

Article

Half Logistic Inverted Nadarajah–Haghighi Distribution under Ranked Set Sampling with Applications

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Abstract: In this paper, we present the half logistic inverted Nadarajah–Haghighi (HL-INH) distribution, a novel extension of the inverted Nadarajah–Haghighi (INH) distribution. The probability density function (PDF) for the HL-INH distribution might have a unimodal, right skewness, or heavy-tailed shape for numerous parameter values; however, the shape forms of the hazard rate function (HRF) for the HL-INH distribution may be decreasing. Four specific entropy measurements were investigated. Some useful expansions for the HL-INH distribution were investigated. Several statistical and computational features of the HL-INH distribution were calculated. Using simple (SRS) and ranked set sampling (RSS), the parameters for the HL-INH distribution were estimated using the maximum likelihood (ML) technique. A simulation analysis was executed in order to determine the model parameters of the HL-INH distribution using the SRS and RSS methods, and RSS was shown to be more efficient than SRS. We demonstrate that the HL-INH distribution is more adaptable than the INH distribution and other statistical distributions when utilizing three real-world datasets.

Keywords: inverted Nadarajah–Haghighi distribution; maximum likelihood estimation; simple random sample; ranked set sampling; half logistic family

MSC: 60E05; 62E15; 62F10; 62P99



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1. Introduction

The concept of probability distribution has made significant contributions to the process of dealing with both small and big datasets. Probability distribution models are very important and are widely utilized in various sectors such as physics, health-care, business management, engineering, and food; it also covers the study of social sciences such as economics and psychology. They are ideally suited for the prediction and forecasting of real-world issue modeling. There are several ways for generalizing distributions, including: beta-G [1]; generalized Kumaraswamy-G [2]; Weibull odd Burr III-G [3]; Gompertz-G [4]; a new power Topp-Leone-G [5]; exponentiated Kumaraswamy-G [6]; type I half logistic Weibull-G [7]; type I half logistic Burr X-G [8]; the transmuted Burr X-G in [9]; odd power Lindley-G [10]; gamma Kumaraswamy-G [11]; new extended cosine-G in [12]; extended-gamma-G [13]; odd Chen-G [14]; Kumaraswamy type I half logistic-G [15]; log-logistic-G [16]; gamma-G [17]; Kumaraswamy Poisson-G [18]; Kumaraswamy Kumaraswamy-G [19]; additive odd-G [20]; beta generalized Marshall–Olkin Kumaraswamy-G [21]; extended alpha power transformed family of distributions [22]; odd

Burr X-G [23]; the Weibull–G in [24]; type II half logistic-G [25]; sec-G [26]; generalized odd linear exponential-G [27]; Stacy-G [28]; odd Perks-G [29]; sine Topp-Leone-G [30]; Kumaraswamy generalized Marshall–Olkin-G [31]; arcsine-exponentiated-X family-G [32]; odd exponentiated half logistic-G [33]; Kumaraswamy Marshall–Olkin-G [34]; and sine-exponentiated Weibull–H [35], among others. Recently, ref. [36] proposed the type I half logistic (HL)-G, and it has the next cumulative distribution function (CDF):

$$F(y; \alpha, \varsigma) = \frac{1 - [1 - G(y; \alpha, \varsigma)]^\alpha}{1 + [1 - G(y; \alpha, \varsigma)]^\alpha}, \quad \alpha > 0, y \in \mathbb{R}, \quad (1)$$

where ς is the vector of parameters for the baseline distribution. The corresponding PDF of the HL-G is provided via:

$$f(y; \alpha, \varsigma) = \frac{2\alpha g(y; \alpha, \varsigma)[1 - G(y; \alpha, \varsigma)]^{\alpha-1}}{[1 + [1 - G(y; \alpha, \varsigma)]^\alpha]^2}. \quad (2)$$

Ref. [37] proposed the Nadarajah–Haghighi (NH) distribution as a generalization version of the exponential distribution. Its PDF allows for decreasing and unimodal forms, but its HRF allows for decreasing, constant, and increasing shapes. Moreover, like the Weibull, generalized exponential, and lifespan models, its PDF, survival function (SF), and HRF have two parameters. The authors also demonstrated how to interpret the NH model as a shortened Weibull distribution. The PDF and CDF of the NH model are provided as below:

$$G(z; \lambda, \gamma) = 1 - e^{1-(1+\lambda z)^\gamma}, \quad z > 0, \lambda, \gamma > 0, \quad (3)$$

and

$$g(z; \lambda, \gamma) = \lambda\gamma(1 + \lambda z)^{\gamma-1}e^{1-(1+\lambda z)^\gamma}, \quad z > 0, \lambda, \gamma > 0, \quad (4)$$

Ref. [38] proposed a novel inverted model called the INH distribution using the transformation $Y = 1/Z$, where Z has an NH distribution. The CDF and PDF of INH distribution are given by:

$$G(y; \lambda, \gamma) = e^{1-(1+\lambda y^{-1})^\gamma}, \quad y > 0, \lambda, \gamma > 0, \quad (5)$$

and

$$g(y; \lambda, \gamma) = \lambda\gamma y^{-2}(1 + \lambda y^{-1})^{\gamma-1}e^{1-(1+\lambda y^{-1})^\gamma}, \quad y > 0, \lambda, \gamma > 0. \quad (6)$$

Many authors have studied the generalization of the INH distribution, such as: Marshall–Olkin INH (MOINH) distribution [39]; transmuted INH distribution [40]; extended odd Weibull INH distribution (EOWINH) [41]; power INH (PINH) distribution [42]; discrete INH distribution [43]; and odd Lomax INH (OLINH) distribution [44].

The main purpose of this paper is to present the HL-INH model as a new extension of the INH model with one scale parameter λ and two shape parameters γ and α . The following factors provide adequate motivation and explanation to explore the HL-INH model. We will characterize them as follows:

1. The HL-INH distribution is more flexible than the INH distribution, and this is recommended in the application part;
2. The curves of the PDF for the HL-INH distribution may be right-skewed, uni-modal, and heavy-tailed, but the HRF may be decreasing;
3. Several basic important mathematical characteristics of the HL-INH distribution, such as the probability-weighted moments (PWMs), ordinary moments, generating function, and various kinds of entropy, such as the Rényi entropy (REN), Tsallis entropy (TEN), Arimoto entropy (AEN), and Havrda and Charvat entropy (HCEN);
4. The quantile function in the HL-INH distribution has a closed shape (QUF). The QUF is used to determine Bowley skewness (BOS) and Moors kurtosis (MOK);

5. The methodology of ML estimation was utilized to estimate the three unknown parameters of the HL-INH distribution using both simple and ranked set sampling;
6. For modeling real-world datasets in several disciplines, the HL-INH distribution fits better than the INH distribution and several other well-known statistical distributions, which is suggested in the application part.

The following describes the way that this manuscript is structured. Section 2 describes how to create the HL-INH model by merging the INH model with the HL generated family of distributions. In Section 3, we develop and analyze various essential expansions for calculating the statistical features of the HL-INH. In Section 4, several entropy measurements are calculated. In Section 5, we investigate various important mathematical characteristics of the HL-INH model. Section 6 discusses RSS. Section 7 investigates estimations of the three unknown parameters utilizing the ML approach under SRS and RSS. Section 8 discusses the simulation results. The relevance and adaptability of the HL-INH distribution are demonstrated in Section 9 by employing three real-world datasets. Furthermore, the concluding remarks are made at the end of the manuscript.

2. The HL-INH Distribution

In this section, we construct three new parameters for continuous statistical distribution called the half logistic inverted Nadarajah–Haghighi (HL-INH) distribution by inserting Equations (5) and (6) into (1) and (2); then, the CDF, PDF, and reliability function (RF) are:

$$F(y; \alpha, \lambda, \gamma) = \frac{1 - \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^\alpha}{1 + \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^\alpha}, \quad (7)$$

$$f(y; \alpha, \lambda, \gamma) = \frac{2\alpha\lambda\gamma y^{-2}(1 + \lambda y^{-1})^{\gamma-1} e^{1 - (1 + \lambda y^{-1})^\gamma} \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^{\alpha-1}}{\left[1 + \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^\alpha\right]^2}, \quad (8)$$

and

$$S(y; \alpha, \lambda, \gamma) = \frac{2 \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^\alpha}{1 + \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^\alpha}.$$

The HRF, reversed HRF, and cumulative HRF are supplied as below:

$$h(y; \alpha, \lambda, \gamma) = \frac{\alpha\lambda\gamma y^{-2}(1 + \lambda y^{-1})^{\gamma-1} e^{1 - (1 + \lambda y^{-1})^\gamma}}{\left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right] \left[1 + \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^\alpha\right]},$$

$$\tau(y; \alpha, \lambda, \gamma) = \frac{2\alpha\lambda\gamma y^{-2}(1 + \lambda y^{-1})^{\gamma-1} e^{1 - (1 + \lambda y^{-1})^\gamma} \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^{\alpha-1}}{1 - \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^{2\alpha}},$$

and

$$H(y; \alpha, \lambda, \gamma) = -\log \left(\frac{2 \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^\alpha}{1 + \left[1 - e^{1 - (1 + \lambda y^{-1})^\gamma}\right]^\alpha} \right).$$

Figure 1 shows the curves of PDF and HRF for the HL-INH distribution with various values of parameters.

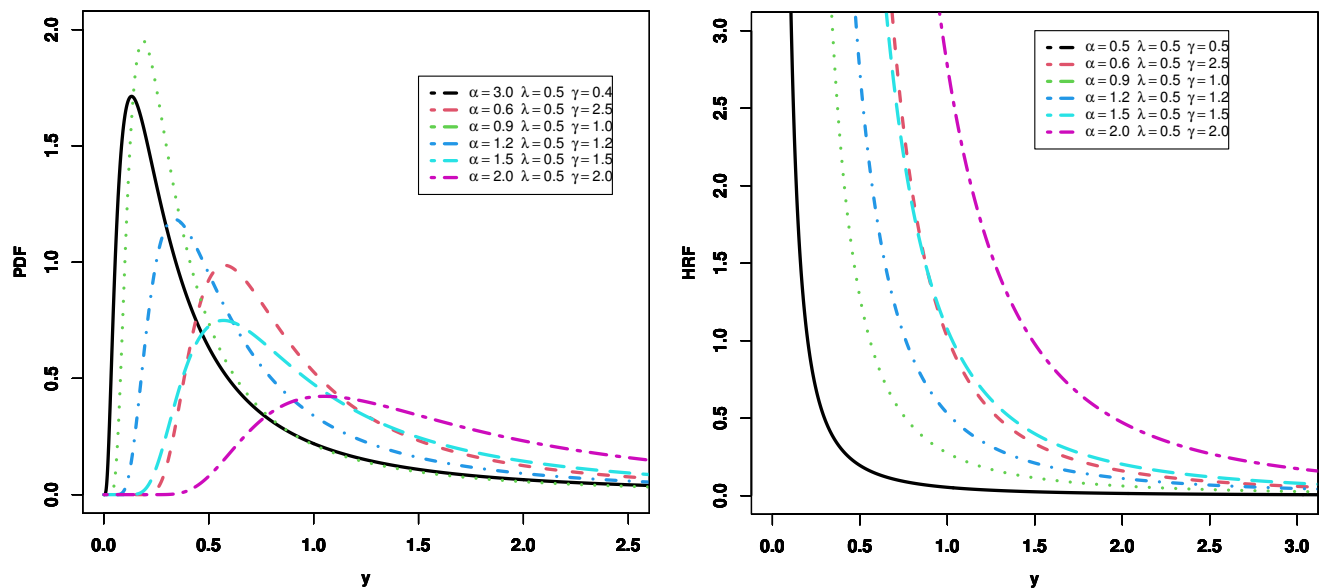


Figure 1. Curves of PDF and HRF for the HL-INH distribution.

3. Some Important Linear Representations

Here, three important linear representations of $f(y; \alpha, \lambda, \gamma)$, $[F(y; \alpha, \lambda, \gamma)]^h$, and $[f(y; \alpha, \lambda, \gamma)]^\eta$ for the HL-INH distribution are investigated to make the calculation of the basic mathematical characteristics simple because we can rewrite the PDF and CDF of the HL-INH distribution as a linear representation of the INH distribution.

At the beginning, we investigate the expansion of $f(y; \alpha, \lambda, \gamma)$ by using the binomial theorem:

$$(1+y)^{-\beta} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta+i-1}{i} y^i. \quad (9)$$

By substituting (9) in (8), then

$$f(y; \alpha, \lambda, \gamma) = 2\alpha\beta\lambda\gamma y^{-2} (1+\lambda y^{-1})^{\gamma-1} e^{1-(1+\lambda y^{-1})^\gamma} \sum_{i=0}^{\infty} (-1)^i (1+i) \left[1 - e^{1-(1+\lambda y^{-1})^\gamma}\right]^{\alpha(i+1)-1}. \quad (10)$$

Again, by utilizing the binomial theorem,

$$(1-y)^\beta = \sum_{j=0}^{\infty} (-1)^j \binom{\beta}{j} y^j. \quad (11)$$

By inserting (11) in (10), then

$$f(y; \alpha, \lambda, \gamma) = \sum_{i,j=0}^{\infty} \mathcal{U}_{i,j} y^{-2} (1+\lambda y^{-1})^{\gamma-1} e^{-(j+1)(1+\lambda y^{-1})^\gamma}, \quad (12)$$

where, $\mathcal{U}_{i,j} = 2\alpha\beta\lambda\gamma e^{j+1} (-1)^{i+j} (1+i) \binom{\alpha(i+1)-1}{j}$.

To investigate the linear representation of $[F(y; \alpha, \lambda, \gamma)]^h$, then

$$[F(y; \alpha, \lambda, \gamma)]^h = \left[\frac{1 - \left[1 - e^{1-(1+\lambda y^{-1})^\gamma}\right]^\alpha}{1 + \left[1 - e^{1-(1+\lambda y^{-1})^\gamma}\right]^\alpha} \right]^h.$$

By utilizing the binomial series two times in the previous equation, then we get

$$[F(y; \alpha, \lambda, \gamma)]^h = \sum_{u,v=0}^h (-1)^{u+v} \binom{h+u-1}{u} \binom{h}{v} \left[1 - e^{1-(1+\lambda y^{-1})^\gamma}\right]^{\alpha(u+v)},$$

Again, by utilizing the binomial theorem, then

$$[F(y; \alpha, \lambda, \gamma)]^h = \sum_{u,v=0}^h \sum_{m=0}^{\infty} \mathfrak{U}_{u,v,m} e^{-m(1+\lambda y^{-1})^\gamma}, \quad (13)$$

$$\text{where } \mathfrak{U}_{i,j,k} = (-1)^{u+v+m} \binom{h+u-1}{u} \binom{h}{v} \binom{\alpha(u+v)}{m} e^m.$$

To compute the linear combination of $[f(y; \alpha, \lambda, \gamma)]^\eta$, then

$$[f(y; \alpha, \lambda, \gamma)]^\eta = \frac{(2\alpha\lambda\gamma e)^\eta y^{-2\eta} (1 + \lambda y^{-1})^{\eta(\gamma-1)} e^{-\eta(1+\lambda y^{-1})^\gamma} \left[1 - e^{1-(1+\lambda y^{-1})^\gamma}\right]^{\eta(\alpha-1)}}{\left[1 + \left[1 - e^{1-(1+\lambda y^{-1})^\gamma}\right]^\alpha\right]^{2\eta}}. \quad (14)$$

By employing the binomial theorem in the above equation, then

$$[f(y; \alpha, \lambda, \gamma)]^\eta = (2\alpha\lambda\gamma e)^\eta y^{-2\eta} (1 + \lambda y^{-1})^{\eta(\gamma-1)} e^{-\eta(1+\lambda y^{-1})^\gamma} \sum_{i=0}^{\infty} (-1)^i \binom{2\eta+i-1}{i} \left[1 - e^{1-(1+\lambda y^{-1})^\gamma}\right]^{\eta(\alpha-1)+\alpha i},$$

Again, by employing the binomial theorem, we get

$$[f(y; \alpha, \lambda, \gamma)]^\eta = y^{-2\eta} (1 + \lambda y^{-1})^{\eta(\gamma-1)} \sum_{i,j=0}^{\infty} \pi_{i,j} e^{-(\eta+j)(1+\lambda y^{-1})^\gamma}, \quad (15)$$

$$\text{where } \pi_{i,j} = (2\alpha\lambda\gamma)^\eta (-1)^{i+j} \binom{2\eta+i-1}{i} \binom{\eta(\alpha-1)+\alpha i}{j} e^{\eta+j}.$$

4. Four Various Measures of Uncertainty

One of the most important measures of uncertainty is entropy. The various types of entropy are useful in determining the risk and reliability analysis.

4.1. Arimoto Entropy

The AEN [45] measure is computed from the equation below:

$$A_\eta = \frac{\eta}{1-\eta} \left[(J_\eta(y))^{\frac{1}{\eta}} - 1 \right], \quad \eta > 0, \eta \neq 1. \quad (16)$$

where $J_\eta(y) = \int_0^\infty [f(y; \alpha, \lambda, \gamma)]^\eta dy$. Now, we need to compute the integral $J_\eta(y)$. Then, by using Equation (15), we get

$$J_\eta(y) = \int_0^\infty [f(y; \alpha, \lambda, \gamma)]^\eta dy = \sum_{i,j=0}^{\infty} \pi_{i,j} \int_0^\infty y^{-2\eta} (1 + \lambda y^{-1})^{\eta(\gamma-1)} e^{-(\eta+j)(1+\lambda y^{-1})^\gamma} dy,$$

Let $v = (1 + \lambda y^{-1})^\gamma$, then

$$J_\eta(y) = \frac{\lambda^{-2\eta+1}}{\gamma} \sum_{i,j=0}^{\infty} \pi_{i,j} \int_0^\infty v^{\eta-\frac{\eta}{\gamma}+\frac{1}{\gamma}-1} \left(1 - v^{\frac{1}{\gamma}}\right)^{2\eta-2} e^{-(\eta+j)v} dv. \quad (17)$$

By using the binomial theory to Equation (17), then

$$J_{\eta}(y) = \frac{\lambda^{-2\eta+1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{k=0}^{2\eta-2} (-1)^k \binom{2\eta-2}{k} \pi_{i,j} \int_0^{\infty} v^{\eta-\frac{\eta}{\gamma}+\frac{k}{\gamma}+\frac{1}{\gamma}-1} e^{-(\eta+j)v} dv.$$

By utilizing the gamma function, then

$$J_{\eta}(y) = \frac{\lambda^{-2\eta+1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{k=0}^{2\eta-2} (-1)^k \binom{2\eta-2}{k} \frac{\pi_{i,j} \Gamma\left(\eta + \frac{k-\eta+1}{\gamma}\right)}{(\eta+j)^{\eta+\frac{k-\eta+1}{\gamma}}}. \quad (18)$$

By inserting Equation (18) into (16), then the AEN of the HL-INH distribution is

$$A_{\eta} = \frac{\eta}{1-\eta} \left[\left(\frac{\lambda^{-2\eta+1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{k=0}^{2\eta-2} (-1)^k \binom{2\eta-2}{k} \frac{\pi_{i,j} \Gamma\left(\eta + \frac{k-\eta+1}{\gamma}\right)}{(\eta+j)^{\eta+\frac{k-\eta+1}{\gamma}}} \right)^{\frac{1}{\eta}} - 1 \right].$$

4.2. Rényi Entropy

The REN [46] measure is computed from the equation below:

$$R_{\eta} = \frac{1}{1-\eta} \log[J_{\eta}(y)], \quad \eta > 0, \eta \neq 1. \quad (19)$$

By inserting Equation (18) into (19), then the REN of the HL-INH distribution is

$$R_{\eta} = \frac{1}{1-\eta} \log \left[\frac{\lambda^{-2\eta+1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{k=0}^{2\eta-2} (-1)^k \binom{2\eta-2}{k} \frac{\pi_{i,j} \Gamma\left(\eta + \frac{k-\eta+1}{\gamma}\right)}{(\eta+j)^{\eta+\frac{k-\eta+1}{\gamma}}} \right].$$

4.3. Tsallis Entropy

The TEN [47] measure is computed from the equation below:

$$T_{\eta} = \frac{1}{\eta-1} [1 - J_{\eta}(y)], \quad \eta > 0, \eta \neq 1. \quad (20)$$

By inserting Equation (18) into (20), then the TEN of the HL-INH distribution is

$$T_{\eta} = \frac{1}{\eta-1} \left[1 - \frac{\lambda^{-2\eta+1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{k=0}^{2\eta-2} (-1)^k \binom{2\eta-2}{k} \frac{\pi_{i,j} \Gamma\left(\eta + \frac{k-\eta+1}{\gamma}\right)}{(\eta+j)^{\eta+\frac{k-\eta+1}{\gamma}}} \right].$$

4.4. Havrda and Charvat Entropy

The HCEN [48] measure is computed from the equation below:

$$HC_{\eta} = \frac{1}{2^{1-\eta}-1} \left[(J_{\eta}(y))^{\frac{1}{\eta}} - 1 \right], \quad \eta > 0, \eta \neq 1. \quad (21)$$

By inserting Equation (18) into (21), then the HCEN of the HL-INH distribution is

$$HC_{\eta} = \frac{1}{2^{1-\eta}-1} \left[\left(\frac{\lambda^{-2\eta+1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{k=0}^{2\eta-2} (-1)^k \binom{2\eta-2}{k} \frac{\pi_{i,j} \Gamma\left(\eta + \frac{k-\eta+1}{\gamma}\right)}{(\eta+j)^{\eta+\frac{k-\eta+1}{\gamma}}} \right)^{\frac{1}{\eta}} - 1 \right].$$

Tables 1 and 2 represent the analysis of the REN, HCEN, TEN, and AEN for the HL-INH distribution at $\lambda = 1.5$.

Table 1. Analysis of the REN, HCEN, TEN, and AEN for the HL-INH distribution at $\lambda = 1.5$ and $\eta = 0.5$ and 0.8 .

α	γ	$\eta = 0.5$				$\eta = 0.8$			
		REN	HCEN	TEN	AEN	REN	HCEN	TEN	AEN
4	0.5	0.270	2.085	0.730	0.864	0.084	0.335	0.198	0.199
	1	0.640	8.122	2.178	3.364	0.477	2.123	1.227	1.263
	1.5	0.830	13.910	3.201	5.762	0.670	3.168	1.809	1.884
	2	0.960	19.606	4.040	8.121	0.802	3.944	2.233	2.346
	2.5	1.059	25.261	4.772	10.463	0.902	4.576	2.573	2.721
	3	1.140	30.894	5.429	12.797	0.982	5.113	2.861	3.041
	3.5	1.208	36.515	6.031	15.125	1.050	5.585	3.110	3.322
	4	1.266	42.128	6.591	17.450	1.109	6.008	3.332	3.573
5	0.5	0.116	0.741	0.287	0.307	−0.040	−0.155	−0.092	−0.092
	1	0.501	5.246	1.563	2.173	0.366	1.576	0.917	0.938
	1.5	0.695	9.541	2.451	3.952	0.563	2.572	1.478	1.530
	2	0.826	13.754	3.176	5.697	0.695	3.307	1.885	1.967
	2.5	0.926	17.930	3.806	7.427	0.795	3.903	2.211	2.322
	3	1.006	22.088	4.372	9.149	0.876	4.411	2.485	2.624
	3.5	1.074	26.234	4.890	10.867	0.944	4.856	2.723	2.888
	4	1.133	30.373	5.370	12.581	1.003	5.254	2.935	3.125
6	0.5	0.004	0.021	0.009	0.009	−0.135	−0.502	−0.301	−0.299
	1	0.402	3.681	1.178	1.525	0.283	1.191	0.697	0.708
	1.5	0.598	7.157	1.982	2.965	0.482	2.152	1.244	1.280
	2	0.730	10.558	2.636	4.373	0.615	2.859	1.638	1.700
	2.5	0.830	13.926	3.203	5.768	0.716	3.431	1.953	2.041
	3	0.911	17.276	3.712	7.156	0.797	3.917	2.218	2.33
	3.5	0.980	20.616	4.177	8.540	0.866	4.343	2.449	2.583
	4	1.038	23.950	4.609	9.920	0.924	4.724	2.653	2.810

Table 2. Analysis of the REN, HCEN, TEN, and AEN for the HL-INH distribution at $\lambda = 1.5$ and $\eta = 1.2$ and 1.5.

α	γ	$\eta = 1.2$				$\eta = 1.5$			
		REN	HCEN	TEN	AEN	REN	HCEN	TEN	AEN
4	0.5	−0.021	−0.062	−0.048	−0.048	−0.065	−0.176	−0.156	−0.154
	1	0.385	1.062	0.813	0.825	0.347	0.797	0.658	0.701
	1.5	0.581	1.545	1.174	1.200	0.543	1.164	0.930	1.023
	2	0.713	1.850	1.400	1.437	0.676	1.382	1.081	1.214
	2.5	0.813	2.071	1.562	1.609	0.776	1.532	1.181	1.346
	3	0.894	2.244	1.688	1.743	0.857	1.646	1.254	1.446
	3.5	0.962	2.386	1.790	1.853	0.925	1.736	1.311	1.525
	4	1.021	2.504	1.876	1.945	0.984	1.810	1.356	1.590
5	0.5	−0.133	−0.403	−0.315	−0.313	−0.172	−0.483	−0.439	−0.424
	1	0.286	0.803	0.617	0.624	0.252	0.600	0.503	0.527
	1.5	0.485	1.311	1.000	1.018	0.451	0.999	0.810	0.878
	2	0.618	1.630	1.238	1.266	0.584	1.233	0.979	1.084
	2.5	0.718	1.861	1.408	1.445	0.685	1.395	1.091	1.226
	3	0.799	2.041	1.540	1.585	0.766	1.517	1.172	1.333
	3.5	0.867	2.187	1.647	1.699	0.834	1.614	1.234	1.418
	4	0.926	2.311	1.736	1.795	0.893	1.694	1.284	1.488
6	0.5	−0.218	−0.676	−0.529	−0.525	−0.255	−0.738	−0.682	−0.649
	1	0.211	0.601	0.463	0.467	0.179	0.438	0.373	0.385
	1.5	0.412	1.129	0.863	0.877	0.380	0.864	0.709	0.760
	2	0.545	1.458	1.110	1.133	0.514	1.113	0.893	0.978
	2.5	0.646	1.697	1.287	1.318	0.615	1.285	1.015	1.129
	3	0.728	1.882	1.423	1.462	0.697	1.414	1.103	1.242
	3.5	0.796	2.033	1.534	1.579	0.765	1.516	1.171	1.332
	4	0.855	2.160	1.627	1.678	0.824	1.600	1.225	1.406

5. Important Mathematical Features of the HL-INH Distribution

This section investigates some important mathematical features of the HL-INH distribution.

5.1. Probability Weighted Moments

The (q, h) th PWMs for the HL-INH distribution, signified as $\rho_{q,h}$, is computed from the equation below:

$$\rho_{q,h} = \int_0^1 y^q f(y; \alpha, \lambda, \gamma) F(y; \alpha, \lambda, \gamma)^h dy. \quad (22)$$

By entering Equation (12) and (13) into (22), then

$$\rho_{q,h} = \sum_{i,j=0}^{\infty} \sum_{u,v=0}^h \sum_{m=0}^{\infty} \mathcal{U}_{u,v,m} \mathcal{V}_{i,j} \int_0^{\infty} y^{q-2} (1 + \lambda y^{-1})^{\gamma-1} e^{-(j+m+1)(1+\lambda y^{-1})^{\gamma}} dy.$$

Let $v = (1 + \lambda y^{-1})^{\gamma}$, then

$$\rho_{q,h} = \frac{\lambda^{q-1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{u,v=0}^h \sum_{m=0}^{\infty} (-1)^{-q} \mathcal{U}_{u,v,m} \mathcal{V}_{i,j} \int_0^{\infty} \left(1 - v^{\frac{1}{\gamma}}\right)^{-q} e^{-(j+m+1)v} dv,$$

By using the binomial expansion, then

$$\rho_{q,h} = \frac{\lambda^{q-1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{u,v=0}^h \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{-q} \binom{q+k-1}{k} \mathfrak{U}_{u,v,m} \mathfrak{U}_{i,j} \int_0^{\infty} v^{\frac{k}{\gamma}} e^{-(j+m+1)v} dv,$$

Then, the PWMs of the HL-INH distribution is

$$\rho_{q,h} = \frac{\lambda^{q-1}}{\gamma} \sum_{i,j=0}^{\infty} \sum_{u,v=0}^h \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{-q} \binom{q+k-1}{k} \mathfrak{U}_{u,v,m} \mathfrak{U}_{i,j} \frac{\Gamma\left(\frac{k}{\gamma} + 1\right)}{(j+m+1)^{\frac{k}{\gamma}+1}}.$$

5.2. Moments

The m th ordinary moment of a random variable Y may be calculated using the following equation:

$$\mu'_m = \int_0^{\infty} y^m f(y; \alpha, \lambda, \gamma) dy. \quad (23)$$

By inserting Equation (12) into (23), then

$$\mu'_m = \sum_{i,j=0}^{\infty} \mathfrak{U}_{i,j} \int_0^{\infty} y^{m-2} \left(1 + \lambda y^{-1}\right)^{\gamma-1} e^{-(j+1)(1+\lambda y^{-1})^{\gamma}} dy$$

Let $v = (1 + \lambda y^{-1})^{\gamma}$, then

$$\mu'_m = \sum_{i,j=0}^{\infty} \frac{(-1)^{-m} \lambda^m \mathfrak{U}_{i,j}}{\gamma} \int_0^{\infty} \left(1 - v^{\frac{1}{\gamma}}\right)^{-m} e^{-(j+1)v} dv,$$

By using the binomial theory, then

$$\mu'_m = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{-m} \lambda^m \mathfrak{U}_{i,j}}{\gamma} \binom{m+k-1}{k} \int_0^{\infty} v^{\frac{k}{\gamma}} e^{-(j+1)v} dv.$$

By using the gamma function ($\Gamma(\cdot, \cdot)$), the m th moment of the HL-INH distribution is

$$\mu'_m = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{-m} \lambda^m \mathfrak{U}_{i,j} \Gamma\left(\frac{k}{\gamma} + 1\right)}{\gamma(j+1)^{\frac{k}{\gamma}+1}} \binom{m+k-1}{k}. \quad (24)$$

Table 3 represents the numerical results of $E(Y)$, $E(Y^2)$, $E(Y^3)$, $E(Y^4)$, the variance (σ^2), the skewness (γ_1), the kurtosis (γ_2), the index of dispersion (ID), and the coefficient of variation (CV).

Table 3. Numerical results of $E(Y)$, $E(Y^2)$, $E(Y^3)$, $E(Y^4)$, σ^2 , γ_1 , γ_2 , CV , and ID for the HL-INH distribution at $\lambda = 1.5$.

α	γ	$E(Y)$	$E(Y^2)$	$E(Y^3)$	$E(Y^4)$	σ^2	γ_1	γ_2	CV	ID
4	0.5	0.428	0.379	0.867	85.693	0.196	6.208	2204.000	1.035	0.458
	1	1.290	2.599	10.443	1493.000	0.934	5.186	1670.000	0.749	0.724
	1.5	2.217	7.095	41.450	8305.000	2.179	4.992	1700.000	0.666	0.983
	2	3.162	13.923	107.132	23,710.000	3.925	4.923	1485.000	0.627	1.241
	2.5	4.114	23.097	220.783	60,880.000	6.171	4.892	1542.000	0.604	1.500
	3	5.070	34.623	395.713	119,219.295	8.916	4.874	1441.120	0.589	1.758
	3.5	6.028	48.502	645.233	228,588.145	12.160	4.864	1485.510	0.578	2.017
	4	6.988	64.735	982.655	374,735.523	15.903	4.857	1419.854	0.571	2.276
5	0.5	0.334	0.201	0.230	0.660	0.089	3.901	56.822	0.892	0.266
	1	1.081	1.628	3.728	16.944	0.459	3.137	38.602	0.627	0.425
	1.5	1.896	4.683	16.409	104.500	1.089	2.992	35.666	0.550	0.574
	2	2.730	9.427	44.663	367.801	1.973	2.941	34.669	0.515	0.723
	2.5	3.572	15.873	94.932	960.751	3.110	2.918	34.213	0.494	0.871
	3	4.419	24.026	173.675	2086.745	4.501	2.905	33.967	0.480	1.019
	3.5	5.267	33.890	287.353	3998.676	6.144	2.897	33.819	0.471	1.166
	4	6.118	45.465	442.428	6998.943	8.040	2.892	33.723	0.463	1.314
6	0.5	0.279	0.129	0.099	0.141	0.051	3.059	28.196	0.807	0.182
	1	0.953	1.188	2.021	5.186	0.280	2.387	18.850	0.556	0.294
	1.5	1.698	3.558	9.584	36.389	0.674	2.259	17.378	0.484	0.397
	2	2.464	7.298	27.045	136.725	1.228	2.214	16.880	0.450	0.498
	2.5	3.238	12.424	58.719	371.439	1.940	2.193	16.653	0.430	0.599
	3	4.016	18.941	108.935	828.107	2.811	2.182	16.530	0.417	0.700
	3.5	4.797	26.852	182.024	1616.653	3.841	2.175	16.457	0.409	0.801
	4	5.579	36.157	282.321	2869.345	5.029	2.170	16.409	0.402	0.901

5.3. Quantile Function and Skewness

The QUF is a useful measure that is utilized in order to produce random samples from the HL-INH distribution. Assume that $Y \sim \text{HL-INH}(\alpha, \lambda, \gamma)$ for $Y > 0$ and $\alpha, \lambda, \gamma > 0$; then, the QUF is provided via

$$y_p = \lambda \left[\left(1 - \ln \left[1 - \left(\frac{1-p}{1+p} \right)^{\frac{1}{\alpha}} \right] \right)^{\frac{1}{\gamma}} - 1 \right]^{-1}, \quad 0 < p < 1. \quad (25)$$

The first quantile ($y_{0.25}$), the second quantile ($y_{0.5}$) (median), and the third quantile ($y_{0.75}$) for the HL-INH distribution are provided as below:

$$y_{0.25} = \lambda \left[\left(1 - \ln \left[1 - \left(\frac{3}{5} \right)^{\frac{1}{\alpha}} \right] \right)^{\frac{1}{\gamma}} - 1 \right]^{-1}, \quad 0 < p < 1,$$

$$y_{0.5} = \lambda \left[\left(1 - \ln \left[1 - \left(\frac{1}{3} \right)^{\frac{1}{\alpha}} \right] \right)^{\frac{1}{\gamma}} - 1 \right]^{-1}, \quad 0 < p < 1,$$

and

$$y_{0.75} = \lambda \left[\left(1 - \ln \left[1 - \left(\frac{1}{7} \right)^{\frac{1}{\alpha}} \right] \right)^{\frac{1}{\gamma}} - 1 \right]^{-1}, \quad 0 < p < 1.$$

The Bowley skewness (β_3) and Moors kurtosis (β_4) are provided via

$$\beta_3 = \frac{y_{0.75} + y_{0.25} - 2y_{0.5}}{y_{0.75} - y_{0.25}}.$$

and

$$\beta_4 = \frac{y_{0.875} - y_{0.625} + y_{0.375} - y_{0.125}}{y_{0.75} - y_{0.25}}.$$

6. Sampling Techniques

The SRS method is the most common method of data collection. In these situations, ranked sampling procedures may be employed to investigate more representative samples from the underlying population and improve the efficacy of the statistical inference. The RSS method was first put forth by [49,50]. Several investigations have demonstrated that, either numerically or theoretically, RSS statistical processes are superior to their SRS analogs. The size of ranking sampling n for the one-cycle $c = 1$ is as follows:

$$\begin{array}{llllll} 1. & \underline{y_{(1:n)}^{(1)}} & y_{(2:n)}^{(1)} & \cdots & y_{(n:n)}^{(1)} & \rightarrow y_{(1)} = y_{(1:n)}^{(1)} \\ 2. & \underline{y_{(1:n)}^{(2)}} & \underline{y_{(2:n)}^{(2)}} & \cdots & y_{(n:n)}^{(2)} & \rightarrow y_{(2)} = y_{(2:n)}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n. & \underline{y_{(1:n)}^{(n)}} & \underline{y_{(2:n)}^{(n)}} & \cdots & \underline{y_{(n:n)}^{(n)}} & \rightarrow y_{(n)} = y_{(n:n)}^{(n)} \end{array}$$

The resulting sample, which has the size n , is known as a one-cycle RSS; it is represented by the equation $\underline{Y} = (y_{(1)}, y_{(2)}, \dots, y_{(n)})$. According to [51], using the perfect judgement ranking as a foundation, $y_{(i)}$ has the same distribution as the order statistic in a set of size n generated from the i th sample with the PDF method.

The initial ranking of n samples of size n for more than one cycle appears as described in Table 4.

Table 4. RSS design with more than one cycle.

Cycle 1				Cycle 2				Cycle c				
$\underline{y_{(1:n)}^{(1)}}$	$y_{(2:n)}^{(1)}$	$\cdots$	$y_{(n:n)}^{(1)}$	$\underline{y_{(1:n)}^{(2)}}$	$y_{(2:n)}^{(2)}$	$\cdots$	$y_{(n:n)}^{(2)}$	$\cdots$	$\underline{y_{(1:n)}^{(c)}}$	$y_{(2:n)}^{(c)}$	$\cdots$	$y_{(n:n)}^{(c)}$
$y_{(2:n)}^{(2)}$	$\underline{y_{(2:n)}^{(2)}}$	$\cdots$	$y_{(n:n)}^{(2)}$	$y_{(2:n)}^{(2)}$	$\underline{y_{(2:n)}^{(2)}}$	$\cdots$	$y_{(n:n)}^{(2)}$	$\cdots$	$y_{(2:n)}^{(c)}$	$\underline{y_{(2:n)}^{(c)}}$	$\vdots$	$y_{(n:n)}^{(c)}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$y_{(2:n)}^{(n)}$	$y_{(2:n)}^{(n)}$	$\cdots$	$y_{(n:n)}^{(n)}$	$y_{(2:n)}^{(n)}$	$y_{(2:n)}^{(n)}$	$\cdots$	$y_{(n:n)}^{(n)}$	$\cdots$	$y_{(2:n)}^{(c)}$	$y_{(2:n)}^{(c)}$	$\cdots$	$y_{(n:n)}^{(c)}$

$$f_{(i)}(y) = \frac{n!}{(i-1)!(n-i)!} [F(y_{(i)j})]^{i-1} [1 - F(y_{(i)j})]^{n-i} f(y_{(i)j}). \quad (26)$$

7. Approach of Maximum Likelihood Estimation

In this part, we derive the SRS and RSS to estimate the parameter estimators of the HL-INH distribution α , γ , and λ .

7.1. ML Estimation under SRS

In this subsection, we must first investigate the ML estimates (MLEs) of α , γ , and λ under SRS. Let Y_i , where $i = (1, \dots, n)$. The likelihood function (LF) under SRS is:

$$\begin{aligned}
L(\alpha, \gamma, \lambda) &= \prod_{i=1}^n f_Y(y_i) \\
&= 2^n \alpha^n \lambda^n \gamma^n e^{n - \sum_{i=1}^n (1 + \lambda y_i^{-1})^\gamma} \prod_{i=1}^n \frac{y_i^{-2} (1 + \lambda y_i^{-1})^{\gamma-1} [1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^{\alpha-1}}{\left[1 + [1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^\alpha\right]^2}.
\end{aligned} \quad (27)$$

The observed samples' ln-LF is

$$\begin{aligned}
\ell(\alpha, \gamma, \lambda) &= n[\ln(2) + \ln(\alpha) + \ln(\lambda) + \ln(\gamma)] + n - \sum_{i=1}^n (1 + \lambda y_i^{-1})^\gamma - 2 \sum_{i=1}^n \ln(y_i) + (\gamma - 1) \sum_{i=1}^n \ln(1 + \lambda y_i^{-1}) + \\
&(\alpha - 1) \sum_{i=1}^n \ln[1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}] - 2 \sum_{i=1}^n \ln \left[1 + [1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^\alpha\right].
\end{aligned} \quad (28)$$

To derive the likelihood equations, the partial derivatives of Equation (28) with respect to α , γ , and λ are

$$\frac{\partial \ell(\alpha, \gamma, \lambda)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln[1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}] - 2 \sum_{i=1}^n \frac{[1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^\alpha \ln[1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]}{1 + [1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^\alpha}, \quad (29)$$

$$\begin{aligned}
\frac{\partial \ell(\alpha, \gamma, \lambda)}{\partial \gamma} &= \frac{n}{\gamma} - \sum_{i=1}^n (1 + \lambda y_i^{-1})^\gamma \ln(1 + \lambda y_i^{-1}) + \sum_{i=1}^n \ln(1 + \lambda y_i^{-1}) + \\
&(\alpha - 1) \sum_{i=1}^n \frac{e^{1 - (1 + \lambda y_i^{-1})^\gamma} (1 + \lambda y_i^{-1})^\gamma \ln(1 + \lambda y_i^{-1})}{1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}} - \\
&2 \sum_{i=1}^n \frac{[1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^{\alpha-1} e^{1 - (1 + \lambda y_i^{-1})^\gamma} (1 + \lambda y_i^{-1})^\gamma \ln(1 + \lambda y_i^{-1})}{1 + [1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^\alpha},
\end{aligned} \quad (30)$$

and

$$\begin{aligned}
\frac{\partial \ell(\alpha, \gamma, \lambda)}{\partial \lambda} &= \frac{n}{\lambda} - \gamma \sum_{i=1}^n y_i^{-1} (1 + \lambda y_i^{-1})^{\gamma-1} + (\gamma - 1) \sum_{i=1}^n \frac{y_i^{-1}}{1 + \lambda y_i^{-1}} + \\
&(\alpha - 1) \gamma \sum_{i=1}^n \frac{y_i^{-1} (1 + \lambda y_i^{-1})^{\gamma-1} e^{1 - (1 + \lambda y_i^{-1})^\gamma}}{1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}} - \\
&2 \alpha \gamma \sum_{i=1}^n \frac{y_i^{-1} (1 + \lambda y_i^{-1})^{\gamma-1} e^{1 - (1 + \lambda y_i^{-1})^\gamma} [1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^{\alpha-1}}{1 + [1 - e^{1 - (1 + \lambda y_i^{-1})^\gamma}]^\alpha}.
\end{aligned} \quad (31)$$

To obtain the MLE under the SRS method, it is necessary to simultaneously solve three non-linear Equations (29)–(31) by equating them to zero, respectively. The “optim” function of the “stats” package, which uses the “Nelder–Mead (NM)” strategy of maximization in the MLE computations, can be used in R language to obtain the MLEs α , γ , and λ for any given data set.

7.2. ML Estimation under on RSS

Let the c -cycle of RSS from the HL-INH $(\alpha, \gamma, \lambda)$ be $y_{(i)j}, i = 1, \dots, n_j, j = 1, \dots, c$. Equation (26) yields the following function under RSS:

$$g_{i:n_j}(y_{(i)j}; \alpha, \gamma, \lambda) = C_{i:n_j} F(y_{(i)j}; \alpha, \gamma, \lambda)^{i-1} [1 - F(y_{(i)j}; \alpha, \gamma, \lambda)]^{n_j-i} f(y_{(i)j}; \alpha, \gamma, \lambda), \quad (32)$$

where $C_{i:n_j} = \frac{n_j!}{(i-1)!(n_j-i)!}$. By using Equation (26), the LF under RSS is

$$\begin{aligned} L(\alpha, \gamma, \lambda; y_{(i)j}) &= \prod_{j=1}^c \prod_{i=1}^{n_j} g_{i:n_j}(y_{(i)j}; \alpha, \gamma, \lambda) \\ &= \prod_{j=1}^c \prod_{i=1}^{n_j} C_{i:n_j} \left(\frac{1 - \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha}{1 + \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha} \right)^{i-1} \left(\frac{2 \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha}{1 + \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha} \right)^{n_j-i} \times \\ &\quad \prod_{j=1}^c 2^{n_j} \alpha^{n_j} \lambda^{n_j} \gamma^{n_j} e^{n_j - \sum_{i=1}^{n_j} (1 + \lambda y_{(i)j}^{-1})^\gamma} \prod_{i=1}^{n_j} \frac{y_{(i)j}^{-2} (1 + \lambda y_{(i)j}^{-1})^{\gamma-1} \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^{\alpha-1}}{\left[1 + \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha \right]^2}. \end{aligned} \quad (33)$$

where $N = \sum_{j=1}^c n_j$. The ln-LF of the HL-INH distribution in this situation is given by

$$\begin{aligned} \ell(\alpha, \gamma, \lambda, y_{(i)j}) &= \sum_{j=1}^c \sum_{i=1}^{n_j} \ln(C_{i:n_j}) + \sum_{j=1}^c \sum_{i=1}^{n_j} (i-1) \ln \left(1 - \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha \right) \\ &\quad - \sum_{j=1}^c \sum_{i=1}^{n_j} (i-1) \ln \left(1 + \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha \right) + \sum_{j=1}^c \sum_{i=1}^{n_j} (n_j-i) \ln \left(2 \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha \right) \\ &\quad - \sum_{j=1}^c \sum_{i=1}^{n_j} (n_j-i) \ln \left(1 + \left[1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right]^\alpha \right) + N \ln(2) + N \ln(\alpha) + N \ln(\lambda) \\ &\quad + N \ln(\gamma) + N - \sum_{j=1}^c \sum_{i=1}^{n_j} (1 + \lambda y_{(i)j}^{-1})^\gamma - 2 \sum_{j=1}^c \sum_{i=1}^{n_j} \ln(y_{(i)j}) + (\gamma-1) \sum_{j=1}^c \sum_{i=1}^{n_j} \ln(1 + \lambda y_{(i)j}^{-1}) \\ &\quad + (\alpha-1) \sum_{j=1}^c \sum_{i=1}^{n_j} \ln \left(1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right) - 2 \sum_{j=1}^c \sum_{i=1}^{n_j} \ln \left(1 + \left(1 - e^{1 - (1 + \lambda y_{(i)j}^{-1})^\gamma} \right)^\alpha \right). \end{aligned} \quad (34)$$

With respect to unknown parameters, the partial derivatives of $\ell(\alpha, \gamma, \lambda; y_{(i)j})$ can be written as

$$\begin{aligned}
\frac{\partial \ell(\alpha, \gamma, \lambda, y_{(ij)})}{\partial \alpha} = & - \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(i-1) \ln \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]}{\left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^{-\alpha} - 1} - \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(i-1) \ln \left(1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right)}{1 + \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^{-\alpha}} \\
& + \sum_{j=1}^c \sum_{i=1}^{n_j} (n_j - 1) \ln \left(1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right) - \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(n_j - 1) \ln \left(1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right)}{1 + \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^{-\alpha}} \\
& + \frac{N}{\alpha} + \sum_{j=1}^c \sum_{i=1}^{n_j} \ln \left(1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right) - 2 \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{\ln \left(1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right)}{1 + \left(1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right)^{-\alpha}}.
\end{aligned} \tag{35}$$

$$\begin{aligned}
\frac{\partial \ell(\alpha, \gamma, \lambda, y_{(ij)})}{\partial \gamma} = & -\alpha \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(i-1) \left(1 + \lambda y_{(ij)}^{-1} \right)^\gamma e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^{\alpha-1} \ln \left(1 + \lambda y_{(ij)}^{-1} \right)}{1 - \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^\alpha} \\
& - \alpha \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(i-1) \left(1 + \lambda y_{(ij)}^{-1} \right)^\gamma e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^{\alpha-1} \ln \left(1 + \lambda y_{(ij)}^{-1} \right)}{1 + \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^\alpha} \\
& + \alpha \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(n_j - 1) \left(1 + \lambda y_{(ij)}^{-1} \right)^\gamma \ln \left(1 + \lambda y_{(ij)}^{-1} \right)}{e^{-1+(1+\lambda y_{(ij)}^{-1})^\gamma} - 1} + \frac{N}{\gamma} \\
& - \alpha \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(n_j - 1) \left(1 + \lambda y_{(ij)}^{-1} \right)^\gamma e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^{\alpha-1} \ln \left(1 + \lambda y_{(ij)}^{-1} \right)}{1 + \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^\alpha} \\
& - \sum_{j=1}^c \sum_{i=1}^{n_j} \left(1 + \lambda y_{(ij)}^{-1} \right)^\gamma \ln \left(1 + \lambda y_{(ij)}^{-1} \right) + \sum_{j=1}^c \sum_{i=1}^{n_j} \ln \left(1 + \lambda y_{(ij)}^{-1} \right) \\
& + (\alpha - 1) \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{\left(1 + \lambda y_{(ij)}^{-1} \right)^\gamma e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \ln \left(1 + \lambda y_{(ij)}^{-1} \right)}{1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma}} \\
& - 2\alpha \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{\left(1 + \lambda y_{(ij)}^{-1} \right)^\gamma e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \left[1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right]^{\alpha-1} \ln \left(1 + \lambda y_{(ij)}^{-1} \right)}{1 + \left(1 - e^{1-(1+\lambda y_{(ij)}^{-1})^\gamma} \right)^\alpha},
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\frac{\partial \ell(\alpha, \gamma, \lambda, y_{(i)j})}{\partial \lambda} = & -\alpha \gamma \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(i-1) \left(1 + \lambda y_{(i)j}^{-1}\right)^{\gamma-1} \left[1 - e^{1-(1+\lambda y_{(i)j}^{-1})^\gamma}\right]^{\alpha-1}}{y_{(i)j} e^{-1+(1+\lambda y_{(i)j}^{-1})^\gamma} \left(1 - \left[1 - e^{1-(1+\lambda y_{(i)j}^{-1})^\gamma}\right]^\alpha\right)} + \frac{N}{\lambda} \\
& -\alpha \gamma \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(i-1) \left(1 + \lambda y_{(i)j}^{-1}\right)^{\gamma-1} \left[1 - e^{1-(1+\lambda y_{(i)j}^{-1})^\gamma}\right]^{\alpha-1}}{y_{(i)j} e^{-1+(1+\lambda y_{(i)j}^{-1})^\gamma} \left(1 + \left[1 - e^{1-(1+\lambda y_{(i)j}^{-1})^\gamma}\right]^\alpha\right)} \\
& + \alpha \gamma \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(n_j-1) \left(1 + \lambda y_{(i)j}^{-1}\right)^{\gamma-1}}{y_{(i)j} \left[e^{-1+(1+\lambda y_{(i)j}^{-1})^\gamma} - 1\right]} - \gamma \sum_{j=1}^c \sum_{i=1}^{n_j} y_{(i)j}^{-1} \left(1 + \lambda y_{(i)j}^{-1}\right)^{\gamma-1} \\
& -\alpha \gamma \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{(n_j-1) \left(1 + \lambda y_{(i)j}^{-1}\right)^{\gamma-1} \left[1 - e^{1-(1+\lambda y_{(i)j}^{-1})^\gamma}\right]^{\alpha-1}}{y_{(i)j} e^{-1+(1+\lambda y_{(i)j}^{-1})^\gamma} \left(1 + \left[1 - e^{1-(1+\lambda y_{(i)j}^{-1})^\gamma}\right]^\alpha\right)} + (\gamma-1) \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{1}{y_{(i)j} + \lambda} \\
& + \gamma(\alpha-1) \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{\left(1 + \lambda y_{(i)j}^{-1}\right)^{\gamma-1}}{y_{(i)j} \left(e^{-1+(1+\lambda y_{(i)j}^{-1})^\gamma} - 1\right)} \\
& - 2\alpha \gamma \sum_{j=1}^c \sum_{i=1}^{n_j} \frac{\left(1 + \lambda y_{(i)j}^{-1}\right)^{\gamma-1} \left(1 - e^{1-(1+\lambda y_{(i)j}^{-1})^\gamma}\right)^{\alpha-1}}{y_{(i)j} e^{-1+(1+\lambda y_{(i)j}^{-1})^\gamma} \left(1 + \left(1 - e^{1-(1+\lambda y_{(i)j}^{-1})^\gamma}\right)^\alpha\right)}.
\end{aligned} \tag{37}$$

The MLEs can be derived by numerically solving the nonlinear equations $\frac{\partial \ell(\alpha, \gamma, \lambda; y_{(i)j})}{\partial \alpha} = 0$, $\frac{\partial \ell(\alpha, \gamma, \lambda; y_{(i)j})}{\partial \gamma} = 0$, and $\frac{\partial \ell(\alpha, \gamma, \lambda; y_{(i)j})}{\partial \lambda} = 0$.

7.3. Asymptotic Confidence Interval

The Fisher information matrix (FIM) I of parameters α , γ , and λ , which are the negative expectations of the second derivative of $\ln$ -LF, is produced by the Hessian matrix using the “optim” function. The variance–covariance is described by the inverse FIM. When n rises, the asymptotic distribution of the MLE $(\hat{\alpha}, \hat{\gamma}, \hat{\lambda})$ is as follows:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\gamma} \\ \hat{\lambda} \end{pmatrix} \sim \mathbb{N} \left[\begin{pmatrix} \alpha \\ \gamma \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix} \right].$$

The estimates of $\hat{\alpha}$, $\hat{\gamma}$, and $\hat{\lambda}$ of the asymptotic variance–covariance matrix V are obtained by inverting the Hessian matrix.

8. Simulation

Using R packages, we perform numerous calculations in accordance with Monte Carlo simulation (MCS) experiments with different combinations of sample sizes n (as $n = 6, 8, 12, 17$) and cycle sizes c ($c = 1, 3$) as part of our complete effort to assess the effectiveness of the inference estimation methods provided. The following parameters $(\alpha, \gamma, \lambda)$ are used to create a HL-INH sample: $(0.7, 0.5, 0.6)$, $(0.7, 0.5, 3)$, $(0.7, 3, 0.6)$, $(2.5, 0.5, 0.6)$, $(2.5, 0.5, 3)$, $(2.5, 3, 0.6)$, and $(2.5, 3, 3)$. The bias, accompanying mean squared error (MSEs), and

relative efficiency (RE) of the point parameter resultant α , γ , and λ estimators are evaluated as follows:

$$Bias(\alpha) = \frac{1}{L} \sum_{i=1}^L (\hat{\alpha} - \alpha), \quad Bias(\gamma) = \frac{1}{L} \sum_{i=1}^L (\hat{\gamma} - \gamma), \quad Bias(\lambda) = \frac{1}{L} \sum_{i=1}^L (\hat{\lambda} - \lambda),$$

$$MSEs(\alpha) = \frac{1}{L} \sum_{i=1}^L (\hat{\alpha} - \alpha)^2, \quad MSEs(\gamma) = \frac{1}{L} \sum_{i=1}^L (\hat{\gamma} - \gamma)^2, \quad MSEs(\lambda) = \frac{1}{L} \sum_{i=1}^L (\hat{\lambda} - \lambda)^2,$$

and

$$RE1(\alpha) = \frac{MSEs^{SRS}(\alpha)}{MSEs^{RSS\ s=1}(\alpha)}, \quad RE2(\alpha) = \frac{MSEs^{SRS}(\alpha)}{MSEs^{RSS\ s=3}(\alpha)}, \quad RE3(\alpha) = \frac{MSEs^{RSS\ s=1}(\alpha)}{MSEs^{RSS\ s=3}(\alpha)},$$

$$RE1(\gamma) = \frac{MSEs^{SRS}(\gamma)}{MSEs^{RSS\ s=1}(\gamma)}, \quad RE2(\gamma) = \frac{MSEs^{SRS}(\gamma)}{MSEs^{RSS\ s=3}(\gamma)}, \quad RE3(\gamma) = \frac{MSEs^{RSS\ s=1}(\gamma)}{MSEs^{RSS\ s=3}(\gamma)},$$

and

$$RE1(\lambda) = \frac{MSEs^{SRS}(\lambda)}{MSEs^{RSS\ s=1}(\lambda)}, \quad RE2(\lambda) = \frac{MSEs^{SRS}(\lambda)}{MSEs^{RSS\ s=3}(\lambda)}, \quad RE3(\lambda) = \frac{MSEs^{RSS\ s=1}(\lambda)}{MSEs^{RSS\ s=3}(\lambda)},$$

The ACI has been considered to obtain the lower (L-ACI), upper (U-ACI), and coverage probability (CP) for the parameters of the f HL-INH distribution with each sampling techniques as follows:

$$L - ACI(\alpha) = \hat{\alpha} - z_{0.025} \sqrt{Var(\hat{\alpha})}, \quad L - ACI(\gamma) = \hat{\gamma} - z_{0.025} \sqrt{Var(\hat{\gamma})}, \quad L - ACI(\lambda) = \hat{\lambda} - z_{0.025} \sqrt{Var(\hat{\lambda})}$$

$$-1cmU - ACI(\alpha) = \hat{\alpha} + z_{0.025} \sqrt{Var(\hat{\alpha})}, \quad U - ACI(\gamma) = \hat{\gamma} + z_{0.025} \sqrt{Var(\hat{\gamma})}, \quad U - ACI(\lambda) = \hat{\lambda} + z_{0.025} \sqrt{Var(\hat{\lambda})},$$

$$CP(\alpha) = Mean \begin{cases} 1 & \text{if } L - ACI(\alpha) \leq \hat{\alpha} \leq U - ACI(\alpha) \\ 0 & \text{otherwise} \end{cases},$$

$$CP(\gamma) = Mean \begin{cases} 1 & \text{if } L - ACI(\gamma) \leq \hat{\gamma} \leq U - ACI(\gamma) \\ 0 & \text{otherwise} \end{cases},$$

and

$$CP(\lambda) = Mean \begin{cases} 1 & \text{if } L - ACI(\lambda) \leq \hat{\lambda} \leq U - ACI(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

where $Var(.) = MSEs(.) - Bias(.)$.

Using an MCS and the “optim” function in the “stats” R package with “ $L = 10,000$ ” repeats, the performance of the estimations is compared for different sizes n , number of cycles c , and different chosen parameter values. We utilized the “rss” function in the “RSSampling” R package to create an RSS sampling. This function uses ranked set sampling to select samples from a target population.

The results of the simulation investigation for point estimation are presented in Tables 5 and 6. We draw the following conclusions from these tables:

- The bias and MSEs for parameters have a downward trend when n value increases.
- We can see that the results for bias and MSEs under RSS are consistently L-ACI compared with the values under SRS.
- As n or c increases, all recommended estimates perform better.
- When the value of λ increases with sample size > 8 , we note that the bias and MSEs for the parameters have a downward trend.

- When the value of γ increases with sample size > 8 , we note that the bias and MSEs for parameters have a downward trend.
- RE2 is the largest RE when compared with RE1 and RE3 because it is the ratio between the estimated MSEs of the SRS and RSS with $c = 3$.
- All RE values are greater than 1.

Table 5. MLE under the SRS and RSS of parameters $\alpha = 0.75$.

$\alpha = 0.7$			SRS		RSS		RSS $c = 3$				
γ, λ	n		Bias	MSEs	Bias	MSEs	Bias	MSEs	RE1	RE2	RE3
0.5, 0.6	6	α	0.3443	0.5191	0.1421	0.1498	0.0447	0.0312	3.47	16.64	4.80
		γ	0.7664	1.1624	0.4190	0.4167	0.2187	0.1642	2.79	7.08	2.54
		λ	0.2578	1.2897	0.1082	0.5862	0.0186	0.2368	2.20	5.45	2.48
	8	α	0.2496	0.2990	0.0967	0.0860	0.0111	0.0219	3.48	13.64	3.92
		γ	0.5856	0.7719	0.3434	0.3344	0.1879	0.1172	2.31	6.58	2.85
		λ	0.2221	0.9814	0.0256	0.3406	−0.0070	0.2337	2.88	4.20	1.46
	12	α	0.1437	0.1516	0.0214	0.0203	0.0112	0.0129	7.47	11.71	1.57
		γ	0.4265	0.5161	0.2031	0.1269	0.0840	0.0410	4.07	12.60	3.10
		λ	0.1803	0.7201	−0.0616	0.1254	0.0291	0.1215	5.74	5.93	1.03
	17	α	0.0920	0.1089	0.0102	0.0133	0.0056	0.0038	8.18	28.81	3.52
		γ	0.4146	0.4875	0.1645	0.0998	0.0458	0.0173	4.89	28.10	5.75
		λ	0.1722	0.7018	−0.0314	0.1236	−0.0149	0.0384	5.68	18.28	3.22
0.5, 3	6	α	0.3048	0.5806	0.1182	0.0900	0.0420	0.0182	6.45	31.99	4.96
		γ	0.8484	1.8193	0.2502	0.2980	0.1136	0.0703	6.10	25.86	4.24
		λ	−0.5488	2.9406	−0.2142	1.1465	−0.1804	0.6645	2.56	4.43	1.73
	8	α	0.1803	0.1843	0.0867	0.0604	0.0239	0.0094	3.05	19.60	6.43
		γ	0.7025	1.2727	0.2157	0.1979	0.0338	0.0126	6.43	100.78	15.67
		λ	−0.6403	2.7989	−0.3556	1.0682	−0.0482	0.1718	2.62	16.30	6.22
	12	α	0.1009	0.0891	0.0294	0.0137	0.0113	0.0043	6.52	20.59	3.16
		γ	0.4308	0.7504	0.1061	0.0591	0.0198	0.0078	12.70	96.48	7.60
		λ	−0.4068	1.9500	−0.2762	0.5721	−0.0507	0.1176	3.41	16.59	4.87
	17	α	0.0492	0.0402	0.0224	0.0095	0.0106	0.0019	4.24	21.62	5.10
		γ	0.3308	0.4716	0.0593	0.0240	0.0113	0.0037	19.69	128.37	6.52
		λ	−0.4043	1.9017	−0.0726	0.4903	−0.0161	0.0232	3.88	81.84	21.10
3, 0.6	6	α	0.4053	0.9390	0.1444	0.1197	0.0588	0.0227	7.84	41.31	5.27
		γ	0.4822	1.2508	0.1614	0.3104	0.0821	0.2842	4.03	4.40	1.09
		λ	0.4696	1.1572	0.1716	0.2398	0.1046	0.1307	4.83	8.86	1.83
	8	α	0.2490	0.2469	0.1115	0.0729	0.0278	0.0113	3.39	21.79	6.43
		γ	0.3691	0.9175	0.1912	0.2735	0.0388	0.0873	3.35	10.51	3.13
		λ	0.3245	0.6920	0.0957	0.1002	0.0375	0.0417	6.91	16.59	2.40
	12	α	0.1537	0.1282	0.0479	0.0173	0.0168	0.0052	7.40	24.51	3.31
		γ	0.1790	0.8533	0.0942	0.2374	−0.0159	0.0730	3.59	11.69	3.25
		λ	0.3046	0.6896	0.0670	0.0733	0.0448	0.0377	9.41	18.28	1.94
	17	α	0.0894	0.0594	0.0271	0.0078	0.0124	0.0022	7.58	26.57	3.51
		γ	0.1956	0.6849	0.0637	0.0973	0.0122	0.0483	7.04	14.17	2.01
		λ	0.2218	0.5910	0.0242	0.0155	0.0135	0.0075	38.20	78.97	2.07
3, 3	6	α	0.3724	0.9002	0.1315	0.1000	0.0525	0.0191	9.00	47.13	5.24
		γ	1.1994	3.0371	0.5578	1.0798	0.2948	0.5427	2.81	5.60	1.99
		λ	0.5140	2.2446	0.1707	1.0838	0.1353	0.8042	2.07	2.79	1.35
	8	α	0.2266	0.2063	0.1059	0.0669	0.0253	0.0103	3.09	19.99	6.48
		γ	0.9439	2.3814	0.3788	0.8480	0.1962	0.2866	2.81	8.31	2.96
		λ	0.2949	1.6875	0.1159	0.5516	−0.0177	0.3418	3.06	4.94	1.61
	12	α	0.1298	0.1020	0.0444	0.0160	0.0136	0.0045	6.36	22.54	3.54
		γ	0.6487	1.6606	0.2613	0.4754	0.0892	0.1343	3.49	12.36	3.54
		λ	0.1906	1.4016	0.0609	0.5100	0.0018	0.1746	2.75	8.03	2.92
	17	α	0.0751	0.0447	0.0261	0.0073	0.0119	0.0020	6.09	21.86	3.59
		γ	0.5969	1.5632	0.1162	0.1451	0.0351	0.0842	10.78	18.56	1.72
		λ	0.1289	1.0484	0.0164	0.0894	0.0141	0.0712	11.72	14.72	1.26

Table 6. MLE under the SRS and RSS of parameters $\alpha = 2.5$.

$\alpha = 2.5$			SRS		RSS		RSS $c = 3$				
γ, λ	n		Bias	MSEs	Bias	MSEs	Bias	MSEs	RE1	RE2	RE3
0.5, 0.6	6	α	0.2604	0.8575	0.0223	0.1264	0.0189	0.1158	6.7855	7.4056	1.0914
		γ	0.4325	0.4284	0.1856	0.1222	0.0788	0.0322	3.5066	13.3014	3.7933
		λ	−0.1203	0.2945	−0.0973	0.0999	−0.0295	0.0673	2.9472	4.3754	1.4846
	8	α	0.1747	0.8002	0.0212	0.1148	−0.0130	0.0185	6.9722	43.2561	6.2041
		γ	0.3321	0.2750	0.1841	0.1204	0.0388	0.0126	2.2835	21.7690	9.5332
		λ	−0.1062	0.2510	−0.0390	0.0915	−0.0393	0.0261	2.7423	9.6146	3.5060
	12	α	0.1247	0.5156	−0.0361	0.1138	−0.0227	0.0167	4.5287	30.8814	6.8191
		γ	0.2146	0.2258	0.0901	0.0329	0.0290	0.0089	6.8563	25.2393	3.6812
		λ	−0.0556	0.1729	−0.0773	0.0615	−0.0267	0.0257	2.8093	6.7252	2.3939
	17	α	−0.0442	0.1900	0.0273	0.0333	0.0018	0.0073	5.7116	25.9408	4.5417
		γ	0.1783	0.1125	0.0332	0.0098	0.0142	0.0026	11.4873	43.5174	3.7883
		λ	−0.1014	0.0970	−0.0208	0.0338	−0.0198	0.0088	2.8716	11.0455	3.8464
0.5, 3	6	α	0.6944	1.9515	0.2678	0.6547	0.1566	0.1978	2.9808	9.8672	3.3103
		γ	0.2469	0.3202	0.0582	0.0281	0.0265	0.0054	11.3793	58.8420	5.1710
		λ	−0.5350	1.1013	−0.1765	0.2926	−0.0710	0.1151	3.7638	9.5679	2.5421
	8	α	0.6125	1.4721	0.2408	0.4607	0.0355	0.0950	3.1954	15.4941	4.8489
		γ	0.1385	0.0904	0.0315	0.0100	0.0109	0.0032	9.0464	27.9550	3.0902
		λ	−0.3491	0.7967	−0.0751	0.2688	−0.0672	0.0613	2.9644	12.9908	4.3822
	12	α	0.4017	0.9124	0.1375	0.2151	0.0521	0.0912	4.2426	10.0075	2.3588
		γ	0.0966	0.0701	0.0225	0.0047	0.0103	0.0022	14.9603	31.6954	2.1186
		λ	−0.1601	0.7151	−0.0800	0.2146	−0.0208	0.0612	3.3321	11.6835	3.5064
	17	α	0.1839	0.5261	0.0379	0.0418	0.0295	0.0351	12.5962	14.9778	1.1891
		γ	0.0808	0.0489	0.0048	0.0011	0.0067	0.0009	43.2188	55.2868	1.2792
		λ	−0.2746	0.5518	−0.0198	0.0059	−0.0142	0.0056	93.9679	98.4008	1.0472
3, 0.6	6	α	1.0252	3.5461	0.4545	1.1531	0.2426	0.4563	3.0753	7.7706	2.5268
		γ	0.3068	0.5923	0.1484	0.1587	0.0743	0.0625	3.7324	9.4770	2.5391
		λ	0.1418	0.2958	0.0224	0.0220	0.0150	0.0079	13.4350	37.6476	2.8022
	8	α	0.8179	2.3824	0.2373	0.4484	0.0977	0.1972	5.3132	12.0812	2.2738
		γ	0.2169	0.3602	0.0322	0.1222	0.0198	0.0554	2.9488	6.4973	2.2034
		λ	0.0938	0.1905	0.0204	0.0211	0.0106	0.0073	9.0362	26.1180	2.8904
	12	α	0.5830	1.5779	0.1865	0.3327	0.0263	0.0566	4.7429	27.9030	5.8831
		γ	0.1375	0.1397	0.1285	0.1204	0.0067	0.0028	1.1602	50.0694	43.1558
		λ	0.0860	0.1281	−0.0005	0.0118	−0.0003	0.0011	10.8728	113.4134	10.4309
	17	α	0.3814	0.8503	0.1094	0.1541	0.0616	0.0455	5.5170	18.6987	3.3893
		γ	0.1448	0.1142	0.0427	0.0500	−0.0460	0.0027	2.2844	42.9291	18.7925
		λ	0.0717	0.1150	0.0044	0.0053	0.0236	0.0010	21.6820	119.6383	5.5179
3, 3	6	α	1.6285	9.5929	0.6978	2.6152	0.2730	0.4711	3.6681	20.3630	5.5514
		γ	1.1681	3.1433	0.3365	0.5741	0.2832	0.5630	5.4754	5.5834	1.0197
		λ	−0.0241	2.0158	0.0113	0.5014	0.0162	0.4629	4.0204	4.3549	1.0832
	8	α	1.1045	4.4207	0.4881	1.4793	0.1131	0.2350	2.9884	18.8080	6.2936
		γ	0.8590	2.1616	0.4007	0.5079	0.0973	0.1101	4.2558	19.6414	4.6152
		λ	0.0133	1.8999	−0.0649	0.4786	−0.0332	0.1325	3.9694	14.3346	3.6113
	12	α	0.6665	2.4806	0.1988	0.3392	0.0702	0.1148	7.3127	21.6071	2.9547
		γ	0.5576	1.3683	0.1616	0.1661	0.0371	0.0910	8.2362	15.0356	1.8256
		λ	0.0179	1.2303	−0.0489	0.1638	0.0237	0.1158	7.5125	10.6255	1.4144
	17	α	0.3954	1.0998	0.1170	0.1698	0.0521	0.0430	6.4765	25.5833	3.9502
		γ	0.3646	0.7632	0.3281	0.1583	0.0154	0.0169	4.8222	45.1428	9.3615
		λ	0.0301	0.9246	−0.0221	0.9495	0.0091	0.0150	0.9738	61.4602	63.1151

The results of the simulation investigation for point estimation are presented in Tables 7 and 8. We draw the following conclusions from these tables:

- The L-ACI and U-ACI for the parameters have convergence when n value increases.
- The length of ACI (U-ACI-L-ACI) have a downward trend when n value increases.

- We can see that the results for L-ACI and U-ACI under RSS are consistently lower than the values under SRS.
- As n or c increases, all recommended estimates perform better.
- The CP for all estimates are approximately 95%.

Table 7. Confidence intervals with CP for the parameters of the HL-INH distribution: $\alpha = 0.7$.

$\alpha = 0.7$			SRS			RSS			RSS $c = 3$		
γ, λ	n		L-ACI	U-ACI	CP	L-ACI	U-ACI	CP	L-ACI	U-ACI	CP
0.5, 0.6	6	α	0.0982	2.2869	94.67%	0.1354	1.5488	94.33%	0.4092	1.0802	96.67%
		γ	0.1222	2.7552	97.67%	0.2045	1.8830	98.00%	0.0490	1.3885	94.33%
		λ	0.2314	3.0292	94.67%	0.2780	2.1963	94.33%	0.2735	1.5106	96.67%
	8	α	0.1056	1.9047	93.67%	0.2532	1.3401	95.00%	0.4212	1.0010	96.00%
		γ	0.1320	2.3715	97.33%	0.1407	1.7569	95.67%	0.1261	1.2498	96.33%
		λ	0.3073	2.7176	93.67%	0.3152	1.7704	95.00%	0.3559	1.5419	96.00%
	12	α	0.1332	1.5542	96.00%	0.4450	0.9979	94.33%	0.4889	0.9335	96.67%
		γ	0.1421	2.0614	95.67%	0.1286	1.2776	94.00%	0.2225	0.9456	96.67%
		λ	0.3185	2.4083	96.00%	0.3515	1.2229	94.33%	0.3528	1.5380	96.67%
	17	α	0.1699	1.4142	94.67%	0.4845	0.9359	95.67%	0.5854	0.8258	94.33%
		γ	0.1588	2.0175	96.00%	0.1352	1.1939	95.67%	0.3034	0.7882	95.67%
		λ	0.3212	2.7524	94.67%	0.3615	1.2896	95.67%	0.2015	0.9687	94.33%
0.5, 3	6	α	0.1366	2.3759	96.00%	0.2769	1.3595	94.67%	0.4907	0.9933	96.00%
		γ	0.1710	3.4071	94.67%	0.2025	1.7028	93.33%	0.1430	1.0842	96.33%
		λ	0.2738	5.6408	96.00%	0.7261	4.8455	94.67%	1.2589	4.3803	96.00%
	8	α	0.1155	1.6451	94.33%	0.3352	1.2382	94.33%	0.5394	0.9083	95.33%
		γ	0.2853	2.9356	95.00%	0.3048	1.4795	95.00%	0.3234	0.7442	96.67%
		λ	0.2967	5.3942	94.33%	0.7392	4.5497	94.33%	2.1437	3.7600	95.33%
	12	α	0.2494	1.3524	96.00%	0.5073	0.9515	96.00%	0.5841	0.8385	94.67%
		γ	0.3054	2.4062	93.00%	0.1766	1.0356	95.67%	0.3511	0.6885	97.67%
		λ	0.3029	5.2159	96.00%	1.3414	4.1062	96.00%	2.2835	3.6151	94.67%
	17	α	0.3677	1.1307	95.33%	0.5365	0.9084	95.33%	0.6285	0.7926	94.33%
		γ	0.3508	2.0124	93.00%	0.2787	0.8400	97.33%	0.3944	0.6282	98.33%
		λ	0.3108	5.2684	95.33%	1.0670	4.7879	95.33%	2.6863	3.2815	94.33%
3, 0.6	6	α	0.1623	2.8334	96.67%	0.2269	1.4618	94.67%	0.4862	1.0314	95.67%
		γ	1.5010	5.4634	92.33%	2.1145	4.2084	93.00%	2.0479	4.1163	91.67%
		λ	0.1831	2.9697	96.67%	0.1929	1.6720	94.67%	0.1253	1.3840	95.67%
	8	α	0.1048	1.7932	94.67%	0.3286	1.2943	94.67%	0.5261	0.9295	96.00%
		γ	1.6337	5.1045	94.33%	2.0938	4.2886	92.00%	2.4636	3.6140	94.33%
		λ	0.1958	2.4284	94.67%	0.1032	1.2881	94.67%	0.2433	1.0317	96.00%
	12	α	0.2189	1.4886	95.67%	0.5073	0.9886	95.00%	0.5787	0.8549	95.33%
		γ	1.7295	5.0634	92.33%	2.1558	4.0326	92.33%	2.1681	3.8001	91.00%
		λ	0.2078	2.6759	95.67%	0.1520	1.1819	95.00%	0.2738	1.0159	95.33%
	17	α	0.3441	1.2347	95.33%	0.5616	0.8927	95.33%	0.6227	0.8020	95.00%
		γ	1.8617	4.7743	90.00%	2.4641	3.6634	94.00%	2.5813	3.4432	94.00%
		λ	0.3062	2.2669	95.33%	0.3846	0.8637	95.33%	0.4457	0.7812	95.00%
3, 3	6	α	-0.6409	2.7856	97.00%	0.2668	1.3962	94.67%	0.5014	1.0035	97.00%
		γ	1.7172	6.6816	95.33%	1.8365	5.2790	94.67%	1.9693	4.6202	94.33%
		λ	0.7512	6.2769	97.00%	1.1545	5.1869	94.67%	1.3948	4.8757	97.00%
	8	α	0.1537	1.6995	94.67%	0.3426	1.2691	95.33%	0.5321	0.9185	96.33%
		γ	1.5470	6.3408	96.00%	1.7309	5.0267	94.00%	2.2182	4.1741	92.67%
		λ	0.8111	5.7787	94.67%	1.6756	4.5562	95.33%	1.8351	4.1295	96.33%
	12	α	0.2567	1.4028	96.33%	0.5114	0.9773	95.33%	0.5842	0.8429	95.00%
		γ	1.4626	5.8347	94.00%	2.0086	4.5140	93.00%	2.3913	3.7871	93.33%
		λ	0.8967	5.4846	96.33%	1.5455	4.5763	95.33%	2.1813	3.8222	95.00%
	17	α	0.3870	1.1631	96.00%	0.5659	0.8863	95.00%	0.6262	0.7975	94.33%
		γ	1.4401	5.7537	93.33%	2.4041	3.8283	96.33%	2.4696	3.6007	92.00%
		λ	0.7508	5.5071	96.00%	2.4302	3.6026	95.00%	2.3578	3.7245	94.33%

Table 8. Confidence intervals with CP for the parameters of the HL-INH distribution: $\alpha = 2.5$.

$\alpha = 2.5$			SRS			RSS			RSS $c = 3$		
γ, λ	n		L-ACI	U-ACI	CP	L-ACI	U-ACI	CP	L-ACI	U-ACI	CP
0.5, 0.6	6	α	1.0158	4.5050	96.67%	1.8258	3.2188	93.33%	1.8289	3.3493	93.67%
		γ	0.0932	1.8971	95.67%	0.1041	1.2671	94.33%	0.2623	0.8954	95.33%
		λ	0.0559	1.5186	96.67%	0.0877	1.0932	93.33%	0.0645	1.0765	93.67%
	8	α	0.7396	4.6098	99.67%	1.2830	3.9548	96.67%	2.2212	2.7528	96.00%
		γ	0.0353	1.6290	96.33%	0.1575	1.9437	99.33%	0.3318	0.7459	96.33%
		λ	0.1089	1.8802	99.67%	0.2020	1.3239	96.67%	0.2530	0.8684	96.00%
	12	α	1.2364	4.0130	92.33%	1.8053	3.1226	94.33%	1.9713	2.9834	94.33%
		γ	0.1251	1.5469	97.33%	0.2808	0.8994	97.00%	0.3523	0.7058	95.33%
		λ	0.1265	1.3534	92.33%	0.0599	0.9855	94.33%	0.2544	0.8922	94.33%
	17	α	1.6045	3.3071	91.67%	2.1733	2.8813	97.00%	2.3339	2.6698	96.67%
		γ	0.1207	1.2359	94.33%	0.3503	0.7162	96.67%	0.4184	0.6100	95.33%
		λ	0.1308	1.0768	91.67%	0.2206	0.9377	97.00%	0.4003	0.7600	96.67%
0.5, 3	6	α	0.8146	5.5742	95.33%	1.2688	4.2668	93.67%	1.8394	3.4738	94.33%
		γ	0.1253	1.7466	92.67%	0.2494	0.8671	96.00%	0.3914	0.6617	95.67%
		λ	0.6925	4.2375	95.33%	1.8196	3.8273	93.67%	2.2777	3.5803	94.33%
	8	α	1.0563	5.1687	94.67%	1.4949	3.9867	94.67%	1.9344	3.1366	95.33%
		γ	0.1144	1.1626	95.67%	0.3451	0.7178	97.00%	0.4013	0.6205	97.33%
		λ	1.0381	4.2637	94.67%	1.9178	3.9319	94.67%	2.4648	3.4007	95.33%
	12	α	1.2002	4.6031	94.67%	1.7680	3.5070	95.00%	1.8879	3.2164	95.33%
		γ	0.1126	1.0806	96.33%	0.3955	0.6494	96.00%	0.4203	0.6004	96.00%
		λ	0.7575	4.9224	94.67%	2.0242	3.8159	95.00%	1.8514	4.1071	95.33%
	17	α	1.3063	4.0614	97.33%	2.1437	2.9322	93.67%	2.1661	2.8929	95.00%
		γ	0.1766	0.9851	95.67%	0.4394	0.5702	96.00%	0.4498	0.5636	94.00%
		λ	1.3703	4.0804	97.33%	2.8349	3.1255	93.67%	2.4258	3.4903	95.00%
3, 0.6	6	α	0.4241	6.6264	96.67%	1.0445	4.8645	94.33%	1.5047	3.9804	96.33%
		γ	1.9210	4.6926	92.33%	2.4225	3.8742	93.67%	2.6056	3.5429	93.33%
		λ	0.2890	1.7727	96.67%	0.3344	0.9104	94.33%	0.4435	0.7865	96.33%
	8	α	0.7481	5.8878	95.33%	1.5079	3.9666	97.00%	1.7472	3.4481	96.67%
		γ	2.1182	4.3156	94.00%	2.3489	3.7155	95.00%	2.5591	3.4804	94.33%
		λ	0.2914	1.5307	95.33%	0.3001	1.2825	97.00%	0.4213	0.7999	96.67%
	12	α	0.8985	5.2675	95.67%	1.6150	3.7581	96.00%	2.0623	2.9903	94.67%
		γ	1.9301	4.3449	94.00%	2.4331	3.8240	96.00%	2.9039	3.1096	94.33%
		λ	0.0039	1.3681	95.67%	0.3864	0.8126	96.00%	0.5337	0.6657	94.67%
	17	α	1.2333	4.5296	95.67%	1.8692	3.3496	95.33%	2.1608	2.9624	95.67%
		γ	1.9013	4.3882	93.33%	2.6118	3.4737	95.67%	2.4575	3.4505	94.67%
		λ	0.0211	1.3223	95.67%	0.4617	0.7471	95.33%	0.4777	0.7696	95.67%
3, 3	6	α	0.0439	9.3009	95.33%	0.3336	6.0620	94.67%	1.5367	4.0093	94.67%
		γ	1.5497	6.7866	97.00%	2.0038	4.6693	94.00%	1.9190	4.6473	97.00%
		λ	0.1889	5.7629	95.33%	1.6213	4.4013	94.67%	1.4596	4.5728	94.67%
	8	α	0.0920	7.1169	93.00%	0.8010	5.1752	95.33%	1.6875	3.5386	96.00%
		γ	1.5164	6.2016	97.67%	1.8420	4.9595	96.00%	2.4746	3.7199	96.67%
		λ	0.3073	5.7192	93.00%	1.1990	4.6713	95.33%	2.2550	3.6785	96.00%
	12	α	0.3648	5.9681	96.33%	1.6240	3.7736	96.67%	1.9194	3.2210	95.33%
		γ	1.5388	5.5764	97.00%	2.4270	3.8962	96.00%	2.4493	3.6248	95.00%
		λ	0.8405	5.1953	96.33%	2.1624	3.7398	96.67%	2.1940	3.8534	95.33%
	17	α	0.9885	4.8024	95.67%	1.8413	3.3928	95.67%	2.1581	2.9461	95.67%
		γ	1.8060	4.9232	96.33%	1.6634	4.9928	98.67%	2.7619	3.2689	94.00%
		λ	1.1432	4.9170	95.67%	1.0653	4.8905	95.67%	2.7689	3.2492	95.67%

9. Application

The analysis of three real-world datasets, one of which is linked to COVID-19 data and another to yearly flood flows, are discussed in this part to confirm that the HL-INH distribution is the best model fitting present in the literature. The models used for comparison are OLINH; PINH; INH; EOWINH; MOINH; extended Weibull (EW), which

was discussed by [52]; exponential Lomax (EL), which was discussed by [53]; Weibull-Lomax (WL), which was discussed by [54]; and Gumbel-Lomax (GL), which was discussed by [55].

Furthermore, we calculate the estimates with the standard error (SE) and eight measures of goodness of fit statistics (GFS) for each statistical model, with the eight measures consisting of “Akaike Information (AI) as M1, corrected AI as M2, Bayesian information as M3, Hannan—Quinn information as M4, Cramer-von-Mises as M5, Anderson—Darling as M6, Kolmogorov-Smirnov distance as M7, and p -value as M8”. The model generally regarded as the best is the one with the smaller M1, M2, M3, M4, M5, M6, and M7 statistics and the higher M8 for the GFS. It should be emphasized that all findings were obtained using the MLE technique.

Flood data: The North Saskatchewan River in Edmonton’s annual flood flows in units of 1000 f3/second over a 47-year period make up the first dataset, according to [56]. Figure 2 discussed the flood data description by box plots, strip plots, and violin plots. The violin plot is used to show how the numerical data is distributed. Violin plots provide summary statistics as well as the density of each variable unlike box plots, which can only show summary statistics. This flood data set has a right-skewed shape, which corresponds to the nature of the distribution form (see Figure 1). The MLE for various competitive statistical distributions with various measures of GFS for the flood data are discussed in Table 9. Figure 3 displays the fitted CDF with an empirical CDF in I, the fitted PDF with flood data histograms in II, a QQ plot in III, and a PP plot for HL-INH in IV. Figure 4 shows the profile likelihood of the estimators of parameters for the HL-INH distribution using the flood data. We confirm that the HL-INH distribution is the best model to fit the flood data, and the estimates have uniqueness and a maximum point of log likelihood.

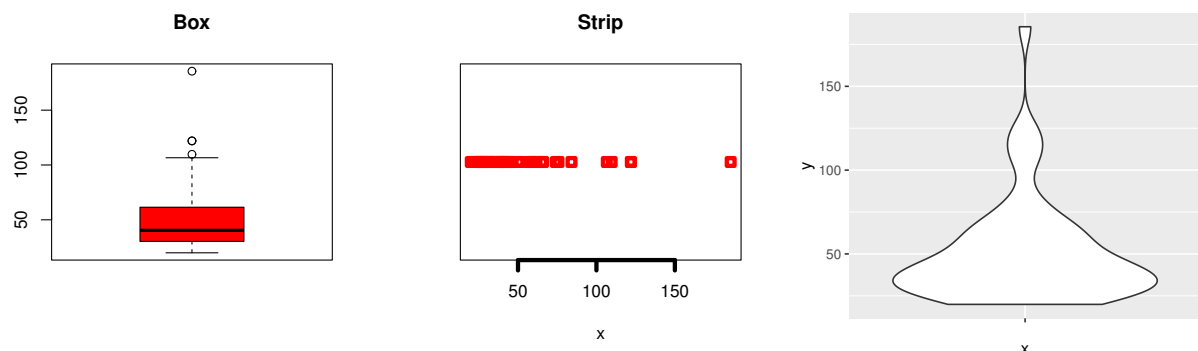


Figure 2. Data description for the flood data using box, strip, and violin plots.

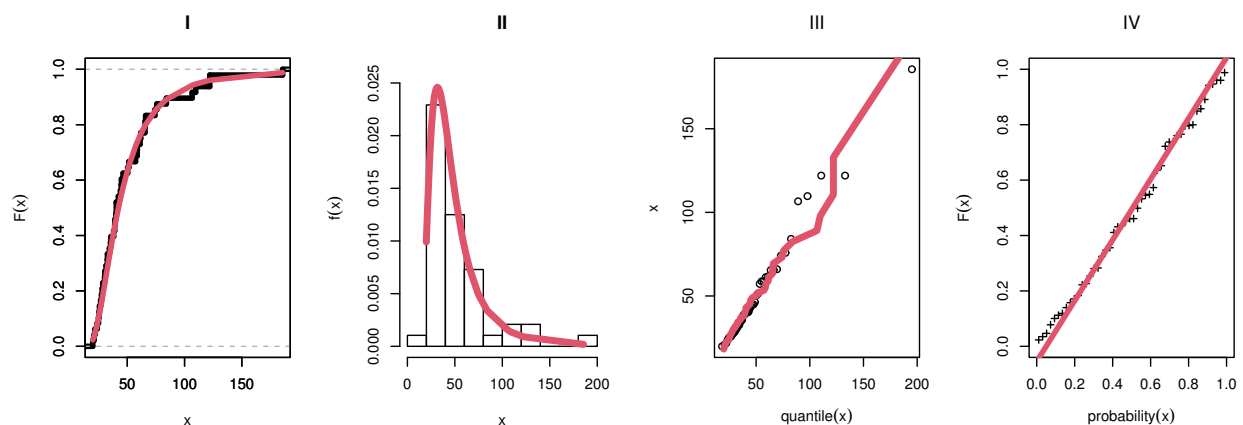
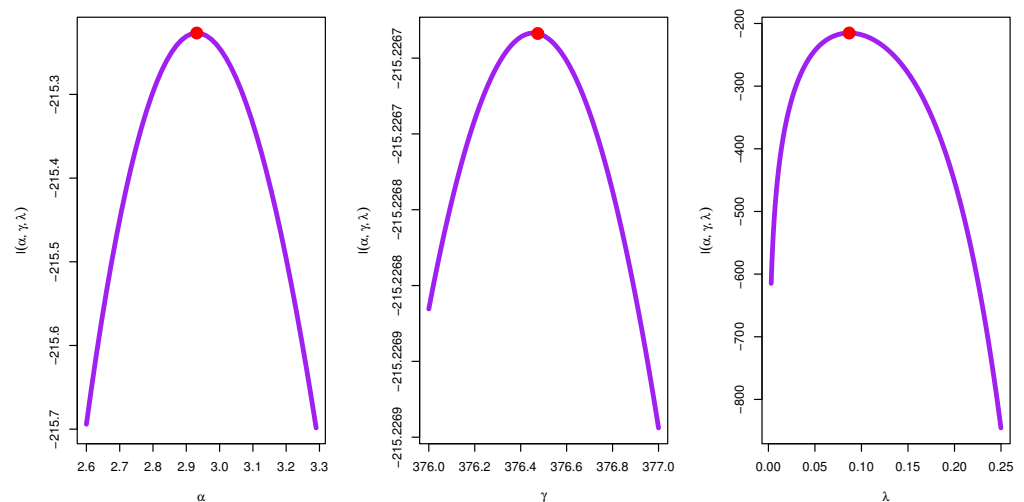


Figure 3. Fitting of (I) ecdf, (II) epdf, (III) QQ and (IV) PP plots for the flood data.

Table 9. MLE for each model with different measures: flood data.

		Estimates	SE	M1	M2	M3	M4	M5	M6	M7	M8
HL-INH	α	2.9313	0.4873								
	γ	376.4744	6103.8330	436.4053	436.9987	442.0669	438.5747	0.0228	0.1608	0.0598	0.9954
	λ	0.0868	1.4100								
OLINH	α	2.8705	1.8291								
	β	1.2768	1.2160								
	λ	0.5966	0.5223	437.9109	438.8411	445.3957	440.7394	0.0237	0.1726	0.0643	0.9888
	θ	59.5522	51.3453								
PINH	α	2.1418	1.1729								
	β	562.8134	638.9703	436.4633	437.0096	442.0269	438.6346	0.0243	0.1669	0.0670	0.9823
	γ	2.0611	0.2153								
INH	γ	868.5419	3765.7212								
	λ	0.0322	0.1398	453.6735	453.9402	457.4159	455.0878	0.0283	0.3006	0.2438	0.0067
EOWINH	α	1.8063	0.8672								
	β	0.8104	0.8693								
	γ	19.0763	7.2102	439.4286	440.3589	446.9134	442.2572	0.0273	0.1922	0.0610	0.9941
	λ	1.2052	4.5071								
MOINH	α	5.3684	8.3880								
	β	8.7657	16.5036	439.9131	440.4585	445.5267	442.0345	0.0555	0.3617	0.0700	0.9727
	γ	0.1751	0.1106								
EW	α	0.2227	0.0072								
	β	0.0050	0.0013	466.2669	466.8123	471.8805	468.3883	0.4023	2.4214	0.1827	0.0813
	θ	0.0562	0.0279								
EL	α	47.0107	92.6907								
	β	3.4559	1.6865	437.1066	437.6520	442.7202	439.2279	0.0263	0.1874	0.0614	0.9935
	θ	17.5988	28.6909								
GL	α	1.1037	1.7143								
	β	8.5911	49.4043								
	θ	0.4355	0.4307	438.2232	439.1534	445.7080	441.0517	0.0239	0.1658	0.0699	0.9731
	λ	41.2152	498.9307								
WL	α	25.3627	93.1384								
	β	3.7405	1.4798								
	θ	0.1651	0.1322	452.5347	453.4650	460.0195	455.3633	0.1897	1.2234	0.1264	0.4270
	λ	7.6128	10.6009								

**Figure 4.** Profile likelihood α , γ , and λ for the flood data.

COVID-19 data: The COVID-19 data covers Britain over 82 days, ranging from 1 May to 16 July 2021. The data shown below are derived from daily new deaths, daily cumulative cases, and daily cumulative deaths. These data have been obtained by [57] as follows: “0.0023, 0.0023, 0.0023, 0.0046, 0.0065, 0.0067, 0.0069, 0.0069, 0.0091, 0.0093, 0.0093, 0.0093, 0.0111, 0.0115, 0.0116, 0.0116, 0.0119, 0.0133, 0.0136, 0.0138, 0.0138, 0.0159, 0.0161, 0.0162, 0.0162, 0.0162, 0.0163, 0.0180, 0.0187, 0.0202, 0.0207, 0.0208, 0.0225, 0.0230, 0.0230, 0.0239, 0.0245, 0.0251, 0.0255, 0.0255, 0.0271, 0.0275, 0.0295, 0.0297, 0.0300, 0.0302, 0.0312, 0.0314, 0.0326, 0.0346, 0.0349, 0.0350, 0.0355, 0.0379, 0.0384, 0.0394, 0.0394, 0.0412, 0.0419, 0.0425, 0.0461, 0.0464, 0.0468, 0.0471, 0.0495, 0.0501, 0.0521, 0.0571, 0.0588, 0.0597, 0.0628, 0.0679, 0.0685, 0.0715, 0.0766, 0.0780, 0.0942, 0.0960, 0.0988, 0.1223, 0.1343, 0.1781”. These data have been cited in <https://covid19.who.int/> (accessed on 15 February 2023). Figure 5 discusses the data description using box, strip, and violin plots. This COVID-19 dataset has a right-skewed shape, which corresponds to the nature of the distribution form (see Figure 1). The MLE for various models with various statistics of GFS for the COVID-19 data of Britain are discussed in Table 10. Figure 6 displays the fitted CDF with an empirical CDF in I, the fitted PDF with COVID-19 data histograms in II, a QQ plot in III, and a PP plot for the HL-INH distribution in IV. Figure 7 shows the profile likelihood for the parameters of the HL-INH distribution based on British COVID-19 data. We confirm that the HL-INH distribution is the best model for the fitting of the COVID-19 data, and the estimates have uniqueness and a maximum point of log likelihood.

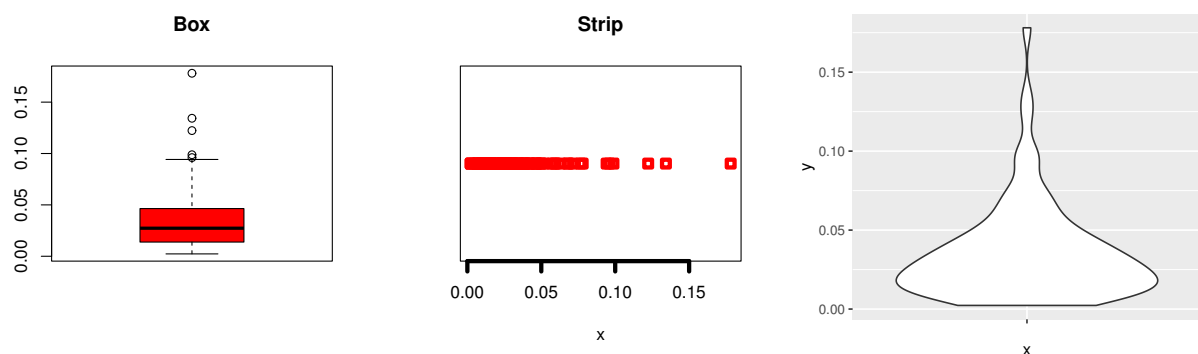


Figure 5. Data description for the COVID-19 data using box, strip, and violin plots.

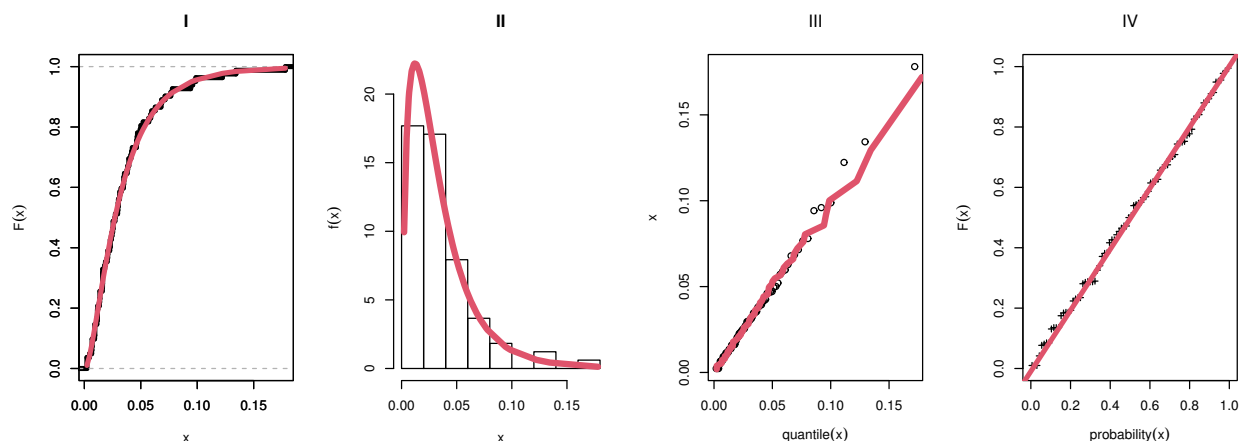
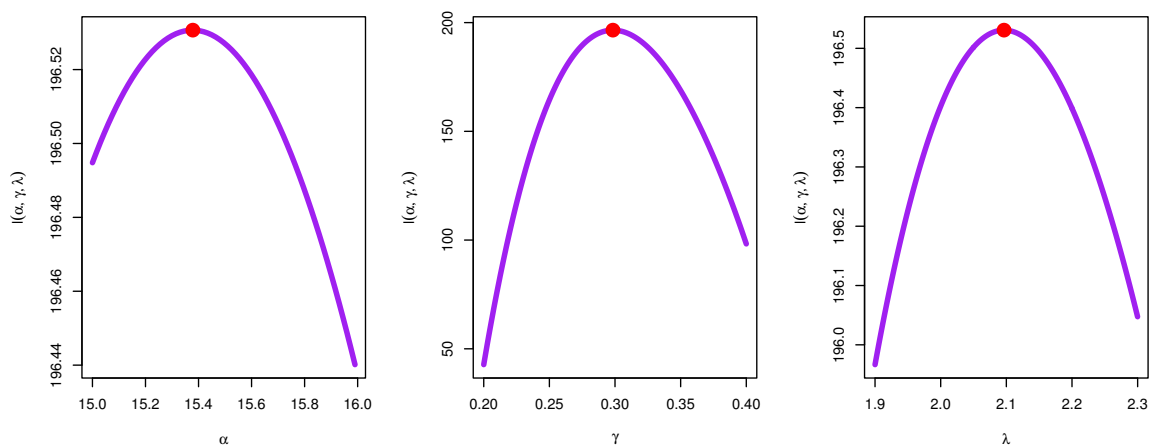


Figure 6. Fitting of (I) ecdf, (II) epdf, (III) QQ and (IV) PP plots for plots for the COVID-19 data.

Table 10. MLE for each model with different measures: COVID-19 data.

		Estimates	SE	M1	M2	M3	M4	M5	M6	M7	M8
HL-INH	α	15.3785	29.9414								
	γ	0.2984	0.1454	−387.0614	−386.7537	−379.8412	−384.1626	0.0135	0.1287	0.0398	0.9995
	λ	2.0965	8.1945								
OLINH	α	20.0128	35.5663								
	β	11.3299	17.0058	−383.8202	−383.3007	−374.1933	−379.9552	0.0286	0.2465	0.0573	0.9509
	λ	0.1102	0.0949								
	θ	0.4862	0.0876								
PINH	α	0.1511	0.0954								
	β	0.0001	0.0008	−385.8691	−385.5614	−378.6489	−382.9703	0.0231	0.2134	0.0451	0.9963
	γ	3.4485	1.0951								
INH	γ	0.7808	0.1283	−363.0467	−362.8948	−358.2333	−361.1142	0.2382	1.6234	0.1364	0.0944
	λ	0.0242	0.0074								
EOWINH	α	1.6714	1.1690								
	β	0.3299	0.4791	−385.0412	−384.5217	−375.4143	−381.1761	0.0137	0.1299	0.0417	0.9988
	γ	0.3332	0.2501								
	λ	0.1235	0.1712								
MOINH	α	0.2889	0.0238								
	β	14.3635	9.7515	−373.1607	−372.8530	−365.9406	−370.2619	0.1703	1.1157	0.0725	0.7813
	γ	0.0051	0.0022								
EW	α	1.2518	0.1027								
	β	108.8249	5.0239	−383.0809	−382.7732	−375.8607	−380.1821	0.0552	0.3909	0.0572	0.9510
	θ	191.7996	62.5018								
EL	α	1.9735	0.4825								
	β	8.0262	8.3354	−386.9522	−386.6445	−379.7320	−384.0534	0.0143	0.1301	0.0432	0.9980
	θ	0.1650	0.2126								
GL	α	20.2646	26.5200								
	β	0.1918	0.3158	−384.8166	−384.2971	−375.1897	−380.9515	0.0180	0.1631	0.0418	0.9988
	θ	2.0804	0.6499								
	λ	2.4390	0.9729								
WL	α	6.6349	19.1434								
	β	1.8551	0.4084	−384.8288	−384.3093	−375.2019	−380.9638	0.0144	0.1297	0.0417	0.9988
	θ	0.2559	0.2911								
	λ	0.0156	0.0113								

**Figure 7.** Profile likelihood α , γ , and λ for the COVID-19 data.

We observed random samples of varied sizes for analysis using the SRS and RSS techniques with different set sizes and cycle counts. The performance of the estimations was then compared after we computed the MLE for the observed SRS and RSS with different cycle c values. As demonstrated in Table 11, we were able to obtain RSS data for the first batch of data when $n = 12$ and $n = 15$ with various cycles as $c = 1$ and $c = 3$. Table 12 displays the MLE under SRS and RSS with various c for the flood data. As demonstrated in Table 13, we were able to obtain RSS data for the second batch of data when $n = 15$ and $n = 18$ with various cycles as $c = 1$ and $c = 3$. Table 14 displays the MLE under SRS and RSS with various cycles for the COVID-19 data. We note that the SE for RSS is smaller than the SE for SRS. The performance of the estimations was then compared after we computed the MLE for the observed SRS and RSS with different cycle c values. Table 15 discusses the summarized measures for each data sets. As demonstrated in Tables 16, and 17, we were discussed MLE for each model with different measures for carbon fibre data and we were able to obtain RSS carbon fibre data for the first batch of data when $n = 28$ with various cycles as $c = 1$ and $c = 4$, respectively. Table 18 displays the MLE under SRS and RSS with various c for the carbon fibre data.

Table 11. SRS and RSS for the flood data.

n		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
17	SRS	21.82	24.888	31.5	31.5	32.6	40.4	42.25	44.02	50.33	61.74	75.8	84.1			
	RSS															
	$c = 1$	21.82	27.5	34.4	32.6	39.2	39.2	57.22	65.597	65.44	84.1	75.8	66			
	$c = 1$	27.5	44.02	32.6	75.8											
	RSS $c = 3$					40.4	44.02	32.6	84.1							
15	SRS	21.82	24.888	27.5	31.5	31.5	32.6	40.4	42.25	44.02	44.02	44.73	50.33	61.74	75.8	84.1
	RSS															
	$c = 1$	23.7	28.1	30.38	30.38	28.1	38.1	40.4	58.8	44.73	44.9	44.02	84.1	75.8	121.97	84.1
	$c = 1$	31.5	32.6	50.33	44.9	58.8										
	RSS $c = 3$						21.82	31.5	42.25	75.8	44.73			27.5	31.5	42.25
	$c = 3$													44.73	75.8	

Table 12. MLE for SRS and RSS of the flood data.

n		SRS		RSS $c = 1$		RSS $c = 3$	
		Estimates	SE	Estimates	SE	Estimates	SE
12	α	3.9030	1.8620	3.2424	0.3937	4.3122	0.2669
	γ	376.4743	208.1903	376.4824	203.5782	376.4811	174.4104
	λ	0.0937	1.2985	0.0985	0.1742	0.0991	0.1691
15	α	4.4195	2.0293	2.9963	0.2329	5.3364	0.0921
	γ	376.4737	121.7994	376.4763	91.5062	376.4545	45.1077
	λ	0.0971	0.8130	0.0905	0.0530	0.1061	0.0255

Carbon fibre data: The third dataset has [58] as its source. It includes the single carbon fibre tensile strength (in GPa). This information is provided by: 0.312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.997, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585.

Table 15 discusses the summarized measures for each dataset, including the minimum (Min), maximum (Max), first quantile (Q1), median, mean, third quantile (Q3), variance (σ^2), skewness (SK), and kurtosis (KT).

Table 13. SRS and RSS for the COVID-19 data.

<i>n</i>	12				18				
	SRS	RSS <i>c</i> = 1	RSS <i>c</i> = 3		SRS	RSS <i>c</i> = 1	RSS <i>c</i> = 3		
i		<i>c</i> = 1	<i>c</i> = 1	<i>c</i> = 2	<i>c</i> = 3	<i>c</i> = 1	<i>c</i> = 1	<i>c</i> = 2	<i>c</i> = 3
1	0.0091	0.0023	0.0162			0.0069	0.0046	0.0091	
2	0.0115	0.0162	0.0116			0.0091	0.0116	0.0162	
3	0.0115	0.0093	0.0162			0.0091	0.0111	0.0207	
4	0.0162	0.0116	0.0588			0.0115	0.0119	0.0349	
5	0.0162	0.0208	0.0521			0.0115	0.0159	0.0394	
6	0.0207	0.0255		0.0116		0.0162	0.0208	0.1343	
7	0.0275	0.0225		0.0346		0.0162	0.0255		0.0069
8	0.0295	0.0295		0.0115		0.0207	0.0275		0.0162
9	0.0346	0.0208		0.0295		0.0275	0.023		0.0207
10	0.0349	0.0275		0.0988		0.0275	0.023		0.0239
11	0.0394	0.0314			0.0091	0.0295	0.0255		0.0521
12	0.0521	0.0412			0.0162	0.0346	0.0379		0.096
13	0.0588	0.0685			0.0521	0.0349	0.0275		0.0115
14	0.0679	0.0394			0.0349	0.0394	0.0468		0.0091
15	0.0988	0.1343			0.0394	0.0521	0.0501		0.0162
16						0.0588	0.0394		0.0163
17						0.0679	0.1343		0.0679
18						0.0988	0.0942		0.1343

Table 14. MLE for the SRS and RSS of the COVID-19 data.

<i>n</i>		SRS		RSS <i>c</i> = 1		RSS <i>c</i> = 3	
		Estimates	SE	Estimates	SE	Estimates	SE
15	α	3.9663	60.8494	10.7857	31.5371	2.4827	1.1644
	γ	0.7396	1.6007	0.3747	0.0420	1.0101	0.0394
	λ	0.0653	0.0838	0.5989	0.0731	0.0251	0.0009
18	α	3.2701	12.0356	3.1813	1.2717	1.8800	0.1719
	γ	0.7406	0.5593	0.8160	0.0386	1.5685	0.0279
	λ	0.0488	0.0153	0.0427	0.0012	0.0109	0.00005

Table 15. Data summarized.

Data	Min	Q1	Median	Mean	Q3	Max	σ^2	γ_1	γ_2
I	19.885	30.335	40.400	51.495	61.335	185.560	1048.259	2.069	7.951
II	0.002	0.014	0.027	0.036	0.046	0.178	0.001	2.021	8.176
III	0.312	1.098	1.478	1.451	1.773	2.585	0.245	−0.028	2.941

The MLE for the various models with various statistics of GFS for the carbon fibre data are discussed in Table 16 and Figure 8. Figure 9 displays the fitted CDF using an empirical CDF in I and the fitted PDF using carbon fibre data histograms in II, a QQ plot in III, and a PP plot for HL-INH in IV. Figure 10 shows the profile likelihood for the parameters of the HL-INH distribution based on the carbon fibre data. We confirm that the HL-INH distribution is the best model for the fitting of the carbon fibre data, and the estimates have uniqueness and a maximum point of log likelihood.

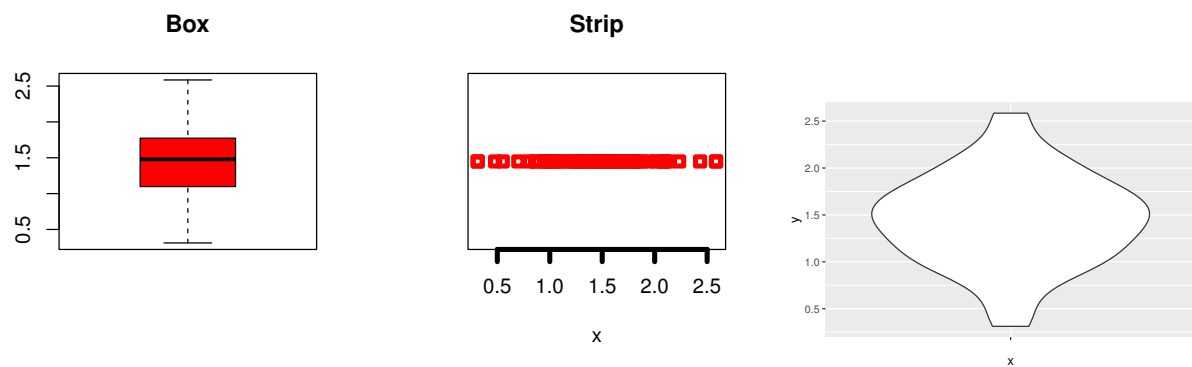


Figure 8. Data description for the carbon fibre data using box, strip, and violin plots.

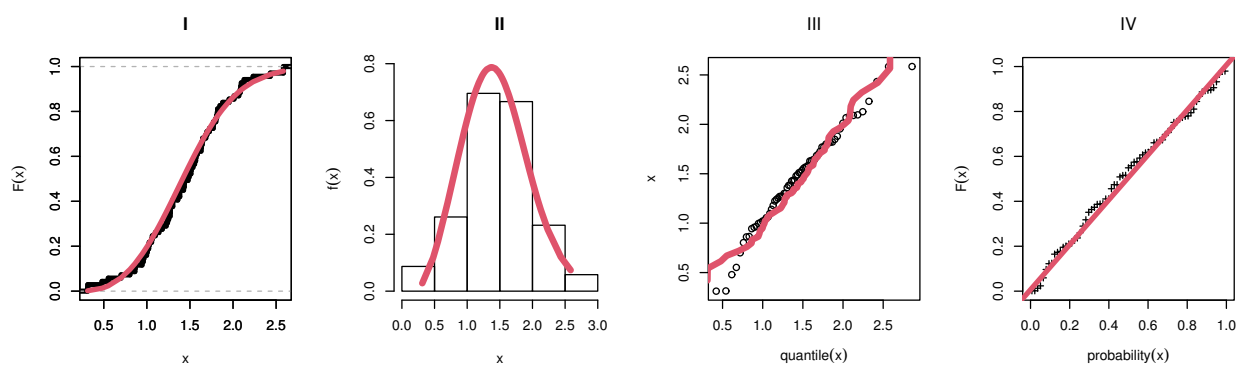


Figure 9. Fitting of (I) ecdf, (II) epdf, (III) QQ and (IV) PP plots for the carbon fibre data.

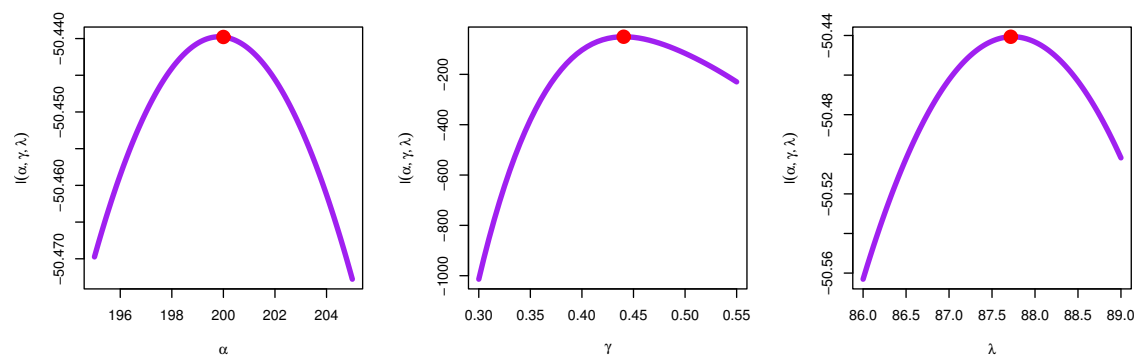


Figure 10. Profile likelihood α , γ , and λ for the carbon fibre data.

Table 16. MLE for each model with different measures: carbon fibre data.

		Estimates	SE	M1	M2	M3	M4	M5	M6	M7	M8
HLINH	α	199.0004	27.8633								
	γ	0.4405	0.0814	106.8813	107.2505	113.5836	109.5403	0.0546	0.4284	0.0614	0.9574
	λ	87.7180	9.2797								
OLINH	α	270.4222	5.1566								
	β	2.2928	0.1464	113.8216	114.4466	122.7580	117.3670	0.1068	0.7676	0.1020	0.4697
	λ	40.2990	7.3892								
PINH	θ	0.5306	0.0288								
	α	0.1867	0.0325	113.8216	114.4466	122.7580	117.3670	0.1068	0.7676	0.1020	0.4697
	β	82.1760	61.6231								
INH	γ	5.9997	0.9364								
	γ	3.6078	2.0237	185.1993	185.3811	189.6675	186.9720	0.8592	5.1773	0.3176	0.0000
	λ	0.2402	0.1612								
MOINH	α	0.6019	0.0488								
	β	30.3594	9.3200	135.2326	135.6019	141.9350	137.8917	0.4046	2.5856	0.1303	0.1917
	γ	0.0037	0.0012								
EL	α	9.1832	2.2274								
	β	41.5432	34.0397	120.7969	121.1661	127.4992	123.4559	0.2216	1.4964	0.1134	0.3374
	θ	20.9149	18.1903								
GL	α	88.4688	85.2253								
	β	14.5011	11.9328	119.7791	120.4041	128.7155	123.3245	0.1903	1.2878	0.1002	0.4926
	θ	2.6888	1.0941								
	λ	13.5403	3.1472								

Table 17. SRS and RSS for the carbon fibre data.

i	SRS	RSS				
		c = 1	c = 1	c = 2	c = 3	c = 4
1	0.861	0.312	0.865			
2	0.861	0.552	0.944			
3	0.861	0.865	1.14			
4	0.944	0.865	1.24			
5	0.944	0.861	1.684			
6	0.944	1.24	1.697			
7	1.021	1.021	1.818			
8	1.021	1.098		0.803		
9	1.027	1.27		0.552		
10	1.253	1.274		1.301		
11	1.272	1.434		1.24		
12	1.274	1.382		2.128		
13	1.301	1.382		1.88		
14	1.382	1.382		2.233		
15	1.426	1.511			0.7	
16	1.49	1.535			1.224	
17	1.566	1.426			1.274	
18	1.57	1.77			1.478	
19	1.57	1.514			1.648	
20	1.586	1.809			1.818	
21	1.586	1.697			2.067	
22	1.726	1.77				0.803
23	1.77	2.012				0.7
24	1.8	1.88				1.24
25	1.848	2.084				1.514
26	2.012	2.096				1.848
27	2.433	1.88				1.697
28	2.585	2.585				2.09

Table 18. MLE for the SRS and RSS of the carbon fibre data.

<i>n</i>		SRS		RSS <i>c</i> = 1		RSS <i>c</i> = 3	
		Estimates	SE	Estimates	SE	Estimates	SE
28	α	204.1923	28.1256	250.0969	25.1562	265.0768	15.1517
	γ	0.5084	0.1055	0.4554	0.0304	0.4422	0.0217
	λ	50.0574	9.1026	84.3680	5.1517	94.8756	2.2562

10. Conclusions

In this paper, we suggested and studied the HL-INH distribution, a new extension of the INH distribution. The shape forms of the PDF for the HL-INH distribution for numerous parameter values are asymmetric. Some useful expansions for the HL-INH distribution were computed. Four alternative entropy measurements were discussed. Several statistical and computational aspects of the HL-INH distribution were computed. Using the SRS and RSS methods, the parameters for the HL-INH distribution were estimated using the ML technique. A simulation analysis was carried out to demonstrate that RSS outperforms SRS. The HL-INH distribution is more flexible than the INH distribution and other distributions such as the OLINH, PINH, INH, EOWINH, MOINH, EW, EL, GL, and WL distributions. The HL-INH distribution provides the greatest fit for the three datasets. The limitation of this paper is that we only studied the statistical inference of the parameters for the HL-INH model using the ML method of estimation under ranked set sampling. For future works, authors can study the statistical inference of the parameters for the new suggested model using different classical and Bayesian estimation approaches; furthermore, they can study the reliability analysis of the new suggested model utilizing different censored schemes.

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