized function), which has have an inverse element S^{-1} for the Fourier convolution. It is known that some generalized functions (distributions) S have an inverse S^{-1} with respect to the convolution, for which the equation

$$(S * S^{-1})(|\mathbf{x}|) = \delta^m(|\mathbf{x}|) \tag{25}$$

is satisfied in the generalized sense. For distributions $M_G(|\mathbf{x}|)$ and $K_G(|\mathbf{x}|)$ the convolution is defined by the equation

$$(M_G * K_G) = (M_G(|\mathbf{x}|) \times K_G(|\mathbf{y}|), \varphi(\mathbf{x} + \mathbf{y})), \tag{26}$$

where $\times$ is direct product, φ is a function belonging to the space of infinitely differentiable finite functions. Equation (26) is valid if at least one of $M_G(|\mathbf{x}|)$ and $K_G(|\mathbf{x}|)$ has compact support. Note that the set of invertible generalized functions (distributions) is an abelian group the Fourier convolution.

4. General Fractional Operators of Riesz Form

Let us now define general fractional operators of Riesz form that are interpreted as general fractional integrals (GFIs) and general fractional derivatives (GFDs).

Below are the definitions are generalizations of the well-known Riesz operators [97] (the Riesz fractional integral and the Riesz fractional derivatives) from the case of operator kernels of the power-law types to operator kernels of general type (belonging to the set $\mathcal{R}^m_{2p}$). The definitions of the general fractional operators proposed below in a sense can be considered as the expansion of the definitions of the general fractional integral and the general fractional derivatives, which are proposed by Luchko for the $[0, \infty)$ in [76], for the m-dimensional Euclidean space $\mathbb{R}^m$. In this consideration, the first-order derivative in the GFD of the Luchko form of GFC is replaced by standard Laplace operator.

Definition 6. *Let kernel pair (M, K) belong to the set $\mathcal{R}^m_{2p}$.*

Then the general fractional integral of the Riesz form is defined by the equation

$$(I_{(M)} f)(\mathbf{x}) = (M * f)(\mathbf{x}) = \int_{\mathbb{R}^m} M(|\mathbf{x} - \mathbf{y}|) \, f(\mathbf{y}) \, d^m\mathbf{y}, \tag{27}$$

where $f(\mathbf{x}) \in \mathcal{C}^0_{-m}(\mathbb{R}^m)$, i.e., $f(\mathbf{x}) \in L_1(\mathbb{R}^m)$ such that $f(\mathbf{x}) \in C_{-m}(\mathbb{R}^m)$.

Definition 7. *Let kernel pair (M, K) belong to the set $\mathcal{R}^m_{2p}$.*

Then the Riesz general fractional derivative of the Caputo (C) type is defined by the equation

$$(D^*_{(K)} f)(\mathbf{x}) = (K * (-\Delta)^p f)(\mathbf{x}) = \int_{\mathbb{R}^m} K(|\mathbf{x} - \mathbf{y}|) \, (-\Delta)^p f(\mathbf{y}) \, d^m\mathbf{y}, \tag{28}$$

where $f(\mathbf{x}) \in \mathcal{C}^{2p}_{-m}(\mathbb{R}^m)$, i.e., $(-\Delta)^p f(\mathbf{x}) \in L_1(\mathbb{R}^m)$ such that $(-\Delta)^p f(\mathbf{x}) \in C_{-m}(\mathbb{R}^m)$.

Definition 8. *Let kernel pair (M, K) belong to the set $\mathcal{R}^m_{2p}$.*

Then the Riesz general fractional derivative of the Riemann–Liouville (RL) type (R-GFD-RL) is defined by the equation

$$(D_{(K)} f)(\mathbf{x}) = (-\Delta)^p \, (K * f)(\mathbf{x}) = (-\Delta)^p \int_{\mathbb{R}^m} K(|\mathbf{x} - \mathbf{y}|) \, f(\mathbf{y}) \, d^m\mathbf{y}, \tag{29}$$

where $f(\mathbf{x}) \in \mathcal{C}^0_{-m}(\mathbb{R}^m)$, i.e., $f(\mathbf{x}) \in L_1(\mathbb{R}^m)$ such that $f(\mathbf{x}) \in C_{-m}(\mathbb{R}^m)$.

The Riesz GF derivative of the Caputo type can be written as

$$(D^*_{(K)} f)(\mathbf{x}) = (I_{(K)} \, (-\Delta)^p f)(\mathbf{x}). \tag{30}$$

The Riesz GF derivative of the RL type can be written as

$$(D_{(K)} f)(\mathbf{x}) = (-\Delta)^p \, (I_{(K)} f)(\mathbf{x}). \tag{31}$$

Example 1. *As an example of the Riesz GFI, one can consider the Riesz potential (Riesz fractional integral) that is defined in [97] by the equation*

$$(I^{\alpha} f)(\mathbf{x}) \;=\; \int_{\mathbb{R}^m} M_{\alpha-m}(|\mathbf{x} - \mathbf{y}|)\, f(\mathbf{y})\, d^m\mathbf{y}, \tag{32}$$

where $0 < \alpha < m$*, and the kernel* $M_{\alpha-m}(|\mathbf{x}|)$ *can be written in the form*

$$M_{\alpha-m}(|\mathbf{x}|) \;=\; \frac{|\mathbf{x}|^{\alpha-m}}{H_m(\alpha)} \;=\; \frac{\Gamma\left(\dfrac{m-\alpha}{2}\right)}{\pi^{m/2}\, 2^{\alpha}\, \Gamma\left(\dfrac{\alpha}{2}\right)}\, |\mathbf{x}|^{\alpha-m}. \tag{33}$$

The parameter α *is called the order of the fractional integral. The Riesz general fractional integral with kernel (33) describes the well-known Riesz potential [1,4,97].*

As an example of the Riesz GFD, one can consider the operator

$$(D^{*}_{(K)} f)(\mathbf{x}) \;=\; \int_{\mathbb{R}^m} M_{2p-\alpha-m}(|\mathbf{x} - \mathbf{y}|)\, (-\Delta)^p\, f(\mathbf{y})\, d^m\mathbf{y}, \tag{34}$$

where $0 < \alpha < m$*,* $0 < 2p - \alpha < m$*, and the kernel* $K(|\mathbf{x}|) = M_{2p-\alpha-m}(|\mathbf{x}|)$ *is defined by Equation (20).*

Remark 1. *In order for some general operators to form a general fractional calculus, they must satisfy some fractional analogues of the first and second fundamental theorems of standard calculus [48]. These theorems allow us to interpret general fractional operators as some analogues of integrals and derivatives of integer order [49]. Note that these fundamental theorems of GFC lead to the fact that at least one of the two operator kernels, which describe the respectively a general integral and a general derivative, should be singular [50,51]. It should be noted that standard Riesz fractional derivative is called the fractional Laplacian of the Riesz form and it is defined by the hyper-singular integral [1].*

If an equation with some integral and differential operators can be presented as a differential equation with a finite number of integer-order derivatives, then these operators cannot describe nonlocality. This statement is based on the fact that integer-order derivatives are determined by properties of differentiable functions only in an infinitely small neighborhood of the considered point. For example, the GF derivatives of the Riesz form are local derivatives, if the kernel pair is defined by Equations (19) and (20).

5. Semi-Group Properties of Riesz GF Integration

Let us describe semi-group property of the Riesz general fractional integrals.

Property 1 (Semi-group property of Riesz GFI)**.** *Let kernels* $M_1 = M_1(|\mathbf{x}|)$ *and* $M_2 = M_2(|\mathbf{x}|)$ *belong to the space* $L_1(\mathbb{R}^m)$ *and to the set* $C_{-m}(\mathbb{R}^m)$ *such that the following condition is satisfied*

$$(M_1 * M_2)(\mathbf{x}) \in C_{-m}(\mathbb{R}^m). \tag{35}$$

Then, the semi-group property is satisfied for the Riesz general fractional integrals in the form

$$(I_{(M_1)}\, I_{(M_2)}\, f)(\mathbf{x}) \;=\; (I_{(M_1 * M_2)}\, f)(\mathbf{x}) \tag{36}$$

for all $\mathbf{x} \in \mathbb{R}^m$*, if the function* $f(\mathbf{x})$ *belongs to the space* $L_q(\mathbb{R}^m)$ *with* $1 \le q \le \infty$*.*

Proof. It is well known that the convolution of the kernels $M_1 = M_1(|\mathbf{x}|)$ and $M_2 = M_2(|\mathbf{x}|)$ belongs to the space $L_1(\mathbb{R}^m)$, i.e.,

$$(M_1 * M_2)(\mathbf{x}) \in L_1(\mathbb{R}^m), \tag{37}$$

if functions $M_1 = M_1(|\mathbf{x}|)$ and $M_2 = M_2(|\mathbf{x}|)$ belong to the space $L_1(\mathbb{R}^m)$.

The semi-group property (36) follows directly from the associativity property of the Fourier convolution and definition of Riesz GFI

$$(I_{(M_1)}\, I_{M_2}\, f)(\mathbf{x}) = (I_{(M_1)}(M_2 * f))(\mathbf{x}) = (M_1 * (M_2 * f))(\mathbf{x}) =$$

$$((M_1 * M_2) * f)(\mathbf{x}) = (I_{(M_1 * M_2)}\, f)(\mathbf{x}). \tag{38}$$

Condition (35) means that the Riesz GF integral $(I_{(M_1 * M_2)}\, f)(\mathbf{x})$ exists, if, for example, $f(\mathbf{x})$ belongs to the space $L_q(\mathbb{R}^m)$ with $1 \leq q \leq \infty$. □

As an example of the semi-group property (36), one can consider the semi-group property for the Riesz potentials. In this case, the kernels has the form

$$M_1 = M_1(|\mathbf{x}|) = M_{\alpha-m}(|\mathbf{x}|), \quad M_2 = M_2(|\mathbf{x}|) = M_{\beta-m}(|\mathbf{x}|) \tag{39}$$

and, property (36) is presented by the equation

$$(I^{\alpha}\, I^{\beta}\, f)(\mathbf{x}) = (I^{\alpha+\beta}\, f)(\mathbf{x}) \tag{40}$$

with $0 < \alpha < m$, and $0 < \beta < m$, where the condition (35), is satisfied if the inequality

$$0 < \alpha + \beta < m, \tag{41}$$

holds (see [97], p. 20).

6. Action of Laplacian on Riesz GF Integrals

Let us consider an action of the Laplacian on the Riesz GF integrals. First consider the action of the Laplacian on the Newtonian potential, which can be considered as the Riesz GFI with the kernel $M = M_{2-m}(|\mathbf{x}|)$.

It is well-known that the Newtonian potential

$$\varphi(\mathbf{x}) = (I^2\, f)(\mathbf{x}) = \int_{\mathbb{R}^m} \frac{|\mathbf{x} - \mathbf{y}|^{2-m}}{H_m(2)}\, f(\mathbf{y})\, d^m\mathbf{y}. \tag{42}$$

is the solution of the equation

$$\Delta\, \varphi(\mathbf{x}) = -f(\mathbf{x}). \tag{43}$$

As a result, the substitution of function (42) into Equation (43) gives that the following property

$$\Delta\, (I^2\, f)(\mathbf{x}) = -f(\mathbf{x}) \tag{44}$$

is satisfied for all $\mathbf{x} \in \mathbb{R}^m$. Using the notations of Riesz GFI, Equation (44) can be presented in the form

$$\Delta\, (I_{M_{2-m}}\, f)(\mathbf{x}) = -f(\mathbf{x}). \tag{45}$$

Using the semi-group property of the Riesz GFI in the form

$$I_{M_{2p-m}} = I^{2p} = (I^2)^p = (I_{M_{2-m}})^p, \tag{46}$$

one can see that Equation (45) gives

$$(-\Delta)^p\, (I_{M_{2p-m}}\, f)(\mathbf{x}) = f(\mathbf{x}), \tag{47}$$

where $p \in \mathbb{N}_0$ and $2p < m$.

Remark 2. *Note that Equation (45) means that the operator $I^2 = I_{(M_{2-m})}$ is the inverse of the operator $(-\Delta)$ (see [97], p. 21), where for $m \neq 2$ the integral $I^2 = I_{(M_{2-m})}$ is described by the equation*

$$(I^2 f)(\mathbf{x}) = \int_{\mathbb{R}^m} \frac{|\mathbf{x} - \mathbf{y}|^{2-m}}{H_m(2)} f(\mathbf{y})\, d^m\mathbf{y}, \tag{48}$$

where

$$H_m(2) = \frac{4\,\pi^{m/2}}{\Gamma\left(\frac{m-2}{2}\right)}. \tag{49}$$

Let us prove the following property that describes the action of the Laplacians on the Riesz GFIs.

Property 2 (Action of Laplacian on Riesz GFI). *Let kernel $M_{2p-m}(|\mathbf{x}|)$ and $M(|\mathbf{x}|)$ belong to the space $L_1(\mathbb{R}^m)$ and to the set $C_{-m}(\mathbb{R}^m)$ such that the following condition is satisfied*

$$(M_{2p-m} * M)(\mathbf{x}) \in C_{-m}(\mathbb{R}^m). \tag{50}$$

Then, the action of the Laplacian (Laplace operator) on the Riesz general fractional integral is described by the equation

$$(-\Delta)^p\, (I_{(M_{2p-m} * M)}\, f)(\mathbf{x}) = (I_{(M)}\, f)(\mathbf{x}). \tag{51}$$

As a special case $p = 1$, Equation (51) has the form

$$\Delta\, (I_{(M_{2-m} * M)}\, f)(\mathbf{x}) = -\,(I_{(M)}\, f)(\mathbf{x}). \tag{52}$$

Proof. The proof of property (52) follows directly from the semi-group property of the Riesz general fractional integrals (36) in the form

$$\Delta\, (I_{(M_{2-m} * M)}\, f)(\mathbf{x}) = \Delta\, (I_{(M_{2-m})}\, I_{(M)}\, f)(\mathbf{x}) =$$

$$(\Delta\, I_{(M_{2-m})}\, (I_{(M)}\, f))(\mathbf{x}) = \Delta\, (I^2\, (I_{(M)}\, f))(\mathbf{x}) = \Delta\, (I^2\, g)(\mathbf{x}) \tag{53}$$

with $g(\mathbf{x}) = (I_{(M)}\, f)(\mathbf{x})$. Using property (44), Equation (53) gives equality (52).

Using the semi-group property of the Riesz GFI in the form

$$I_{(M_{2p-m}) * M)} = I^{2p}\, I_{(M)} = (I^2)^p\, I_{(M)}, \tag{54}$$

the successive repetitions of applying Equation (53) gives equality (51). □

As an example of Equation (52) of Property 2, one can consider the action of the Laplacian on the Riesz potential that is described by the equation

$$\Delta\, (I^{\alpha+2}\, f)(\mathbf{x}) = -\,(I^{\alpha}\, f)(\mathbf{x}). \tag{55}$$

In paper [97], it was proved that the Riesz potential

$$\varphi(\mathbf{x}) = (I^{\alpha}\, f)(\mathbf{x}) = (I_{(M_{\alpha-m})}\, f)(\mathbf{x}) = \int_{\mathbb{R}^m} \frac{|\mathbf{x} - \mathbf{y}|^{\alpha-m}}{H_m(\alpha)} f(\mathbf{y})\, d^m\mathbf{y} \tag{56}$$

is the solution of the equation

$$(D_{(K)}\, \varphi)(\mathbf{x}) = -\,f(\mathbf{x}), \tag{57}$$

where $K(\mathbf{x}) = M_{2p-\alpha-m}(\mathbf{x})$ and $p \in \mathbb{N}_0,\, 0 \leq p < m$.

Remark 3. *The limiting case of a zero-order Riesz fractional integral (the Riesz potential) is described by the equation (see Equation (16) in [97], p. 23) in the form*

$$(I^0 f)(\mathbf{x}) \,=\, \lim_{\alpha \to 0+} (I^{\alpha} f)(\mathbf{x}) \,=\, f(\mathbf{x}). \tag{58}$$

Using Equation (58), one can see that

$$\lim_{\alpha \to 0+} \Delta\, (I^{\alpha+2} f)(\mathbf{x}) \,=\, -\lim_{\alpha \to 0+} (I^{\alpha} f)(\mathbf{x}), \tag{59}$$

gives

$$\Delta\, (I^2 f)(\mathbf{x}) \,=\, -\,(I^0 f)(\mathbf{x}) \,=\, -\,f(\mathbf{x}). \tag{60}$$

7. Fundamental Theorems of GFC in Riesz Form

Let us define a set of functions that is used in the first fundamental theorem of the GFC in the Riesz form.

Definition 9. *Let a function $K = K(|\mathbf{x}|)$ belong to the set $C_{-m}(\mathbb{R}^m)$, and let a function $f(x)$ can be represented in the form*

$$f(\mathbf{x}) \,=\, (I_{(K)}\, \varphi)(\mathbf{x}) \tag{61}$$

for all $\mathbf{x} \in \mathbb{R}$, where $\varphi(\mathbf{x}) \in C_{-m}(\mathbb{R}^m)$.

Then, the set of such functions $f(x)$ is denoted as $C_{-m,(K)}(\mathbb{R}^m)$.

Theorem 1 (First Fundamental Theorem for Riesz GFD of RLtype)**.** *Let (M, K) be a pair of the kernels from the set $\mathcal{R}_{2p}$.*

The Riesz GFD of the Riemann–Liouville type is a left inverse operator to the Riesz GFI and the equation

$$(D_{(K)}\, I_{(M)}\, f)(\mathbf{x}) \,=\, f(\mathbf{x}) \tag{62}$$

holds for all $\mathbf{x} \in \mathbb{R}$, if the function $f(\mathbf{x})$ belongs to the space $C_{-m}(\mathbb{R}^m)$.

Proof. To prove Equation (62), the definition of the Riesz GFD of the Riemann–Liouville type can be written in the form

$$(D_{(K)}\, g)(\mathbf{x}) \,=\, ((-\,\Delta)^p\, I_{(K)}\, g)(\mathbf{x}), \tag{63}$$

where one can use $g(\mathbf{x}) \,=\, (I_{(M)}\, f)(\mathbf{x})$. Then, the left side of Equation (62) takes the form

$$(D_{(K)}\, I_{(M)}\, f)(\mathbf{x}) \,=\, ((-\,\Delta)^p\, I_{(K)}\, I_{(M)}\, f)(\mathbf{x}) \,=\, f(\mathbf{x}). \tag{64}$$

Using that the Riesz GFI can be represented through the Fourier convolution and the fact that the pair (M, K) belongs to the set $\mathcal{R}_{2p}$, one can get

$$(D_{(K)}\, I_{(M)}\, f)(\mathbf{x}) \,=\, (-\,\Delta)^p\, (K * M * f)(\mathbf{x}) \,=\, (-\,\Delta)^p\, (M_{2p-m} * f)(\mathbf{x}) \,=$$

$$(-\,\Delta)^p\, (I_{(M_{2p-m})}\, f)(\mathbf{x}) \,=\, ((-\,\Delta)^p\, I^{2p}\, f)(\mathbf{x}) \,=\, f(\mathbf{x}), \tag{65}$$

where the equality

$$(\Delta\, I^2\, h)(\mathbf{x}) \,=\, -\,h(\mathbf{x})$$

is used p times. □

Theorem 2 (First Fundamental Theorem for Riesz GFD of Caputo type)**.** *Let (M, K) be a pair of the kernels from the set $\mathcal{R}_{2p}$.*

The Riesz GFD of the Caputo type is a left inverse operator to the Riesz GFI and the equation

$$(D^*_{(K)}\, I_{(M)}\, f)(\mathbf{x}) \;=\; f(\mathbf{x}) \tag{66}$$

holds for all $\mathbf{x} \in \mathbb{R}$, *if the function* $f(\mathbf{x})$ *belongs to the set* $C_{-m,(K)}(\mathbb{R}^m)$.

Proof. To prove Equation (66), the definition of the Riesz GFD of the Caputo type can be written in the form

$$(D^*_{(K)}\, g)(\mathbf{x}) \;=\; (I_{(K)}\, (-\Delta)^p\, g)(\mathbf{x}), \tag{67}$$

where one can use $g(\mathbf{x}) = (I_{(M)}\, f)(\mathbf{x})$. Using the fact that $f(\mathbf{x})$ belongs to the set $C_{-m,(K)}(\mathbb{R}^m)$, where

$$f(\mathbf{x}) \;=\; (I_{(K)}\, \varphi)(\mathbf{x}), \tag{68}$$

one can represent the function $g(\mathbf{x})$ in the form

$$g(\mathbf{x}) \;=\; (I_{(M)}\, f)(\mathbf{x}) \;=\; (I_{(M)}\, I_{(K)}\, \varphi)(\mathbf{x}) \;=\; (M * K * \varphi)(\mathbf{x}). \tag{69}$$

Using the fact that the pair (M, K) belongs to the set $\mathcal{R}_{2p}$, i.e., $(M * K) = M_{2p-m}$ one can get

$$g(\mathbf{x}) \;=\; (M_{2p-m} * \varphi)(\mathbf{x}) \;=\; (I_{(M_{2p-m})}\, \varphi)(\mathbf{x}) \;=\; (I^{2p}\, \varphi)(\mathbf{x}). \tag{70}$$

Then, using Equation (70), Equation (67) takes the form

$$(D^*_{(K)}\, g)(\mathbf{x}) \;=\; (I_{(K)}\, (-\Delta)^p\, I^{2p}\, \varphi)(\mathbf{x}) \;=\; (I_{(K)}\, \varphi)(\mathbf{x}) \;=\; f(\mathbf{x}), \tag{71}$$

where the equality$(\Delta\, I^2\, h)(\mathbf{x}) = -h(\mathbf{x})$ is used p times and Equation (68) is taken into account. □

Theorem 3 (Second Fundamental Theorem for Riesz GFD of RL type)**.** *Let* (M, K) *be a pair of the kernels from the set* $\mathcal{R}_{2p}$.

The Riesz GFD of the Riemann–Liouville type is a right inverse operator to the Riesz GFI and the equation

$$(I_{(M)}\, D_{(K)}\, f)(\mathbf{x}) \;=\; f(\mathbf{x}) \tag{72}$$

holds for all $\mathbf{x} \in \mathbb{R}$, *if the function* $f(\mathbf{x})$ *belongs to the space* $C_{-m,(M)}(\mathbb{R}^m)$.

Proof. To prove Equation (72), the definition of the Riesz GFD of the Riemann–Liouville type can be written in the form

$$(D_{(K)}\, f)(\mathbf{x}) \;=\; (-\Delta)^p\, (I_{(K)}\, f)(\mathbf{x}). \tag{73}$$

Using the fact that $f(\mathbf{x})$ belongs to the set $C_{-m,(M)}(\mathbb{R}^m)$, where

$$f(\mathbf{x}) \;=\; (I_{(M)}\, \varphi)(\mathbf{x}), \tag{74}$$

and the representation of the Riesz GFI as the Fourier convolution, the Riesz GFD (73) can be represented in the form

$$(D_{(K)}\, f)(\mathbf{x}) \;=\; (-\Delta)^p\, (I_{(K)}\, I_{(M)}\, \varphi)(\mathbf{x}) \;=\; (-\Delta)^p\, (K * M * \varphi)(\mathbf{x}). \tag{75}$$

Then, using the fact that the pair (M, K) belongs to the set $\mathcal{R}_{2p}$, Equation (75) gives

$$(D_{(K)}\, f)(\mathbf{x}) \;=\; (-\Delta)^p\, (K * M * \varphi)(\mathbf{x}) \;=\; (-\Delta)^p\, (M_{2p-m} * \varphi)(\mathbf{x}) \;=$$

$$(-\Delta)^p\, (I_{(M_{2p-m})}\, \varphi)(\mathbf{x}) \;=\; (-\Delta)^p\, (I^{2p}\, \varphi)(\mathbf{x}) \;=\; \varphi(\mathbf{x}), \tag{76}$$

where the equality $(\Delta\, I^2\, h)(\mathbf{x}) \;=\; -h(\mathbf{x})$ is used p times for $h_k \;=\; (I^{2(p-k)}\varphi)$, $k \;=\; 1, \ldots, p$. Using Equation (76), the left side of Equation (72) takes the form

$$(I_{(M)}\, D_{(K)}\, f)(\mathbf{x}) \;=\; (I_{(M)}\, \varphi)(\mathbf{x}) \;=\; f(\mathbf{x}), \tag{77}$$

where Equation (74) is taken into account. □

Remark 4. *Using Equation (12) in [97], p. 21, in the form*

$$\Delta \frac{|\mathbf{x}|^{\alpha+2-m}}{H_m(\alpha+2)} \;=\; -\frac{|\mathbf{x}|^{\alpha-m}}{H_m(\alpha)}, \tag{78}$$

which can be written as

$$\Delta\, M_{\alpha+2-m}(\mathbf{x}) \;=\; -M_{\alpha-m}(\mathbf{x}), \tag{79}$$

and applying p times the Green's equation (see [97], p. 23) in the form

$$\int_{\mathbb{R}^m} f(\mathbf{y})\, \Delta\, g(\mathbf{y})\, d^m\mathbf{y} \;=\; \int_{\mathbb{R}^m} g(\mathbf{y})\, \Delta\, f(\mathbf{y})\, d^m\mathbf{y}, \tag{80}$$

the following equation is proved in [97] one can get

$$(I^\alpha\, f)(\mathbf{x}) \;=\; (-1)^p\, (I^{\alpha+2p}\, (-\Delta)^p\, f)(\mathbf{x}), \tag{81}$$

where $p \in \mathbb{N}$. Here it is assumed the following conditions: (A) the function $f(\mathbf{x})$ has continuous derivatives of any order $k \leq 2p$; (B) the function $f(\mathbf{x})$ and its derivatives behave at infinity in such a way that the integrals are absolutely convergent and that the integrations by parts are satisfied.

Using Equation (81), one can have the analytic extension of the operator $(I^\alpha\, f)(\mathbf{x})$ for any value of $\alpha > -2p$.

Let us give a formulation of the second fundamental theorem of the Riesz GF calculus.

Theorem 4 (Second Fundamental Theorem for Riesz GFD of C type)**.** *Let (M, K) be a pair of the kernels from the set $\mathcal{R}_{2p}$.*

The Riesz GFD of the Caputo type is a right inverse operator to the Riesz GFI and the equation

$$(I_{(M)}\, D^*_{(K)}\, f)(\mathbf{x}) \;=\; f(\mathbf{x}) \tag{82}$$

holds for all $\mathbf{x} \in \mathbb{R}$, if the function $f(\mathbf{x})$ belongs to the set $C^{2p}_{-m,(K)}(\mathbb{R}^m)$ and conditions of Remark 4 are satisfied for $f(\mathbf{x})$.

Proof. To prove Equation (82), one should use the definition of the Riesz GFD of the Caputo type has the form

$$(D^*_{(K)}\, f)(\mathbf{x}) \;=\; (I_{(K)}\, (-\Delta)^p\, f)(\mathbf{x}). \tag{83}$$

Then, using Equation (83) and the representation of the Riesz GFI as the Fourier convolution, the left side of Equation (82) takes the form

$$(I_{(M)}\, D^*_{(K)}\, f)(\mathbf{x}) \;=\; (I_{(M)}\, I_{(K)}\, (-\Delta)^p\, f)(\mathbf{x}) \;=\; (M * K * ((-\Delta)^p\, f))(\mathbf{x}). \tag{84}$$

Then, using the fact that the pair (M, K) belongs to the set $\mathcal{R}_{2p}$, Equation (84) gives

$$(I_{(M)}\, D^{*}_{(K)}\, f)(\mathbf{x}) \,=\, (M_{2p-m} * ((-\Delta)^{p}\, f))(\mathbf{x}) \,=\, (I_{(M_{2p-m})}\, (-\Delta)^{p}\, f)(\mathbf{x}) \,=\, (I^{2p}\, (-\Delta)^{p}\, f)(\mathbf{x}). \tag{85}$$

Using Equation (17) in [97], p. 23, with $\alpha = 0+$ in the form

$$(I^{\alpha+2p}\, (-\Delta)^{p}\, f)(\mathbf{x}) \,=\, (I^{\alpha}\, f)(\mathbf{x}) \quad (\alpha \,\to\, 0+) \tag{86}$$

for functions $f(\mathbf{x})$ that satisfy the conditions of Remark 4. As a result, Equation (85) gives Equation (82). □

Remark 5. *The assumptions about the properties of functions $f(\mathbf{x})$ and its derivatives, which are used in Remark 4, were put forward for the entire space $\mathbb{R}^m$. Equations become more complicated, if one can admit certain $(m-1)$-surfaces of discontinuity (see [97], p. 24). Let us restrict ourselves to the case where $f(\mathbf{x})$ is identically zero outside a closed surface S, sufficiently regular, while the function $f(\mathbf{x})$ and the derivatives thereof intervene are continuous in the closed domain bounded by this surface, without canceling out on the surface, in general.*

Using the Riesz fractional integral that is defined as

$$(I^{\alpha}\, f)(\mathbf{x}) \,=\, \int_{\Omega} \frac{|\mathbf{x}-\mathbf{y}|^{\alpha-m}}{H_m(\alpha)}\, f(\mathbf{y})\, d^m\mathbf{y}, \tag{87}$$

and the Green's equation, one can get (see Equation (20) in [97], p. 24) the equation

$$(I^{\alpha}\, f)(\mathbf{x}) \,=\, (I^{\alpha+2p}\, (-\Delta)^{p}\, f)(\mathbf{x}) +$$

$$\sum_{k=1}^{p} \int_{S} \Big(\Big((-\Delta)^{k-2}\, f \Big)(\mathbf{y})\, \frac{dM_{\alpha+2k-m}(|\mathbf{x}-\mathbf{y}|)}{dn} \,-\, \frac{d(-\Delta)^{k-1}\, f(\mathbf{y})}{dn}\, M_{\alpha+2k-m}(|\mathbf{x}-\mathbf{y}|) \Big)\, dS, \tag{88}$$

where

$$M_{\alpha+2k-m}(|\mathbf{x}|) \,=\, \frac{|\mathbf{x}|^{\alpha+2k-m}}{H_m(\alpha+2k)}, \tag{89}$$

and n is the normal to the point $\mathbf{y}(y_1, \dots y_m)$ of S directed towards the interior of this surface, and dS the surface element around the point $Q(y_1, \dots y_m)$.

As a special case of Equation (88) one can consider the limit $\alpha \to 0+$, to get

$$(I^{2p}\, (-\Delta)^{p}\, f)(\mathbf{x}) \,=\, f(\mathbf{x}) -$$

$$\sum_{k=1}^{p} \int_{s} \Big(\Big((-\Delta)^{k-2} f \Big)(\mathbf{y})\, \frac{dM_{2k-m}(|\mathbf{x}-\mathbf{y}|)}{dn} \,-\, \frac{d(-\Delta)^{k-1}\, f(\mathbf{y})}{dn}\, M_{2k-m}(|\mathbf{x}-\mathbf{y}|) \Big)\, dS. \tag{90}$$

Taking into account Remark 5, the second fundamental theorem for Riesz GFD of Caputo type can be formulated in the following more general form.

Theorem 5 (Second Fundamental Theorem for Riesz GFD of C type in general form)**.** *Let (M, K) be a pair of the kernels from the set $\mathcal{R}_{2p}$.*

Let function $f(\mathbf{x})$ belongs to the space $C^{2p}_{-m,(K)}(\mathbb{R}^m)$ and let $f(\mathbf{x})$ be identically zero outside a closed surface S, sufficiently regular, while the function $f(\mathbf{x})$ and the derivatives thereof intervene are continuous in the closed domain bounded by this surface.

Then, the action of the Riesz GFD of the Caputo type on the Riesz GFI is described by the equation

$$(I_{(M)}\, D^{*}_{(K)}\, f)(\mathbf{x}) \; = $$

$$f(\mathbf{x}) - \sum_{k=1}^{p} \int_{s} \Big(\Big((-\Delta)^{k-2} f \Big)(\mathbf{y}) \, \frac{dM_{2k-m}(|\mathbf{x} - \mathbf{y}|)}{dn} - \frac{d(-\Delta)^{k-1} f(\mathbf{y})}{dn} \, M_{2k-m}(|\mathbf{x} - \mathbf{y}|) \Big) \, dS \quad (91)$$

holds for all $\mathbf{x} \in \mathbb{R}$.

Proof. To prove Equation (91), one can use the proof of Theorem 4 and repeat the transformation from Equation (83) to Equation (85). Then, using Equation (85) in the form

$$(I_{(M)}\, D^{*}_{(K)}\, f)(\mathbf{x}) \; = \; (I^{2p}\, (-\Delta)^{p}\, f)(\mathbf{x}), \quad (92)$$

where one should be used Equation (90) of Remark 5 to get equality (91). □

8. Conclusions

An extension of the general fractional calculus in Luchko's form to the multi-dimensional case was first proposed in [87]. Then, this calculus was applied to construct nonlocal physical models in [90–92,94] and nonlocal probability theory [93]. However, this extension did not use the entire multi-dimensional Euclidean space $\mathbb{R}^m$.

In this paper, an extension of the general fractional calculus, which takes into account the entire multi-dimensional Euclidean space $\mathbb{R}^m$, is proposed. The suggested extension of the GFC is in fact a generalization of the well-known fractional Riesz calculus [1,97,98] from power-type operator kernels to a wider class of operator kernels. The proposed multi-dimensional form of GFC can also be considered as an extension GFC from positive real line, which is used in [76–85], and the Laplace convolution to the entire m-dimensional Euclidean space and the Fourier convolution.

Let us briefly list the main results of this work.

(a) The general fractional integrals and derivatives are defined as convolution type operators. In these definitions the Fourier convolution on m-dimensional Euclidean space is used instead of the Laplace convolution on positive semi-axis. These operators are called the Riesz general fractional operators.

(b) The sets of operator kernels for general fractional operators in the Riesz form are described. The spaces of functions, for which the proposed operators exist, are also described.

(c) Some basic properties of the proposed Riesz general fractional integrals and the Riesz general fractional operators are considered.

(d) The general fractional analogs of first and second fundamental theorems of calculus are proved for the general fractional operators.

(e) The fractional calculus of the Riesz potential and the Riesz fractional Laplacian are special cases of proposed general fractional calculus in the Riesz form.

Let us describe some possibility of future research in the framework of the proposed approach to building a general fractional calculus of the Riesz type and possible generalizations of the proposed multi-dimensional general fractional calculus.

(M1) It is important, the derive the series representations of kernels that belongs to the set, which will be analogous of the Sonin representations proposed in [52,53] and described by Equations (21)–(23) in Luchko papers [76,77].

(M2) It is important to get various examples of the operator pairs that belongs to the set $\mathcal{R}^m_{2p}$, which can be considered as analogs of the examples of the operator kernels that are given in Table 1 in [90], page 6 and [91], p.15. (see also [93], pp. 22–23, [92], p. 6, [94], p. 11).

(M3) It should be noted that Riesz proposed a fractional calculus not only for the Euclidean space, but also for the Minkowski space, which is actively used in relativistic physics. Therefore it is important to extend the proposed general fractional operators of the Riesz form from the Euclidean space, but also for the Minkowski space by using results of the Riesz paper [97] and the propose work.

(M4) The proposed GF derivatives of the Riesz form can be interpreted as GF Laplacians of the Riesz form. The GF integrals of the Riesz form can be interpreted as GF Riesz potentials. It is interesting to derive an exact discretization of the general fractional Laplacian of the Riesz form by using the approach, which is is proposed in [128]. This exact discrete GF Laplacian can be considered as an extension of the exact discretization of the Riesz fractional Laplacian proposed in [129]. Note that exact finite differences of integer and fractional orders are proposed in [128–130] are a generalization in some sense of the Mickens non-standard finite differences (see papers [131–133] and bookd [134–136]). Note that exact finite differences of integer order satisfy the same characteristic algebraic identities and properties as derivatives of integer order. Note that non-standard finite difference are used for fractional-order differential equations [137]. The exact fractional differences can be considered as an exact discrete analog of the fractional-order operators of the Riesz type, which are connected by the transform operation preserving algebraic structure. Note the exact discretization of the Riesz fractional Laplacian in proposed in [129]. One can assume that exact general fractional differences can also defined as discrete analogs of the proposed general fractional operators of the Riesz form.

(M5) A lattice analog of fractional calculus in the Riesz form is proposed in papers [113–115]. This lattice fractional calculus can also be generalized from power-type operator kernels to a wider class of operator kernels by used the proposed multi-dimensional general fractional calculus. This lattice GFC calculus can be important in application to physical lattice models with long-range interactions, including the lattice models in quantum field theory [120].

Let us describe briefly some important elements for future research in applications of the proposed approach to formulate nonlocal physical models by using general fractional calculus of the Riesz type. Note that the proposed GF derivatives of the Riesz form can be interpreted as GF Laplacians of the Riesz form. The GF integrals of the Riesz form can be interpreted as GF Riesz potentials. These interpretations of the proposed GF operators largely dictate their possible applications in physics and mechanics.

(P1) The proposed GF derivatives of the Riesz form (the GF Laplacians of the Riesz form) can be used to generalize results proposed in papers for fractional gravity theory [117,138,139] and general nonlocal gravity theory [92].

(P2) The proposed GF derivatives of the Riesz form (the GF Laplacians of the Riesz form) can be used to generalize results proposed in papers [140–142], to describe electrodynamics of plasma-like media [143–145] and spatial dispersion in crystal optics [146–148]. Nonlocal models of such media can be considered as a special form of general nonlocal electrodynamics [91].

(P3) The GF Laplacians of the Riesz form (GF derivatives of the Riesz form) can be used to describe wide class of nonlocality in the framework of fractional gradient elasticity models of media with spatial dispersion. The models with GF Laplacian will be generalizations of the fractional gradient elasticity models that are the first time proposed proposed in [149,150] (see aalso [151,152]. These models an also be considered in the framework of the general nonlocal comtinuum mechan that is proposed in [90].

(P4) The GF Laplacians of the Riesz form can be useful to describe chaotic systems with long-range interaction [153,154].

(P5) The proposed general fractional derivatives (GF Laplacians) of the Riesz form can be used to generalized the fractional Schrodinger equation, which are described by Laskin in [155,156] (see also important comments in [157–159]). The general form of operator kenlals can be also important in nonlocal quantum field theory [120].

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